Entropy of Automorphisms of AF Algebras

By

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Here we consider a notion of topological entropy for automorphisms of AF algebras, based on a corresponding measure theoretic entropy of Connes and Størmer [6] for automorphisms of hyperfinite von Neumann algebras with an invariant trace. We show how to compute the entropy of the shift on certain AF algebras associated with topological Markov chains.

If $\mathscr{A}$ is a unital AF algebra, let $T(\mathscr{A})$ denote the normalized traces on $\mathscr{A}$. If $\phi \in T(\mathscr{A})$, and B is a finite dimensional C^*-subalgebra of $\mathscr{A}$, let E_B^ϕ denote the conditional expectation of $\mathscr{A}$ onto B, relative to ϕ. For simplicity, we will always only consider finite dimensional subalgebras, with the same unit as $\mathscr{A}$. If $n \in N$, let S_n denote the maps x from Z^n into $\mathscr{A}_+$ with finite support such that $\sum_{\alpha_i \in Z} x(\alpha_1, ..., \alpha_n) = 1$. For $1 \le l \le n$, $x \in S_n$, $\alpha \in Z$, put

$$x^l(\alpha) = \sum_{\alpha_1, ..., \alpha_{l-1}, \alpha_{l+1}, ..., \alpha_n} x(\alpha_1, ..., \alpha_{l-1}, \alpha, \alpha_{l+1}, ..., \alpha_n).$$

Also, let $\eta(x) = -x \log x$ for $x \in [0, \infty)$. Let $B_1, ..., B_n$ be finite dimensional subalgebras of $\mathscr{A}$. Then following [6], define:

$$H(B_1, ..., B_n) = \sup_{\phi \in T(A)} \sup_{x \in S_n} \{ \sum_{\alpha_i \in Z} \eta \phi x(\alpha_1, ..., \alpha_n)$$

$$- \sum_{l=1}^n \sum_{\alpha \in Z} \phi \eta E_{B_l}^\phi(x^l(\alpha)) \}.$$

The proofs of [6] carry over to give that H satisfies:

(A) $H(A_1, ..., A_n) \le H(B_1, ..., B_n)$ if $A_j \subseteq B_j$, $j = 1, ..., n$.

(B) $H(B_1, ..., B_n, B_{n+1}, ..., B_{n+m}) \le H(B_1, ..., B_n) + H(B_{n+1}, ..., B_{n+m})$.

(C) $B_1, ..., B_n \subseteq B \Rightarrow H(B_1, ..., B_n, B_{n+1}, ..., B_{n+m}) \le H(B, B_{n+1}, ..., B_{n+m})$

(D) For any family of minimal projections of B, $(e_\alpha)_{\alpha \in I}$ such that $\sum e_\alpha = 1$, one has $H(B) = \sup_{\phi \in T(A)} (\sum_{\alpha \in I} \eta \phi(e_\alpha))$.

(E) If $C^*(B_1, ..., B_n)$, the C^*-algebra generated by $B_1, ..., B_n$ is generated by

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pairwise commuting C^*-subalgebras P_j of B_j, then $H(B_1,\ldots,B_n)=H(C^*(B_1,\ldots,B_n))$.

