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Transformation Groups for Soliton Equations —Euclidean Lie Algebras and Reduction of the KP Hierarchy—

By

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This is the last chapter of our series of papers [1], [3], [10], [11] on transformation groups for soliton equations.

In [1] a link between the KdV (Korteweg de Vries) equation and the affine Lie algebra $A_1^{(1)}$ was found: the vertex operator that affords an explicit realization of the basic representation of $A_1^{(1)}$ [2] acts infinitesimally on the τ functions of the KdV hierarchy. It was shown also that this link between the KdV equation and $A_1^{(1)}$ comes from a similar link between the KP (Kadomtsev-Petviashvili) equation and $\mathfrak{gl}(\infty)$[†]; the restriction to the subalgebra $A_1^{(1)}$ in $\mathfrak{gl}(\infty)$ reduces the KP hierarchy to the KdV hierarchy.

$$\begin{tikzcd}
\text{Soliton equations} & \text{Infinitesimal transformations} \\
\text{KP} \arrow[r, leftrightarrow] \arrow[d, "\text{reduction}"] & \mathfrak{gl}(\infty) \\
\text{KdV} \arrow[r, leftrightarrow] & A_1^{(1)} \arrow[u, hook, "\text{subalgebra}"']
\end{tikzcd}$$

In this paper we carry out a detailed study of reduction problems of this sort using three kinds of master equations: the KP equation, the BKP equation and the 2-component BKP equation [3]. We thus obtain several new series of soliton equations along with the explicit forms of N-soliton solutions. The list of soliton equations and the corresponding Euclidean Lie algebras treated in this paper are given in Table 1. Here we extract the known soliton equations in the list:

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[†] See (1.7) as for the definition of $\mathfrak{gl}(\infty)$.

Soliton equations in Hirota's forms	Reduction type	Euclidean Lie algebras
KdV: $(D_1^4-4D_1D_3)\tau\cdot\tau=0$	$(\mathrm{KP})_2$	$A_1^{(1)}$
Boussinesq: $(D_1^4+3D_2^2)\tau\cdot\tau=0$	$(\mathrm{KP})_3$	$A_2^{(1)}$
Sawada-Kotera [4]: $(D_1^6+9D_1D_5)\tau\cdot\tau=0$	$(\mathrm{BKP})_3$	$A_2^{(2)}$
Ramani [5]: $(D_1^6-5D_1^3D_3-5D_3^2)\tau\cdot\tau=0$	$(\mathrm{BKP})_5$	$A_4^{(2)}$
Ito [6]: $(D_3^2+2D_1^3D_3)\tau\cdot\tau=0$	$(\mathrm{BKP})_6$	$D_3^{(2)}$

where we have used Hirota's symbol $P(D_1, D_2,\dots)f\cdot g=P\Big(\dfrac{\partial}{\partial y_1}, \dfrac{\partial}{\partial y_2},\dots\Big)$ $(f(x_1+y_1, x_2+y_2,\dots)g(x_1-y_1, x_2-y_2,\dots))|_{y_1=0,y_2=0,\dots}$.

In [7], M. and Y. Sato proposed the problem of counting the number of Hirota bilinear equatons of weighted homogeneous degree n in a hierarchy of soliton equations. The above mentioned link and the character formula for Euclidean Lie algebras give us a systematic way of carrying out this counting. Our reasoning owes a great deal to [8]. The results are listed in Table 5.

In the language of quantum field theory [9], the main conclusion of our series of papers [1] [3] [10] [11] can be stated as follows: *The space of τ functions for a hierarchy of soliton equations is the orbit of the vacuum vector for the Fock representation of an infinite dimensional Lie algebra.*

Let us explain briefly the statement above in seven steps. The aim and the idea of the present paper will, then, be much clearer. We take the BKP hierarchy as an example [3].

i) Neutral free fermions.

Let us start with an infinite dimensional orthogonal space $W_B=\bigoplus_{n\in\boldsymbol{Z}}\boldsymbol{C}\phi_n$ with the inner product

$$(\sum_m(-)^m f_m\phi_{-m}, \sum_n(-)^n g_n\phi_{-n})=\sum_n(-)^n f_n g_{-n}.$$

The Clifford algebra on W_B, which we denote by $A(W_B)$, is an algebra generated by $\phi_n(n\in\boldsymbol{Z})$ with the defining relation

$$[\phi_m, \phi_n]_+=(-)^m\delta_{m,-n}.$$

The generators $\phi_n(n\in\boldsymbol{Z})$ are called neutral free fermions.

ii) The vacuum vectors and the Fock representation.

The vacuum $|\,\mathrm{vac}>$ and the anti-vacuum $<\mathrm{vac}\,|$ are defined by the following relations.

$$\begin{aligned} &\phi_n|\,\mathrm{vac}> = 0 \qquad && n \leqq -1\,, \\ &<\mathrm{vac}\,|\phi_n = 0 && n \geqq 1\,, \\ &<\mathrm{vac}\,|\,\mathrm{vac}> = 1\,. \end{aligned}$$

The vector space $A(W)|\mathrm{vac}> = A(W)/A(W)(\underset{n\leqq -1}{\oplus} \boldsymbol{C}\phi_n)$ is called the Fock space, and the representation of the Clifford algebra $A(W_B)$ on $A(W_B)|\,\mathrm{vac}>$, by left multiplication, is called the Fock representation.

iii) The vertex operator and a realization of the Fock representation.

Let $x=(x_1, x_3, \ldots)$ be infinitely many time variables. We define a Hamiltonian $H_B(x)$ by

$$H_B(x) = \frac{1}{2} \sum_{\substack{n\in \boldsymbol{Z}\\ l:\mathrm{odd}}} (-)^{n+1} x_l \phi_n \phi_{-n-l}\,.$$

We set $V = \boldsymbol{C}[x_1, x_3, \ldots]$ and $\mathscr{F}(W_B) = V \otimes \boldsymbol{C}[\phi_0] = V \oplus V\phi_0$. We note that ϕ_0 acts on $\mathscr{F}(W_B)$ by multiplication $\left(\phi_0^2 = \frac{1}{2}\right)$. The following linear differential operator is called a vertex operator:

$$X_B(k) = \exp\,(\sum_{l:\mathrm{odd}} x_l k^l)\, \exp\left(\sum_{l:\mathrm{odd}} -\frac{2}{lk^l} \frac{\partial}{\partial x_l}\right) = \sum_{n\in \boldsymbol{Z}} X_{Bn} k^n\,.$$

Then the following (ι, ρ) gives us a realization of the Fock representation in terms of polynomials and linear differential operators acting on them.

$$\begin{tikzcd} \iota\colon A(W_B)|\,\mathrm{vac}> \arrow[r, "\cong"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathscr{F}(W_B) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ |a> \arrow[r, mapsto] & <\mathrm{vac}\,|e^{H_B(x)}|a> \oplus <\mathrm{vac}\,|\phi_0 e^{H_B(x)}|a> \phi_0 \end{tikzcd}$$

$$\begin{tikzcd} \rho\colon A(W_B) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathrm{End}(\mathscr{F}(W_B)) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ \phi_j \arrow[r, mapsto] & \phi_0 X_{Bj} \end{tikzcd}$$

iv) The Clifford group and its Lie algebra.

The quadratic elements $\phi_i \phi_{-j}$ in $A(W_B)$ span an infinite dimensional Lie algebra $\mathfrak{go}(\infty) = \mathfrak{o}(\infty) \oplus \boldsymbol{C}$. This acts on V through (ι, ρ). The corresponding infinite dimensional group is called the even Clifford group.

v) The τ functions.

Let $\mathscr{L}$ be the orbit of the vacuum $|\,\mathrm{vac}>$ by the action of the even Clifford group. For an element $|L> \in \mathscr{L}$ we define a τ function by

$$\tau(x) = <\mathrm{vac}\, | e^{H_B(x)} | L> .$$

vi) Hirota bilinear equations.

The τ function $\tau(x)$ satisfies the following bilinear identity:

$$\int \underline{dk} X_B(k)\tau(x) \cdot X_B(-k)\tau(x') = \tau(x)\tau(x') \tag{0.1}$$

where $\underline{dk} = dk/2\pi ik$. This is equivalent to infinitely many Hirota bilinear equations for $\tau(x)$:

$$\sum_{j \geqq 1} \tilde{p}_j(2y)\tilde{p}_j(-2D_x) e^{\sum_{l:\mathrm{odd}} y_l D_l} \tau(x) \cdot \tau(x) = 0 \tag{0.2}$$

where $e^{\sum_{l:\mathrm{odd}} k^l x_l} = \sum_{j \geqq 0} \tilde{p}_j(x)$.

vii) The linear problem.

We define a wave function by

$$w(x, k) = X_B(k)\tau(x)/\tau(x) .$$

$w(x, k)$ solves the following linear problem (Zakharov-Shabat problem)

$$\frac{\partial w(x, k)}{\partial x_l} = \left(\frac{\partial^l}{\partial x_1^l} + \sum_{m=1}^{l-2} b_{lm}(x) \frac{\partial^m}{\partial x_1^m} \right) w(x, k) \qquad (l = 1, 3, \ldots) . \tag{0.3}$$

The compatibility conditions for this problem are equivalent to the set of Hirota bilinear equation (0.2). The characteristic feature of (0.3) is that only odd time flows enter and that the constant terms $b_{l0}(x)$ vanish.

The first two weighted homogeneous bilinear equations in the hierarchy are:

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)\tau(x) \cdot \tau(x) = 0 ,$$
$$(D_1^8 + 7D_1^5 D_3 - 35D_1^2 D_3^2 - 21D_1^3 D_5 - 42D_3 D_5 + 90D_1 D_7)\tau(x) \cdot \tau(x) = 0 .$$

The N soliton solution is given by

$$\tau(x) = \sum_{n=0}^{N} \sum_{1 \leqq j_1 < \cdots < j_n \leqq N} a_{j_1} \cdots a_{j_n} \prod_{l < l'} c_{j_l j_{l'}} \exp\left(\sum_{l=1}^{n} \tilde{\xi}(x, p_{j_l}) + \tilde{\xi}(x, q_{j_l}) \right) \tag{0.4}$$

where $\bar{\xi}(x, k) = \sum_{l:\mathrm{odd}} x_l k^l$ and $c_{jj'} = \dfrac{(p_j - p_{j'})(p_j - q_{j'})(q_j - p_{j'})(q_j - q_{j'})}{(p_j + p_{j'})(p_j + q_{j'})(q_j + p_{j'})(q_j + q_{j'})}$. This hierarchy is "sub-sub-holonomic" in the sense that it admits an arbitrary function in *two* variables x_1 and x_3 as an initial value. The presence of two kinds of spectral parameters p_j's and q_j's reflects the sub-sub-holonomic nature.

If we restrict ourselves to hierarchies which are described by a single τ function, we know three kinds of sub-sub-holonomic hierarchies: *the KP hierarchy*, *the BKP hierarchy and the 2-component BKP hierarchy.* The corresponding infinite dimensional Lie algebras are $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ and $\mathfrak{go}(2\infty)$ (see §1).

The reduction problem is stated as follows: *Find a sub-holonomic hierarchy by adding an additional constraint on a sub-sub-holonomic hierarchy.*

For example, the Sawada-Kotera hierarchy

$$\begin{aligned}&(D_1^6+9D_1D_5)\tau(x)\cdot\tau(x)=0,\\&(D_1^8-21D_1^3D_5+90D_1D_7)\tau(x)\cdot\tau(x)=0,\\&\cdots\end{aligned}$$

is obtained by imposing the condition

$$\frac{\partial\tau(x)}{\partial x_{3n}}=0, \qquad (n=1, 3,\dots). \tag{0.5}$$

For the N-soliton solution, this amounts to setting $p_j^3=(-q_j)^3$ in (0.4). We call this reduction the 3-reduction of the BKP hierarchy.

In general, we shall consider the l-reduction of the KP hierarchy, the l-reduction of the BKP hierarchy and the (l_1, l_2)-reduction of the 2-component BKP hierarchy (with l_1+l_2: even). We note that the additional constraint for the τ function in the case of even reduction of the BKP types is subtler than (0.5).

In our philosophy, "reduction" means: to find an appropriate subalgebra of $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ and $\mathfrak{go}(2\infty)$. It is remarkable that the reductions above lead us to subalgebras known as Euclidean Lie algebras [12] [13], and that the Fock representation induces their basic representations [2] [14] [15] [16]. We should comment on the references in this direction. As mentioned in the beginning, in [2], an explicit realization of the basic representation of $A_1^{(1)}$ was first constructed by using a vertex operator. The construction was generalized to "most" of the Euclidean Lie algebras in [14]. We owe very much to these constructions. The construction in [15] corresponds to the reduction of the multi-component KP or BKP hierarchies. We do not discuss this problem here. Priority in using the Clifford algebra in the representation theory of Euclidean Lie algebras is attributed to [16]. This work, however, lacks an explicit realization of the Fock representation, which is essential in establishing the link with soliton equations.

This paper is organized as follows: In Section 1 we discuss the structure of infinite dimensional Lie algebras which govern the *KP* hierarchy, etc. The reductions are discussed in Section 2. An explicit realization of $A_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_n^{(1)}$ in $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ and $\mathfrak{go}(2\infty)$ is given in the appendix.

In Section 3 we solve the problem of M. and Y. Sato on the number of Hirota bilinear equations.

It is a pleasure to express our thanks to M. and Y. Sato whose work led us to the present study. We are grateful to J. Lepowsky for sending us several preprints. Among them [8] and [14] are very helpful to this work. We thank also Y. Tanaka for his clear explanation of the Kac-Moody Lie algebras. We benefited by discussions with H. Flaschka, R. Hirota and J. Satsuma.

§1. The Infinite Dimensional Lie Algebras

In [1], we showed that $\mathfrak{gl}(\infty)$ operates on the space of τ-functions of the KP-hierarchy through the vertex operator. In this section, we shall discuss this action more precisely. The Lie algebra $\mathfrak{gl}(\infty)$ operates on the space of functions in $x_1, x_2, x_3, \dots$. For the sake of definiteness, we shall restrict this representation to the space $V=\boldsymbol{C}[x_1, x_2, x_3, \dots]$ of polynomials. Modifying the vertex operator in [1] [11] by a constant, we set

$$\begin{aligned} Z(p, q) &= \frac{q}{p-q}\left(e^{\xi(x,p)-\xi(x,q)}e^{-\xi(\tilde{\partial},p^{-1})+\xi(\tilde{\partial},q^{-1})}-1\right) \\ &= \sum Z_{ij}p^i q^{-j}\,. \end{aligned} \tag{1.1}$$

Here $\xi(x, p)=\sum x_j p^j$ and $\tilde{\partial}=\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \dots\right)$. Since $\exp(\xi(x, p)-\xi(x, q))\exp(-\xi(\tilde{\partial}, p^{-1})+\xi(\partial, q^{-1}))-1$ vanishes when $p=q$, $Z(p, q)$ and $Z_{i,j}$ are well-defined.

