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Quasi-Periodic Solutions of the Orthogonal KP Equation

—Transformation Groups for Soliton Equations V—

By

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§0. Introduction

In this note we study quasi-periodic solutions of the BKP hierarchy introduced in [1]. Our main result is the Theorem in Section 2, which states that quasi-periodic τ-functions for the BKP hierarchy are the theta functions on the Prym varieties of algebraic curves admitting involutions with two fixed points.

The rational and soliton solutions of the BKP hierarchy were studied in part IV [2] together with its operator formalism. We also showed that the BKP hierarchy is the compatibility condition for the following system of linear equations for $w(x)$, $x=(x_1, x_3, x_5,\ldots)$;

$$\frac{\partial w}{\partial x_l}=B_l w, \qquad l=1, 3, 5,\ldots \tag{1}$$

where B_l is a linear ordinary differential operator with respect to x_1 without constant term.

$$B_l=\frac{\partial^l}{\partial x_1^l}+\sum_{m=1}^{l-2} b_{lm}(x)\frac{\partial^m}{\partial x_1^m}.$$

One of the specific properties of the BKP hierarchy is the fact that squares of τ-functions for the BKP hierarchy are τ-functions for the KP hierarchy with $x_{2j}=0$.

Now we explain why the Prym varieties and the theta functions on them appear in our present study.

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One derivation is given through the examination of the geometrical properties of the wave functions associated with soliton solutions. We regard these functions as defined on rational curves with double points. The curves in this case have involutions with two fixed points. The divisors of the wave functions belong to the Prym varieties. This is an immediate consequence of the fact that the wave functions for the BKP hierarchy admit time evolutions only with respect to x_{odd}. Further we find that the pole divisor of the wave function belongs to a translation of the Prym variety which is tangent to the theta divisor in the Jacobian variety. This is a reflection of the above mentioned fact $\tau_{BKP}(x)^2 = \tau_{KP}(x)|_{x_2=x_4=\cdots=0}$.

On the other hand, by the result of Krichever [3], quasi-periodic τ-functions for the KP hierarchy are the theta functions on the Jacobian varieties of arbitrary curves (Riemann's theta functions). Therefore the quasi-periodic BKP τ-function must be the square root of Riemann's theta function. Such a function appears in connection with the Prym variety. If the relevant curve has an involution with two fixed points, then Riemann's theta function is known to reduce to the square of the theta function on the associated Prym variety when its arguments are restricted to a translation of the Prym part (see, for example, [4]). This fact agrees with our observation for the soliton case.

Prym varieties are mentioned by several authors [5] [6] in their study of quasi-periodic solutions of soliton type equations. But the location of the pole divisors in their work differs from what we have described above for the BKP hierarchy. This seems to be the reason why in their work the theta functions on the Prym varieties do not appear.

Section 1 is devoted to the study of wave functions associated with soliton solutions. Here we rephrase the linear constraint (1) for the wave functions in terms of their pole divisors. In Section 2, we construct quasi-periodic wave functions for the BKP hierarchy by using the theory of abelian integrals. An explicit formula is given in terms of the theta functions on the Prym varieties.

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§ 1. Construction on the Rational Curve

Before proceeding to the construction of quasi-periodic solution of the BKP hierarchy, let us first examine the geometrical meaning of known soliton solutions. In the notation of [2] the N soliton τ function is

$$\tau(x)=\sum_{r=0}^{N}\sum_{i_1<\cdots<i_r} a_{i_1}\cdots a_{i_r}c_{i_1\cdots i_r}e^{\xi_{i_1}+\cdots+\xi_{i_r}}$$
$$=1+\sum_{i=1}^{N} a_i e^{\xi_i}+\sum_{i<j} a_i a_j c_{ij} e^{\xi_i+\xi_j}+\cdots,$$