If σ is an automorphism of $\mathscr{A}$, and B a finite dimensional C^*-subalgebra of $\mathscr{A}$, let $H(B,\sigma)=\lim_{k\to\infty} H(B,\sigma B,\ldots,\sigma^k B)/k$. Then define the (topological) entropy of σ to be $H(\sigma)=\sup H(B,\sigma)$, where the supremum is taken over all finite dimensional subalgebras B of $\mathscr{A}$. If $\mathscr{A}$ is also commutative, i.e. $\mathscr{A}=C(X)$, where X is a totally disconnected compact Hausdorff space, and if B is a finite dimensional subalgebra with minimal projections $(e_\alpha)_{\alpha\in I}$, and $\sum e_\alpha=1$, then by D, $H(B)=\sup_{\phi\in T(A)}\{\sum_\alpha \eta\phi(e_\alpha)\}=\log|I|$. It then becomes clear from E that if σ is an automorphism induced by a homeomorphism $\hat\sigma$ of X, then $H(\sigma)$ is the same as the classical topological entropy of $\hat\sigma$, [1]. Moreover, suppose σ is an automorphism of an AF algebra $\mathscr{A}$ with invariant trace ϕ. Then let (π,H,ξ) be the GNS decomposition of ϕ, and let $\tilde\sigma$ be the induced automorphism of $\tilde{\mathscr{A}}=\pi(\mathscr{A})''$, with invariant faithful vector trace $\tilde\phi(\cdot)=<(\cdot)\xi,\xi>$. Then it follows from the Kolmogorov-Sinai theorem of [6] that $H(\sigma)\geq H_{\tilde\phi}(\tilde\sigma)$, the latter being the measure-theoretic entropy of $\tilde\sigma$ on $\tilde{\mathscr{A}}$ with respect to $\tilde\phi$.

Proposition 1. *If $\mathscr{A}=\overline{\cup B_n}$, with $B_n\subseteq B_{n+1}$ finite dimensional C^*-subalgebras, and σ an automorphism of A, then $H(\sigma)=\lim_{n\to\infty} H(B_n,\sigma)$.*

Proof. Define for B and C finite dimensional C^*-subalgebras of $\mathscr{A}$:

$$H(B|C)=\sup_{\phi\in T(A)}\sup_{x\in S_1}\left(\sum_i\{\phi\eta E_C^\phi x(i)-\phi\eta E_B^\phi(x(i))\}\right).$$

Say that $B\overset{\delta}{\subset}C$ for $\delta>0$, if for each b in B, there exists c in C such that $\|b-c\|\leq\delta\|b\|$. Claim that if B is fixed and $\varepsilon>0$, there exists $\delta>0$, such that if $B\overset{\delta}{\subset}C$, then $H(B|C)<\varepsilon$. An examination of the proof of [6, Lemma 10] shows that if B is fixed and $\varepsilon>0$, then there exists $\varepsilon'>0$, such that if v is a unitary in $\mathscr{A}$ with $\|v-1\|<\varepsilon'$, then $H(B|vBv^*)<\varepsilon$. Now by [5, Theorem 6.4], given $\varepsilon'>0$, there exists $\delta>0$ such that if $B\overset{\delta}{\subset}C$, then there exists a unitary v in $\mathscr{A}$ such that $vBv^*\subseteq C$ and $\|v-1\|<\varepsilon'$. Thus for $B\overset{\delta}{\subset}C$,

$$H(B|C)\leq H(B|vBv^*)+H(vBv^*|C) \quad\text{from definitions}$$
$$\leq H(B|vBv^*) \quad\text{as H is increasing in first variable}$$
$$<\varepsilon.$$

Thus if B is any finite dimensional C^*-subalgebra of $\mathscr{A}$, choose B_n such that $B\overset{\delta}{\subset}B_n$, $H(B|B_n)<\varepsilon$, and

$$H(B,\sigma)=\lim_k\frac{1}{k}H(B,\sigma B,\ldots,\sigma^k B)$$

$$\leq \lim_k \frac{1}{k} H(B_n, \sigma B_n, \ldots, \sigma^k B_n) + \varlimsup_k \frac{1}{k} \sum_{j=0}^k H(\sigma^j(B)|\sigma^j(B_n))$$

<div align="right">from definitions</div>

$$\leq H(B_n, \sigma) + \varepsilon.$$

The proposition follows.