The commutation relations among the $Z_{i,j}$'s are calculated in [1]:

$$[Z_{i,j}, Z_{i',j'}]=\delta_{i',j}Z_{i,j'}-\delta_{i,j'}Z_{i',j}+\delta_{ij'}\delta_{ji'}(Y_+(j)-Y_+(i)) \tag{1.2}$$

where $Y_+(j)=0$ for $j<0$ and $=1$ for $j\geqq 0$. For a positive integer r, we set $V_r=\{f\in V;\ \deg f\leqq r\}$. Here, we count the degree of x_j as j. Then, an easy calculation leads us to

$$\begin{aligned} &Z_{i,j}V_r\subset V_{r+i-j} \text{ and} \\ &Z_{i,j}|_{V_r}=0 \text{ except when } i\geqq -r,\ j<r,\ i-j>-r\,. \end{aligned} \tag{1.3}$$

Hence, if we define $\tilde{\mathfrak{g}}$ to be the vector space spanned by 1 and $\{\sum a_{i,j}Z_{ij};\ a_{ij}=0$ for $|i-j|\gg 0\}$, then $\tilde{\mathfrak{g}}$ is a Lie algebra operating on V.

Equating the coefficients of $Z(p, p)$ (1.1), we obtain

$$\begin{aligned} &jx_j=\Lambda_{-j},\ \partial/\partial x_j=\Lambda_j \qquad \text{for } j=1, 2, \dots \\ &\text{and } \Lambda_0=0, \end{aligned} \tag{1.4}$$

where $A_n = \sum_{\nu \in \boldsymbol{Z}} Z_{\nu,\nu+n}$. From $\partial Z(p, q)/\partial p\,|_{p=q}$, we obtain

$$\sum jx_j \frac{\partial}{\partial x_j} = -\sum_{\nu \in \boldsymbol{Z}} \nu Z_{\nu,\nu}\,. \tag{1.5}$$

The structure of $\tilde{\mathfrak{g}}$ is clarified and formulated as follows. We set

$$\mathfrak{pgl}(\infty) = \{\sum a_{ij}E_{i,j};\ a_{ij}=0 \quad \text{for } |i-j| \gg 0\} \tag{1.6}$$

where $E_{i,j}$ is a matrix of infinite size whose entries are 0 but for a 1 at the (i, j)-place. Sometimes, it is convenient to identify $\boldsymbol{C}^{\infty}$ with $W = \boldsymbol{C}[k, k^{-1}]$ and $E_{ij}k^{-\nu} = \delta_{j,\nu}k^{-i}$. Then $A_n = \sum_{\nu} E_{\nu,\nu+n}$ is nothing but the multiplication by k^n.

We define the central extension $\mathfrak{gl}(\infty) = \mathfrak{pgl}(\infty) \oplus \boldsymbol{C}z$ by

$$[A \oplus cz,\ A' \oplus c'z] = [A, A'] \oplus c(A, A')z \tag{1.7}$$

where $c(A, A')$ is the skew-symmetric bilinear form on $\mathfrak{pgl}(\infty)$ given by

$$c(\sum a_{ij}E_{ij}, \sum a'_{ij}E_{ij}) = \sum a_{ij}a'_{ji}(Y_+(j) - Y_+(i))\,. \tag{1.8}$$

Then we obtain

Proposition 1.1. *$\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{gl}(\infty)$ as a Lie algebra by the correspondence $Z_{ij} \leftrightarrow E_{ij}$ and $1 \leftrightarrow z$.*

Remark. The choice of $Z(p, q)$ and $c(A, A')$ in this section are neatly explained in terms of field operators. As shown in III [11], the charged fermion fields ψ_j, ψ_j^* $(j \in \boldsymbol{Z})$ operate on $V' = \boldsymbol{C}[u, u^{-1}, x_1, x_2, x_3, \ldots]$. Then $\tilde{\mathfrak{g}}$ is nothing but the Lie algebra generated by 1 and $\sum a_{ij}\psi_i\psi_j^*$ $(a_{ij}=0$ for $|i-j| \gg 0)$ which operates on $V \subset V'$. We decompose $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{C} \cdot 1$ where $\mathfrak{g} = \{\sum a_{ij} :\psi_i\psi_j^*:;\ a_{ij}=0$ for $|i-j| \gg 0\}$. Then Z_{ij} coincides with $\psi_i\psi_j^*$ and $c(E_{ij}, E_{i'j'})$ coincides with the expectation value of $[:\psi_i\psi_j^*:,\ :\psi_{i'}\psi_{j'}^*:]$.

The considerations on the KP-hierarchy given above apply as well to the BKP-hierarchy. Set $V_B = \boldsymbol{C}[x_1, x_3, x_5, \ldots]$ and consider the vertex operator

$$\begin{aligned} Z_B(p, q) &= \frac{p-q}{2(p+q)}\left(e^{\tilde{\xi}(x,p)+\tilde{\xi}(x,q)}e^{-2\tilde{\xi}(\tilde{\partial},p^{-1})-2\tilde{\xi}(\tilde{\partial},q^{-1})} - 1\right) \\ &= \sum Z_{B,ij}p^iq^{-j}\,. \end{aligned} \tag{1.9}$$

Here $\tilde{\xi}(x, p) = \sum_{j\,\mathrm{odd}>0} x_jp^j$ and $\tilde{\partial} = \left(\frac{\partial}{\partial x_1}, \frac{1}{3}\frac{\partial}{\partial x_3}, \frac{1}{5}\frac{\partial}{\partial x_5}, \ldots\right)$. Then $Z_{B,ij}$ satisfies

$$Z_{B,ij} = -Z_{B,-j,-i} \tag{1.10}$$

$$[Z_{B,ij}, Z_{B,i'j'}]=(-)^j\delta_{j,i'}Z_{B,ij'}-(-)^j\delta_{j,-j'}Z_{B,i,-i'} \\ -(-)^i\delta_{i,-i'}Z_{B,-j,j'}-(-)^{j'}\delta_{i,j'}Z_{B,j'i'} \\ +(-)^{i+j}(\delta_{i'j}\delta_{ij'}-\delta_{i,-i'}\delta_{j,-j'})(Y_B(j)-Y_B(i)). \tag{1.11}$$

Here $Y_B(j)=1$ for $j>0$, $=1/2$ for $j=0$ and $=0$ for $j<0$. Thus, the vector space $\tilde{\mathfrak{g}}_B$ spanned by 1 and $\{\sum a_{ij}Z_{B,ij};\ a_{ij}=0 \text{ for } |i-j|\gg 0\}$ becomes a Lie algebra operating on V_B. Set $\mathfrak{o}(\infty)=\{\sum a_{ij}E_{ij};\ a_{ij}=(-)^{i+j+1}a_{-j,-i}$ for any i, j and $a_{ij}=0$ for $|i-j|\gg 0\}$. Then $\mathfrak{o}(\infty)$ is the orthogonal Lie algebra on $W=\boldsymbol{C}^\infty =\{f(k)=\sum \xi_i k^{-i}\in\boldsymbol{C}[k, k^{-1}]\}$ equipped with the inner product $(f, f)=\sum (-)^i\xi_i\xi_{-i}=\operatorname{Res}_{k=0}f(k)f(-k)dk/k$. We define the central extension $\mathfrak{go}(\infty)=\mathfrak{o}(\infty)\oplus\boldsymbol{C}z$ of $\mathfrak{o}(\infty)$ by the formula

$$[A\oplus cz, A'\oplus c'z]=[A, A']\oplus c_B(A, A'). \tag{1.12}$$

Here $c_B(A, A')$ is the skew-symmetric bilinear form on $\mathfrak{o}(\infty)$ given by

$$c_B(\sum a_{ij}E_{ij}, \sum a'_{ij}E_{ij})=\frac{1}{2}\sum a_{ij}a'_{ji}(Y_B(j)-Y_B(i)). \tag{1.13}$$

Proposition 1.2. *$\tilde{\mathfrak{g}}_B$ is isomorphic to $\mathfrak{go}(\infty)$ by the correspondence $Z_{ij}\leftrightarrow(-)^jE_{ij}-(-)^iE_{-j,-i}$ and $1\leftrightarrow z$.*

By calculating $Z(p, -p)$ and $\partial Z(p, q)/\partial q|_{p=-q}$ we obtain

$$jx_j=2\Lambda_{-j} \quad\text{and}\quad \partial/\partial x_j=\Lambda_j \qquad \text{for } j=1, 3,\dots$$

where

$$\Lambda_j=\sum E_{\nu,\nu+j} \tag{1.14}$$

and

$$\sum_{\substack{j>0\\ \text{odd}}} jx_j\partial/\partial x_j=-\sum_{\nu\in\boldsymbol{Z}}\nu E_{\nu,\nu}. \tag{1.15}$$

Although the discussions above apply to multi-component KP- and BKP-hierarchies, we shall here treat the two-component BKP-hierarchy. Otherwise, we should need a different kind of vertex operator, such as $e^{\xi(x^{(\alpha)},p)-\xi(x^{(\beta)},q)} \times e^{-\xi(\tilde{\partial}^{(\alpha)},p^{-1})+\xi(\tilde{\partial}^{(\beta)},q^{-1})}u_\alpha u_\beta$ (see III [11]).

Prepare two series of independent variables x_1, x_3, x_5,... and y_1, y_3, y_5,... . Set $V_{2B}=\boldsymbol{C}[x_1, x_3, x_5,\dots, y_1, y_3, y_5,\dots]$. Consider the vertex operators

$$Z^{11}(pq)=\frac{p-q}{2(p+q)}(e^{\tilde{\xi}(x,p)+\tilde{\xi}(x,q)}e^{-2\tilde{\xi}(\tilde{\partial}_x,p^{-1})-2\tilde{\xi}(\tilde{\partial}_x,q^{-1})}-1) \\ =\sum Z^{11}_{ij}p^iq^{-j} \tag{1.16}$$

$$Z^{22}(pq)=\frac{p-q}{2(p+q)}(e^{\tilde{\xi}(y,p)+\tilde{\xi}(y,q)}e^{-2\tilde{\xi}(\tilde{\partial}_y,p^{-1})-2\tilde{\xi}(\tilde{\partial}_y,q^{-1})}-1)$$
$$=\sum Z_{ij}^{22}p^iq^{-j}$$
$$Z^{12}(p,\,q)=\frac{1}{2\sqrt{-1}}e^{\tilde{\xi}(x,p)+\tilde{\xi}(y,q)}e^{-2\tilde{\xi}(\tilde{\partial}_x,p^{-1})-2\tilde{\xi}(\tilde{\partial}_y,q^{-1})}$$
$$=\sum Z_{ij}^{12}p^iq^{-j}$$

and

$$Z^{21}(p,\,q)=-Z^{12}\,(q,\,p)=\sum Z_{ij}^{21}p^iq^{-j}\,.$$

Let $\tilde{\mathfrak{g}}_{2B}$ be the vector space spanned by 1 and

$$\{\sum_{\alpha,\beta=1,2}\sum_{i,j}a_{ij}^{\alpha\beta}Z_{ij}^{\alpha\beta};\quad a_{ij}^{\alpha\beta}=0\quad\text{for}\quad|i-j|\gg 0\}\,.$$

Then $\tilde{\mathfrak{g}}_{2B}$ becomes a Lie algebra operating on V_{2B}. Set $V_{2B}=\mathbb{C}[k_1,\,k_1^{-1}]\oplus\mathbb{C}[k_2,\,k_2^{-1}]$ and let $E_{ij}^{\alpha\beta}$ denote the endomorphism of V_{2B} defined by $E_{ij}^{\alpha\beta}k_\gamma^{-l}=\delta_{\beta,\gamma}\delta_{j,l}k_\alpha^{-i}$ for α, β, $\gamma=1$, 2 and i, j, $l\in\mathbb{Z}$. We define

$$\mathfrak{o}(2\infty)=\{\sum_{\substack{\alpha,\beta=1,2\\ i,j\in\mathbb{Z}}}a_{ij}^{\alpha\beta}E_{i,j}^{\alpha\beta}\,;\,a_{ij}^{\alpha\beta}=(-)^{i+j+1}a_{-j,-i}^{\alpha\beta}\quad\text{and}\quad a_{ij}^{\alpha\beta}=0\quad\text{for}\quad|i-j|\gg 0\}\,.\tag{1.17}$$

Then $\mathfrak{o}(2\infty)$ is the orthogonal Lie algebra on V_{2B} with the inner product $(f(k_1)\oplus g(k_2))^2=\mathrm{Res}_{\,k_1=0}f(k_1)f(-k_1)dk_1/k_1+\mathrm{Res}_{k_2=0}\,g(k_2)g(-k_2)dk_2/k_2$.

Let $\mathfrak{go}(2\infty)=\mathfrak{o}(2\infty)\oplus\mathbb{C}z$ be the central extension of $\mathfrak{o}(2\infty)$ defined by

$$[A\oplus cz,\,A'\oplus c'z]=[A,\,A']\oplus c_{2B}(A,\,A')z\tag{1.18}$$

for A, $A'\in\mathfrak{o}(2\infty)$ and c, $c'\in\mathbb{C}$. Here $c_{2B}(A,\,A')$ is the skew-symmetric bilinear form on $\mathfrak{o}(2\infty)$ given by

$$c_{2B}(\sum a_{ij}^{\alpha\beta}E_{ij}^{\alpha\beta},\,\sum a_{ij}'^{\alpha\beta}E_{ij}^{\alpha\beta})=\frac{1}{2}\sum_{\alpha,\beta,i,j}a_{ij}^{\alpha\beta}a_{ji}'^{\alpha\beta}(Y_B(j)-Y_B(i))\tag{1.19}$$

where $Y_B(i)$ is as in (1.11).

Then as in the earlier cases, we obtain

Proposition 1.3. *The Lie algebra* $\tilde{\mathfrak{g}}_{2B}$ *is isomorphic to* $\mathfrak{go}(2\infty)$ *by the correspondence* $Z_{ij}^{\alpha\beta}\longleftrightarrow(-)^jE_{ij}^{\alpha\beta}-(-)^iE_{-j,-i}^{\beta\alpha}$ *and* $1\longleftrightarrow z$.

§ 2. Reductions and Euclidean Lie Algebras

In the previous papers [1] [3] [11] we have mainly treated hierarchies of "sub-sub-holonomic" nonlinear equations and their transformation groups. Now we shall discuss the problem of reducing them into "sub-holonomic" hierarchies by suitably imposing additional constraints.

