Publ. RIMS, Kyoto Univ.
18 (1982), 1121–1136

On Unbounded Derivations Commuting with a Compact Group of *-Automorphisms

By

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Abstract

Let $\mathfrak{A}$ be a C^*-algebra with identity, α a continuous action of a compact abelian group G as *-automorphisms of $\mathfrak{A}$, $\mathfrak{A}^\alpha(\gamma)$ the spectral subspace of α corresponding to γ in the dual $\hat{G}$ of G and $\mathfrak{A}^\alpha(=\mathfrak{A}^\alpha(0))$ the fixed point algebra of α. Let δ be a closed symmetric derivation of $\mathfrak{A}$ which commutes with α and generates a one-parameter group of *-automorphisms of $\mathfrak{A}^\alpha$. We assume that the linear span of $\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ is dense in $\mathfrak{A}^\alpha$ for each $\gamma\in\hat{G}$ and then deduce that δ is a generator on $\mathfrak{A}$. Some relevant material on covariant representations is also given.

§1. Introduction

Let δ be a closed (symmetric) derivation of C^*-algebra $\mathfrak{A}$ which commutes with a continuous action α of a topological group G as *-automorphisms of $\mathfrak{A}$. Several authors [1] [2] [3] [4] [5] recently derived conditions on $\mathfrak{A}$, G, and δ, which ensure that δ is a generator, i.e., the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$. For example, if G is compact abelian, and δ vanishes on the fixed point algebra $\mathfrak{A}^\alpha$ of α, then this result is established in [4]. If, alternatively, δ is an inner derivation of $\mathfrak{A}^\alpha$ it follows from this result, and perturbation theory, that δ is a generator. But bounded derivations are generators of uniformly continuous groups and hence this can be viewed as an extension result; if G is compact abelian, δ commutes with α, and δ generates a uniformly continuous one-parameter group τ^0 of inner automorphisms of the fixed point algebra $\mathfrak{A}^\alpha$ then τ^0 extends to a strongly continuous group τ, with generator δ, on $\mathfrak{A}$. Example 6.1 of [4] also establishes that this result does not necessarily extend to the case that δ generates a strongly continuous

Communicated by H. Araki, November 25, 1981.
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group of *-automorphisms of $\mathfrak{A}^\alpha$. Nevertheless in this note we demonstrate that strong continuity of τ^0 suffices if, in addition, $\mathfrak{A}^\alpha=\overline{\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)}$ for each $\gamma\in\hat{G}$, where the bar denotes the closed linear span. (Here, and throughout the sequel, we adopt the notation of [4]. In particular $\mathfrak{A}^\alpha(\gamma)$ denotes the spectral subspace of α corresponding to γ in the dual group $\hat{G}$). Thus we aim to establish the following;

Theorem 1. *Let $\mathfrak{A}$ be a C^*-algebra with identity, G a compact abelian group, and α a continuous action of G as *-automorphisms of $\mathfrak{A}$. Furthermore let δ be a closed symmetric derivation satisfying;*

1. $\alpha_g\circ\delta=\delta\circ\alpha_g$, $g\in G$,
2. $\delta_0=\delta|_{\mathfrak{A}^\alpha}$ *is a generator on* $\mathfrak{A}^\alpha$.

Finally assume that the closed linear span of $\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ equals $\mathfrak{A}^\alpha$ for each $\gamma\in\hat{G}$.

It follows that δ is a generator.

In this theorem we do not know whether the assumption on $\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ can be weakened, e.g., to the assumption that $\overline{\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)}$, an ideal of $\mathfrak{A}^\alpha$, is invariant under the automorphism group generated by δ_0, for each $\gamma\in\hat{G}$, which is apparently necessary for δ to be a generator. (In the example in [4] we refered to above, this weaker assumption is violated.) We want to point out two typical cases where the assumption on $\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ is satisfied. One is the case where each $\mathfrak{A}^\alpha(\gamma)$ contains a unitary. For example, for a C^*-algebra B with identity with action β of a discrete abelian group Γ, let $\mathfrak{A}$ be the crossed product $B\times_\beta\Gamma$ and α the dual action $\hat{\beta}$ of $G=\hat{\Gamma}$. Then for the system $(\mathfrak{A}, G, \alpha)$, $\mathfrak{A}^\alpha(\gamma)$ contains a unitary. The other is the case where $\mathfrak{A}^\alpha$ is simple, e.g., the Cuntz algebras O_n with the gauge action of $\mathbb{T}$.

The general lines of proof of this theorem are very similar to those of [4]. There are two basic arguments. First one proves that δ is a generator of a group of bounded operators on each $\mathfrak{A}^\alpha(\gamma)$ and second one argues that this is sufficient for δ to be a generator on $\mathfrak{A}$. This second step is independent of the assumption on $\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ and is based upon the construction and exploitation of appropriate covariant representations. Hence we begin with the discussion of this latter lifting procedure in Section 2 and then return to the proof of Theorem 1, and discussion of the action of δ on the spectral subspaces $\mathfrak{A}^\alpha(\gamma)$, in Section 3. Relevant information about covariant representations is collected in an appendix.

§ 2. Generators and Spectral Subspaces

In this section we examine the generator problem under the assumption that δ_γ, the restriction of δ to the spectral subspace $\mathfrak{A}^\alpha(\gamma)$, is a generator for each $\gamma \in \hat{G}$. In fact we need information on δ under slightly weaker assumptions on the δ_γ but we will state this as a corollary of the proof of the following general result.