Let A be an aperiodic $n \times n$ matrix with entries in $\{0, 1\}$ and let $\Sigma = \{1, 2, \ldots, n\}$. Let $\mathscr{F}^A$ be the associated two sided unital AF algebra with dimension group $\underrightarrow{\lim} (\mathbf{Z}^{n^2}, A \times A^t)$ and dimension range

$$\lim_{m \to \infty} \Gamma_m, \text{ where } \Gamma_m = \{x \in \mathbf{Z}^{n^2};\ 0 \leq x \leq (A \times A^t)^m(A)\},$$

and $\mathbf{Z}^{n^2}$ is given the simplicial ordering (see [10, 7, 9]). Thus let X_A denote the compact space $\{(x_i) \in \Sigma^{\mathbf{Z}}: A(x_i, x_{i+1}) = 1\}$, and $\mathscr{G}$ be the group of all uniformly finite dimensional homeomorphisms of X_A which change only finitely many coordinates. Let $\mathscr{A}$ denote the C^*-crossed product $C^*(X_A, \mathscr{G})$ and let U denote the canonical representation of $\mathscr{G}$ in $\mathscr{A}$. If u is a uniformly finite dimensional homeomorphism of a compact open subset B onto a compact open subset C of X_A, let $\tilde{u}$ denote the corresponding partial isometry in $\mathscr{A}$ with initial projection $P(B)$, the characteristic function of B in $C(X_A)$, and final projection $P(C)$. Let $\mathscr{I}$ denote the ideal generated by $U(g)P(B) - U(h)P(B)$, where B is a compact open subset of X_A and where $g, h \in \mathscr{G}$ agree on $\mathscr{B}$. Then let $\mathscr{F}^A = \mathscr{A}/\mathscr{I}$. The shift σ on X_A, normalises $\mathscr{G}$ and induces an automorphism of $\mathscr{F}^A$ also denoted by σ. The AF algebra $\mathscr{F}^A$ can be described as follows:

Let $\mathscr{M}_{rs}^{ij} = \{(i_{-r}, \ldots, i_{-1}, i_0, \ldots, i_s): i_k \in \Sigma,\ A(i_k, i_{k+1}) = 1,\ i_{-r} = i,\ i_s = j\}$ for $i, j \in \Sigma$, $r, s \in \mathbf{N}$. Let $\mathscr{M}_{rs} = \bigcup_{i,j} \mathscr{M}_{rs}^{ij}$, $\mathscr{M}_s^{ij} = \mathscr{M}_{ss}^{ij}$, $\mathscr{M}_s = \mathscr{M}_{ss}$. If $\mu \in \mathscr{M}_{rs}^{ij}$, let $Z(\mu) = \{x \in X_A: (x_i)_{-r}^s = \mu\}$. If $\mu, \nu \in \mathscr{M}_{rs}^{ij}$, let $u_{\mu\nu}$ denote the u.f.d. homeomorphism of $Z(\nu)$ onto $Z(\mu)$ given by $(u_{\mu\nu}(x))_i = x_i$ if $i < -r$, or $i > s$, and put $e_{\mu\nu} = \tilde{u}_{\mu\nu}$, $e_\mu = e_{\mu\mu}$. Then $\{e_{\mu\nu}: \mu, \nu \in \mathscr{M}_s^{ij}\}$ are matrix units and generate a finite dimensional matrix algebra M_s^{ij}. Then if $M_s = \bigoplus_{i,j} M_s^{ij}$, $M_s \subseteq M_{s+1}$, $\mathscr{F}^A = \overline{\bigcup_s M_s}$, and the embedding of M_s in M_{s+1} is given by $A \times A^t$. The gauge action of $\mathbf{T}^n$ on $\mathscr{F}^A$ is defined as follows. If $t = (t_i)_{i=1}^n \in \mathbb{C}^n$, and $\mu = (i_{-s}, \ldots, i_s) \in \mathscr{M}_s$ let $t_\mu = t_{i_{-s}} \ldots t_{i_s} \in \mathbb{C}$. Then let $\mathbf{T}^n$ act on $\mathscr{F}^A$ by $t: e_{\mu\nu} \to t_\mu \overline{t_\nu} e_{\mu\nu}$, for $\mu, \nu \in \mathscr{M}_s$, $t \in \mathbf{T}^n$. Let $\mathscr{C}^A$ denote the gauge invariant or observable algebra $(\mathscr{F}^A)^{\mathbf{T}^n}$, and let σ_0 denote the restriction of σ to $\mathscr{C}^A$. Then $\mathscr{C}^A$ is an AF algebra, which can be described as follows [3, 4]: If $(r_i)_{i=1}^n \in \mathbf{N}^n$, with $\sum_{i=1}^n r_i = 2s+1$, let $\Lambda_{ij}^s(r_1, \ldots, r_n)$ denote the set of μ in $\mathscr{M}_s^{ij}$, such that k in Σ occurs in μ exactly r_k times.