For the sake of definiteness, we consider first the case of the KP hierarchy defined through the linear problem [17]

$$(2.1) \qquad L(x, \partial)w = kw, \; L(x, \partial) = \partial + a_2(x)\partial^{-1} + a_3(x)\partial^{-2} + \cdots$$

$$(2.2) \qquad \frac{\partial}{\partial x_n} w = B_n(x, \partial)w, \; B_n(x, \partial) = [L(x, \partial)^n]_+ \quad (n = 1, 2, 3, \ldots).$$

Here $\partial = \frac{\partial}{\partial x_1}$, and $[L(x, \partial)^n]_+$ denotes the differential operator part of $L(x, \partial)^n$. For a fixed positive integer l, we impose the constraint

$$(2.3)_l \qquad L(x, \partial)^l = \text{a differential operator} = B_l(x, \partial).$$

In other words, we consider along with (2.2) the linear constraint $\frac{\partial w}{\partial x_l} = k^l w$, i.e. the linear eigenvalue problem

$$(2.4)_l \qquad B_l(x, \partial)w = k^l w.$$

For example, the cases $l = 2$ or $l = 3$,

$$(2.4)_2 \qquad (\partial^2 + 2a_2(x))w = k^2 w$$

$$(2.4)_3 \qquad (\partial^3 + 3a_2(x)\partial + 3(a_3(x) + \partial a_2(x)))w = k^3 w$$

together with (2.2) give rise to the hierarchy of the higher order KdV or Boussinesq equations, respectively. The general case of $(2.4)_l$ will be called the l-reduced KP hierarchy.

Recall that the KP τ function $\tau(x)$ is related to the formal solution of (2.1) and (2.2) through $w(x, k) = e^{\xi(x,k)} e^{-\xi(\tilde{\partial}, k^{-1})} \tau(x)/\tau(x)$[1]. In terms of $\tau(x)$, the condition $(2.4)_l$ is restrated simply as

$$(2.5)_l \qquad \frac{\partial}{\partial x_l} \tau(x) = \text{const. } \tau(x).$$

We remark that, once $(2.3)_l$ or $(2.4)_l$ is satisfied for an $l \in \boldsymbol{Z}$, then they are valid for all integral multiples of l:

$$(2.3)'_l \qquad L(x, \partial)^j = \text{a differential operator}, \; j \equiv 0 \mod l, \; j > 0$$

$$(2.4)'_l \qquad B_j(x, \partial)w = k^j w, \qquad j \equiv 0 \mod l, \; j > 0.$$

Making use of the freedom $\tau(x) \mapsto e^{\sum_{j=1}^{\infty} c_j x_j} \tau(x)$, $w(x, k) \mapsto f(k)w(x, k)$ ($f(k)$

$=\exp(-\sum_{j=1}^{\infty}\frac{1}{j}c_jk^j)=1+O(k^{-1}))$ in the choice of $w(x, k)$ and $\tau(x)$, we can thus rephrase $(2.5)_l$ as

$$\frac{\partial}{\partial x_j}\tau(x)=0, \quad j\equiv 0 \mod l, j>0. \tag*{$(2.5)'_l$}$$

In terms of the Grassmann formulation of Sato [17], this means that the corresponding point $A=(a'_{\mu\nu})_{\mu\in\boldsymbol{Z}, \nu<0}$ of the Grassmann manifold satisfies this condition:

there exist $S_j\in GL(\infty)$ such that

$(S_j)_{\mu\nu}=0 \quad (\mu\geqq 0, \nu<0) \quad$ and $\quad \Lambda^jA=AS_j, \quad j\equiv 0 \mod l$

where $\Lambda=(\delta_{\mu+1,\nu})_{\mu,\nu\in\boldsymbol{Z}}$. Consequently, if $P(D)\tau\cdot\tau=0$ is one of the KP bilinear equations, then by setting $P(D)_l=P(D)|_{D_l=0,D_{2l}=0,\ldots}$ we obtain a bilinear equation $P(D)_l\tau\cdot\tau=0$ for the l-reduced KP hierarchy. Likewise a modified KP equation $Q(D)\tau_{[k]}\cdot\tau=0$ gives us an l-reduced one $Q(D)_l\tau_{[k]}\cdot\tau=0$, where $Q(D)_l=Q(D)|_{D_l=k^l,D_{2l}=k^2\ldots}$.

The transformation group of the l-reduced KP hierarchy consists of transformations that are compatible with the constraint $(2.5)'_l$. Let $X\in\mathrm{gl}(\infty)$ be an infinitesimal transformation of the KP hierarchy. Then X preserves $(2.5)'_l$ provided

$$\left[\frac{\partial}{\partial x_j}, X\right]=0, \quad [x_j, X]=0, \quad j\equiv 0 \mod l \tag*{$(2.6)_l$}$$

where $\frac{\partial}{\partial x_j}=\Lambda_j=\sum_{i\in\boldsymbol{Z}}E_{i,i+j}$ and $jx_j=\Lambda_{-j}=\sum_{i\in\boldsymbol{Z}}E_{i,i-j}$ (see (1.4)). We denote by $\mathrm{gl}(\infty)_l$ the Lie subalgebra of $\mathrm{gl}(\infty)$ whose elements satisfy $(2.6)_l$. If we write $X=A\oplus cz$ with $A=\sum_{i,j\in\boldsymbol{Z}}a_{ij}E_{ij}\in\mathrm{gl}(\infty)$ and $c\in\boldsymbol{C}$, condition $(2.6)_l$ implies that $a_{i+l,j+l}=a_{ij}$. Thus the block partition $A=(A_{\mu\nu})_{\mu,\nu\in\boldsymbol{Z}}$, $A_{\mu\nu}=(a_{i+\mu l,j+\nu l})_{0\leqq i,j\leqq l-1}$, has the structure $A_{\mu\nu}=A_{0,\nu-\mu}$. This allows us to identify A with an $l\times l$ matrix of Laurent polynomials

$$A\longleftrightarrow A(t)=\sum_{\nu\in\boldsymbol{Z}}A_{0\nu}t^{\nu}. \tag{2.7}$$

In fact, $(2.6)_l$ says that, when viewed as a linear transformation on $\boldsymbol{C}[k, k^{-1}]$, A commutes with multiplication by $t=k^l$ (see page 1083). Hence it may be regarded as an element of $\mathrm{End}_{\boldsymbol{C}[t,t^{-1}]}(\boldsymbol{C}[k, k^{-1}])\cong\mathrm{gl}(l; \boldsymbol{C}[t, t^{-1}])$ (this is what (2.7) means). With this identification (2.7), the skew-symmetric bilinear form (1.8) reduces to

$$c(A, A')=\mathrm{Res\,trace}\frac{dA}{dt}(t)A'(t)=\sum_{\nu\in\boldsymbol{Z}}\nu\ \mathrm{trace}\,A_{0\nu}A'_{0-\nu}. \tag{2.8}$$

From this we see, in particular, that trace $A_{0\nu}=0$ $(\nu\in\boldsymbol{Z})$ for $A\oplus cz\in\mathfrak{gl}(\infty)_l$. Therefore

$$\mathfrak{gl}(\infty)_l\cong\mathfrak{sl}(l;\ \boldsymbol{C}[t,\ t^{-1}])\oplus\boldsymbol{C}Z\,. \tag{2.9}$$

Here the right hand side singnifies the central extension of $\mathfrak{sl}(l;\ \boldsymbol{C}[t,\ t^{-1}])$ through (2.8), known in the literature [14] as the Euclidean Lie algebra $A_{l-1}^{(1)}$.

The representation of $\mathfrak{gl}(\infty)$ via the vertex operator (1.1) carries over to the present situation. Since $\left[\frac{\partial}{\partial x_j},\ Z(p,\ q)\right]=(p^j-q^j)Z(p,\ q)$ and $[x_j,\ Z(p,\ q)]=j^{-1}(p^{-j}-q^{-j})Z(p,\ q)$ $(p\neq q)$, the specialization $p^l=q^l$ of $Z(p,\ q)$ belongs to $\mathfrak{gl}(\infty)_l$:

$$\begin{aligned}Z(p,\ \omega p)&=\frac{\omega}{1-\omega}\Big(\exp\big(\sum_{j\geqq 1}(1-\omega^j)p^jx_j\big)\\&\qquad\times\exp\Big(-\sum(1-\omega^{-j})p^{-j}\frac{1}{j}\frac{\partial}{\partial x_j}\Big)-1\Big)\\&=\sum Z_j(\omega)p^j\qquad(\omega^l=1,\ \omega\neq 1)\,.\end{aligned}\tag{2.10}$$

Conversely, $\mathfrak{gl}(\infty)_l$ is spanned by the homogeneous components $Z_j(\omega)$ of (2.10) and $1, x_j$ and $\frac{\partial}{\partial x_j}$ $(j\equiv 0 \mod l)$. Thus the l-reduction procedure naturally leads to a linear representation of $A_{l-1}^{(1)}$ on the space $\boldsymbol{C}[x_j\ (j\not\equiv 0 \mod l)]$ by means of the vertex operator (2.10). This representation is identical with the one obtained by Kac-Kazhdan-Lepowsky-Wilson [14]. The explicit form of (2.10) is of interest from the viewpoint of soliton theory, for the functional form of the N-soliton τ functions $\tau_N(x)$ is immediately read off from (2.10). In the present case of the l-reduced KP hierarchy, we have

$$\tau_N(x)=\tau_{N,\mathrm{KP}}\left(x;\ \begin{matrix}a_1\\p_1,\ q_1\end{matrix},\dots,\begin{matrix}a_N\\p_N,\ q_N\end{matrix}\right)\Bigg|_{p_1^l=q_1^l,\dots,p_N^l=q_N^l}$$

where $\tau_{N,\mathrm{KP}}$ denotes the N soliton KP τ function (5) in [1].

With minor changes, the considerations above apply also to the BKP and the two-component BKP hierarchy. Let us work out first the corresponding transformation groups. We set

(2.11)

$$\begin{aligned}&\mathfrak{go}(\infty)_l=\{A\oplus cz\in\mathfrak{go}(\infty)\,|\,A=\sum_{i,j\in\boldsymbol{Z}}a_{ij}E_{ij},\ a_{ij}=(-)^{i+j+1}a_{-j,-i}=a_{i+l,j+l}\}\\&\mathfrak{go}(2\infty)_{l_1,l_2}=\{A\oplus cz\in\mathfrak{go}(2\infty)\,|\,A=\sum_{\alpha,\beta=1,2}\ \sum_{i,j\in\boldsymbol{Z}}a_{ij}^{\alpha\beta}E_{ij}^{\alpha\beta}\,,\\&\qquad\qquad a_{ij}^{\alpha\beta}=(-)^{i+j+1}a_{-j,-i}^{\beta\alpha}=a_{i+l_\alpha,j+l_\beta}^{\alpha\beta}\}\,.\end{aligned}$$

In the second line of (2.11) we assume that l_1+l_2 is even (otherwise $\mathfrak{go}(2\infty)_{l_1,l_2}$ splits into $\mathfrak{go}(\infty)_{l_1}\oplus\mathfrak{go}(\infty)_{l_2}$). Note also that, for odd l, l_1 and l_2, (2.11) can be rephased as

$$(2.11)' \quad \begin{aligned} &\mathfrak{go}(\infty)_l=\{X\in\mathfrak{go}(\infty)\,|\,[X,\,A_{\nu l}]=0 \qquad \text{for all} \quad \nu:\text{odd}\}\,, \\ &\mathfrak{go}(2\infty)_{l_1,l_2}=\{X\in\mathfrak{go}(2\infty)\,|\,[X,\,A^{11}_{\nu l_1}+A^{22}_{\nu l_2}]=0 \qquad \text{for all} \quad \nu:\text{odd}\}\,. \end{aligned}$$

(In the case of even l, l_1 and l_2, $A_{\nu l}$ or $A^{\alpha\alpha}_{\nu l_\alpha}$ do not belongs to $\mathfrak{go}(\infty)$ or $\mathfrak{go}(2\infty)$).

Just as for KP, $\mathfrak{go}(\infty)_l$ and $\mathfrak{go}(2\infty)_{l_1,l_2}$ have realizations as Lie algebras over Laurent polynomials (see e.g. [14]). Recall that $\mathfrak{o}(\infty)$ is the orthogonal Lie algebra on the linear space $\boldsymbol{C}[k,\,k^{-1}]$ with respect to the inner product given in page 1084. We define for $f,\,g\in\boldsymbol{C}[k,\,k^{-1}]$,

$$(2.12) \qquad \begin{aligned} \langle f,\,g\rangle_l(t)&=\sum_{\nu\in\boldsymbol{Z}}(A_{-\nu l}f,\,g)t^\nu \\ &=\sum_{\nu\in\boldsymbol{Z}}\oint k^{-\nu l}f(k)g(-k)\underline{dk}\cdot t^\nu\in\boldsymbol{C}[t,\,t^{-1}]\,. \end{aligned}$$

We have then $\langle A_{\nu l}f,\,g\rangle_l(t)=t^\nu\langle f,\,g\rangle_l(t)$. Moreover, for even l, $\langle f,\,g\rangle_l(t)$ is symmetric, and for odd l, is "Hermitian" in the sense that $\langle g,f\rangle_l(t)=\langle f,\,g\rangle_l(-t)$. Since $\mathfrak{go}(\infty)_l$ preserves the bilinear form (2.12), we have

$$(2.13) \qquad \mathfrak{go}(\infty)_l\cong\begin{cases} \mathfrak{su}(l;\,\boldsymbol{C}[t,\,t^{-1}])\oplus\boldsymbol{C}z & (l\text{ odd}) \\ \mathfrak{o}(l;\,\boldsymbol{C}[t,\,t^{-1}])\oplus\boldsymbol{C}z & (l\text{ even})\,. \end{cases}$$

More explicitly, the right hand side means

$$\begin{aligned} &\mathfrak{su}(l;\,\boldsymbol{C}[t,\,t^{-1}])=\{A(t)\in\mathfrak{sl}(l;\,\boldsymbol{C}[t,\,t^{-1}])\,|\,J_l(t)A(-t)+{}^tA(t)J_l(t)=0\} \\ &\mathfrak{o}(l;\,\boldsymbol{C}[t,\,t^{-1}])=\{A(t)\in\mathfrak{sl}(l;\,\boldsymbol{C}[t,\,t^{-1}])\,|\,J_l(t)A(t)+{}^tA(t)J_l(t)=0\} \end{aligned}$$

where we have used the identification (2.7), and

$$J_l(t)=(\langle k^{-i},\,k^{-j}\rangle(t))_{0\leqq i,j\leqq l-1}=\left(\begin{array}{c|cccc} 1 & & & & \\ \hline & & & & (-)^{l-1}t^{-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & t^{-1} & & \\ & -t^{-1} & & & \end{array}\right).$$

The rule of the central extension in the right hand side of (2.13) is given through

$$(2.14) \qquad c_B(A,\,A')=\frac{1}{2}\operatorname{Res}\operatorname{trace}\frac{dA}{dt}(t)A'(t)dt\,.$$

Likewise, for $A\oplus cz\in\mathfrak{go}(2\infty)_{l_1,l_2}$, $A=\sum_{\alpha,\beta=1,2}\sum_{i,j\in\boldsymbol{Z}}a^{\alpha\beta}_{ij}E^{\alpha\beta}_{ij}$, we set $A_{\mu\nu}=\begin{pmatrix} A^{11}_{\mu\nu} & A^{12}_{\mu\nu} \\ A^{21}_{\mu\nu} & A^{22}_{\mu\nu} \end{pmatrix}$,

$A^{\alpha\beta}_{\mu\nu}=(a^{\alpha\beta}_{i+\mu l,j+\nu l})_{\substack{0\leqq i\leqq l_\alpha\\0\leqq j\leqq l_\beta}}$. Then under the identification (2.7) we have

$$\text{(2.15)}\qquad \mathfrak{go}(2\infty)_{l_1,l_2}\cong\{A(t)\in\mathfrak{sl}(l_1+l_2;\,\boldsymbol{C}[t,t^{-1}])|\\ (J_{l_1}(t)\oplus J_{l_2}(t))A(\pm t)+{}^tA(t)(J_{l_1}(t)\oplus J_{l_2}(t))=0\}\oplus\boldsymbol{C}z$$

according as $(-)^{l_1}=(-)^{l_2}=\pm1$. The role of the central extension is again given by (2.14).