Proposition 2. *Let $\mathfrak{A}$ be a C^*-algebra, G a compact group, and α a continuous action of G as *-automorphisms of $\mathfrak{A}$. Furthermore let δ be a closed symmetric derivation of $\mathfrak{A}$ satisfying,*

1. $\alpha_g \circ \delta = \delta \circ \alpha_g$, $g \in G$,
2. $\delta_\gamma = \delta|_{\mathfrak{A}^\alpha(\gamma)}$ *is the generator of a strongly continuous one-parameter group of bounded operators on the Banach space $\mathfrak{A}^\alpha(\gamma)$ for each $\gamma \in \hat{G}$.*

*It follows that δ is the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$.*

Remarks. 1. This result is valid for non-abelian G too.

2. A weaker version of this proposition is given in [4] Lemma 4.2, where it is further assumed that $\mathfrak{A}^\alpha \subset D(\delta)$, but this domain requirement is in fact irrelevant. The following proof via covariant representations is an 'integrated' version of the 'infinitesimal' proof of Lemma 4.2 in [4]. It is the elimination of infinitesimal methods which avoids the domain requirements.

Proof. Let 0 denote the trivial representation of G. Since $\mathfrak{A}^\alpha$ $(=\mathfrak{A}^\alpha(0))$ is a C^*-subalgebra it follows that δ_0 generates a *-automorphism group τ^0. Next define a projection P from $\mathfrak{A}$ onto the fixed point algebra $\mathfrak{A}^\alpha$ by

$$P(x) = \int_G dg \alpha_g(x),$$

where dg is the normalized Haar measure on G. Now for any state ω_0 of $\mathfrak{A}^\alpha$ define a state ω of $\mathfrak{A}$ by

$$\omega(x) = \frac{1}{2}\int_{-\infty}^{\infty} dt e^{-|t|}\omega_0(\tau_t^0(P(x)))\,.$$

Thus for $x \in \mathfrak{A}^\alpha$

$$\omega(\tau_s^0(x^*x)) = \frac{1}{2}\int_{-\infty}^{\infty} dt e^{-|t-s|}\omega_0(\tau_t^0(x^*x))$$

and hence

$$\omega(\tau_s^0(x^*x)) \le e^{|s|}\omega(x^*x) .$$

Consequently if $(\mathscr{H}, \pi, \Omega)$ denotes the cyclic representation associated with ω one can define a strongly continuous one-parameter group of bounded linear operators acting on the subspace $\mathscr{H}_0 = [\pi(\mathfrak{A}^\alpha)\Omega]$ by

$$T_s\pi(x)\Omega = \pi(\tau_s^0(x))\Omega ,$$

and in fact one has $\|T_s\| \le \exp\{|s|/2\}$. Moreover

$$T_s\pi(x)T_s^{-1} = \pi(\tau_s^0(x))$$

for all $x \in \mathfrak{A}^\alpha$. Next we argue that by multiplication with an element of $(\pi(\mathfrak{A}^\alpha)|_{\mathscr{H}_0})'$ the group T may be arranged to be unitary without affecting this covariant implementation law.

Since τ^0 is a group of *-automorphisms

$$T_{-s}^*\pi(x)T_s^* = \pi(\tau_s^0(x))$$

for all $x \in \mathfrak{A}^\alpha$ and hence $T_s^*T_s \in (\pi(\mathfrak{A}^\alpha)|_{\mathscr{H}_0})'$. Next let L_0 be the generator of T and remark that

$$|s|^{-1}(\|T_s\psi\|^2 - \|\psi\|^2) \le |s|^{-1}(e^{|s|} - 1)\|\psi\|^2$$

for all $\psi \in \mathscr{H}_0$ by the above estimate on $\omega\circ\tau^0\circ P$. Hence

$$|(L_0\psi, \psi) + (\psi, L_0\psi)| \le \|\psi\|^2$$

for all $\psi \in D(L_0)$. It follows that

$$|(L_0\psi, \phi) + (\psi, L_0\phi)| \le \|\phi\| \, \|\psi\|$$

for all $\phi, \psi \in D(L_0)$. Hence $D(L_0) \subseteq D(L_0^*)$ and

$$|(\psi, (L_0^* + L_0)\phi)| \le \|\phi\| \, \|\psi\| .$$

Therefore $(L_0^* + L_0)/2$ has a bounded self-adjoint extension h_0 with $\|h_0\| \le \frac{1}{2}$. But L_0 generates the strongly continuous one-parameter group T on $\mathscr{H}_0$ and hence by perturbation theory $iH_0 = L_0 - h_0$ generates a similar group. Since H_0 is symmetric on $D(L_0)$, it is automatically self-adjoint. Now if $U_s = \exp\{iH_0 s\}$ the Trotter product formula implies that

$$U_s = \lim_{n\to\infty} (T_{s/n}e^{-h_0 s/n})^n .$$

Finally since $T_s^*T_s \in (\pi(\mathfrak{A}^\alpha)|_{\mathscr{H}_0})'$ it follows that $h_0 \in ((\pi(\mathfrak{A}^\alpha)|_{\mathscr{H}_0})'$ and hence

$$U_s\pi(x)U_s^* = \pi(\tau_s^0(x))$$

for all $x \in \mathfrak{A}^{\alpha}$.

Thus τ^0 is covariantly implemented on $\mathfrak{A}^{\alpha}$ by either T or U and consequently δ is spatially implemented on $\mathfrak{A}^{\alpha}$ either by L_0 or iH_0. Specifically

$$i[H_0, \pi(x)] = \pi(\delta(x))$$

for all $x \in \mathfrak{A}^{\alpha} \cap D(\delta)$. Our next aim is to derive a similar spatial implementation law for δ on $\mathfrak{A}$ and for this we begin by extending h_0 and H_0 to $\mathscr{H}$.

