*Publ. RIMS, Kyoto Univ.*
**18** (1982), 1163–1173

# Projectivity of the Space of Divisors on a Normal Compact Complex Space

By

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## Introduction

For any complex space we shall denote by $D_X$ the Douady space of compact complex subspaces of $X$ [1]. Let $Z_X \subseteq D_X \times X$ be the universal subspace so that for each $d \in D_X$, the corresponding subspace of $X$ is given by $Z_{X,d} := Z_X \cap (\{d\} \times X) \subseteq \{d\} \times X = X$. Recall that a Cartier divisor on $X$ is a complex subspace of $X$ whose sheaf of ideals is generated locally by a single element which is not a zero divisor. Let $\mathrm{Div}\, X = \{d \in D_X;\ Z_{X,d}$ is a Cartier divisor on $X\}$. Then $\mathrm{Div}\, X$ is Zariski open in $D_X$, and in fact is a union of connected components of $D_X$ when $X$ is nonsingular. Then the purpose of this paper is to prove the following:

**Theorem 1.** *For any normal compact complex space X every connected component of* Div *X is compact and projective.*

When $X$ is nonsingular, the proof actually gives a more precise structure theorem of $\mathrm{Div}\, X$ (cf. Proposition in §1 below). The motivation for this theorem comes from Fischer-Forster [2] where they proved that there exist only a finite number of reduced divisors on any compact complex manifold $X$ which are mapped surjectively onto $Y$ where $f: X \to Y$ is an algebraic reduction of $X$ (cf. §1); this implies that almost all the divisors on $X$ are obtained as the pull-backs of those on $Y$ which is projective. Theorem 1 reveals a striking contrast to the case of codimension $> 1$, where in order to obtain the compactness even of the irreducible components of $D_X$ in general, it is necessary to assume that $X$ is Kähler or more generally that $X$ is in $\mathscr{C}$ (cf. [9]). Indeed, the analogy of [2] fails in codimension $> 1$ as was shown by Campana [0].

Though we prove the compactness and projectivity at the same time in Theorem 1, there is an easy alternative proof for the projectivity once the com-

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Commuincated by S. Nakano, January 21, 1982.
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pactness is established. Namely we shall also show the following theorem, stimulated by a result of Ohsawa ([8] Theorem 2) (cf. Remark 2).

**Theorem 2.** *Let $X$ be a connected normal compact complex space and $A$ an analytic subset of $X$. Then for any irreducible component $D_\alpha$ of* Div $X$ *the subset $D_\alpha(A) := \{d \in D_\alpha;\ Z_{X,d} \cap A \neq \emptyset\}$ is a support of an ample divisor on $D_\alpha$ if $D_\alpha(A) \neq D_\alpha$.*

The projectivity is used in [3] to get a local projectivity of a model of a relative algebraic reduction for a fiber space in $\mathscr{C}$.

*Convention.* For any complex space $B$, $B_{\mathrm{red}}$ denotes the underlying reduced subspace. *A complex variety* is a reduced and irreducible complex space. A morphism $f\colon X \to Y$ of complex varieties is called a *fiber space* if $f$ is proper and the general fiber of $f$ is irreducible.

## §1. Preliminary Reductions

Let $f\colon X \to Y$ be a morphism of complex varieties. Let $Z \subseteq X$ be a Cartier divisor on $X$. Then we call $Z$ a *relative divisor* over $Y$ if the following equivalent conditions are satisfied: 1) $Z$ is flat over $Y$. 2) $Z$ contains no irreducible component of the fibers of $f$ (cf. [5], 21.15).

Conversely, if $Z \subseteq X$ is a subspace which is flat over $Y$ and if $Z_y$ is a Cartier divisor on $X_y$ for every $y \in Y$, then $Z$ is a relative divisor over $Y$ ([5]).

Thus if we set $Z(X) = Z_X \cap (\mathrm{Div}\, X \times X) \subseteq \mathrm{Div}\, X \times X$, $Z(X)$ is a relative divisor over Div $X$, $Z_X$ being flat over $D_X$. Further Div $X$ has the following universal property. Let $Z \subseteq T \times X$ be a relative divisor over $T$ with respect to the natural morphism $\rho_T\colon Z \to T$ where $T$ is any complex space. Then there exists a unique morphism $\tau\colon T \to \mathrm{Div}\, X$ such that $Z = (\tau \times id_X)^{-1}(Z(X))$, and hence, that $\rho_T$ is induced from the universal morphism $\rho\colon Z(X) \to \mathrm{Div}\, X$. So we shall call $Z(X)$ the *universal divisor* associated to $X$.

