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On the Instability of a Minimal Surface in a 4-Manifold Whose Curvature Lies in the Interval (1/4, 1]

By

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Introduction

Let M be a compact manifold minimally immersed in a Riemannian manifold. M is said to be *unstable* if the second variation $\delta^2(u)$ of the volume of M for some normal vector field u is strictly negative.

The purpose of this paper is to prove a generalization of the following result of Aminov [1].

Theorem (Aminov). *Let M be a surface minimally immersed in an orientable Riemannian manifold of dimension* 4, *whose curvature lies in the interval* (1/4, 1]. *Suppose that M is homeomorphic to the* 2*-sphere S^2. Then M is unstable.*

This theorem is related to the conjecture of Lawson and Simons [5], i.e., every minimal current in a complete simply connected Riemannian manifold whose curvature lies in the interval (1/4, 1] is unstable.

The proof of this theorem can be roughly outlined as follows: First he shows that if a non-trivial cross-section u of the normal bundle ν of M satisfies a certain differential equation (($*$) in §1 in our terminology), then $\delta^2(u)+\delta^2(Ju)<0$, where J is the complex structure defined by the orientations of M and the ambient manifold. Secondly he constructs a solution of ($*$) on M which is homeomorphic to S^2.

In this paper we investigate the dimension of the solution space H of ($*$) for a general immersed surface and obtain the following lemma.

Lemma 2. *Let $\chi(\nu)$ and $g(M)$ denote the Euler number of the normal*

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bundle ν of M and the genus of M respectively. Then

$$\dim H \geqq \chi(\nu) + 1 - g(M).$$

We note that all the above arguments work well for J and $-J$ equally. Hence we can assume that the Euler number $\chi(\nu)$ is non-negative. For details, see Remark below the proof of Proposition 3.

As a corollary we get the following generalization of the theorem of Aminov [1].

Theorem. *Let M be a compact orientable surface minimally immersed in a orientable Riemannian manifold of dimension 4, whose curvature lies in the interval* $(1/4, 1]$. *Suppose that the Euler number of the normal bundle is greater than or equal to the genus of M. Then M is unstable.*

To prove Lemma 2, we give a structure of a holomorphic line bundle to the normal bundle ν of M whose holomorphic cross-sections u coincide with the solutions of $(*)$, and we apply the Riemann-Roch theorem.

The paper is divided into three sections. In the first section we represent the first step of Aminov's argument in our terminology and reduce the instability problem to the differential equation $(*)$. In Section 2, we investigate the dimension of the solution space of $(*)$, and prove Theorem. In Section 3, we give examples.

Throughout this paper all manifolds and maps are to be differentiable of class C^∞ unless otherwise stated.

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§ 1. The Second Variation and the Equation $J \circ \nabla u = \nabla u \circ I$

Let M be an oriented surface minimally immersed in an oriented Riemannian manifold N of dimension 4. We can define a complex structure J of the normal bundle ν of M by the orientations of M and N. Let ∇ denotes the covariant differentiation of ν associated to the induced metric.

Proposition 1. *The complex structure J is parallel.*

Proof. Let n be a unit cross-section of ν around a point p of M. Then $\{n, Jn\}$ is an orthonormal frame of ν around p which satisfies the following equations:

$$\nabla_X n = \langle \nabla_X n, Jn \rangle Jn,$$
$$\nabla_X(Jn) = -\langle \nabla_X n, Jn \rangle n,$$

for every tangent vector X to M at p. Since

$$\nabla_X(Jn) = -\langle \nabla_X n, Jn \rangle n = J \nabla_X n,$$
$$\nabla_X(JJn) = -\nabla_X n = J(\nabla_X(Jn)),$$

we have $\nabla_X J = 0$. Thus the proposition is proved.

Let R be the curvature of ν. For every cross-section u of ν, let us define a function $S(u)$ on M as follows:

$$S(u)(p) = \langle R(e, Ie)u(p), Ju(p) \rangle,$$

where e is a unit tangent vector to M at p, and I is the complex structure defined by the orientation and the induced metric of TM.