Actually the Lie algebras $\mathfrak{go}(\infty)_l$ and $\mathfrak{go}(2\infty)_{l_1,l_2}$ are isomorphic to one of the Euclidean Lie algebras $A^{(2)}_{2n}$, $D^{(2)}_{n+1}$, $A^{(2)}_{2n-1}$ and $D^{(1)}_n$ (see Table 1). For the reader's reference we give below the definition of Kac-Moody Lie algebras, among which the Euclidean algebras form an important class. For details see [12], [13], [14].

In general a Kac-Moody Lie algebra is a complex Lie algebra, defined by giving 3(n+1) generators e_i, f_i, h_i $(0\leqq i\leqq n)$ and defining relations of the following form:

$$\text{(2.16)}\qquad \begin{aligned}&[e_i,f_j]=\delta_{ij}h_i, && [h_i,h_j]=0,\\ &[h_i,e_j]=C_{ij}e_j, && [h_i,f_j]=-C_{ij}f_j,\\ &(ad\ e_i)^{1-C_{ij}}e_j=0, && (ad\ f_i)^{1-C_{ij}}f_j=0.^{\dagger)}\end{aligned}$$

Here the C_{ij}'s are integers such that $C_{ii}=2$, $C_{ij}\leqq0$ $(i\neq j)$ and $C_{ij}=0$ if $C_{ji}=0$. The matrix $C=(C_{ij})_{0\leqq i,j\leqq n}$ is called a (generalized) Cartan matrix. In Table 1 the matrix of C is tabulated for $A^{(1)}_n$, $A^{(2)}_{2n}$, $D^{(2)}_{n+1}$, $A^{(2)}_{2n-1}$ and $D^{(1)}_n$. In order to identify $\mathfrak{gl}(\infty)_l$, $\mathfrak{gl}(\infty)_l$ and $\mathfrak{gl}(2\infty)_{l,l'}$ with these algebras, it is sufficient to know the canonical generators e_i, f_i and h_i. In Table 2 we have given a choice of them in terms of infinite matrices in $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ and $\mathfrak{go}(2\infty)$. Using this one can verify the commutation relations (2.16) by a direct calculation.

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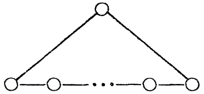
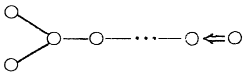
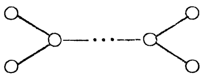
†) $(\mathrm{ad}\ X)^nY=\overbrace{[X,\ldots,[X,[X,Y]],\ldots]}^{n\ \text{times}}$.

Table 1.

reduction	Lie algebra	Cartan matrix	Dynkin diagram†	realization
$(\mathrm{KP})_{n+1}$	$A_n^{(1)}$	$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$		$\mathfrak{sl}(n+1;\mathbf{C}[t,t^{-1}])\oplus\mathbf{C}z$
$(\mathrm{BKP})_{2n+1}$	$A_{2n}^{(2)}$	$\begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & \ddots & 2 & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}$		$\mathfrak{su}(2n+1;\mathbf{C}[t,t^{-1}])\oplus\mathbf{C}z$
$(\mathrm{BKP})_{2n+2}$	$D_{n+1}^{(2)}$	$\begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$		$\mathfrak{o}(2n+2;\mathbf{C}[t,t^{-1}])\oplus\mathbf{C}z$
$(\mathrm{BKP\ II})_{2r+1,2s+1}$ $(n=r+s+1)$	$A_{2n-1}^{(2)}$	$\begin{pmatrix} 2 & & -1 & & & \\ & 2 & -1 & & & \\ -1 & -1 & 2 & -1 & & \\ & & -1 & 2 & \ddots & \\ & & & \ddots & \ddots & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}$		$\mathfrak{su}(2r+1,2s+1;\mathbf{C}[t,t^{-1}])\oplus\mathbf{C}z$
$(\mathrm{BKP})\ \mathrm{II}_{2r,2s}$ $(n=r+s)$	$D_n^{(1)}$	$\begin{pmatrix} 2 & & -1 & & & & \\ & 2 & -1 & & & & \\ -1 & -1 & 2 & \ddots & & & \\ & & -1 & \ddots & -1 & & \\ & & & \ddots & 2 & -1 & -1 \\ & & & & -1 & 2 & \\ & & & & -1 & & 2 \end{pmatrix}$		$\mathfrak{o}(2r,2s;\mathbf{C}[t,t^{-1}])\oplus\mathbf{C}z$

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†) This diagram represents the Cartan matrix (C_{ij}). There are $n+1$ nodes $p_0,\dots,p_n$ altogether, each pair (p_i,p_j) being joined by $C_{ij}C_{ji}$ lines. An arrow from p_i to p_j indicates $|C_{ij}|<|C_{ji}|$.

Table 2. For a fixed l or (l_1, l_2) we use the convention

$$\bar{E}_{ij}=\sum_n E_{i+nl,j+nl}, \quad \bar{E}^{\alpha\beta}_{ij}=\sum_n E^{\alpha\beta}_{i+nl\alpha,j+nl\beta} \qquad (\alpha, \beta=1, 2).$$

$(\mathrm{KP})_{n+1}$ $(l=n+1)$

$$e_i=\bar{E}_{i-1,i}, \quad f_i=\bar{E}_{i,i-1}, \quad h_i=\bar{E}_{i-1,i-1}-\bar{E}_{ii}+\delta_{i0}z \quad (0\leqq i\leqq n).$$

$(\mathrm{BKP})_{2n+1}$ $(l=2n+1)$

$$e_0=\sqrt{2}(\bar{E}_{-10}+\bar{E}_{01}), \quad e_i=\bar{E}_{ii+1}+\bar{E}_{-i-1,-i} \quad (1\leqq i\leqq n-1), \quad e_n=\bar{E}_{nn+1},$$
$$f_0=\sqrt{2}(\bar{E}_{0,-1}+\bar{E}_{10}), \quad f_i=\bar{E}_{i+1,i}+\bar{E}_{-i,-i-1} \quad (1\leqq i\leqq n-1), \quad f_n=\bar{E}_{n+1n},$$
$$h_0=2(\bar{E}_{-1,-1}-\bar{E}_{11})+z, \quad h_i=\bar{E}_{ii}-\bar{E}_{i+1,i+1}-\bar{E}_{-i,-i}+\bar{E}_{-i-1,-i-1}$$
$$(1\leqq i\leqq n-1), \quad h_n=\bar{E}_{nn}-\bar{E}_{-n,-n}.$$

$(\mathrm{BKP})_{2n+2}$ $(l=2n+2)$

$$e_0=\sqrt{2}(\bar{E}_{-10}+\bar{E}_{01}), \quad e_i=\bar{E}_{ii+1}+\bar{E}_{-i-1,-i} \quad (1\leqq i\leqq n-1),$$
$$e_n=\sqrt{2}(\bar{E}_{nn+1}+\bar{E}_{n+1,n+2}),$$
$$f_0=\sqrt{2}(\bar{E}_{0,-1}+\bar{E}_{10}), \quad f_i=\bar{E}_{i+1,i}+\bar{E}_{-i,-i-1} \quad (1\leqq i\leqq n-1),$$
$$f_n=\sqrt{2}(\bar{E}_{n+1,n}+\bar{E}_{n+2,n+1}),$$
$$h_0=2(\bar{E}_{-1,-1}-\bar{E}_{11})+z, \quad h_i=\bar{E}_{ii}-\bar{E}_{i+1,i+1}-\bar{E}_{-i,-i}+\bar{E}_{-i-1,-i-1}$$
$$(1\leqq i\leqq n-1), \quad h_n=2(\bar{E}_{nn}-\bar{E}_{-n,-n}).$$

$(\mathrm{BKP\ II})_{2r+1,2s+1}$ $(n=r+s+1, l_1=2r+1, l_2=2s+1)$

$$\left.\begin{matrix}e_0\\ e_1\end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{11}_{01}+\bar{E}^{11}_{-10}\mp i(\bar{E}^{21}_{01}+\bar{E}^{12}_{-10})), \quad e_i=\bar{E}^{11}_{i-1,i}+\bar{E}^{11}_{-i,-i+1}$$
$$(2\leqq i\leqq r),$$
$$e_{r+1}=\bar{E}^{12}_{r1}+(-)^r\bar{E}^{21}_{-1,-r}, \quad e_{j+r}=\bar{E}^{22}_{j-1,j}+\bar{E}^{22}_{-j,-j+1} \quad (2\leqq j\leqq s),$$
$$e_n=\bar{E}^{22}_{s,s+1},$$
$$\left.\begin{matrix}f_0\\ f_1\end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{11}_{10}+\bar{E}^{11}_{0,-1}\pm i(\bar{E}^{12}_{10}+\bar{E}^{21}_{0,-1})), \quad f_i=\bar{E}^{11}_{i,i-1}+\bar{E}^{11}_{-i+1,-i} \quad (2\leqq i\leqq r),$$
$$f_{r+1}=\bar{E}^{21}_{1r}+(-)^r\bar{E}^{12}_{-r,-1}, \quad f_{j+r}=\bar{E}^{22}_{jj-1}+\bar{E}^{22}_{-j+1,-j} \quad (2\leqq j\leqq s), \quad f_n=\bar{E}^{22}_{s+1,s},$$
$$\left.\begin{matrix}h_0\\ h_1\end{matrix}\right\}=\bar{E}^{11}_{-1,-1}-\bar{E}^{11}_{11}\pm i(\bar{E}^{12}_{00}-\bar{E}^{21}_{00})+\frac{1}{2}z,$$
$$h_i=\bar{E}^{11}_{i-1,i-1}-\bar{E}^{11}_{ii}-\bar{E}^{11}_{-i+1,-i+1}+\bar{E}^{11}_{-i-i} \quad (2\leqq i\leqq r),$$
$$h_{r+1}=\bar{E}^{11}_{r,r}-\bar{E}^{11}_{r+1,r+1}-\bar{E}^{22}_{11}+\bar{E}^{22}_{-1,-1},$$
$$h_{j+r}=\bar{E}^{22}_{j-1,j-1}-\bar{E}^{22}_{jj}-\bar{E}^{22}_{-j+1,-j+1}+\bar{E}^{22}_{-j,-j} \quad (2\leqq j\leqq s),$$
$$h_n=\bar{E}^{22}_{ss}-\bar{E}^{22}_{s+1,s+1}.$$

$(\mathrm{BKP\ II})_{2r,2s}$ $\quad (n=r+s,\ l_1=2r,\ l_2=2s) \quad \varepsilon=\sqrt{(-)^{r+s-1}}$

$$\left.\begin{matrix} e_0 \\ e_1 \end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{11}_{01}+\bar{E}^{11}_{-1,0}\mp i(\bar{E}^{21}_{01}+\bar{E}^{12}_{-1,0})), \quad e_i=\bar{E}^{11}_{i-1,i}+\bar{E}^{11}_{-i,-i+1}$$

$(2\leqq i\leqq r-1)$,

$$e_r=\bar{E}^{12}_{r-1,1}+(-)^{r-1}\bar{E}^{21}_{-1,-r+1}, \quad e_{j+r-1}=\bar{E}^{22}_{j-1,i}+\bar{E}^{22}_{-j,-j+1} \quad (2\leqq j\leqq s-1),$$

$$\left.\begin{matrix} e_{n-1} \\ e_n \end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{22}_{s-1,s}+\bar{E}^{22}_{s,s+1}\pm\frac{1}{\varepsilon}\bar{E}^{12}_{r,s+1}\mp\varepsilon\bar{E}^{21}_{s-1,r}),$$

$$\left.\begin{matrix} f_0 \\ f_1 \end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{11}_{10}+\bar{E}^{11}_{0,-1}\pm i(\bar{E}^{12}_{10}+\bar{E}^{21}_{0,-1})), \quad f_i=\bar{E}^{11}_{i,i-1}+\bar{E}^{11}_{-i+1,-i}$$

$(2\leqq i\leqq r-1)$,

$$f_r=\bar{E}^{21}_{1,r-1}+(-)^{r-1}\bar{E}^{12}_{-r+1,-1}, \quad f_{j+r-1}=\bar{E}^{22}_{jj-1}+\bar{E}^{22}_{-j+1,-j} \quad (2\leqq j\leqq s-1),$$

$$\left.\begin{matrix} f_{n-1} \\ f_n \end{matrix}\right\}=\frac{1}{\sqrt{2}}(\bar{E}^{22}_{s,s-1}+\bar{E}^{22}_{s+1,s}\mp\frac{1}{\varepsilon}\bar{E}^{12}_{r,s-1}\pm\varepsilon\bar{E}^{21}_{s+1,r}),$$

$$\left.\begin{matrix} h_0 \\ h_1 \end{matrix}\right\}=\bar{E}^{11}_{-1,-1}-\bar{E}^{11}_{11}\pm i(\bar{E}^{12}_{00}-\bar{E}^{21}_{00})+\frac{1}{2}z,$$

$$h_i=\bar{E}^{11}_{i-1,i-1}-\bar{E}^{11}_{-i+1,-i+1}-\bar{E}^{11}_{ii}+\bar{E}^{11}_{-i,-i} \quad (2\leqq i\leqq r-1),$$

$$h_r=\bar{E}^{11}_{r-1,r-1}-\bar{E}^{11}_{-r+1,-r+1}-\bar{E}^{22}_{11}+\bar{E}^{22}_{-1,-1},$$

$$h_{j+r-1}=\bar{E}^{22}_{j-1,j-1}-\bar{E}^{22}_{-j+1,-j+1}-\bar{E}^{22}_{jj}+\bar{E}^{22}_{-j,-j} \quad (2\leqq j\leqq s-1),$$

$$\left.\begin{matrix} h_{n-1} \\ h_n \end{matrix}\right\}=\bar{E}^{22}_{s-1,s-1}-\bar{E}^{22}_{s+1,s+1}\pm\frac{1}{\varepsilon}\bar{E}^{12}_{rs}\pm\varepsilon\bar{E}^{21}_{sr}.$$