Define h on $\pi(\mathfrak{A})\Omega$ by

$$h\pi(x)\Omega = \pi(x)h_0\Omega .$$

Since ω is α-invariant one has

$$\begin{aligned}\|\pi(x)h_0\Omega\|^2 &= (h_0\Omega, \pi(x^*x)h_0\Omega)\\ &= (h_0\Omega, \pi(P(x^*x))h_0\Omega)\\ &= (\pi(P(x^*x))^{\frac{1}{2}}\Omega, h_0^2\pi(P(x^*x))^{\frac{1}{2}}\Omega)\\ &\leq \|h_0\|^2(\Omega, P(x^*x)\Omega)\\ &= \|h_0\|^2\|\pi(x)\Omega\|^2 .\end{aligned}$$

Hence h is well-defined and extends by continuity to a bounded operator with $\|h\| \leq \|h_0\| \leq \frac{1}{2}$. A number of properties of h follow straightforwardly, e.g.,

$$\begin{aligned}&h = h^* \in \pi(\mathfrak{A})', \quad \|h\| = \|h_0\| ,\\ &E_\gamma h = hE_\gamma, \quad hE_0 = h_0 ,\end{aligned}$$

where

$$E_\gamma = [\pi(\mathfrak{A}^{\alpha}(\gamma))\Omega] .$$

Next define H by

$$\begin{aligned}iH\pi(x)\Omega &= \pi(\delta(x))\Omega - h\pi(x)\Omega\\ &= \pi(\delta(x))\Omega - \pi(x)h_0\Omega\end{aligned}$$

for $x \in D(\delta)$. If $\pi(x)\Omega = 0$ and $y \in D(\delta)$ one calculates that

$$\begin{aligned}(\pi(y)\Omega, \pi(\delta(x))\Omega) &= \omega(y^*\delta(x))\\ &= \omega(\delta(y^*x)) - \omega(\delta(y^*)x)\\ &= \omega(\delta(y^*x)) .\end{aligned}$$

But for $z \in D(\delta)$ one has

$$\begin{aligned}\omega(\delta(z)) &= \omega(P(\delta(z)))\\ &= \omega(\delta(P(z)))\\ &= (\Omega, L_0\pi(P(z))\Omega)\\ &= (L_0^*\Omega, \pi(P(z))\Omega)\\ &= 2(h_0\Omega, \pi(P(z))\Omega)\end{aligned}$$

where we have used $L_0^* = -L_0 + 2h_0$ and $L_0\Omega = 0$. Combining these two observations one concludes that

$$(\pi(y)\Omega,\ \pi(\delta(x))\Omega) = 2(h_0\Omega,\ \pi(y^*x)\Omega) = 0$$

and hence H is well-defined. But for $x, y \in D(\delta)$

$$\begin{aligned}(\pi(y)\Omega,\ iH\pi(x)\Omega) &= \omega(y^*\delta(x)) - (\pi(y)\Omega,\ h\pi(x)\Omega)\\ &= \omega(\delta(y^*x)) - \omega(\delta(y^*)x) - (h_0\Omega,\ \pi(y^*x)\Omega)\\ &= (h_0\Omega,\ \pi(y^*x)\Omega) - \omega(\delta(y^*)x)\\ &= -(iH\pi(y)\Omega,\ \pi(x)\Omega),\end{aligned}$$

i.e., H is symmetric. Moreover

$$\begin{aligned}i[H,\ \pi(x)]\pi(y)\Omega &= iH\pi(xy)\Omega - \pi(x)iH\pi(y)\Omega\\ &= \pi(\delta(xy))\Omega - \pi(x\delta(y))\Omega\\ &= \pi(\delta(x))\pi(y)\Omega,\end{aligned}$$

i.e., δ is implemented by iH. Next we prove that H is essentially self-adjoint. It is at this point we use the assumption that δ_γ is the generator of a group of bounded operators.

Set $L = iH + h$ and note that if $x \in D(\delta) \cap \mathfrak{A}^\alpha(\gamma)$ then

$$(I + \beta L)\pi(x)\Omega = \pi((I + \beta\delta_\gamma)(x))\Omega$$

for all real β. This demonstrates that $I + \beta L$ leaves $E_\gamma\mathscr{H}$ invariant and since δ_γ generates a strongly continuous group of bounded operators on $\mathfrak{A}^\alpha(\gamma)$ it also establishes that there is a β_γ such that $R((I + \beta L)E_\gamma)$ is dense in $E_\gamma\mathscr{H}$ for all $|\beta| < \beta_\gamma$. Thus in this range of β, $(I + \beta L)^{-1}E_\gamma$ is well defined. But

$$\begin{aligned}\|(I + \beta L)\pi(x)\Omega\| &\geq \mathrm{Re}\,(\pi(x)\Omega,\ (1 + \beta L)\pi(x)\Omega)/\|\pi(x)\Omega\|\\ &= (\pi(x)\Omega,\ (1 + \beta h)\pi(x)\Omega)/\|\pi(x)\Omega\|\\ &\geq (1 - |\beta|\ \|h\|)\|\pi(x)\Omega\|\end{aligned}$$

and hence

$$\|(I + \beta L)^{-1}E_\gamma\| \leq (1 - |\beta|\ \|h\|)^{-1}.$$

Now define H_γ as the restriction of H to $E_\gamma\mathscr{H}$. It follows from perturbation theory that $(I + i\beta H_\gamma)^{-1}$ is defined as a bounded operator for all sufficiently small β. But this establishes that H_γ is essentially self-adjoint and hence $R(I + i\beta H_\gamma)$ is dense for all real β. Since this is true for all $\gamma \in \hat{G}$ it follows that H is essentially self-adjoint on $\mathscr{H}$.