Proposition 2. *If a cross-section u of ν is a solution of the differential equation*

$$(*) \qquad J \circ \nabla u = \nabla u \circ I,$$

then u satisfies the equation

$$\langle \triangle u, u \rangle + S(u) = 0,$$

where $\triangle = \nabla^* \circ \nabla$ is the Laplacian in ν.

Proof. Let e be a unit tangent vector to M at p, E be a local vector field around p such that $E(p) = e$, $(\nabla E)(p) = 0$. Then

$$\begin{aligned} R(e, Ie)u &= \nabla_e \nabla_{IE} u - \nabla_{Ie} \nabla_E u \\ &= \nabla_e (J \nabla_E u) + \nabla_{Ie} (J \nabla_{IE} u) \\ &= -J \triangle u. \end{aligned}$$

Thus the proposition is proved.

Let $\bar{R}$ be the curvature of N. For every cross-section u of ν, let us define a function $\bar{S}$ on M as follows:

$$\bar{S}(u)(p) = \langle \bar{R}(u(p), e)e, u(p) \rangle + \langle \bar{R}(u(p), Ie)Ie, u(p) \rangle,$$

where e is a unit tangent vector to M at p.

As is well known, the second variation $\delta^2(u)$ of the volume of M in the direction of a normal vector field u is written as follows:

$$\delta^2(u) = \int_M (\langle \triangle u, u \rangle - \bar{S}(u) - \|A^u\|^2) dV,$$

where $\|A^{u(p)}\|$ is the norm of the second fundamental form of M in the direction of $u(p)$, and dV is the volume form of M (cf. [7] p. 73. Our $\triangle$ is $-\nabla^2$ in [7].).

Proposition 3. *If the sectional curvature of N lies in the interval* $(1/4, 1]$, *and if $u \not\equiv 0$ is a solution of* $(*)$, *then*

$$\delta^2(u)+\delta^2(Ju)<0 .$$

Hence either $\delta^2(u)$ or $\delta^2(Ju)$ is negative.

Proof. We express the integrand as follows:

$$\begin{aligned}&\langle \triangle u, u\rangle+\langle \triangle(Ju), Ju\rangle-\bar{S}(u)-\bar{S}(Ju)-\|A^u\|^2-\|A^{Ju}\|^2\\ &=[\langle \triangle u, u\rangle+S(u)+\langle \triangle(Ju), Ju\rangle+S(Ju)]\\ &\quad+[-S(u)-S(Ju)-\bar{S}(u)-\bar{S}(Ju)-\|A^u\|^2-\|A^{Ju}\|^2].\end{aligned}$$

Since the first parenthesis vanishes by Proposition 2, it suffices to show that the second parenthesis is strictly negative when $u(p)\neq 0$. Let e be a unit tangent vector to M at p. By the equation of Ricci, we have

$$\begin{aligned}S(u)(p)=&\langle \bar{R}(e, Ie)u(p), Ju(p)\rangle+\langle A^{Ju(p)}(e), A^{u(p)}(Ie)\rangle\\ &-\langle A^{u(p)}(e), A^{Ju(p)}(Ie)\rangle .\end{aligned}$$

Hence the second parenthesis at p is

$$\begin{aligned}&[-2\langle A^{Ju(p)}(e), A^{u(p)}(Ie)\rangle+2\langle A^{u(p)}(e), A^{Ju(p)}(Ie)\rangle-\|A^{u(p)}\|^2-\|A^{Ju(p)}\|^2]\\ &\quad+[-2\langle \bar{R}(e, Ie)u(p), Ju(p)\rangle-\bar{S}(u)(p)-\bar{S}(Ju)(p)] .\end{aligned}$$

The first parenthesis of the above is not positive. The inequality of Berger [2] implies that if the sectional curvature of N lies in the interval $[a, 1]$ for some positive a, then

$$(**)\qquad \begin{aligned}&2|\langle \bar{R}(e, Ie)u(p), Ju(p)\rangle|-[\bar{S}(u)(p)+\bar{S}(Ju)(p)]\\ &\leqq\left[\frac{4}{3}(1-a)-4a\right]\cdot\|u(p)\|^2\\ &=\frac{4}{3}(1-4a)\cdot\|u(p)\|^2 .\end{aligned}$$

Thus the proposition is proved.