The realization of these elements as Laurent polynomials is given by the following rule: $\bar{E}_{ij}\leftrightarrow E_{\bar{\imath},\bar{\jmath}}\cdot t^{\nu}$, where $i\equiv\bar{\imath},\ j\equiv\bar{\jmath} \mod l\ (0\leqq\bar{\imath},\ \bar{\jmath}\leqq l-1)$ and $[(j-i)/l]=\nu$. For example, in the case of $(\mathrm{KP})_{n+1}$, we have

$$e_0=\begin{pmatrix} & & \\ & & \\ t & & \end{pmatrix}, \quad e_i=\begin{array}{cc} & \begin{matrix} & \underbrace{i}_{} & \end{matrix} \\ i-1) & \begin{pmatrix} & | & \\ - & 1 & - \\ & | & \end{pmatrix} \end{array} \quad (1\leqq i\leqq n)$$

$$f_0=\begin{pmatrix} & & t^{-1} \\ & & \\ & & \end{pmatrix}, \quad f_i=\begin{array}{cc} & \begin{matrix} & \underbrace{i-1}_{} & \end{matrix} \\ i) & \begin{pmatrix} & | & \\ - & 1 & - \\ & | & \end{pmatrix} \end{array} \quad (1\leqq i\leqq n)$$

$$h_0=\begin{pmatrix} -1 & & \\ & & \\ & & 1 \end{pmatrix}+z, \quad h_i=\begin{array}{cc} & \begin{matrix} & \underbrace{i-1}_{} & \underbrace{i}_{} & \end{matrix} \\ \begin{matrix} i-1) \\ i) \end{matrix} & \begin{pmatrix} & | & | & \\ - & 1 & & \\ - & & -1 & - \\ & | & | & \end{pmatrix} \end{array} \quad (1\leqq i\leqq n).$$

As in the KP case, reduction of the vertex operators $Z_B(p, q)$ or $Z_B^{\alpha\beta}(p, q)$ leads to linear representations of $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_n^{(1)}$. In the case of $\mathfrak{go}(\infty)_l$, the homogeneous components of

$$(2.17)\qquad Z_B(p, -\omega p)=\frac{1}{2}\frac{1+\omega}{1-\omega}\Big(\exp\big(\sum_{j:\mathrm{odd}\geqq 1}(1-\omega^j)p^jx_j\big)$$
$$\times\exp\Big(-2\sum_{j:\mathrm{odd}\geqq 1}(1-\omega^{-j})p^{-j}\frac{1}{j}\frac{\partial}{\partial x_j}\Big)-1\Big)$$
$$\omega^l=1,\ \omega\neq 1$$

span $\mathfrak{go}(\infty)_l$ together with 1, x_j and $\frac{\partial}{\partial x_j}$ (j: odd$\geqq 1$, $j\not\equiv 0 \bmod l$). In the case of $\mathfrak{go}(2\infty)_{l_1,l_2}$, there appear two series of variables x_j and y_j(j: odd). In the case l_1, l_2=even, $\mathfrak{go}(2\infty)_{l_1,l_2}$ is spanned by the homogeneous components of

$$(2.18)\qquad Z_B^{11}(p, -\omega p)=\frac{1}{2}\frac{1+\omega}{1-\omega}\Big(\exp\big(\sum_{j:\mathrm{odd}>0}(1-\omega^j)p^jx_j\big)$$
$$\times\exp\Big(-2\sum_{j:\mathrm{odd}>0}(1-\omega^{-j})p^{-j}\frac{1}{j}\frac{\partial}{\partial x_j}\Big)-1\Big)$$
$$(\omega^{l_1}=1,\ \omega\neq 1)$$
$$Z_B^{22}(p, -\eta p)=\frac{1}{2}\frac{1+\eta}{1-\eta}\Big(\exp\big(\sum_{j:\mathrm{odd}>0}(1-\eta^j)p^jy_j\big)$$
$$\times\exp\Big(-2\sum_{j:\mathrm{odd}>0}(1-\eta^{-j})p^{-j}\frac{1}{j}\frac{\partial}{\partial y_j}\Big)-1\Big)$$
$$(\eta^{l_2}=1,\ \eta\neq 1)$$
$$Z_B^{12}(\omega p^{l_2}, -\eta p^{l_1})=\frac{1}{2i}\exp\big(\sum_{j:\mathrm{odd}>0}(\omega^jp^{l_2j}x_j-\eta^jp^{l_1j}y_j)\big)$$
$$\times\exp\Big(-2\sum_{j:\mathrm{odd}>0}\frac{1}{j}\Big(\omega^{-j}p^{-l_2j}\frac{\partial}{\partial x_j}-\eta^{-j}p^{-l_1j}\frac{\partial}{\partial y_j}\Big)\Big)$$
$$(\omega^{l_1}=1,\ \eta^{l_2}=1)$$

together with 1, x_j, $\frac{\partial}{\partial x_j}$, y_j and $\frac{\partial}{\partial y_j}$ (j: odd). In the case l_1, l_2=odd, we note that x_j, $\frac{\partial}{\partial x_j}$ ($j\equiv 0 \bmod l_1$) and y_j, $\frac{\partial}{\partial y_j}$ ($j\equiv 0 \bmod l_2$) are absent in Z_B^{11}, Z_B^{22}, whereas they appear in Z_B^{12} only through the combinations $x_{l_1\nu}-y_{l_2\nu}$ and $\frac{1}{l_1\nu}\frac{\partial}{\partial x_{l_1\nu}}-\frac{1}{l_2\nu}\frac{\partial}{\partial y_{l_2\nu}}$. In correspondence with this fact, the basis of $\mathfrak{go}(2\infty)_{l_1,l_2}$ is formed by the homogeneous components of (2.18) and 1, x_j, $\frac{\partial}{\partial x_j}$ ($j\not\equiv 0 \bmod l_1$, j: odd), y_j, $\frac{\partial}{\partial y_j}$ ($j\not\equiv 0 \bmod l_2$, j: odd), $x_{l_1\nu}-y_{l_2\nu}$ and $\frac{1}{l_1\nu}\times\frac{\partial}{\partial x_{l_1\nu}}-\frac{1}{l_2\nu}\frac{\partial}{\partial y_{l_2\nu}}$ (ν: odd). Thus the representation space of $A_n^{(1)},\dots, D_n^{(1)}$ is the polynomial algebra $\boldsymbol{C}[\{x_i\}]$ in the following infinitely many variables:

Table 3.

$(\mathrm{KP})_{n+1}$	$A_n^{(1)}$	$x_i: i \not\equiv 0 \mod n+1$
$(\mathrm{BKP})_{2n+1}$	$A_{2n}^{(2)}$	$x_i: i$ odd, $i \not\equiv 0 \mod 2n+1$
$(\mathrm{BKP})_{2n}$	$D_{n+1}^{(2)}$	$x_i: i$ odd
$(\mathrm{BKP\ II})_{2r+1,2s+1}$ $(n=r+s+1)$	$A_{2n-1}^{(2)}$	$x_i: i$ odd, $i \not\equiv 0 \mod 2s+1$ $y_j: j$ odd, $j \not\equiv 0$ mnd $2r+1$ $x_{(2r+1)\nu} - y_{(2s+1)\nu}: \nu:$ odd
$(\mathrm{BKP\ II})_{2r,2s}$ $(n=r+s)$	$D_n^{(1)}$	$x_i: i$ odd $y_j: j$ odd

In the case of BKP II_{l_1,l_2}, we count the homogeneous degree of x_i, y_i to be $\deg x_i = il_2/d$, $\deg y_j = jl_1/d$ where $d=$ g. c. m. (l_1, l_2).

These representations are actually the same†) as the basic representations given by Kac-Kazhdan-Lepowsky-Wilson [14]. To recall the terminology, let $\mathfrak{g}$ be a Kac-Moody Lie algebra and let λ be a linear form on $\mathfrak{h} = \bigoplus_{i=0}^{n} \mathbb{C}h_i \subset \mathfrak{g}$ such that each $\lambda(h_i)$ is a nonnegative integer for $i=0, 1, \ldots, n$. An irreducible $\mathfrak{g}$-module V is called the standard module with the highest weight λ, if there exists a nonzero vector $v_\lambda \in V$ satisfying

$$(2.19) \qquad \begin{aligned} &hv_\lambda = \lambda(h)v_\lambda \qquad \text{for any} \quad h \in f, \\ &e_0 v_\lambda = 0, \quad e_1 v_\lambda = 0, \ldots, \quad e_n v_\lambda = 0. \end{aligned}$$

The vector v_λ is unique up to constant multiple, and is called the highest weight vector. The representation associated with the simplest choice of $\lambda = \Lambda$ defined by

$$(2.20) \qquad \Lambda(h_0) = 1, \quad \Lambda(h_1) = 0, \ldots, \quad \Lambda(h_n) = 0$$

is called the basic representation.(*) In the present situation, the constant function 1 in the polynomial algebra $V = \mathbb{C}[\{x_i\}]$ is the highest weight vector. Using the realization of canonical generators in Table 2, one can explicitly verify (2.19) and (2.20) by noting

$$z \cdot 1 = 1, \quad Z_{B,00}^{12} \cdot 1 = \frac{1}{2i}$$

---

†) For BKP II_{l_1,l_2}, their vertex operators correspond to the choice $l_2=1$ or $l_2=2$.

(*) We have reversed the enumeration of $\mathfrak{h}_i$'s for $A_{2n}^{(2)}$[14].

$$Z_{ij}\cdot 1=0,\quad Z_{B,ij}\cdot 1=0,\quad Z^{\mu\nu}_{B,ij}\cdot 1=0 \qquad \text{for} \quad i<0 \quad \text{or} \quad j>0.$$

Now we return to the standpoint of soliton theory. We define the space of τ functions of the l-reduced BKP hierarchy (resp. (l_1, l_2)-reduced BKP II hierarchy) to be the orbit of 1 by the group action of $\mathfrak{go}(\infty)_l$ (resp. $\mathfrak{go}(2\infty)_{l_1,l_2}$). It is then clear that, for odd l or (l_1, l_2), we have the constraint

$$\frac{\partial}{\partial x_j}\tau_{\mathrm{BKP}}(x)=0, \quad j\equiv 0 \bmod l,\ j:\text{odd} \tag{2.21}$$

$$\left(\frac{\partial}{\partial x_j}+\frac{\partial}{\partial y_j}\right)\tau_{\mathrm{BKP\,II}}(x, y)=0, \quad j\equiv 0 \bmod \frac{l_1 l_2}{d},\ j:\text{odd}. \tag{2.22}$$

In [1], [3] we obtained bilinear identities for the wave functions of KP, BKP and BKP II which are equivalent to bilinear equations for the corresponding τ functions.

One can also derive extra bilinear identities for the wave functions of the reduced hierarchies. For KP_l, BKP_l and $\mathrm{BKP\ II}_{l_1,l_2}$, they read as follows:

$$\oint w_{\mathrm{KP}}(x, k)w^*_{\mathrm{KP}}(x', -k)k^{l+1}\underline{dk}=0 \tag{2.23}$$

$$\oint w_{\mathrm{BKP}}(x, k)w_{\mathrm{BKP}}(x', -k)k^{l}\underline{dk}=0 \tag{2.24}$$

$$\begin{aligned}\oint w^{(1)}_{\mathrm{BKP\,II}}(x, y, k)w^{(1)}_{\mathrm{BKP\,II}}(x', y', -k)k^{l_1}\underline{dk}\\ -\oint w^{(2)}_{\mathrm{BKP\,II}}(x, y, k)w^{(2)}_{\mathrm{BKP\,II}}(x', y', -k)k^{l_2}\underline{dk}=0.\end{aligned} \tag{2.25}$$

Here $\underline{dk}=dk/2\pi ik$, and the integration is taken over a small circuit around $k=\infty$. The wave functions in (2.23)–(2.25) are related to the τ functions through $w_{\mathrm{KP}}(x, k)=e^{\xi(x,k)}e^{-\xi(\tilde{\partial},k^{-1})}\tau_{\mathrm{KP}}(x)/\tau_{\mathrm{KP}}(x)$, $w^*_{\mathrm{KP}}(x, k)=e^{-\xi(x,k)}e^{\xi(\tilde{\partial},k^{-1})}.$ $\tau_{\mathrm{KP}}(x)/\tau_{\mathrm{KP}}(x)$, $w_{\mathrm{BKP}}(x, k)=e^{\tilde{\xi}(x,k)}e^{-2\tilde{\xi}(\tilde{\partial},k^{-1})}\tau_{\mathrm{BKP}}(x)/\tau_{\mathrm{BKP}}(x)$, $w^{(1)}_{\mathrm{BKP\,II}}(x, y, k)=e^{\tilde{\xi}(x,k)}e^{-2\tilde{\xi}(\tilde{\partial}_x,k^{-1})}\tau_{\mathrm{BKP\,II}}(x, y)/\tau_{\mathrm{BKP\,II}}(x, y)$ and $w^{(2)}_{\mathrm{BKP\,II}}(x, y, k)=e^{\tilde{\xi}(y,k)}e^{-2\tilde{\xi}(\tilde{\partial}_y,k^{-1})}.$ $\tau_{\mathrm{BKP\,II}}(x, y)/\tau_{\mathrm{BKP\,II}}(x, y)$, respectively [10] [3]. To derive (2.23) we use the operator expression (18) of [11] for w_{KP} and w^*_{KP}. We set $\psi_j g=\sum_{i\in\boldsymbol{Z}} g\psi_i a_{ij}$, $\psi^*_j g=\sum_{i\in\boldsymbol{Z}} g\psi^*_i a^*_{ij}$ with $\sum_{j\in\boldsymbol{Z}} a_{ij}a^*_{i'j}=\delta_{ii'}$. For the l-reduced hierarchy, we have $a_{i+l,j+l}=a_{ij}$. Therefore by following the argument in [11] we have

$$\begin{aligned}&\oint \tau_{\mathrm{KP}}(x)w_{\mathrm{KP}}(x, k)\tau_{\mathrm{KP}}(x')w^*_{\mathrm{KP}}(x', k)k^{l+1}\underline{dk}\\ &=\sum_{i,i',j\in\boldsymbol{Z}}\langle\mathrm{vac}\,|\varphi_1(x)g\psi_i|\,\mathrm{vac}\rangle a_{ij}\langle\mathrm{vac}\,|\varphi_{-1}(x)g\psi^*_{i'}|\,\mathrm{vac}\rangle a^*_{i',j+l}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i,i'\in\boldsymbol{Z}}\langle\mathrm{vac}\,|\varphi_1(x)g\psi_i|\,\mathrm{vac}\rangle\,\langle\mathrm{vac}\,|\varphi_{-1}(x)g\psi_{i'}^*|\,\mathrm{vac}\rangle\delta_{i,i'-l}\\
&=0\,.
\end{aligned}
$$

Similar arguments show (2.24) and (2.25).