Now if $\bar{H}$ denotes the self-adjoint closure of H then

$$x \in \mathscr{L}(\mathscr{H}) \longmapsto \tau_t(x) = e^{it\bar{H}} x e^{-it\bar{H}} \in \mathscr{L}(\mathscr{H})$$

defines a σ-weakly continuous group of isometries of $\mathscr{L}(\mathscr{H})$ such that

$$\frac{d}{dt}\tau_t(\pi(x)) = \tau_t(\pi(\delta(x))), \quad x \in D(\delta).$$

It follows from semigroup theory that

$$\|\pi((I+\beta\delta)(x))\| \geq \|\pi(x)\|$$

for all real β and all $x \in D(\delta)$. Since by varying ω_0 one can construct a faithful family of covariant states ω one then concludes that

$$(*) \qquad \|(I+\beta\delta)(x)\| \geq \|x\|$$

for all real β and $x \in D(\delta)$. Finally since δ, and hence δ_γ, is implemented by the self-adjoint operator $\bar{H}$ the δ_γ must generate groups of isometries. Therefore

$$R(I+\beta\delta_\gamma) = \mathfrak{A}^\alpha(\gamma)$$

and since this is true for all $\gamma \in \hat{G}$

$$(**) \qquad R(I+\beta\delta) = \mathfrak{A}.$$

The two properties $(*)$ and $(**)$ imply, however, that δ is a generator.

In the above proof we have not used all the assumptions on δ. The first part of the proof relies upon the assumption that δ_0 is a generator but the second part uses less information about the δ_γ.

Corollary 3. *Let* $(\mathfrak{A}, G, \alpha)$ *be as in Proposition* 2 *and let* δ *be a closed symmetric derivation of* $\mathfrak{A}$ *satisfying*

1. $\alpha_g \circ \delta = \delta \circ \alpha_g$, $g \in G$,

2a. δ_0 *is a generator on* $\mathfrak{A}^\alpha$,

b. *For each non-zero* $\gamma \in \hat{G}$ *there is a* $\beta_\gamma > 0$ *such that* $R(I+\beta\delta_\gamma)$ *is dense in* $\mathfrak{A}^\alpha(\gamma)$ *for all* $|\beta| < \beta_\gamma$.

It follows that δ *is the generator of a strongly continuous one-parameter group of *-automorphisms of* $\mathfrak{A}$.

§3. Proof of Theorem 1

The proof of Theorem 1 is based upon verification of the assumptions of Corollary 3. This relies upon algebraic arguments, similar to those employed to prove Theorem 5.1 of [4], combined with perturbation theoretic techniques.

An essential part of the perturbation argument is summarized in the next lemma.

Lemma 4. *Let X be a Banach space and $X_1, X_2, \dots, X_n$ closed subspaces such that $X=X_1+X_2+\cdots+X_n$. Furthermore let δ be a closed operator and $\delta_1, \delta_2, .., \delta_n$ bounded operators on X. Assume that for $i=1, 2, \dots, n$*

$$(\delta+\delta_i)(X_i \cap D(\delta)) \subseteq X_i$$

and that $\delta+\delta_i$ is the generator of a semigroup of bounded operators on X_i.
It follows that $R(I+\beta\delta)$ is dense in X for all sufficiently small β.

Proof. Let $\hat{X}=X_1 \oplus X_2 \cdots \oplus X_n$ with the norm of $\hat{x}=(x_1, x_2, \dots, x_n)$ defined by

$$\|\hat{x}\| = \sum_{i=1}^{n} \|x_i\| .$$

Thus $\hat{X}$ is a Banach space. Next consider the linear map ϕ from $\hat{X}$ to X with the action

$$\phi(\hat{x}) = \sum_{i=1}^{n} x_i .$$

This map is continuous and since $X=X_1+\cdots+X_n$ its range is equal to X. Thus the quotient space $\hat{X}/\ker\phi$, with the quotient norm $\|\cdot\|_\phi$, is canonically isomorphic to X. Hence there is a $c>0$ such that

$$\|\hat{x}\|_\phi \leq c\|\phi(\hat{x})\|$$

for all $\hat{x} \in \hat{X}$. Thus for any $x \in X$ one may choose $x_i \in X_i$ such that

$$x = \sum_{i=1}^{n} x_i$$

and

$$\sum_{i=1}^{n} \|x_i\| < M\|x\|$$

where M is a constant slightly larger than c.

Next for $x \in X$ choose $x_i \in X_i$ with the foregoing properties. Then by the assumption that $\delta+\delta_i$ is a generator on X_i one may choose $y_i \in X_i \cap D(\delta)$ such that

$$y_i + \beta(\delta+\delta_i)(y_i) = x_i$$

for β sufficiently small. Therefore by semigroup theory there are constants c_i, $d_i>0$ such that

$$\|x_i\| \geq c_i\|y_i\|(I-|\beta|d_i)$$

for $|\beta|d_i<1$. Thus setting

$$y=\sum_{i=1}^{n} y_i$$

one has

$$\begin{aligned}\|y+\beta\delta(y)-x\| &= \|\sum_{i=1}^{n}(y_i+\beta(\delta+\delta_i)(y_i)-x_i)-\sum_{i=1}^{n}\beta\delta_i(y_i)\| \\ &\leq |\beta|\sum_{i=1}^{n}\|\delta_i\|\,\|y_i\| \\ &\leq \sum_{i=1}^{n}\frac{|\beta|\,\|\delta\|}{c_i(I-|\beta|d_i)}\|x_i\| \\ &\leq \|x\|\max_{1\leq i\leq n}\frac{|\beta|\,\|\delta_i\|M}{c_i(I-|\beta|d_i)} \\ &< \|x\|/2\end{aligned}$$

for all sufficiently small β. But if $R(I+\beta\delta)$ is not dense in X then for any $\varepsilon>0$ there is an $x'\in X$ such that

$$\|y+\beta\delta(y)-x'\|>\|x'\|(1-\varepsilon)$$

for all $y\in D(\delta)$. Since this contradicts the previous estimate one concludes that $R(I+\beta\delta)$ is dense in X for all sufficiently small β.