Remark. If we consider $-J$ instead of J, $S(u)$ and $S(Ju)$ should be replaced by $-S(u)$ and $-S(Ju)$ respectively in Proposition 2 and Proposition 3. But the inequality $(**)$ remains valid even though J is replaced by $-J$. Hence Proposition 3 is true for J and $-J$ equally.

§2. The Solution Space of $J\circ\nabla u=\nabla u\circ I$

In this section we investigate the solution space of the equation $(*)$.

Lemma 1. *The normal bundle ν has a structure of a holomorphic line bundle whose holomorphic cross-sections coincide with the solutions of the equation* $(*)$.

Proof. First of all we assert that for every point p of M there exists a non-vanishing C^∞ solution u_p of $(*)$ around p. Let (x, y) be the local coordinate obtained from the complex one $z=x+iy$. Then $(*)$ is equivalent to the equation

$$(***)\qquad J\nabla_{\frac{\partial}{\partial x}}u=\nabla_{\frac{\partial}{\partial y}}u\,.$$

We seek two C^∞ functions f, g such that $u_p=fn+gJn$ is a non-vanishing solution of $(*)$, where $\{n, Jn\}$ is a local orthonormal frame of ν. For this purpose we write $(***)$ as follows:

$$\frac{\partial f}{\partial y}-g\langle\nabla_{\frac{\partial}{\partial y}}n,\, Jn\rangle=-\frac{\partial g}{\partial x}-f\langle\nabla_{\frac{\partial}{\partial x}}n,\, Jn\rangle\,,$$

$$\frac{\partial f}{\partial x}-g\langle\nabla_{\frac{\partial}{\partial x}}n,\, Jn\rangle=\frac{\partial g}{\partial y}+f\langle\nabla_{\frac{\partial}{\partial y}}n,\, Jn\rangle\,.$$

Denoting $\langle\nabla_{\frac{\partial}{\partial y}}n, Jn\rangle$ and $-\langle\nabla_{\frac{\partial}{\partial x}}n, Jn\rangle$ by α and β respectively, the above equations are equivalent to the following:

$$\frac{\partial}{\partial\bar{z}}(f+ig)=\frac{1}{2}(\alpha+i\beta)(f+ig)\,.$$

The complex function theory enables us to solve the equation

$$\frac{\partial}{\partial\bar{z}}F=\frac{1}{2}(\alpha+i\beta)$$

locally (for example cf. Morrow and Kodaira [6] Lemma 6.2.). So, taking $f+ig=\exp F$, we get a non-vanishing solution $u_p=fn+gJn$ of $(*)$ around p, and the assertion is proved.

Since $\{u_p, Ju_p\}$ defines a frame of ν around p, we can define a bundle chart $\varphi_p\colon U_p\times\boldsymbol{C}\to\pi^{-1}(U_p)$ of ν by setting

$$\varphi_p(z,\, a+ib)=au_p(z)+bJu_p(z)\,,$$

where U_p is the domain of u_p, and π is the projection of ν.

If $U_p\cap U_q\neq\emptyset$, $p, q\in M$, we can write $u_q=mu_p+nJu_p$, where m and n are

$\mathbf{R}$-valued C^∞ functions. Since u_q satisfies $(*)$, for every tangent vector X to M at $z \in U_p \cap U_q$, we have

$$\begin{aligned}\nabla_{IX}u_q - J\nabla_X u_q &= m(\nabla_{IX}u_p - J\nabla_X u_p) + nJ(\nabla_{IX}u_p - J\nabla_X u_p)\\ &\quad + [(IX)m + Xn]u_p + [(IX)n - Xm]Ju_p\\ &= 0.\end{aligned}$$

Since u_p also satisfies $(*)$, we have

$$Xm-(IX)n=0,$$
$$(IX)m+Xn=0.$$

These equations show that the $\mathbf{C}$-valued function $m+in$ is holomorphic on $U_p \cap U_q$.