Let $X$ be a compact complex space. Let $\mathrm{Pic}\, X = H^1(X, \mathcal{O}_X^*)$ be the Picard variety of $X$, which has the natural structure of a commutative complex Lie group [6]. Let $c_1\colon H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$ be the first chern class map and $NS(X) = \mathrm{Im}\, c_1$, the Nerson-Severi group of $X$. For $\gamma \in NS(X)$ we set $\mathrm{Pic}_\gamma X = c_1^{-1}(\gamma)$ so that we have

$$\mathrm{Pic}\, X = \coprod_{\gamma \in NS(X)} \mathrm{Pic}_\gamma X .$$

In particular $\mathrm{Pic}_0 X = \mathrm{Ker}\, c_1$ is the identity component of $\mathrm{Pic}\, X$.

Now $Z(X)$ defines a line bundle $[Z(X)]$ on $\mathrm{Div}\, X \times X$ and then by the universality of $\mathrm{Pic}\, X$ we have the natural morphism $\mu_X$: $\mathrm{Div}\, X \to \mathrm{Pic}\, X$, which eventually associates to each $d \in \mathrm{Div}\, X$ the corresponding line bundle $[Z(X)_d]$ (cf. [3] and [4], exposé 234, §4). Further, we know that $\mu_X$ is projective and the fiber over $L \in \mathrm{Pic}\, X$ is naturally identified with the projective space $\boldsymbol{P}(\Gamma(X,L))$: $=(\Gamma(X, L)-\{0\})/\boldsymbol{C}^*$, i.e., the linear system on $X$ associated to $L$, if it is not empty (cf. [3], [4]). Thus if we set $(\mathrm{Div}\, X)^-: = \mu_X(\mathrm{Div}\, X) \subseteqq \mathrm{Pic}\, X$, the following lemma holds.

**Lemma 1.** $(\mathrm{Div}\, X)^-_{\mathrm{red}} = \{p \in \mathrm{Pic}\, X;\ \dim \Gamma(X, L_p) > 0\} - \{e\}$ *where $e$ is the identity of* $\mathrm{Pic}\, X$ *and $L_p$ is the line bundle on $X$ corresponding to $p$.*

Note that $(\mathrm{Div}\, X)^-$ is an analytic subspace of $\mathrm{Pic}\, X$ as $\mu_X$ is proper, though this also follows from Lemma 1 easily. We set $D_\gamma = \mu_X^{-1}(\mathrm{Pic}_\gamma X)$ and $\mu_\gamma = \mu_X | D_\gamma$: $D_\gamma \to \mathrm{Pic}_\gamma X$. Let $Z_\gamma \subseteqq D_\gamma \times X$ be the restriction of $Z(X)$ over to $D_\gamma$. We have of course $\mathrm{Div}\, X = \coprod_\gamma D_\gamma$. Let $\bar{D}_\gamma = \mu_\gamma(D_\gamma)$, so that $(\mathrm{Div}\, X)^- = \coprod_\gamma \bar{D}_\gamma$. Then by virtue of the above description of $\mathrm{Div}\, X$ we see that our task is to show that every connected component of $\bar{D}_\gamma$ is projective.

Let $X$ be a compact connected complex manifold. Let

$$(*) \qquad \begin{array}{ccc} X^* & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

be a holomorphic model of algebraic reduction of $X$. Namely $X^*$ is a compact complex manifold, $\varphi$ is a bimeromorphic morphism, $Y$ is a projective manifold and $f$ is a fiber space which induces an isomorphism $f^*$: $\boldsymbol{C}(Y) \cong \boldsymbol{C}(X^*)$ of the meromorphic function fields of $X^*$ and $Y$. Let $f^*$: $\mathrm{Pic}_0 Y \to \mathrm{Pic}_0 X^*$ and $\varphi^*$: $\mathrm{Pic}_0 X \to \mathrm{Pic}_0 X^*$ be the natural homomorphisms. Then we know that $\varphi^*$ is isomorphic and $f^*$ is injective (since $f_* \mathcal{O}_{X^*} \cong \mathcal{O}_Y$). Thus we get an injective homomorphism $\varphi^{*-1} f^*$: $\mathrm{Pic}_0 Y \to \mathrm{Pic}_0 X$. We omit the proof of the following lemma, which is standard.