At this stage we are prepared to prove Theorem 1.

Corollary 3 establishes that it is sufficient to show that for each $\gamma\in\hat{G}$ there is a $\beta_\gamma>0$ such that $R(I+\beta\delta_\gamma)$ is dense in $\mathfrak{A}^\alpha(\gamma)$ for all $|\beta|<\beta_\gamma$.

Fix $\gamma\in\hat{G}$. Since $D_\gamma=D(\delta)\cap\mathfrak{A}^\alpha(\gamma)$ is dense in $\mathfrak{A}^\alpha(\gamma)$ the closed linear span of $D_\gamma^*D_\gamma$ is dense in $\mathfrak{A}^\alpha$. This follows from the final assumption of Theorem 1. Moreover $\mathfrak{A}^\alpha$ contains the identity $\mathbb{1}$. Hence there exists a finite number m of x_i, $y_i\in D_\gamma$ such that

$$\|\sum_{i=1}^{m} x_i^*y_i-\mathbb{1}\|<\frac{1}{2}$$

and consequently

$$\sum_{i=1}^{m}(x_i^*x_i+y_i^*y_i)\geq\sum_{i=1}^{m}(x_i^*y_i+y_i^*x_i)\geq\mathbb{1}\,.$$

Thus we may suppose that there are $n(=2m)$ elements $y_i\in D_\gamma$ with the property

$$\sum_{i=1}^{n} y_i^*y_i\geq\mathbb{1}\,.$$

Similarly there are a finite number n' of $z_j\in D_\gamma$ such that

$$\sum_{j=1}^{n'} z_jz_j^*\geq\mathbb{1}$$

because $D_\gamma D_\gamma^*$ is dense in $\mathfrak{A}^\alpha$. Now define

$$a=\sum_{i=1}^{n} y_i y_i^* + \sum_{i=1}^{n'} z_i z_i^*$$

and set $x_i = a^{-\frac{1}{2}} y_i$ and $x_{n+i} = a^{-\frac{1}{2}} z_i$. It follows that $a^{-\frac{1}{2}} \in \mathfrak{A}^\alpha \cap D(\delta)$, $x_i \in D_\gamma$, and

$$\sum_{i=1}^{N} x_i x_i^* = \mathbb{1}$$

where $N = n + n'$. Furthermore

$$\sum_{i=1}^{N} x_i^* x_i \geq \sum_{i=1}^{n} y_i^* a^{-1} y_i \geq \|a\|^{-1} \sum_{i=1}^{n} y_i^* y_i \geq \|a\|^{-1} \mathbb{1} .$$

Next consider the system $(\mathfrak{A}_N = \mathfrak{A} \otimes M_N, G, \bar{\alpha})$ where M_N is the $N \times N$ matrix algebra and $\bar{\alpha}_g = \alpha_g \otimes \iota$. Here ι denotes the trivial action. Further let $\bar{\delta} = \delta \otimes \iota$ with $D(\bar{\delta}) = D(\delta) \otimes M_N$. Thus $\bar{\alpha}$ and $\bar{\delta}$ satisfy the same properties as α and δ. Now define

$$v = \begin{pmatrix} x_1, x_2, \dots, x_N \\ 0 \end{pmatrix} \in \mathfrak{A}_N^{\bar{\alpha}}(\gamma) \cap D(\bar{\delta}) .$$

It follows from the above construction that $vv^* = e_{11}$, where e_{11} is the matrix unit with $(e_{11})_{ij} = \delta_{i1}\delta_{j1}\mathbf{1}$, and $\bar{\delta}(vv^*) = 0$. Now for $b \in \mathfrak{A}_N^{\bar{\alpha}}(\gamma) v^* v \cap D(\bar{\delta})$ one has

$$\begin{aligned} \bar{\delta}(b) &= \bar{\delta}(bv^*v) \\ &= \bar{\delta}_0(bv^*)v + bv^*\bar{\delta}(v) \end{aligned}$$

where $\bar{\delta}_0 = \delta_0 \otimes \iota$ with $D(\bar{\delta}_0) = D(\delta_0) \otimes M_N$. Therefore

$$\begin{aligned} (\bar{\delta} + \delta_{v^*\bar{\delta}(v)})(b) &= \bar{\delta}_0(bv^*)v + v^*\bar{\delta}(v)b \\ &= \{\bar{\delta}_0(bv^*) + v^*\bar{\delta}(v)bv^*\}v \end{aligned}$$

where δ_u denotes the derivation with action $\delta_u(b) = ub - bu$.

Now the map from $b \in \mathfrak{A}_N^{\bar{\alpha}}(\gamma) v^* v$ to $bv^* \in \mathfrak{A}_N^{\bar{\alpha}} vv^*$ is an isomorphism from the Banach space $\mathfrak{A}_N^{\bar{\alpha}}(\gamma) v^* v$ onto the Banach space $\mathfrak{A}_N^{\bar{\alpha}} vv^*$. But since $\bar{\delta}_0(vv^*) = 0$ the restriction of $\bar{\delta}_0$ to $\mathfrak{A}_N^{\bar{\alpha}} vv^*$ is also a generator. Moreover the operator of left multiplication by $v^*\bar{\delta}(v)$ is bounded and leaves $\mathfrak{A}_N^{\bar{\alpha}} vv^*$ invariant. Therefore $\bar{\delta}_0 + v^*\bar{\delta}(v)$ is the generator of a group of bounded operators on $\mathfrak{A}_N^{\bar{\alpha}} vv^*$. Hence $\bar{\delta} + \delta_{v^*\bar{\delta}(v)}$ is a generator on $\mathfrak{A}_N^{\bar{\alpha}}(\gamma) v^* v$.