Remark. In the case of $(\mathrm{BKP})_l$ with l even, also, $L(x, \partial)^l$ is a differential operator, since it should belong to $(\mathrm{KP})_l$ when evolved with respect to $x_2, x_4, \ldots$. By applying $L(x, \partial_x)^l$ and $L(x', \partial_{x'})^l$ to (2.24) we see that their constant terms must be absent.

In terms of τ functions, (2.23)–(2.25) turn into

$$\sum_{i\geqq 0} p_i(2u)p_{i+l+1}(-\tilde{D})e^{\langle u,D\rangle}\tau_{\mathrm{KP}}\cdot\tau_{\mathrm{KP}}=0 \tag{2.26}$$

$$\sum_{i\geqq 0} \tilde{p}_i(2u)\tilde{p}_{i+l}(-2\tilde{D})e^{\langle u,D\rangle}\tau_{\mathrm{BKP}}\cdot\tau_{\mathrm{BKP}}=0 \tag{2.27}$$

$$\begin{aligned}&(\sum_{i\geqq 0} \tilde{p}_i(2u)\tilde{p}_{i+l_1}(-2\tilde{D}_x)-\sum_{i\geqq 0} \tilde{p}_i(2v)\tilde{p}_{i+l_2}(-2\tilde{D}_y))\\ &\qquad\qquad\times e^{\langle u,D_x\rangle+\langle v,D_y\rangle}\tau_{\mathrm{BKP\,II}}\cdot\tau_{\mathrm{BKP\,II}}=0\end{aligned} \tag{2.28}$$

respectively, where $\langle u, D\rangle=\sum_{j=1}^{\infty} u_jD_j$ (for KP), $=\sum_{j:\mathrm{odd}\geqq 1} u_jD_j$ (for BKP and BKP II), and u_i, v_i are auxiliary parameters. In the case of even reduction of BKP or BKP II, (2.27) or (2.28) certainly give additional bilinear equations which are not contained in the unreduced hierarchies. In the remaining cases, bilinear equations for the reduced hierarchies are obtained from the original hierarchies of bilinear equations by setting $D_j=0$ ($j\equiv 0 \bmod l$) or $D_{x_{l_2'j}}+D_{y_{l_1'j}}=0$ ($j\equiv 0 \bmod l_1l_2/d$). We expect that (2.26)–(2.28) are exhausted by this procedure.

By a similar calculation we obtain the following equations for the modified τ functions

$$(\sum_{i\geqq 0} p_i(-2u)p_{i+l}(\tilde{D})e^{\langle u,D\rangle}-k^le^{-\langle u,D\rangle})\tau_{\mathrm{KP}[k]}\cdot\tau_{\mathrm{KP}}=0\,. \tag{2.29}$$

$$(\sum_{i\geqq 0} \tilde{p}_i(2u)\tilde{p}_{i+l}(-2\tilde{D})e^{\langle u,D\rangle}-2(-k)^le^{-\langle u,D\rangle})\tau_{\mathrm{BKP}[k]}\cdot\tau_{\mathrm{BKP}}=0\,. \tag{2.30}$$

$$\begin{aligned}&\{(\sum_{i\geqq 0} \tilde{p}_i(2u)\tilde{p}_{i+l_1}(-2\tilde{D}_x)+\sum_{i\geqq 0} \tilde{p}_i(2v)\tilde{p}_{i+l_2}(-2\tilde{D}_y))e^{\langle u,D_x\rangle+\langle v,D_y\rangle}\\ &\qquad -2(-k)^{l_\alpha}e^{-\langle u,D_x\rangle-\langle v,D_y\rangle}\}\tau^{(\alpha)}_{\mathrm{BKP\,II}[k]}\cdot\tau_{\mathrm{BKP\,II}}=0 \qquad (\alpha=1,\,2)\,.\end{aligned} \tag{2.31}$$

Example. We give below the first few bilinear equations for the reduced hierarchies.

$(\mathrm{KP})_2(=\mathrm{KdV})$, $A_1^{(1)}$: x_j, j: odd $\geqq 1$.

$(D_1^4-4D_1D_3)\tau\cdot\tau=0,\quad (D_1^6+4D_1^3D_3-32D_3^2)\tau\cdot\tau=0,\ldots$

$$(D_1^2-k^2)\tau_{[k]}\cdot\tau=0,\quad (4D_3-D_1^3-3k^2D_1)\tau_{[k]}\cdot\tau=0,\dots$$

$(\mathrm{KP})_3$(=Boussinesq), $A_2^{(1)}$: x_j, $j\equiv 1, 2 \mod 3$.

$$(D_1^4+3D_2^2)\tau\cdot\tau=0.\quad (D_1^3D_2-3D_1D_4)\tau\cdot\tau=0,\dots$$
$$(D_2-D_1^2)\tau_{[k]}\cdot\tau=0,\quad (D_1^3+3D_1D_2-4k^3)\tau_{[k]}\cdot\tau=0,\dots$$

$(\mathrm{BKP})_3$(=Sawada-Kotera [4]), $A_2^{(2)}$: x_j, $j\equiv \pm 1 \mod 6$.

$$(D_1^6+9D_1D_5)\tau\cdot\tau=0,\quad (D_1^8-21D_1^3D_5+90D_1D_7)\tau\cdot\tau=0,\dots$$
$$(D_1^3-k^3)\tau_{[k]}\cdot\tau=0,\quad (6D_5-D_1^5-5k^3D_1^2)\tau_{[k]}\cdot\tau=0,\dots$$

$(\mathrm{BKP})_5$(=Ramani [5]), $A_4^{(2)}$: x_j, j odd, $j\not\equiv 0 \mod 5$.

$$(D_1^6-5D_1^3D_3-5D_3^2)\tau\cdot\tau=0,\quad (D_1^8+7D_1^5D_3-35D_1^2D_3^2+90D_1D_7)\tau\cdot\tau=0,\dots$$
$$(D_3-D_1^3)\tau_{[k]}\cdot\tau=0,\quad (5D_3D_1^2+D_1^5-6k^5)\tau_{[k]}\cdot\tau=0,\dots$$

$(\mathrm{BKP})_6^{(\dagger)}$ (=Ito [6]), $D_3^{(2)}$: x_j, j: odd$\geqq 1$.

$$(D_3^2+2D_3D_1^3)\tau\cdot\tau=0,\quad (2D_1^5D_3-5D_1^2D_3^2-12D_3D_5)\tau\cdot\tau=0,\dots$$
$$(D_3^2+2D_3D_1^3-3k^6)\tau_{[k]}\cdot\tau=0,$$
$$(72D_5D_1^2+55D_3^2D_1+10D_3D_1^4-2D_1^7-135k^6D_1)\tau_{[k]}\cdot\tau=0,\dots$$

$(\mathrm{BKP\ II})_{5,1}^{(\dagger)}$, $A_5^{(2)}$: x_j, j: odd$\geqq 1$. (We replace $x_{5j}-y_j$ by x_{5j})

$$(D_3-D_1^3)D_5\tau\cdot\tau=0,\quad (6D_5-5D_3D_1^2-D_1^5)D_5\tau\cdot\tau=0,\dots$$
$$D_1(D_5-k^5)\tau_{[k]}^*\cdot\tau=0,\quad (2D_3+D_1^3)(D_5-k^5)\tau_{[k]}^*\cdot\tau=0,\dots\ (\tau_{[k]}^*=\tau_{[k]}^{(1)} \text{ or } \tau_{[k^5]}^{(2)})$$

$(\mathrm{BKP\ II})_{6,2}^{(\dagger)}$, $D_4^{(1)}$: x_j, $y_j\equiv x'_{3j}$; j: odd$\geqq 1$.

$$(D_3-D_1^3)D'_3\tau\cdot\tau=0,\quad (D_3^2+2D_1^3D_3-3D_3'^2)\tau\cdot\tau=0,\dots$$
$$(D_3^2+2D_1^3D_3+3D_3'^2-3k^6)\tau_{[k]}^*\cdot\tau=0,\dots\ (\tau_{[k]}^*=\tau_{[k]}^{(1)} \quad\text{or}\quad \tau_{[k^3]}^{(2)}).$$

Remark. The 4-reduced BKP hierarchy ($(\mathrm{BKP})_4$, "$D_2^{(2)}$") is equivalent to the 2-reduced KP hierarchy ($(\mathrm{KP})_2=\mathrm{KdV}$, $A_1^{(1)}$). In fact, by a change of variables

$$x_j \longmapsto \varepsilon_j\sqrt{2}x_j,\ D_j \longmapsto \varepsilon_j\sqrt{2}^{-1}D_j,\ j=1, 3, 5,\dots$$
$$(\varepsilon_j\equiv 1,\ -1,\ -1,\ +1, \qquad \text{for}\quad j\equiv 1, 3, 5, 7 \mod 8, \text{ respectively})$$

we see that the vertex operators and Hirota's bilinear equations for $(\mathrm{BKP})_4$ reduce to those for $(\mathrm{KP})_2$, respectively. Likewise we have

$$(\mathrm{BKP\ II})_{3,1}(\text{"}A_3^{(2)}\text{"})=(\mathrm{BKP})_6(D_3^{(2)})$$
$$(\mathrm{BKP\ II})_{4,2}(\text{"}D_3^{(1)}\text{"})=(\mathrm{KP})_4(A_3^{(1)}).$$

In each case, soliton solutions are immediately obtained from the form of the (reduced) vertex operators. Take as an example the case of $(\mathrm{BKP\ II})_{l,1}$ (l: odd). For convenience we adopt the renaming $x_{l\nu}-y_\nu\mapsto x_{l\nu}$, $\frac{1}{l\nu}\frac{\partial}{\partial x_{l\nu}}$

---

†) τ satisfies also the bilinear equations for BKP with respect to x_1, x_3, x_5, $\dots$.

$-\frac{1}{\nu}\frac{\partial}{\partial y_{\nu}} \mapsto \left(1+\frac{1}{l}\right)\frac{\partial}{\partial x_{l\nu}}$. Then by calculating $e^{a_1 Z_1 + a_2 Z_2}\cdot 1$ ($a_i \in \mathbb{C}$, $Z_i = Z^{11}(p, -\omega p)$ with $\omega^l = 1$, or $= Z^{12}(p, -p^l)$) we obtain the following different types of 2 soliton solutions:

$$1+a_1e^{\xi_1}+a_2e^{\xi_2}+a_1a_2\frac{p_1-p_2}{p_1+p_2}\;\frac{\omega_1p_1+p_2}{\omega_1p_1-p_2}\;\frac{p_1+\omega_2p_2}{p_1-\omega_2p_2}\;\frac{\omega_1p_1-\omega_2p_2}{\omega_1p_1+\omega_2p_2}\;e^{\xi_1+\xi_2}$$

$$1+a_1e^{\eta_1}+a_2e^{\eta_2}+a_1a_2\frac{p_1-p_2}{p_1+p_2}\;\frac{p_1^l-p_2^l}{p_1^l+p_2^l}\,e^{\eta_1+\eta_2},$$

$$1+a_1e^{\xi_1}+a_2e^{\eta_2}+a_1a_2\frac{p_1-p_2}{p_1+p_2}\;\frac{\omega_1p_1+p_2}{\omega_1p_1-p_2}\,e^{\xi_1+\eta_2}.$$

where $\xi_i = \sum_{j \not\equiv 0 \bmod l} (1-\omega^j)p_i^j x_j$, $\eta_i = \sum_{j:\mathrm{odd}} p_i^j x_j$ and $\omega_i^l = 1 (i=1, 2)$.

§ 3. Enumeration of Bilinear Equations

In this section we study the Hirota bilinear equations from the viewpoint of the representation theory of Kac-Moody Lie algebras.

For a hierarchy S of soliton type equations, we are interested in the number of linearly independent Hirota bilinear equations of given degree which τ-functions τ_S of the hierarchy S satisfy. Namely, we want to count the dimension of the space

$$H_S(m) = \{P \in \mathbb{C}[\partial_x] \mid P(D_x)\tau_S \cdot \tau_S = 0, \quad \text{for any } \tau_S, \deg P = m\}.$$

Here $x = (x_i)_{i \in I_S}$ is the set of independent time variables attached to S.

This problem was proposed by M. and Y. Sato [7]. For the KdV hierarchy, they showed

$$\text{(3.1)} \qquad \begin{aligned} \dim H_{\mathrm{KdV}}(m) = \#\{(m_1,\dots, m_k) \mid m_i&\text{: positive odd integer} \\ & m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^k m_i = m\} \\ -\#\{(m_1,\dots, m_k) \mid m_i\text{: positive even integer}, &\ m_1 < \cdots < m_k, \\ & \textstyle\sum_{i=1}^k m_i = m\}. \end{aligned}$$

They also counted the dimension for the MKdV and the nonlinear Schrödinger hierarchies. For the KP and the Sawada-Kotera (SK) hierarchies they conjectured the following:

$$\text{(3.2)} \qquad \begin{aligned} \dim H_{\mathrm{KP}}(m) = \#\{(m_1,\dots, m_k) \mid m_i&\text{: positive integer}, \\ m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^k & m_i = m-1\}, \end{aligned}$$

$$\text{(3.3)} \qquad \begin{aligned} \dim H_{\mathrm{SK}}(m) = \#\{(m_1,\dots, m_k) \mid m_i&\text{: positive integer}, \\ m_1 \leqq \cdots \leqq m_k,\ m_i \equiv \pm 1 & \text{ (mod. 6)},\ \textstyle\sum_{i=1}^k m_i = m\} \end{aligned}$$

$$-\#\{(m_1,\dots,m_k)\,|\,m_i\text{: positive integer, } m_1\leqq\cdots\leqq m_k,$$
$$m_i\equiv\pm2\ (\text{mod. }10),\ \textstyle\sum_{i=1}^k m_i=m\}.$$

In the previous section we showed that to each of the Euclidean Lie algebras $A_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_{n+1}^{(2)}$, there corresponds a sub-holonomic hierarchy of soliton type equations obtained as the reduction of the KP, the BKP and the two-component BKP hierarchies. The correspondence is given in Table 1. The associated set of independent time variables is listed in Table 3. Hereafter we mean by S one of such hierarchies. We also denote by $\mathfrak{g}_S$ the corresponding Euclidean Lie algebras.

In the sequel, we count $\dim H_S(m)$ by using the character formula for $\mathfrak{g}_S$. The dimension formulas (3.2), (3.12) of $H_{\mathrm{KP}}(m)$ and $H_{\mathrm{BKP}}(m)$, then, follows as a consequence.

The use of the character formula is based on the following observation. We put $H_S=\oplus_m H_S(m)$.