Then let $N^s_{ij}(r_1,\ldots,r_n)$ denote the matrix algebra generated by $\{e_{\mu\nu}\colon \mu, \nu \in \Lambda^s_{ij}(r_1,\ldots,r_n)\}$, and put $N_s = \bigoplus\limits_{\substack{i,j \\ r_1,\ldots,r_n}} N^s_{ij}(r_1,\ldots,r_n)$. Then $N_s = (M_s)^{\mathrm{T}^n}$, $N_s \subseteq N_{s+1}$, and $\mathscr{C}^A = \overline{\bigcup\limits_s N_s}$.

We first show the following which may be of independent interest. The use of the theory of the chemical potential for the GICAR algebra was first advocated by G. A. Elliott.

Proposition 2. *If ϕ is a σ_0 invariant, faithful extremal tracial state on $\mathscr{C}^A$, then there exist ρ_i, u_i, v_i, θ in $(0, \infty)$ with $\sum \rho_r A(r, i)u_r = \theta u_i$, $\sum A(j, t)\rho_t v_t = \theta v_j$ such that*

(F) $\phi(e_{\mu\nu}) = \rho_\mu u_i v_j / \theta^{-2s}\delta_{\mu\nu}$, *for $\mu, \nu \in \Lambda^s_{ij}(r_1,\ldots,r_n)$ where $\rho_\mu = \rho_1^{r_1}\rho_2^{r_2}\cdots\rho_n^{r_n}$.*

Proof. It is easy to check that σ is norm asymptotically abelian on $\mathscr{F}^A$, i.e. $\lim\limits_{m\to\infty} \|\sigma^m(a)b - b\sigma^m(a)\| = 0$, for all a, b in $\mathscr{F}^A$. Hence by [2], there exist $h_1,\ldots,h_n$ in $\mathbf{R}$ such that if $\gamma(t) = (\exp(ih_jt))^n_{j=1} \in T^n$, for $t \in \mathbf{R}$, then ϕ is the restriction of a state ψ on $\mathscr{F}^A$ which is KMS with respect to the gauge action $\{\gamma(t)\colon t \in \mathbf{R}\}$ on $\mathscr{F}^A$ at inverse temperature $+1$. By uniqueness of KMS states on full matrix algebras, there exist scalars $a^s_{ij} > 0$ such htat

$$\psi e_{\mu\nu} = \rho_\mu a^s_{ij}\delta_{\mu\nu}, \qquad \text{for} \quad \mu, \nu \in \mathscr{M}^{ij}_s$$

and where $(\rho_i) = (\exp(-h_i)) \in (0, \infty)^n$. Then by considering the embedding of M_s in M_{s+1},