(2) $$\xi_i=\tilde{\xi}(x;p_i)+\tilde{\xi}(x;q_i),\quad \tilde{\xi}(x;k)=\sum_{j>0,odd} x_j k^j$$

$$c_{i_1\cdots i_r}=\prod_{1\leq\mu<\nu\leq r} c_{i_\mu i_\nu},\quad c_{ij}=\frac{(p_i-p_j)(p_i-q_j)(q_i-p_j)(q_i-q_j)}{(p_i+p_j)(p_i+q_j)(q_i+p_j)(q_i+q_j)},$$

while the associated wave function reads

(3) $$w(x;k)=\tau_{[k]}(x)/\tau(x).$$

Here, by definition, $\tau_{[k]}(x)e^{-\tilde{\xi}(x,k)}$ is obtained from (2) by replacing the parameters a_i by $a_i\dfrac{k-p_i}{k+p_i}\dfrac{k-q_i}{k+q_i}$. For geometrical interpretation we prefer to modify (3) as

$$v(x;k)=\prod_{i=1}^{N}(k+p_i)(k+q_i)\cdot\prod_{\nu=1}^{2N}(k-c_\nu)^{-1}\cdot w(x;k),$$

requiring

(4) $$v(x;p_i)=v(x;-q_i),\ v(x;q_i)=v(x;-p_i),\quad i=1,\dots,N.$$

Let C (resp. $\bar{C}$) denote the rational curve with the coordinate k (resp. k^2), obtained by identifying the points $k=p_i$ with $k=-q_i$ and $k=q_i$ with $k=-p_i$ (resp. $k^2=p_i^2$ with $k^2=q_i^2$), $i=1,\dots,N$. We have a double covering map $\pi\colon C\ni k\mapsto k^2\in\bar{C}$ and the involution of sheet change $\iota(k)=-k$. Condition (4) says that $v(x;k)$ is defined on the curve C. The Jacobian varieties for these singular curves (with the multiplicative group law) are $J=GL(1)^{2N}$ and $\bar{J}=GL(1)^N$, respectively. The Abel map is defined to be

(5) $$\mathscr{A}\colon C\longrightarrow J,\ k\longmapsto(\alpha_1(k),\dots,\alpha_N(k),\beta_1(k),\dots,\beta_N(k))=(\alpha(k),\beta(k))$$
$$\alpha_i(k)=\frac{k-p_i}{k+q_i}=\beta_i(-k)^{-1},\quad \beta_i(k)=\frac{k-q_i}{k+p_i}=\alpha_i(-k)^{-1}.$$

The natural extension of (5) to the r-th symmetric product $\mathfrak{S}^rC\to J$, $(k_1,\dots,k_r)\mapsto(\prod_{\nu=1}^{r}\alpha_1(k_\nu),\dots,\prod_{\nu=1}^{r}\alpha_N(k_\nu),\prod_{\nu=1}^{r}\beta_1(k_\nu),\dots,\prod_{\nu=1}^{r}\beta_N(k_\nu))$ will also be denoted by $\mathscr{A}$.

The involution ι carries over to J as $\iota(\alpha_1,\dots,\alpha_N,\beta_1,\dots,\beta_N)=(\beta_1^{-1},\dots,\beta_N^{-1},\alpha_1^{-1},\dots,\alpha_N^{-1})$. The "even" and "odd" parts of J under ι are the Jacobian $\bar{J}$ of $\bar{C}$ and the "Prym variety" P respectively:

$$\bar{J}=\{(\alpha,\beta)\in J\mid\alpha_1=\beta_1^{-1},\dots,\alpha_N=\beta_N^{-1}\}$$
$$P=\{(\alpha,\beta)\in J\mid\alpha_1=\beta_1,\dots,\alpha_N=\beta_N\}.$$

Setting $v(x;k)=\prod_{\nu=1}^{2N}\frac{(k-c_\nu(x))}{(k-c_\nu)}e^{\xi(x;k)}$ and writing (4) down we see immediately

$$\prod_{\nu=1}^{2N}\frac{\alpha_i(c_\nu(x))}{\alpha_i(c_\nu)}=\prod_{\nu=1}^{2N}\frac{\beta_i(c_\nu(x))}{\beta_i(c_\nu)}=e^{-\xi(x;p_i)-\xi(x;q_i)} \qquad (i=1,\dots,N). \tag{6}$$