Recalling the definition of the Hirota blinear operator:

$$P(D_x)f\cdot g=P(\partial_y)f(x+y)g(x-y)\,|_{y=0},$$

we write the product $\tau_S(x+y)\tau_S(x-y)$ as

$$\tau_S(x+y)\tau_S(x-y)=\sum F_i(x;\,\tau_S)G_i(y;\,\tau_S)$$

where in the right hand side we take the shortest representation (cf. [7]). We introduce a pairing of $\boldsymbol{C}[\partial_{y_i};\,i\in I_S]$ and $\boldsymbol{C}[y_i;\,i\in I_S]$ by

$$\langle a,\,b\rangle=a(\partial_y)b(y)|_{y=0},\ a\in\boldsymbol{C}[\partial_{y_i};\,i\in I_S],\ b\in\boldsymbol{C}[y_i;\,i\in I_S]. \tag{3.4}$$

Then with respect to this pairing, the space H_S is the orthogonal complement of

Ω' = the vector space spanned by $\{G_i(y;\,\tau_S)|\text{for any }\tau_S\text{ and }i\}$.

Recall that the totality of (polynomial) τ-functions is the orbit space of the highest weight vector 1 under the basic representation of $\mathfrak{g}_S$ on $V(\Lambda)=\boldsymbol{C}[x_i;\,i\in I_S]$ (see §2). Here Λ is the highest weight of the basic representation (§2). Therefore the irreducible representation space $V(2\Lambda)$ with the highest weight 2Λ is the space

linear hull of $\{\tau_S(x^{(1)})\tau_S(x^{(2)})\}$

where $x^{(1)}$ and $x^{(2)}$ are two copies of the time variables attached to S.

In the space $V(2\Lambda)$ we consider the subspace

$$\begin{aligned}\Omega &= \{v \in V(2\Lambda) | (\partial/\partial x_i^{(1)} + \partial/\partial x_i^{(2)})v = 0, \ \forall i \in I_S\} \\ &= \{v \in V(2\Lambda) | v\colon \text{function of } y\}\end{aligned}$$

where we have set

$$2x_i = x_i^{(1)} + x_i^{(2)}, \quad 2y_i = x_i^{(1)} - x_i^{(2)}.$$

The space Ω was introduced in Lepowsky-Wilson [8] for the standard modules of $A_1^{(1)}$ in their study of the Rogers-Ramanujan identities. The following relation among $V(\Lambda)$, $V(2\Lambda)$ and Ω (Prop. 1) is given in [8] for $A_1^{(1)}$. Their arguments are valid for our case.

In Section 2, we gave a realization of $\mathfrak{g}_S$ as differential and multiplication operators on $V(\Lambda) = \mathbf{C}[x_i;\ i \in I_s]$ (the basic representation). In this realization, the operators x_i, $\partial/\partial x_i$, $i \in I_s$ and 1 form an infinite dimensional Heisenberg subalgebra. The corresponding subalgebra in $\mathfrak{g}_S$ is called the principal subalgebra ([2], [14]). We denote this algebra by $\mathfrak{s}$. The above space Ω is the $\mathfrak{s}$-highest weight space of the $\mathfrak{s}$-module $V(2\Lambda)$. The complete reducibility of $\mathfrak{s}$-modules implies that $V(\Lambda)$ and $V(2\Lambda)$ are related in the following manner as $\mathfrak{s}$-modules

Proposition 1 ([8], p. 16, p. 7).

$$V(2\Lambda) \cong V(\Lambda) \otimes_C \Omega.$$

Now we have

Proposition 2. $\Omega' = \Omega$.

Proof. First we show the inclusion $\Omega'^{\perp} \subset \Omega^{\perp}$, where $^{\perp}$ denotes the orthogonal complements with respect to the pairing (3.4). Take $P \in \Omega'^{\perp}$. By the definition of $\Omega'^{\perp}$, we have

$$P(\partial_y) G_i(y;\ \tau_s)|_{y=0} = 0$$

for any τ_s and i. This implies

$$P(\partial_y) \tau_s(x+y) \tau_s(x-y)|_{y=0} = 0$$

and consequently for any $v \in V(2\Lambda)$

$$P(\partial_y) v(x, y)|_{y=0} = 0.$$

In particular, this equality holds for $v \in \Omega$. Hence we have

$$P \in \Omega^{\perp}.$$

Now we show the converse inclusion. By Proposition 1 we have

$$V(2\Lambda)=V(\Lambda)\otimes_{\boldsymbol{C}}\Omega .$$

On the other hand, the definition of Ω' implies

$$V(2\Lambda)=\{\text{linear hull of } F_i\text{'s}\}\otimes\Omega' .$$

Using the complete reducibility of $\mathfrak{s}$-modules we have

$$\{\text{linear hull of } F_i\text{'s}\}=\boldsymbol{C}[x_i|i\in I_s] .$$

Therefore we have

$$1\otimes\Omega'\subset V(2\Lambda). \qquad \text{q.e.d.}$$

Let

$$\Omega_{-m}=\{P\in\Omega|\deg P=m\} .$$

In view of the considerations above, we have

$$\dim H_S(m)=\dim\{P\in\boldsymbol{C}[x_i,\ i\in I_S]\,|\deg P=m\}-\dim\Omega_{-m} . \tag{3.5}$$

The dimension of the space $\{P\in\boldsymbol{C}[x_i,\ i\in I_S]|\deg P=m\}$ is equal to

$$\sharp\{(m_1,\dots,m_k)|m_i\in I_S,\ m_1\leqq\cdots\leqq m_k,\ \textstyle\sum_{i=1}^k m_i=m\} .$$

Therefore our problem is reduced to the calculation of $\dim\Omega_{-m}$.

A method for calculating $\dim\Omega_{-m}$ is supplied by the character formula for the standard modules of $\mathfrak{g}_S$.

We explain the character formula, restricting ourselves to the principally specialized characters. For details we refer to [14], [18], [19].

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of rank $n+1$. For $a=(a_0, a_1,\dots, a_n)\in\boldsymbol{Z}_+^{n+1}$, $a\neq0$, the $\boldsymbol{Z}$-gradation of $\mathfrak{g}$ of type a is defined by assigning the degrees

$$\deg e_i=-\deg f_i=a_i,\ \deg h_i=0,\ i=0,\ 1,\dots,\ n$$

to the generators of $\mathfrak{g}$ (see [14] and Section 2). We denote this gradation by $\mathfrak{g}(a)=\oplus_m\mathfrak{g}(a)_m$. The principal gradation of $\mathfrak{g}$ is the $\boldsymbol{Z}$-gradation of type $\mathbf{1}=(1,\ 1,\dots,\ 1)$ and we write $\mathfrak{g}_m$ for $\mathfrak{g}(\mathbf{1})_m$.

Let $V(\lambda)$ be a standard module with the highest weight λ as defined in Section 2. If we set

$$V(\lambda)_m=\textstyle\sum_{m=m_1+\cdots+m_k}\mathfrak{g}_{m_1}\cdots\mathfrak{g}_{m_k}v_\lambda ,$$

$V(\lambda)$ is endowed with the gradation called the principal gradation of $V(\lambda)$.

Let W be a graded subspace of $V(\lambda)$ with respect to the principal gradation. The subspace Ω of $V(2\lambda)$ is an example of such a graded subspace.

The principally specialized character of W is the function

$$\mathrm{ch}_q(W)=\textstyle\sum_{m=0}^{\infty}(\dim W_{-m})q^m$$

where q is an indeterminate and W_{-m} denotes the subspace of W consisting of elements of principal degree $-m$. In other words, the principally specialized character is the generating function of dimensions of homogeneous parts of W.

To state the character formula, we further need the concept of dual Kac-Moody Lie algebras. For a Kac-Moody Lie algebra $\mathfrak{g}$ with its Cartan matrix C, the dual Kac-Moody Lie algebra $\mathfrak{g}^{\vee}$ is defined to be the Kac-Moody Lie algebra with the Cartan matrix ${}^{t}C$.

For $a=(a_0, a_1,\dots, a_n)\in \mathbf{Z}_{+}^{n+1}$, $a\neq 0$,
we put

$$D(\mathfrak{g};\, a)=\textstyle\prod_{m\geq 1}(1-q^m)^{\dim \mathfrak{g}(a)_m}.$$

A Cartan matrix C is called symmetrizable if there exists a nondegenerate diagonal matrix A such that AC is a symmetric matrix.

Then, for a Kac-Moody Lie algebra with a symmetrizable Cartan matrix, the principally specialized character of a standard module $V(\lambda)$ of $\mathfrak{g}$ is calculated in the following way.

Proposition 3. *Let* $a_i=\lambda(h_i)\in \mathbf{Z}_{+}$, $i=0, 1,\dots, n$, *and* $a=(a_0, a_1,\dots, a_n)$. *Then*

$$\mathrm{ch}_q(V(\lambda))=D(\mathfrak{g}^{\vee};\, a+\mathbf{1})/D(\mathfrak{g}^{\vee};\, \mathbf{1}).$$

The Euclidean Lie algebras $\mathfrak{g}_S$ of concern here are $A_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_{n+1}^{(2)}$. As is seen in Table 1, their Cartan matrices are symmetrizable. Therefore for $\mathfrak{g}_S$, Proposition 3 can be applied. Their duals are

$$A_n^{(1)\vee}=A_n^{(1)},\ D_n^{(1)\vee}=D_n^{(1)},\ A_{2n}^{(2)\vee}=A_{2n}^{(2)},\ A_{2n-1}^{(2)\vee}=B_n^{(1)}\ \text{ and }\ D_{n+1}^{(2)\vee}=C_n^{(1)}.$$

On the other hand, by using Proposition 2, we have

Proposition 4 ([8]).

$$\mathrm{ch}_q(V(2\lambda))=\mathrm{ch}q(V(\lambda))\cdot \mathrm{ch}_q(\Omega).$$

In view of Proposition 3, 4, the principally specialized character of Ω is given by

$$\begin{aligned}\text{(3.6)}\qquad \mathrm{ch}_q(\Omega)&=\textstyle\sum_{m=0}^{\infty}(\dim \Omega_{-m})q^m\\ &=D(\mathfrak{g}_S^{\vee};\, (3, 1,\dots, 1))/D(\mathfrak{g}_S^{\vee};\, (2, 1,\dots, 1)).\end{aligned}$$

Since the principal degree of x_i regarded as an operator on $V(\Lambda)=\mathbf{C}[x_i, j\in I_S]$ coincides with its degree given in Table 3, but with the opposite sign, the principally specialized character of Ω (3.6) gives us the desired answer. We list the results in Table 5.

Below we explain how to calculate $D(\mathfrak{g}_S^{\vee}; (a, 1,\dots, 1))$ explicitly. The dimensions $\dim \mathfrak{g}_S^{\vee}((a, 1,\dots, 1))_m$ can be calculated by using a realization of $\mathfrak{g}_S^{\vee}$ as a Lie algebra over Laurent polynomials. Here we employ the realization given in [14], which slightly differs from the one given in Section 2.

We first treat the affine Lie algebras $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$. Let us denote by L one of the simple Lie algebras $\mathfrak{sl}(n+1, \boldsymbol{C})(=A_n)$, $\mathfrak{o}(2n+1, \boldsymbol{C})(=B_n)$ $\mathfrak{sp}(n, \boldsymbol{C})(=C_n)$ and $\mathfrak{o}(2n, \boldsymbol{C})(=D_n)$. Let t and z be two indeterminates. Then the affine Lie algebra $L^{(1)}$ is realized as the complex vector space

$$L^{(1)}=\boldsymbol{C}[t, t^{-1}]\otimes L\oplus \boldsymbol{C}z$$

provided with the bracket

$$[P_1(t)\oplus c_1 z, P_2(t)\oplus c_2 z]=[P_1(t), P_2(t)]\oplus k\left(\operatorname*{Res}_{t=0} \operatorname{trace}\frac{dP_1(t)}{dt} P_2(t)\right)z \tag{3.7}$$

where, $P_1, P_2 \in \boldsymbol{C}[t, t^{-1}]\otimes L$, $c_1, c_2 \in \boldsymbol{C}$ (the central extension of $\boldsymbol{C}[t, t^{-1}]\otimes L$). Here $[\ ,\ ]$ denotes the usual bracket in $\boldsymbol{C}[t, t^{-1}]\otimes L$ and $k=1$ for $L=A_n, D_n$, $k=1/2$ for $L=B_n, C_n$.

In this realization, the generators e_i, f_i, h_i $0\leqq i\leqq n$ of $\mathfrak{g}_S$ are described in the following way. Let $\mathfrak{h}_0$ be a Cartan subalgebra of L. Let $L=\mathfrak{h}_0\oplus\sum_{\alpha\in\Delta_L} L_\alpha$ be the root space decomposition of L with respect to $\mathfrak{h}_0$, where Δ_L denotes the set of roots of L. We fix a system of positive roots $\Delta_{L,+}$. Let $\alpha_1,\dots, \alpha_n$ be simple roots and let $\tilde{\alpha}$ be the highest root. Choose $E_i \in L_{\alpha_i}$ and $F_i \in L_{-\alpha_i}$ so that $\alpha_i(H_i)=2$, $H_i=[E_i, F_i]$ holds (E_i, F_i, H_i: the canonical generators of L). Further we choose $E_0 \in L_{-\tilde{\alpha}}$, and $F_0 \in L_{\tilde{\alpha}}$ by the condition

$$[H_0, F_0]=2E_0,\ [H_0, F_0]=-2F_0, \text{ where } H_0=[E_0, F_0]. \tag{3.8}$$

Then, we have

$$\begin{aligned} &e_0=t\otimes E_0,\ e_i=1\otimes E_i,\ i=1,\dots, n\\ &f_0=t^{-1}\otimes F_0, f_i=1\otimes F_i,\ i=1,\dots, n\\ &h_0=1\otimes H_0\oplus z,\ h_i=1\otimes H_i,\ i=1,\dots, n. \end{aligned} \tag{3.9}$$

The vector space $L^{(1)}$ has the following direct sum decomposition

$$L^{(1)}=\boldsymbol{C}z \bigoplus_{(k,\alpha)} t^k\otimes L_\alpha \tag{3.10}$$

where (k, α) ranges over the set $\boldsymbol{Z}\times(\Delta_L\cup\{0\})$ and $L_0=\mathfrak{h}_0$. This decomposition can be regarded as the "root space decomposition" of $L^{(1)}$. But we do not discuss this concept here (see, for example, [15], [19]).