$$\sum_{r,t} A(r, i)A(j, t)\rho_r\rho_t a^{s+1}_{rt} = a^s_{ij}.$$

The $n^2 \times n^2$ matrix $\{A(r, i)A(j, t)\rho_r\rho_t\colon (i, j), (r, t) \in \Sigma^2\}$ is irreducible, and so by the theorem of Perron-Frobenius, it possesses a strictly positive eigenvector, say $\{b(i, j)\colon (i, j) \in \Sigma^2\}$ which is unique up to scalar multiplication. If ρ denotes the diagonal matrix $[\rho_i\delta_{ij}]$, let θ denote the spectral radius of ρA which is also the spectral radius of $A\rho$, as $\sigma(\rho A) \cup \{0\} = \sigma(A\rho) \cup \{0\}$. Then let (u_i) and (v_i) be the (essentially) unique strictly positive eigenvectors satisfying $\sum \rho_r A(r, i)u_r = \theta u_i$, $\sum A(j, t)\rho_t v_t = \theta v_j$. Then $\sum\limits_{r,t} A(r, i)A(j, t)\rho_r\rho_t u_r v_t = \theta^2 u_i v_j$. Thus by uniqueness, and rescaling if necessary, $b(i, j) = u_i v_j$. Then by [9, Theorem 6.1], extended to the case of irreducible matrices, with not necessarily integral entries, it follows that $a^s_{ij} = b(i, j)/\theta^{-2s}$, after rescaling b if necessary. Thus $\psi(e_{\psi\nu}) = \rho_\mu u_i v_j / \theta^{-2s}\delta_{\mu\nu}$, for $\mu, \nu \in \mathscr{M}^{ij}_s$.

Theorem. $H(\sigma_0) = \log \lambda$, *where λ is the spectral radius of A.*

Proof.   Let $B = C^*(e_{\mu\nu} : e_{\mu\nu} \in \mathscr{C}^A, \ \mu, \ \nu \in \mathscr{M}^{ij}_{s,s+k}, \ (i,j) \in \Sigma^2)$ for $s, \ k \in \mathbb{N}$.
Then $\sigma_0^i(N_s) \subset B$ for $i = 0, 1, 2, \ldots, k$. Then

$$H(N_s, \sigma_0(N_s), \ldots, \sigma_0^k(N_s)) \leq H(B) \qquad \text{by} \quad C$$

$$= \sup_{\phi \in T(\mathscr{C}^A)} \{ \textstyle\sum \eta\phi(e_\mu) : \mu \in \mathscr{M}_{s,s+k} \} \quad \text{by} \quad D$$

$$\leq \log |\mathscr{M}_{s,s+k}| = \log \left(\textstyle\sum_{i,j} A^{2s+k+1}(i,j) \right).$$

Hence $H(N_s, \sigma) \leq \lim\limits_{k \to \infty} \dfrac{1}{k} \log \left(\sum A^{2s+k+1}(i,j) \right) = \log \lambda$,   by   a   computation   [8,
p. 128] of the classical entropy of σ on X_A.   Taking the partition of unity given
by

$$\mu_0, \mu_1, \ldots, \mu_k \longrightarrow e_{\mu_0}\sigma_0(e_{\mu_1}) \cdots \sigma_0^k(e_{\mu_k}), \quad \mu_i \in \mathscr{M}_s,$$

we have

$$H(N_s, \sigma_0(N_s), \ldots, \sigma_0^k(N_s)) \geq \sup_{\phi \in T(C^A)} \{ \sum_{\mu_j} \eta\phi(e_{\mu_0}\sigma_0(e_{\mu_1}) \cdots \sigma_0^k(e_{\mu_k})) \}$$

$$= \sup_{\phi \in T(C^A)} \{ \sum_\mu \eta\phi(e_\mu) : \mu \in \mathscr{M}_{s,s+k} \}.$$

Let ϕ_0 be the (normalised) trace on $\mathscr{C}^A$ given by F, with $\rho_i = 1/n$, $i \in \Sigma$, and
where $u_i, v_i > 0$ satisfy $\sum A(r,i)u_r = \lambda u_i$, $\sum A(j,t)v_t = \lambda v_j$, and where λ is the
spectral radius of A.   Then again by considering the entropy of σ on X_A, with
respect to the invariant measure $\phi_0|_{\mathscr{C}(X_A)}$ [8, p. 69, 129] it is seen that

$$H(N_s, \sigma) \geq \varlimsup_k \frac{1}{k} \left(\sum_n \eta\phi_0(e_\mu) \right) \text{ (summation over } \mu \in \mathscr{M}_{s,s+k}) = \log \lambda.$$

The result follows from Proposition 1.