Here we have used $\xi(x,-k)=-\xi(x,k)$. This (6) shows that the divisor $(c_1(x))+\cdots+(c_{2N}(x))-(c_1)-\cdots-(c_{2N})$ of $v(x;k)$ belongs to, and moves linearly within, the Prym variety P. Moreover if we rewrite (4) by using (3) and eliminating a_i's, we obtain the following relations among the poles $c_1,\dots,c_{2N}$ of $v(x;k)$.

$$\prod_{\nu=1}^{2N}\alpha_i(c_\nu)=\kappa_i\prod_{\nu=1}^{2N}\beta_i(c_\nu), \quad \kappa_i=\frac{p_i^2}{q_i^2}\prod_{j(\neq i)}\frac{(p_i^2-p_j^2)(p_i^2-q_j^2)}{(q_i^2-p_j^2)(q_i^2-q_j^2)} \qquad (i=1,\dots,N). \tag{7}$$

In other words the image $\mathscr{A}(\delta)$ of the pole divisor $\delta=(c_1)+\cdots+(c_{2N})$ belongs to the translation of the Prym variety

$$P'=\{(\alpha_1,\dots,\alpha_{2N})\in J\mid \alpha_1=\kappa_1\alpha_2,\dots,\alpha_{2N-1}=\kappa_N\alpha_{2N}\}.$$

The numbers κ_i are related to the canonical divisor K_C of C as $\mathscr{A}(K_C)=\left(-\frac{q_1}{p_1}\kappa_1,\ -\frac{p_1}{q_1}\kappa_1^{-1},\dots,\ -\frac{q_N}{p_N}\kappa_N,\ -\frac{p_N}{q_N}\kappa_N^{-1}\right)$. We have also $\mathscr{A}((0)\cdot(\infty))=\left(-\frac{p_1}{q_1},\ -\frac{q_1}{p_1},\dots,\ -\frac{p_N}{q_N},\ -\frac{q_N}{p_N}\right)$. Hence (7) is alternatively stated as a relation of divisors

$$\delta+\iota\delta=(0)+(\infty)+K_C. \tag{8}$$

Example ($N=1$). The configurations of $\bar{J}$, P and P' for $N=1$ are shown in Figure 1. The theta divisor $\Theta=\mathscr{A}(\mathfrak{S}^{2N-1}C)$ is given by $\Theta=\{(\alpha,\beta)\in J\mid(p-q)(\alpha\beta+1)+2q\alpha-2p\beta=0\}$. One can see explicitly that Θ is tangent to both P and P'. Correspondingly the zeros of the associated τ function $\tau_{KP}(x)$ *in the sense of the KP hierarchy* are all double when restricted on $x_2=x_4=\cdots=0$. This comes from the fact that $\tau_{KP}(x_1,0,x_3,0,\dots)=\tau_{BKP}(x_1x_3,\dots)^2$, where $\tau_{BKP}(x)$ is the τ function *in the sense of the BKP hierarchy* (IV [2]).

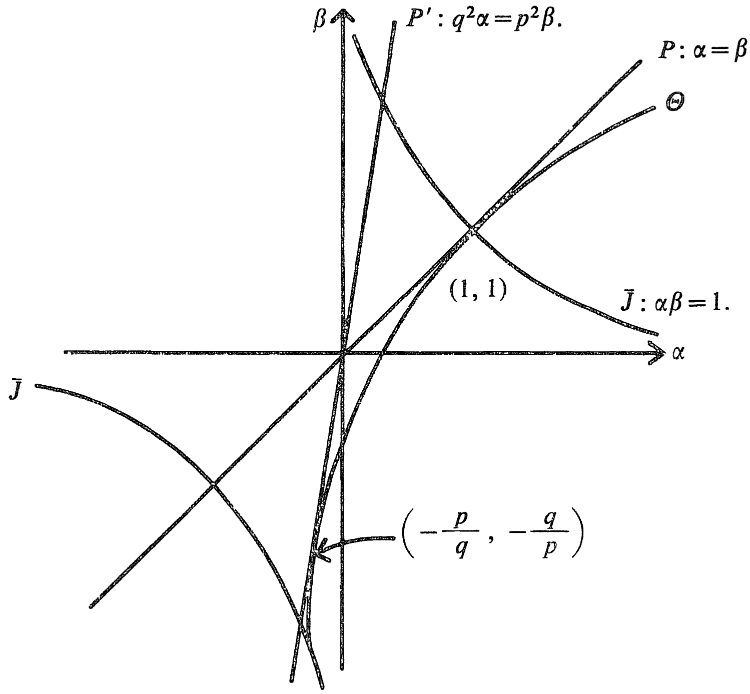

Figure 1.