Next we repeat this argument with matrices $v_i(\sigma)$ whose elements are zero except in the *i-th* row which is given by

$$\sigma(i)x_i, \ \sigma(i+1)x_{i+1}, \dots, \sigma(N)x_N, \ \sigma(1)x_1, \dots, \sigma(i-1)x_{i-1}$$

where the $\sigma(j)$ take values ± 1. Then $v_i(\sigma) \in \mathfrak{A}_N^{\bar{\alpha}}(\gamma) \cap D(\bar{\delta})$ and

$$v_i(\sigma)v_i(\sigma)^* = e_{ii}.$$

By the above reasoning $\bar{\delta} + \delta_{v_i(\sigma)^*\bar{\delta}(v_i(\sigma))}$ is a generator on $\mathfrak{A}_N^{\bar{\alpha}}(\gamma)v_i(\sigma)^*v_i(\sigma)$. But

$$2^{-N} \sum_{\sigma} v_i(\sigma)^* v_i(\sigma) = \begin{pmatrix} x_i^* x_i & & & 0 \\ & x_{i+1}^* x_{i+1} & & \\ & & \ddots & \\ 0 & & & x_{i-1}^* x_{i-1} \end{pmatrix}$$

and

$$2^{-N} \sum_{i=1}^{N} \sum_{\sigma} v_i(\sigma)^* v_i(\sigma) = \sum_{i=1}^{N} x_i^* x_i \mathbb{1}_N \geq \|a\|^{-1} \mathbb{1}$$

where $\mathbb{1}_N$ is the identity of M_N. Therefore

$$\mathfrak{A}_N^{\bar{\alpha}}(\gamma) = \sum_{i=1}^{N} \sum_{\sigma} \mathfrak{A}_N^{\bar{\alpha}}(\gamma) v_i(\sigma)^* v_i(\sigma)$$

and we can apply Lemma 4 to the family

$$X = \mathfrak{A}_N^{\bar{\alpha}}(\gamma), \quad X_i = \mathfrak{A}_N^{\bar{\alpha}}(\gamma) v_i(\sigma)^* v_i(\sigma),$$

and the bounded operators $\delta_{v_i(\sigma)^*\bar{\delta}(v_i(\sigma))}$ and conclude that $(I+\beta\bar{\delta})(\mathfrak{A}_N^{\bar{\alpha}}(\gamma) \cap D(\bar{\delta}))$ is dense in $\mathfrak{A}_N^{\bar{\alpha}}(\gamma)$ for sufficiently small β. Since $\mathfrak{A}_N^{\bar{\alpha}}(\gamma) = \mathfrak{A}^{\alpha}(\gamma) \otimes M_N$ this implies that $(I+\beta\delta)(\mathfrak{A}^{\alpha}(\gamma) \cap D(\delta))$ is dense in $\mathfrak{A}^{\alpha}(\gamma)$ for small β and hence δ is a generator by Corollary 3.

Appendix

Covariant Representations

Throughout this appendix $(\mathfrak{A}, \tau, \omega)$ denotes a C^*-algebra $\mathfrak{A}$, a strongly continuous one-parameter group of *-automorphisms τ of $\mathfrak{A}$, and a state ω over $\mathfrak{A}$. Furthermore $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes the cyclic representation of $\mathfrak{A}$ associated with ω. It follows from the proof of Proposition 2 that the state

$$\omega_e = \frac{1}{2} \int_{-\infty}^{\infty} dt e^{-|t|} \omega \circ \tau_t$$

generates a covariant representation, i.e., there exists a strongly continuous one-parameter group of unitary operators U_{ω_e} on $\mathscr{H}_{\omega_e}$ which implements the automorphisms τ,

$$\pi_{\omega_e}(\tau_t(A)) = U_{\omega_e}(t) \pi_{\omega_e}(A) U_{\omega_e}(t)^{-1}.$$

The purpose of this appendix is to further analyze this phenomenon by proving the following.

Theorem A1. *Let f be an almost everywhere positive integrable function over $\boldsymbol{R}$ with total integral one.*

It follows that the state

$$\omega_f = \int dt f(t) \omega \circ \tau_t$$

generates a covariant representation.

Remark. If the Fourier transform $\hat{f}$ of f has compact support this result is a corollary of a spectral theorem of Arveson (Theorem 5.3. of [6]).

The proof of Theorem A1 will be divided into two pieces each of which have a separate interest. The first piece of information extends a construction used by Tomita in the decomposition theory of states (see [7] Chapter 4, in particular Lemma 4.1.21). In the following $E_{\mathfrak{A}}$ will denote the state space of $\mathfrak{A}$ equipped with the weak*-topology.

Proposition A2. *Let μ be a regular probability measure on $E_{\mathfrak{A}}$ with barycentre ω and let f be a non-negative μ-integrable function over $E_{\mathfrak{A}}$. Define the positive sesquilinear form s_f over $\mathscr{H}_\omega$ by $D(s_f) = \pi_\omega(\mathfrak{A})\Omega_\omega$ and*

$$s_f(\pi_\omega(x)\Omega_\omega, \pi_\omega(y)\Omega_\omega) = \int_{E_{\mathfrak{A}}} d\mu(\omega') f(\omega') \omega'(x^* y).$$

It follows that s_f is closable and the positive self-adjoint operator S_f associated with the closure $\bar{s}_f$ of s_f is affiliated with the commutant $\pi_\omega(\mathfrak{A})'$ of $\pi_\omega(\mathfrak{A})$. Moreover if f is positive μ-almost everywhere then S_f is invertible.