The decomposition (3.10) permits us to express $D(L^{(1)}; (a, 1,\dots, 1))$ it terms of the root system Δ_L of L (cf. [21]). For $\alpha \in \Delta_L$, we define its height $l(\alpha)$ by

$$l(\alpha)=\sum_{i=1}^{n} k_i, \text{ where } \alpha=\sum_{i=1}^{n} k_i\alpha_i .$$

In view of the relations (3.9), the indeterminate t has degree $a+l(\tilde{\alpha})$ in the $\mathbb{Z}$-gradation of $L^{(1)}$ of type $(a, 1,\dots, 1)$. Therefore we have the equality between vector spaces

$$L^{(1)}((a, 1,\dots, 1))_m = \bigoplus_{(k,\alpha)} t^k \otimes L_\alpha$$

where in the right hand side, (k, α) ranges over the set

$$\{(k, \alpha) \in \mathbb{Z} \times (\Delta_L \cup \{0\}) \mid k(a+l(\tilde{\alpha}))+l(\alpha)=m\} .$$

Noting that

$$\dim t^k \otimes L_\alpha = \begin{cases} 1 \text{ if } \alpha \neq 0 \\ n \text{ otherwise}, \end{cases}$$

we have

$$\begin{aligned}(3.11) \qquad D(L^{(1)}, (a, 1,\dots, 1)) &= \textstyle\prod_{k\geqq 1} (1-q^{k(a+l(\tilde{\alpha}))})^n \\ &\times \textstyle\prod_{k\geqq 1} \prod_{\alpha\in\Delta_{L,+}} (1-q^{k(a+l(\tilde{\alpha}))+l(\alpha)-(a+l(\tilde{\alpha}))}) \\ &\times \textstyle\prod_{k\geqq 1} \prod_{\alpha\in\Delta_{L,+}} (1-q^{k(a+l(\tilde{\alpha}))-l(\alpha)}) .\end{aligned}$$

An analogous expression for $D(A_{2n}^{(2)}; (a, 1,\dots, 1))$ is also possible. We explain this briefly. A realization of $A_{2n}^{(2)}$ is given as follows. On $\mathfrak{sl}(2n+1, \mathbb{C})$ consider the involution

$$l \longmapsto -{}^t l \quad l \in \mathfrak{sl}(2n+1, \mathbb{C}) .$$

Let L_0 (resp. L_1) be the 1 (resp. -1)-eigenspace of this involution. Then L_0 is isomorphic to $\mathfrak{o}(2n+1, \mathbb{C})$ $(=B_n)$ and L_1 is an irreducible L_0-module. The Euclidean Lie algebra $A_{2n}^{(2)}$ is realized as

$$A_{2n}^{(2)}=\mathbb{C}[t^2, t^{-2})\otimes L_0 \oplus t\mathbb{C}[t^2, t^{-2}]\otimes L_1 \oplus \mathbb{C}z$$

with the bracket (3.7), in which P_i $(i=1, 2)$ belongs to

$$\mathbb{C}[t^2, t^{-2}]\otimes L_0 \oplus t\mathbb{C}[t^2, t^{-2}]\otimes L_1 \text{ and } k=1/2 .$$

We fix the root space decomposition of L_0:

$$L_0=\mathfrak{h}_0 \oplus \sum_{\alpha\in\Delta_0} L_{0,\alpha}$$

with respect to a Cartan subalgebra $\mathfrak{h}_0$ of L_0. Here Δ_0 denotes the set of roots of L_0. We also fix a system of positive roots $\Delta_{0,+}$ and let $\alpha_1,\dots, \alpha_n$ be the system of simple roots of L_0. Then L_1 is an irreducible L_0-module with the highest weight $\tilde{\alpha}=2\alpha_1+\cdots 2\alpha_n$. The set of weights Δ_1 of the L_0-module L_1 is

$$\Delta_0 \cup \{\pm 2(\alpha_i+\cdots+\alpha_n) \mid 1 \leqq i \leqq n\}.$$

Let

$$L_1=\oplus_{\gamma\in\Delta_{L_1}} L_{1,\gamma}$$

be the weight space domoposition of L_1.

Then corresponding to (3.10), we have the following decomposition of $A_{2n}^{(2)}$:

$$A_{2n}^{(2)}=\boldsymbol{C}z \oplus_{(k,\alpha)} t^{2k}L_{0,\alpha} \oplus_{(l,\gamma)} t^{2l-1}L_{1,\gamma}$$

where $L_{0,0}=\mathfrak{h}_0$ and (k, α) ranges over the set $\boldsymbol{Z}\times(\Delta_0 \cup \{0\})$ and (l, γ) ranges over $\boldsymbol{Z}\times\Delta_1$.

The realization of the generators e_i, f_i, h_i, $i=0, 1,\dots, n$ of $A_{2n}^{(2)}$ is given by (3.9), in which E_0 (resp. F_0) is chosen from $L_{1,-\tilde{\alpha}}$ (resp. $L_{1,\tilde{\alpha}}$) so that the relation (3.6) holds and the relation $h_0=1\otimes H_0 \oplus z$ is replaced by

$$h_0=1\otimes H_0 \oplus 2^{-1}z.$$

Then arguments to those used in the case of affine Lie algebras give

$$\begin{aligned}(3.12)\qquad D(A_{2n}^{(2)}; (a, 1,\dots, 1)) &= \textstyle\prod_{k\geqq 1}(1-q^{k(a+l(\tilde{\alpha}))})^n \\ &\times \textstyle\prod_{k\geqq 1}\prod_{\alpha\in\Delta_{0,+}}(1-q^{2k(a+l(\tilde{\alpha}))+l(\alpha)-2(a+l(\tilde{\alpha}))}) \\ &\times \textstyle\prod_{k\geqq 1}\prod_{\alpha\in\Delta_{0,+}}(1-q^{2k(a+l(\tilde{\alpha}))-l(\alpha)}) \\ &\times \textstyle\prod_{k\geqq 1}\prod_{\gamma\in\Delta_{1,+}}(1-q^{(2k-1)(a+l(\tilde{\alpha}))+l(\alpha)})(1-q^{(2k-1)(a+l(\tilde{\alpha}))-l(\alpha)})\end{aligned}$$

where $\Delta_{1,+}$ denote the set of positive weights of L_1.

The remaining task is to count the number of elements in $\Delta_{L,+}$ or $\Delta_{0,+}\cup\Delta_{1,+}$ of given height. This is achieved by consulting the Table in [20]. We list the results in Table 4.

Using Table 4 and (3.12), (3.12), we obtain the results given in Table 5. Using Table 5 and (3.5), $\dim H_S(m)$ is calculated as follows. For the KdV $(=(\mathrm{KP})_2)$ hierarchy, we have

$$\begin{aligned}\dim H_{\mathrm{KdV}}(m) &= \#\{(m_1,\dots, m_k) \mid m_i\in\boldsymbol{Z}_+, \text{ odd } m_1\leqq\cdots\leqq m_k, \textstyle\sum_{i=1}^k m_i=m\} \\ &\quad -\#\{(m_1,\dots, m_k) \mid m_i\in\boldsymbol{Z}_+, m_i\equiv 2 \ (\mathrm{mod.}\ 4), m_1\leqq\cdots\leqq m_k, \textstyle\sum_{i=1}^k m_i=m\}.\end{aligned}$$

A theorem of Euler states that

$$\begin{aligned}&\#\{(m_1,\dots, m_k) \mid m_i\in\boldsymbol{Z}_+, m_i\equiv 2 \ (\mathrm{mod.}\ 4), m_1\leqq\cdots\leqq m_k, \textstyle\sum_{i=1}^k m_i=m\} \\ &\quad =\#\{(m_1,\dots, m_k) \mid m_i\text{: positive even integer}, m_1<\cdots<m_k, \textstyle\sum_{i=1}^k m_i=m\}.\end{aligned}$$

Hence, in a accordance with the result of M. and Y. Sato, we have (3.1).

For the $(\mathrm{KP})_{n+1}$ hierarchy, we have

$$\begin{aligned}\dim H_{(\mathrm{KP})_{n+1}}(m) &= \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i \not\equiv 0 \ (\mathrm{mod.}\ n+1), \\ &\qquad 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\} \\ &\quad - \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i \not\equiv 0,\ \pm 1 (\mathrm{mod.}\ n+3),\ 1 \leqq m_1 \leqq \cdots \leqq m_k, \\ &\qquad \textstyle\sum_{i=1}^{k} m_i = m\}.\end{aligned}$$

For the Sawada-Kotera $(=(\mathrm{BKP})_3)$ hiererachy, we have (3.3), proving the conjecture of M. and Y. Sato.

For the $(\mathrm{BKP})_{2n+1}$ and the $(\mathrm{BKP})_{2n+2}$ hierarchies, we have

$$\begin{aligned}\dim H_{(\mathrm{BKP})_{2n+1}}(m) &= \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i\colon \text{odd}, \\ &\qquad m_i \not\equiv 0 \ (\mathrm{mod.}\ 2n+1),\ 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\} \\ &\quad - \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i\colon \text{even},\ m_i \equiv 2j,\ 1 \leqq j \leqq 2n+2, \\ &\qquad j \neq n+1,\ n+2 \ (\mathrm{mod.}\ 4n+6),\ 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\},\end{aligned}$$

$$\begin{aligned}\dim H_{(\mathrm{BKP})_{2n+2}}(m) &= \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i\colon \text{odd}, \\ &\qquad 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\} \\ &\quad - \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ m_i\colon \text{even},\ m_i \not\equiv 0 \ (\mathrm{mod.}\ 2n+2), \\ &\qquad 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\}.\end{aligned}$$

For the $(\mathrm{BKP\,II})_{2r+1,2s+1}$ hierarchy, the result is the same as that of the $(\mathrm{BKP})_{2n}$ hierarchy with $n = r+s+1$.

For the $(\mathrm{BKP\,II})_{2r,2s}$ hierarchy $(n = r+s)$, we have

$$\begin{aligned}&\dim H_{(\mathrm{BKP\,II})_{2r,2s}}(m) \\ &\quad = \sharp\{(a_1, a_3,\dots, a_{2n-3}, b) \in \sharp\boldsymbol{Z}_+^n \mid m = \textstyle\sum_{i=1}^{n-1} (2i-1) a_{2i-1} + (n-1) b\}, \\ &\qquad - \sharp\{(a_2, a_4,\dots, a_{2n-2}, b) \in \boldsymbol{Z}_+^n \mid m = \textstyle\sum_{i=1}^{n-1} 2i a_{2i} + nb\}.\end{aligned}$$

Finally we mention the result of the calculation of $\dim H_{\mathrm{KP}}(m)$ and $\dim H_{\mathrm{BKP}}(m)$:

$$\begin{aligned}\dim H_{\mathrm{KP}}(m) &= \lim_{n\to\infty} \dim H_{(\mathrm{KP})_n}(m), \\ \dim H_{\mathrm{BKP}}(m) &= \lim_{n\to\infty} \dim H_{(\mathrm{BKP})_n}(m).\end{aligned}$$

In view of the results above, we have

$$\begin{aligned}\dim H_{\mathrm{KP}}(m) &= \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\} \\ &\quad - \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ 2 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m\} \\ &= \sharp\{(m_1,\dots, m_k) \mid m_i \in \boldsymbol{Z}_+,\ 1 \leqq m_1 \leqq \cdots \leqq m_k,\ \textstyle\sum_{i=1}^{k} m_i = m-1\},\end{aligned}$$

which proves the conjecture (3.2) of M. and Y. Sato.

For the BKP hierarchy, as announced in [11, IV], we have

$$\dim H_{\mathrm{BKP}}(m)=\sharp\{(m_1,\dots,m_k)|m_i\in \boldsymbol{Z}_+,\ m_i\text{: odd},\ 1\leqq m_1\leqq\cdots\leqq m_k,\ \textstyle\sum_{i=1}^k m_i=m\}$$
$$-\sharp\{(m_1,\dots,m_k)|m_i\in \boldsymbol{Z}_+,\ m_i\text{: even},\ 1\leqq m_1\leqq\cdots\leqq m_k,\ \textstyle\sum_{i=1}^k m_i=m\}.$$

Table 4.

type of $\mathfrak{g}_S^\vee$	$A_n^{(1)}$, $\Delta_{A_n,+}$	$B_n^{(1)}$, $C_n^{(1)}$ $\Delta_{B_n,+}$, $\Delta_{C_n,+}$	$D_n^{(1)}$ $\Delta_{D_n,+}$	
number of roots (weights) of length i	$n-i+1$ $(1\leqq i\leqq n)$	$n-[i/2]$ $(1\leqq i\leqq 2n-1)$	$n-[i/2]$ $(1\leqq i\leqq n)$	$n-[i/2]-1$ $(n\leqq i\leqq 2n-3)$

type of $\mathfrak{g}_S^\vee$	$A_{2n}^{(2)}$	
	$\Delta_{0,+}$	$\Delta_{1,+}$
number of roots (weights) of length i	$n-[i/2]$ $(1\leqq i\leqq 2n-1)$	$n-[(i-1)/2]$ $(1\leqq i\leqq 2n)$

In this Table, [ ] denotes Gauss's symbol.

Table 5.

Type of S	$(\mathrm{KP})_{n+1}$	$(\mathrm{BKP\ II})_{2r+1,2s+1}$	$(\mathrm{BKP})_{2n+2}$
$\mathfrak{g}_S$	$A_n^{(1)}$	$A_{2n-1}^{(2)},\ n=r+s+1$	$D_{n+1}^{(2)}$
$\mathrm{ch}_q\Omega$	$\prod_{k\geqq 1}\prod_{i=2}^{n+1}(1-q^{(n+3)k-i})^{-1}$	$\prod_{k\geqq 1}\prod_{i=1}^{n}(1-q^{(2n+2)k-2i})^{-1}$	
$D(\mathfrak{g}_S^\vee:(1,1,\dots,1))$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n}(1-q^{(n+1)k-i})$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n}(1-q^{2nk-(2i-1)})$	
$D(\mathfrak{g}_S^\vee:(2,1,\dots,1))$	$\varphi(q)^n$		
$D(\mathfrak{g}_S^\vee:(3,1,\dots,1))$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n+1}(1-q^{(n+3)k-i})^{-1}$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n}(1-q^{(2n+2)k-2i})^{-1}$	

Continue

$(\mathrm{BKP\ II})_{2r,2s}$	$(\mathrm{BKP})_{2n+1}$
$D_n^{(1)}$, $n=r+s$	$A_{2n}^{(2)}$
$\prod_{k\geqq 1}\prod_{i=1}^{n-1}(1-q^{2nk-2i})^{-1}\prod_{k\geqq 1}(1-q^{2nk-n})^{-1}$	$\prod_{k\geqq 1}\prod_{\substack{1\leqq i\leqq 2n+2\\ i\neq n+1,n+2}}(1-q^{(4n+6)k-2i})^{-1}$
$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n-1}(1-q^{(2n-2)k-(2i-1)})$ $\times\prod_{k\geqq 1}(1-q^{(2n-2)k-(n-1)})$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{\substack{1\leqq i\leqq 2n+1\\ i\neq n+1}}(1-q^{(4n+2)k-(2i-1)})$
	$\varphi(q)^n$
$\varphi(q)^n\prod_{k\geqq 1}\prod_{i=1}^{n-1}(1-q^{2nk-2i})^{-1}$ $\times\prod_{k\geqq 1}(1-q^{2nk-n})^{-1}$	$\varphi(q)^n\prod_{k\geqq 1}\prod_{\substack{1\leqq i\leqq 2n+2\\ 1\neq n+1,n+2}}(1-q^{(4n+6)k-2i})^{-1}$

In this Table, $\varphi(q)=\prod_{k\geqq 1}(1-q^k)$.

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