On can also show directly from (8) that $v(x;k)$ satisfies the defining constraint (30) [2] for the BKP hierarchy. Set

$$\omega=\frac{\prod_{\nu=1}^{2N}(k^2-c_\nu^2)}{\prod_{i=1}^{N}(k^2-p_i^2)(k^2-q_i^2)}\frac{dk^2}{k^2}.$$

Then (7) implies $\mathrm{Res}_{k^2=p_i^2}\,\omega=-\mathrm{Res}_{k^2=q_i^2}\,\omega$. In other words there exists a 1-form ω on $\bar{C}$, or rather $\pi^*\omega$ on C, with zeros at $k=\pm c_\nu$ and simple poles at $k=0,\infty$ (This is what (8) means). Since $v(x;k)v(x;-k)\omega$ is holomorphic everywhere except at the poles $k=0,\infty$, the residue theorem yields

$$0=\sum_{k=0,\infty}\mathrm{Res}_k\,v(x;k)v(x;-k)\omega=(v(x;0)^2-1)\times\text{const.},$$

that is

$$v(x;0)=\pm 1.$$

This shows that the constant function ± 1 solves the linear equations $\left(\frac{\partial}{\partial x_n}-B_n(x;\partial)\right)v=0\left(\partial=\frac{\partial}{\partial x_1}\right)$, and hence the zeroth order term of $B_n(x;\partial)$

should be absent ([2]). The same argument will be employed in the construction of quasi-periodic solutions discussed in Section 2.

§2. Quasi-Periodic Solutions

In the KP hierarchy case, the construction of quasi-periodic solutions is done through the construction of wave functions (the Baker-Akhiezer functions) (cf. [3]). Here we proceed in the same manner. The point is to translate the condition (30) [2] on the BKP wave function to constraints on its pole divisor (9) below.

Let C be a non-singular algebraic curve of genus g admitting an involution ι with two fixed points q_0, q_∞ (which correspond to 0, ∞ in the preceding section). Take a local coordinate k^{-1} around q_∞ such that $k\circ\iota=-k$. Let δ be a positive divisor of degree g on C such that

$$\iota\delta+\delta=K_C+q_0+q_\infty \tag{9}$$

where K_C is the canonical divisor on C.

By standard arguments it can be shown that there exists a unique function $w(x, p)$, $x=(x_1, x_3,\ldots)$, $p\in C$ (a Baker-Akhiezer function) with the following properties:

i) w is meromorphic on $C-\{q_\infty\}$ and its pole divisor is δ,

ii) around q_∞, w behaves like

$$w(x, p)=(1+O(k^{-1}))\exp\big(\sum_{j>0,odd} k^j x_j\big).$$

Let ω be a differential of the third kind on C with the pole divisor $(\omega)=\delta+\iota\delta-q_0-q_\infty$. Then $w(x, p)w(x, \iota p)\omega$ is a meromorphic differential on C with poles only at q_0, q_∞. By using the residue theorem and noting that $w(x, p)w(x, \iota p)|_{p=q_\infty}=1$, we have

$$w(x, q_0)=\pm 1.$$

Therefore by the same reasoning as in Section 1, $w(x, p)$ is a wave function for the BKP hierarchy.

Next we express the Baker-Akhiezer function $w(x, p)$ in terms of Abelian integrals and the theta function on the Prym variety. For this purpose we need some general facts from the theory of Abelian integrals and theta functions. For details we refer to Fay [4].

By the Riemann-Hurwitz relation, the genus g of C is even, $g=2\bar{g}$. We can take a canonical homology basis α_j, β_j, $1 \leqq j \leqq g$, of C with the property $\iota\alpha_j + \alpha_{\bar{g}+j} = \iota\beta_j + \beta_{\bar{g}+j} = 0$, $1 \leqq j \leqq \bar{g}$. Let ω_j, $1 \leqq j \leqq g$, be the corresponding normalized basis of holomorphic 1-forms on C: $\int_{\alpha_j} \omega_i = \delta_{ij}$, $1 \leqq i, j \leqq g$. Then $\iota^*\omega_j + \omega_{\bar{g}+j} = 0$, $1 \leqq j \leqq \bar{g}$.