Proof. Define f_n by $f_n(x) = \min(f(x), n)$. Thus the f_n form an increasing family of positive functions which converges pointwise to f. Next let $\kappa_\mu(f_n) \in \pi_\omega(\mathfrak{A})'$ denote the bounded operators defined by

$$(\Omega_\omega, \kappa_\mu(f_n)\pi_\omega(x)\Omega_\omega) = \int d\mu(\omega') f(\omega') \omega'(x)$$

(see [7] Lemma 4.1.21). Now introduce the increasing family of bounded quadratic forms

$$t_n(\psi) = (\psi, \kappa_\mu(f_n)\psi), \quad \psi \in \mathscr{H}_\omega$$

and their monotone limit

$$t(\psi)=\sup_n t_n(\psi)=\lim_{n\to\infty} t_n(\psi)$$

where $D(t)$ is the family of $\psi\in\mathscr{H}_\omega$ for which the supremum is finite. It follows from [8] Lemma 5.2.13 that t is closed. But

$$\begin{aligned} t(\pi_\omega(x)\Omega_\omega) &= \int d\mu(\omega')f(\omega')\omega'(x^*x) \\ &= s_f(\pi_\omega(x)\Omega_\omega) \end{aligned}$$

for all $x\in\mathfrak{A}$. Thus t is a closed extension of s_f, i.e., s_f is closable.

Now $\pi_\omega(\mathfrak{A})\Omega_\omega$ is automatically a core for $S_f^{\frac{1}{2}}$. Moreover

$$\begin{aligned} \|S_f^{\frac{1}{2}}\pi_\omega(x)\pi_\omega(y)\Omega_\omega\|^2 &= \int d\mu(\omega')f(\omega')\omega'(y^*x^*xy) \\ &\le \|x\|^2\int d\mu(\omega')f(\omega')\omega'(y^*y) \\ &= \|x\|^2\|S_f^{\frac{1}{2}}\pi_\omega(y)\Omega_\omega\|^2 . \end{aligned}$$

Thus it follows that $\pi_\omega(\mathfrak{A})D(S_f^{\frac{1}{2}})\subseteq D(S_f^{\frac{1}{2}})$. Moreover one concludes from the identity

$$\begin{aligned} (S_f^{\frac{1}{2}}\pi_\omega(xy)\Omega_\omega,\ S_f^{\frac{1}{2}}\pi_\omega(z)\Omega_\omega) &= \int d\mu(\omega')f(\omega')\omega'(y^*x^*z) \\ &= (S_f^{\frac{1}{2}}\pi_\omega(y)\Omega_\omega,\ S_f^{\frac{1}{2}}\pi_\omega(x^*z)\Omega_\omega) \end{aligned}$$

by a double approximation procedure that

$$(S_f^{\frac{1}{2}}\pi_\omega(x)\phi,\ S_f^{\frac{1}{2}}\psi)=(S_f^{\frac{1}{2}}\phi,\ S_f^{\frac{1}{2}}\pi_\omega(x^*)\psi)$$

for all ϕ, $\psi\in D(S_f)\subseteq D(S_f^{\frac{1}{2}})$. But the left hand side is continuous in ϕ and the right hand side is continuous in ψ. Hence one deduces that $\pi_\omega(\mathfrak{A})D(S_f)\subseteq D(S_f)$ and

$$\begin{aligned} (S_f\pi_\omega(x)\phi,\ \psi) &= (S_f\phi,\ \pi_\omega(x^*)\psi) \\ &= (\pi_\omega(x)S_f\phi,\ \psi) . \end{aligned}$$

Thus S_f is affiliated with $\pi_\omega(\mathfrak{A})'$.

Next suppose f is positive μ-almost everywhere. The approximants f_n introduced above then have this property. Moreover since $f\ge f_n>0$ it follows that

$$S_f\ge\kappa_\mu(f_n)\ge 0$$

where the operator ordering is in the sense of quadratic forms. Thus to prove that S_f is invertible it suffices to prove that $\kappa_\mu(f_n)$ is invertible and this effectively reduces the problem to the examination of bounded f. Therefore we now assume f is bounded.

Next define

$$\theta_n = \left\{\omega' \in E_{\mathfrak{A}}\,;\, f(\omega') \geq \frac{1}{n}\right\}$$

and consider the bounded sesquilinear forms t_n over $\mathscr{H}_\omega \times \mathscr{H}_\omega$ with the property

$$t_n(\pi_\omega(x)\Omega_\omega,\ \pi_\omega(y)\Omega_\omega) = \int_{\theta_n} d\mu(\omega')\omega'(x^*y)\,.$$

Since $nf(\omega') \geq 1$ on θ_n

$$\begin{aligned} t_n(\pi_\omega(x)\Omega_\omega,\ \pi_\omega(x)\Omega_\omega) &\leq n \int d\mu(\omega') f(\omega')\omega'(x^*y) \\ &= n\|\pi_\omega(x)S_f^{\frac{1}{2}}\Omega_\omega\|^2\,. \end{aligned}$$