Since the transition function $g_{qp}\colon U_p \cap U_q \to GL(1, \mathbf{C})$ is written as follows:

$$g_{qp}(z) = [(m+in)]^{-1},$$

the bundle charts defined above give a structure of a holomorphic line bundle to the normal bundle ν.

The fact that holomorphic cross-sections of this structure coincide with the solutions of $(*)$ is a straight consequence of the definition of the bundle charts. Thus the lemma is proved.

Let $\chi(\nu)$ and $g(M)$ denote the Euler number of the normal bundle ν and the genus of M respectively. Then by the Riemann-Roch theorem we have

$$\dim H^0(M, \nu) - \dim H^1(M, \nu) = \chi(\nu) + 1 - g(M).$$

Hence by Lemma 1, we get the following lemma.

Lemma 2. *Let H denote the solution space of the equation* $(*)$. *Then we have*

$$\dim H \geqq \chi(\nu) + 1 - g(M).$$

Now the proof of Theorem is clear by Proposition 3 and Lemma 2.

As remarked in the introduction and Section 1, we can choose the orientation of the normal bundle ν so that $\chi(\nu)$ is not negative. Hence we get the following corollary which is one of the main result of Aminov [1].

Corollary 1. *If M is homeomorphic to the 2-sphere S^2, then M is unstable.*

Taking $g(M)=1$ in Theorem, we get the following corollary.

Corollary 2. *If M is homeomorphic to the torus T^2, and if the normal bundle is non-trivial, then M is unstable.*

§3. Examples

In this section we give examples other than S^2 which satisfy the assumption of Theorem.

Proposition 4. *Let M be a minimal submanifold of (N, g), $g^*=\rho^2 g$ be a new metric for some positive C^∞ function ρ on N. Then M is minimal with respect to g^* if and only if the restriction of the vector field* $\operatorname{grad}(\log\rho)$ *to M is tangent to M.*

Proof. Let ∇^* and ∇ denote the covariant differentiation with respect to g^* and g respectively. Then we have

$$\nabla^*_X Y=\nabla_X Y+(d\log\rho)(X)Y+(d\log\rho)(Y)X-g(X, Y)\operatorname{grad}(\log\rho)$$

for all vector fields X and Y (cf. Chen [3] p. 23, (5.1)). By this formula the proposition can be proved by the straightforward calculation.

Let us consider a compact complex submanifold M of the n-dimensional complex projective space CP^n with the Fubini-Study metric g_0 of holomorphic sectional curvature 1. M is minimal because a complex submanifold of a Kaehler manifold is minimal.

We shall show that we can change the metric g_0 conformally around M, so that the sectional curvature lies in the interval $(\delta/4, \delta]$ for some $\delta>0$, and M is minimal with respect to the new metric.

Making use of the identification of the tubular neighborhood U of M with the total space of ν, consider a C^∞ function ρ_ε on U defined by $\rho_\varepsilon(x)=1-\varepsilon\|x\|^2$, $x\in U$, for sufficiently small $\varepsilon>0$.

If we define a new metric g_ε on U by $g_\varepsilon=(\rho_\varepsilon)^2 g_0$, M is minimal with respect to g_ε by Proposition 4.

Proposition 5. *For sufficiently small $\varepsilon>0$, the sectional curvature of g_ε at every point of M lies in the interval* $(1/4+\varepsilon, 1+4\varepsilon]$.

Proof. Let X and Y be orthonormal tangent vectors of CP^n at $p\in M$. Let $K_\varepsilon(X\wedge Y)$ and $K_0(X\wedge Y)$ denote the sectional curvatures of g_ε and g_0 respectively, for the plane section $X\wedge Y$ spaned by X and Y. Then we have (cf. Chen [3] p. 24 (5.5))

$$K_\varepsilon(X \wedge Y)=K_0(X \wedge Y)-g_0(s(Y, Y)X, X)+g_0(s(X, Y)Y, X) \\ -g_0(Y, Y)g_0(SX, X)+g_0(X, Y)g_0(SY, X),$$

where

$$s(X, Y)=(\nabla_X d\log\rho_\varepsilon)(Y)-(X\rho_\varepsilon)(Y\rho_\varepsilon)+\frac{1}{2}\|\operatorname{grad}\rho_\varepsilon\|^2 g_0(X, Y),$$