Put $\tau_{ij} = \int_{\beta_j} \omega_i$, $1 \leqq i, j \leqq g$, $\pi_{lm} = \int_{\beta_m} (\omega_l + \omega_{\bar{g}+l})$, $1 \leqq l, m \leqq \bar{g}$, then $T = (\tau_{ij})$ and $\Pi = (\pi_{lm})$ are symmetric matrices with positive definite imaginary parts. The Jacobian variety $J(C)$ of C is defined to be the complex torus $J(C) = \boldsymbol{C}^g/(I_g, T)$ where I_g denotes the identity matrix of order g. The Abel map $\mathscr{A}$: $C \to J(C)$ is defined by $\mathscr{A}(p) = (\mathscr{A}_1(p), \dots, \mathscr{A}_g(p))$, $p \in C$, $\mathscr{A}_j(p) = \int_{q_0}^{p} \omega_j$. We extend this map to the divisor group by linearity. The induced map ι: $J(C) \to J(C)$ is given by

$$\iota(u_1, \dots, u_g) = -(u_{\bar{g}+1}, \dots, u_g, u_1, \dots, u_{\bar{g}}).$$

The Prym variety associated with ι is defined to be the set $P_\iota = \{u \in J(C) \mid \iota(u) = -u\} = \{u \in \boldsymbol{C}^g \mid u = (u_1, \dots, u_{\bar{g}}, u_1, \dots, u_{\bar{g}})\}/(I_g, T)$. It is known that P_ι is a principally polarized Abelian variety of dimension $\bar{g}$ isomorphic to the complex torus $P = \boldsymbol{C}^{\bar{g}}/(I_{\bar{g}}, \Pi)$. The isomorphism σ: $P \to P_\iota$ is given by $\sigma(v_1, \dots, v_{\bar{g}}) = (v_1, \dots, v_{\bar{g}}, v_1, \dots, v_{\bar{g}})$. We define the map $\mathscr{P}$: $C \to P$ by $\mathscr{P}(p) = (\mathscr{P}_1(p), \dots, \mathscr{P}_{\bar{g}}(p))$, $p \in C$, $\mathscr{P}_j(p) = \int_{q_0}^{p} (\omega_j + \omega_{\bar{g}+j})$. The theta function $\vartheta_P(v)$ on P is defined by

$$\vartheta_P(v) = \textstyle\sum_{m \in \boldsymbol{Z}^g} \exp(2\pi\sqrt{-1}v^t m + \pi\sqrt{-1} m \Pi^t m), \quad v \in \boldsymbol{C}^{\bar{g}}.$$

The following property of ϑ_P is basic for our purpose. For $c \in \boldsymbol{C}^{\bar{g}}$, if $\vartheta_P(\mathscr{P}(p) - c)$ does not vanish identically, then the zero divisor ζ of $\vartheta_P(\mathscr{P}(p) - c)$ on C is of degree g and satisfies the relation

$$\sigma(c) = \mathscr{A}(\zeta) - 2^{-1}\mathscr{A}(q_\infty) - 2^{-1}\mathscr{A}(K_C) \quad \text{in} \quad J(C). \tag{10}$$

Now we give an explicit formula for the wave function $w(x, p)$ constructed at the beginning of this section. Let $\omega^{(j)}$ be the normalized differential of the second kind with its only pole at q_∞, where it has the form $dk^j +$(holomorphic part), and put $U_i^{(j)} = \frac{1}{2\pi\sqrt{-1}} \int_{\beta_i} \omega^{(j)}$. Since $\iota^*\omega^{(j)} = (-1)^j \omega^{(j)}$ and $\iota\beta_i = -\beta_{\bar{g}+i}$, we have

$$U_{\bar{g}+i}^{(j)} = (-1)^{j+1} U_i^{(j)} \quad 1 \leqq i \leqq \bar{g}. \tag{11}$$