Hence there is a sequence of positive bounded operators S_n on the range $\mathscr{H}_n$ of $E_n = [\pi_\omega(\mathfrak{A})S_f^{\frac{1}{2}}\Omega_\omega]$ such that

$$t_n(\pi_\omega(x)\Omega_\omega,\ \pi_\omega(y)\Omega_\omega) = (\pi_\omega(x)S_f^{\frac{1}{2}}\Omega_\omega,\ S_n\pi_\omega(y)S_f^{\frac{1}{2}}\Omega_\omega)$$

and S_n is in the commutant $\pi_\omega(\mathfrak{A})'$ restricted to $\mathscr{H}_n$. But $E_n \in \pi_\omega(\mathfrak{A})'$ and hence $S_n = S_nE_n = E_nS_nE_n$ may be regarded as an operator in $\pi_\omega(\mathfrak{A})'$ acting on $\mathscr{H}_\omega$. Moreover the family of forms associated with t_n is monotone increasing and

$$\begin{aligned} \lim_{n\to\infty} t_n(\pi_\omega(x)\Omega_\omega,\ \pi_\omega(x)\Omega_\omega) &= \lim_{n\to\infty} (\pi_\omega(x)\Omega_\omega,\ S_f^{\frac{1}{2}}S_nS_f^{\frac{1}{2}}\pi_\omega(x)\Omega_\omega) \\ &= (\pi_\omega(x)\Omega_\omega,\ \pi_\omega(x)\Omega_\omega)\,. \end{aligned}$$

Thus $S_f^{\frac{1}{2}}S_nS_f^{\frac{1}{2}}$ converges weakly, hence strongly, to the identity.

Finally suppose $S_f\phi = 0$. Then

$$(\phi,\ S_f^{\frac{1}{2}}S_nS_f^{\frac{1}{2}}\phi) = 0\,.$$

But this contradicts the previous convergence result unless $\phi = 0$, i.e., S_f is invertible.

Next we compare the representations generated by the states obtained from two probability measures on the state space $E_{\mathfrak{A}}$.

Proposition A3. *Let μ_1 and μ_2 be two regular probability measures on $E_{\mathfrak{A}}$ with barycentres ω_1 and ω_2 respectively.*

If μ_1 is absolutely continuous with respect to μ_2 then π_{ω_1} is unitarily *equivalent to a subrepresentation of* π_{ω_2}, *and if* μ_1 *and* μ_2 *are mutually absolutely continuous then* π_{ω_1} *and* π_{ω_2} *are unitarily equivalent.*

Proof. If μ_1 is absolutely continuous with respect to μ_2 there is a non-negative $f \in L^1(\mu_2)$ such that $d\mu_1 = f d\mu_2$. Now define S_f on $\mathscr{H}_{\omega_2}$ by the construction of Proposition A2. Thus

$$(S_f^{\frac{1}{2}}\Omega_{\omega_2}, \pi_{\omega_2}(x)S_f^{\frac{1}{2}}\Omega_{\omega_2}) = \int d\mu_2(\omega')f(\omega')\omega'(x)$$
$$= \int d\mu_1(\omega')\omega'(x) = \omega_1(x)$$

and S_f is affiliated to $\pi_{\omega_2}(\mathfrak{A})'$. Next define an operator from $\mathscr{H}_{\omega_1}$ to $R(S_f^{\frac{1}{2}})$, the closure of the range of $S_f^{\frac{1}{2}}$, such that

$$U\pi_{\omega_1}(x)\Omega_{\omega_1} = \pi_{\omega_2}(x)S_f^{\frac{1}{2}}\Omega_{\omega_2}.$$

Note that

$$\|U\pi_{\omega_1}(x)\Omega_{\omega_1}\|^2 = \|\pi_{\omega_2}(x)S_f^{\frac{1}{2}}\Omega_{\omega_2}\|^2$$
$$= \omega_1(x^*x) = \|\pi_{\omega_1}(x)\Omega_{\omega_1}\|^2.$$

Hence U extends to a well defined isometry. But then one readily calculates that

$$U\pi_{\omega_1}(x)U^* = \pi_{\omega_2}(x)\,|_{R(S_f^{\frac{1}{2}})}$$

i.e., $\pi_{\omega_1}(\mathfrak{A})$ is unitarily equivalent to the subrepresentation of $\pi_{\omega_2}(\mathfrak{A})$ acting on $R(S_f^{\frac{1}{2}})$.

Finally if μ_1 and μ_2 are mutually absolutely continuous then f is positive μ_2-almost everywhere and S_f is invertible by Proposition A2. Thus $R(S_f^{\frac{1}{2}}) = \mathscr{H}_{\omega_2}$ and $(\mathscr{H}_{\omega_1}, \pi_{\omega_1})$ and $(\mathscr{H}_{\omega_2}, \pi_{\omega_2})$ are unitarily equivalent.

Now we are in a position to prove Theorem A1.

Proof of Theorem A1. Let E be an arbitrary Borel set in $E_{\mathfrak{A}}$. Since τ is strongly continuous one can define a unique regular Borel measure μ_f such that

$$\mu_f(E) = \int_{\omega\circ\tau_t \in E} dt f(t).$$

But f has total integral one and hence μ_f is a probability measure. Moreover

$$\int d\mu_f(\omega')\omega' = \int dt f(t)\omega\circ\tau_t = \omega_f.$$

Similarly for e; $e(t) = \exp\{-|t|\}/2$ one can introduce a measure μ_e and a state ω_e. But since f and e are almost everywhere positive the measures μ_f and μ_e

are mutually absolutely continuous and ω_f and ω_e generate unitarily equivalent representations by Proposition A3. But the representation associated with ω_e is covariant, by the proof of Proposition 2, and hence the representation associated with ω_f is also covariant.

Finally we remark that the observation that ω_e generates a covariant representation can be used to reestablish a result of Borchers [9]; the representation $(\mathscr{H}_\omega, \pi_\omega)$ extends to a covariant representation if, and only if, $t \to \omega \circ \tau_t$ is norm continuous. The necessity of the continuity condition is straightforward. The sufficiency follows by noting that ω is the norm limit of the sequence of states

$$\omega_n = \frac{1}{2} \int dt e^{-|t|} \omega \circ \tau_{t/n}$$

and hence π_ω is quasi-contained in the direct sum of the covariant representations π_{ω_n}. In fact Borchers obtains his result for general locally compact groups of automorphisms.

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