$$g_0(SX, Y)=s(X, Y).$$

Since $(d\log\rho_\varepsilon)(p)=0$, we have

$$K_\varepsilon(X \wedge Y)=K_0(X \wedge Y)-[(\nabla_X d\log\rho_\varepsilon)(X)+(\nabla_Y d\log\rho_\varepsilon)(Y)] \\ =K_0(X \wedge Y)-[X(\tilde{X}\rho_\varepsilon)+Y(\tilde{Y}\rho_\varepsilon)],$$

where $\tilde{X}$ (resp. $\tilde{Y}$) is any vector field with $\tilde{X}(p)=X$ (resp. $\tilde{Y}(p)=Y$). Let us decompose X to the tangent component X^T and the normal component X^N. Let $\widetilde{X^T}$ and $\widetilde{X^N}$ be vector fields around p with $\widetilde{X^T}(p)=X^T$ and $\widetilde{X^N}(p)=X^N$ such that $\widetilde{X^T}$ is tangent to the hypersurface $\rho_\varepsilon\equiv$constant, and the integral curve $c(t)$ of $\widetilde{X^N}$ with $c(0)=p$ is a geodesic of (CP^n, g_0). Then we have

$$X(\tilde{X}\rho_\varepsilon)=X^N(\widetilde{X^N}\rho_\varepsilon).$$

Applying the same process to Y, we get the identity

$$K_\varepsilon(X \wedge Y)=K_0(X \wedge Y)-[X^N(\widetilde{X^N}\rho_\varepsilon)+Y^N(\widetilde{Y^N}\rho_\varepsilon)] \\ =K_0(X \wedge Y)+2\varepsilon[\|X^N\|^2+\|Y^N\|^2].$$

Clearly the maximum value of $K_\varepsilon(X \wedge Y)$ for all plane sections at p is $1+4\varepsilon$. When X and Y span the real subspace of $T_p(CP^n)$, $K_0(X \wedge Y)$ takes the minimum value $1/4$, and $\|X^N\|^2+\|Y^N\|^2$ is equal to 1. Hence the minimum value of $K_\varepsilon(X \wedge Y)$ is greater than $1/4+\varepsilon$ for sufficiently small $\varepsilon>0$. Thus the proposition is proved.

Since M is compact, it follows that for sufficiently small $\varepsilon>0$, there exists a neighborhood V of M in CP^n such that the sectional curvature lies in the interval $(\delta/4, \delta]$ for some positive $\delta>0$.

Now consider the nonsingular algebraic curve M defined by

$$z_0^3+z_1^3+z_2^3=0,$$

where z_0, z_1, and z_2 are the homogeneous coordinates of CP^2. The genus of M is 1 (cf. Morrow and Kodaira [6]), and M is homeomorphic to the torus T^2. Suppose that the normal bundle ν of $M=T^2$ is trivial. Then $T(T^2)\oplus\nu=i^{-1}(T(CP^2))$ is trivial, where $i^{-1}(T(CP^2))$ is the bundle induced by the inclusion i: $T^2\to CP^2$. Hence the Chern class

$$c_1(i^{-1}(T(\mathbb{C}P^2)))=i^*(c_1(T(\mathbb{C}P^2)))$$

is equal to zero. This contradicts the facts that $c_1(T(\mathbb{C}P^2))\neq 0$ and $i^*\colon H^2(\mathbb{C}P^2; \boldsymbol{Z})\to H^2(T^2; \boldsymbol{Z})$ is injective. Hence ν is not trivial.

By the preceding argument and Theorem, we can conclude that the minimal surface T^2 of (V, g_ε) is unstable for sufficiently small $\varepsilon>0$.

Remark. In fact the genus of the nonsingular algebraic curve M defined by the homogeneous equation $z_0^d+z_1^d+z_2^d=0$ is $\frac{1}{2}d(d-3)+1$, and we can show that the Euler number of the normal bundle of M is d^2. Since d^2 is greater than $\frac{1}{2}d(d-3)+1$ for every positive integer d, we get many more examples.

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