We put $U^{(j)} = (U_1^{(j)}, \dots, U_{\bar{g}}^{(j)})$. The zero divisor $\delta(x)$ of $w(x, p)$ is of degree g and satisfies the relation

(12) $$\mathscr{A}(\delta(x)-\delta)+\sum_{j>0,odd} x_j(U^{(j)}, U^{(j)})=0 \quad \text{in} \quad J(C).$$

On the other hand, by using (9), we have

$$\iota(\mathscr{A}(\delta)-2^{-1}\mathscr{A}(q_\infty)-2^{-1}\mathscr{A}(K_C))=-(\mathscr{A}(\delta)-2^{-1}\mathscr{A}(q_\infty)-2^{-1}\mathscr{A}(K_C)) \quad \text{in } J(C).$$

Namely $\mathscr{A}(\delta)-2^{-1}\mathscr{A}(q_\infty)-2^{-1}\mathscr{A}(K_C)\in P_\iota$. We put

$$K_\delta=\sigma^{-1}(\mathscr{A}(\delta)-2^{-1}\mathscr{A}(q_\infty)-2^{-1}\mathscr{A}(K_C)).$$

Comparing (10) and (12), we see that $w(x, p)$ is expressed as

$$w(x, p)=\frac{\vartheta_P(\mathscr{P}(p)+\sum_{j>0,odd} x_j U^{(j)}-K_\delta)\vartheta_P(\mathscr{P}(q_\infty)-K_\delta)}{\vartheta_P(\mathscr{P}(p)-K_\delta)\vartheta_P(\mathscr{P}(q_\infty)+\sum_{j>0,odd} x_j U^{(j)}-K_\delta)} \times\exp\left(\sum_{j>0,odd} x_j\int_{q_0}^{p}(\omega^{(j)}-c_j)\right)$$

where the constant c_j is chosen so that $\int_{q_0}^{p}(\omega^{(j)}-c_j)-k^j=O(k^{-1})$ $(p\to q_\infty)$ holds.

The remaining task is to find the τ-function corresponding to $w(x, p)$. Expanding $\mathscr{P}_i(p)=\mathscr{A}_i(p)+\mathscr{A}_{\bar{g}+i}(p)$ around q_∞ in a Taylor series in k^{-1}, using the relation

$$U_i^{(j)}=\frac{1}{2\pi\sqrt{-1}}\int_{\beta_i}\omega^{(j)}=-\frac{1}{(j-1)!}\frac{d^j\mathscr{A}_i}{dz^j}\bigg|_{z=0}, \quad z=k^{-1},$$

which is derived from Riemann's bilinear relation, and (11), we easily see that

$$\vartheta_P\left(\mathscr{P}(q_\infty)+\sum_{j>0,odd}\left(x_j-\frac{2}{jk^j}U^{(j)}-K_\delta\right)\right) =\vartheta_P(\mathscr{P}(p)+\sum_{j>0,odd} x_j U^{(j)}-K_\delta)$$

holds. Hence in view of the transformation $w(x; k)=e^{\xi(x,k)}e^{-2\xi(\tilde{\partial},k^{-1})}\tau(x)/\tau(x)$ ((7) [2] cf. (3)) which transforms a τ-function $\tau(x)$ to a wave function $w(x; k)$, we have

$$\tau(x)=\exp(\text{quadratic function in } x)\cdot\vartheta_P(\mathscr{P}(q_\infty)+\sum_{i>0,odd} x_j U^{(j)}-K_\delta).$$

Theorem. *The theta functions on the Prym varieties of algebraic curves admitting involutions with two fixed points are τ-functions for the BKP hierarchy up to exponential of quadratic functions of x.*

Finally we note that the relation

$$\vartheta_P(v)^2=const.\ \vartheta_J(\sigma(v)+a), \quad a=(a_1,\dots, a_g),$$
$$a_j=-a_{\bar{g}+j}=\frac{1}{2}\int_{q_0}^{q_\infty}\omega_j, \qquad (1\leqq j\leqq \bar{g})$$

holds [4] where ϑ_J is the theta function on $J(C)$. This relation agrees with the relation between τ_{KP} and τ_{BKP} (8) [2].

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