Remark 1.   The theorem shows that the entropy of σ_0 is the same as the
classical entropy of its restriction to X_A [11].   Similarly, $H(\sigma) = \log \lambda$.

Remark 2.   Consider the case of the full n-shift, where $A(i,j) \equiv 1$.   Then
every extremal trace corresponds to $\Delta = \{(\rho_i)^n_{i=1} = \rho_i \geq 0, \ \sum \rho_i = 1$ and all are invar-
iant under σ_0.   If $(\rho_i) \in \Delta$, the corresponding trace is given by $e_{\mu\nu} \to \rho_\mu \delta_{\mu\nu}$.   This
is essentially a classical Hausdorff moment problem.   Given $\{a^s_{r_1 \ldots r_n} \geq 0 : s = 1, 2,$
$\ldots, r_i \in \mathbb{N}, \ \sum\limits_{i=1}^n r_i = s\}$ such that $a^s_{r_1 \ldots r_n} = \sum\limits_{i=1}^n a^{s+1}_{r_1 \ldots r_i+1 \ldots r_n}$, there exists a finite
measure θ on Δ such that $a^s_{r_1 \ldots r_n} = \int_\Delta \rho_1^{r_1} \cdots \rho_n^{r_n} d\theta(\rho_1, \ldots, \rho_n)$.   This can be proved
by classical methods (see e.g. [13, Theorems 1.5, 1.6] for the cases $n = 2, 3$) or
alternatively by looking at $K_0(\mathscr{C}^A)$.   Now $K_0(\mathscr{C}^A)$ can be identified with the
ring $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ of polynomials in indeterminates $t_1, \ldots, t_{n-1}$ with integer
coefficients, such that a non-zero p in $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ lies in the positive cone if

and only if $p(\rho_1, \ldots, \rho_{n-1}) > 0$ whenever $(\rho_1, \ldots, \rho_{n-1}, 1 - \sum_{i=1}^{n-1} \rho_i)$ lies in the interior of Δ, [12]. Then if ϕ is the trace on $\mathscr{C}^A$ corresponding to $(\rho_1, \ldots, \rho_n) \in \Delta$, it follows as in the above theorem that $H_{\tilde\phi}(\tilde\sigma_0) = -\sum_{i=1}^{n} \rho_i \log \rho_i$, (cf. the Bernoulli shift defined by $\{\rho_1, \ldots, \rho_n\}$ in the sense of [6, Definition 2]). Thus $H_{\tilde\phi}(\tilde\sigma_0) < H(\sigma_0)$, unless $\rho_i \equiv 1/n$, or $\phi = \phi_0$. Thus by convexity, $H_{\tilde\phi}(\tilde\sigma_0) < H(\sigma_0)$, for any trace ϕ on $\mathscr{C}^A$ distinct from ϕ_0. Hence ϕ_0 is the unique trace with maximal entropy. In the more general situation of the main theorem, it can be shown that $H_{\tilde\phi}(\tilde\sigma_0) < H(\sigma_0)$ if ϕ is any trace satisfying the conditions of Proposition 2, unless $\phi = \phi_0$, the canonical trace used to compute the topological entropy.

Remark 3. Let $A(l_2) = C^*(a(h): h \in l_2)$ be the CAR algebra generated by a representation a of the canonical anti-commutation relations of the complex Hilbert space l_2. Let u denote the two-sided shift on l_2 and θ be the associated Bogoliubov automorphism of $A(l_2)$, and θ_0 its restriction to the current algebra. Then as in the proof of the Theorem, $H(\theta) = \log 2 = H(\theta_0)$, but it is unclear whether θ, (respectively θ_0) is conjugate to the shift on $\overset{\infty}{\underset{-\infty}{\otimes}} M_2$ (respectively the restriction of the shift to $(\overset{\infty}{\underset{-\infty}{\otimes}} M_2)^{\mathbf{T}^2}$, with $\mathbf{T}^2$ acting by the usual product action).

Acknowledgements

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