**Lemma 2.** *The abelian (group) subvariety $P^a$ of* $\mathrm{Pic}_0 X$*, which is by definition the image of* $\mathrm{Pic}_0 Y$ *via $\varphi^{*-1} f^*$, is independent of the choice of a holomorphic model of algebraic reduction of $X$ as above and depends only on $X$.*

Each $\mathrm{Pic}_\gamma X$ is naturally a principal homogeneous space under $\mathrm{Pic}_0 X$.

We denote by $P_{\gamma}^{a} \subseteqq \mathrm{Pic}_{\gamma} X$ any orbit of the induced action of $P^{a} \subseteqq \mathrm{Pic}_{0} X$ on $\mathrm{Pic}_{\gamma} X$.

**Lemma 3.** *Let $\gamma \in NS(X)$. Let $C$ be any complex subspace of* $\mathrm{Pic}_{\gamma} X$ *such that $C_{\mathrm{red}}$ is contained in an orbit $P_{\gamma}^{a}$ of $P^{a}$ in* $\mathrm{Pic}_{\gamma} X$. *Then $C$ is projective.*

*Proof.* Clearly we may assume that $\mathrm{Pic}_{\gamma} X = \mathrm{Pic}_{0} X$ and $P_{\gamma}^{a} = P^{a}$. Let $\bar{P}^{a} = (\mathrm{Pic}_{0} X)/P^{a}$ be the quotient Lie group and $q\colon \mathrm{Pic}_{0} X \to \bar{P}_{a}$ the natural homomorphism. Let $\bar{e} = q(e)$. Since $q$ is a holomorphic fiber bundle with typical fiber $P^{a}$ there exists a neighborhood $V$ of $\bar{e}$ such that $q$ is projective (in fact trivial) over $V$. In particular any infinitesimal neighborhood of $P^{a}$ in $\mathrm{Pic}_{0} X$, e.g., the space $P_{(n)}^{a} = (P^{a}, \mathcal{O}_{\mathrm{Pic}_{0} X}/\mathcal{I}^{n+1})$ where $\mathcal{I}$ is the defining ideal of $P^{a}$ in $\mathrm{Pic}_{0} X$, is projective. Since $C \subseteqq P_{(n)}^{a}$ for some $n > 0$ by our assumption, $C$ also is projective. q.e.d.

By this lemma, if we prove the next proposition, Theorem would follow in the case $X$ is connected and nonsingular, in view of the projectivity of $\mu_{\gamma}$.

**Proposition.** *Let $X$ be a compact complex manifold. Let $\gamma \in NS(X)$. Then any connected component of $\bar{D}_{\gamma,\mathrm{red}} := \mu_{\gamma}(D_{\gamma,\mathrm{red}})$ is contained in an orbit $P_{\gamma}^{a}$ of $P^{a}$ on* $\mathrm{Pic}_{\gamma} X$.

## § 2. Proof of Proposition

First we shall fix some notations. Let $X$ be a complex space.

Let $n\colon \widetilde{\mathrm{Div}}\, X \to (\mathrm{Div}\, X)_{\mathrm{red}}$ be the normalization of $(\mathrm{Div}\, X)_{\mathrm{red}}$ and $\tilde{\rho}\colon \tilde{Z}(X) \to \widetilde{\mathrm{Div}}\, X$ be the pull-back of the universal family to $\widetilde{\mathrm{Div}}\, X$. Let $T$ be any normal complex space and $Z \subseteqq T \times X$ a relative divisor over $T$. Then by the universal property of $\mathrm{Div}\, X$ and the normality of $T$ we can find a morphism $\tau\colon T \to \widetilde{\mathrm{Div}}\, X$ (not necessarily unique) such that $\rho_{T}\colon Z \to T$ is induced from $\tilde{\rho}$ via $\tau$. We call any such morphism also a *universal map associated to* $\rho_{T}$. For any irreducible component $D_{\alpha}$ of $(\mathrm{Div}\, X)_{\mathrm{red}}$ we denote by $\tilde{D}_{\alpha}$ the corresponding irreducible component of $\widetilde{\mathrm{Div}}\, X$ and by $\tilde{Z}_{\alpha} \to \tilde{D}_{\alpha}$, $\tilde{Z}_{\alpha} \subseteqq \tilde{D}_{\alpha} \times X$, the pull-back of the universal family to $\tilde{D}_{\alpha}$. Then $n_{\alpha} := n|_{\tilde{D}_{\alpha}}\colon \tilde{D}_{\alpha} \to D_{\alpha}$ is the normalization of $D_{\alpha}$.

Let $'\mathrm{Div}\, X$ (resp. $'\widetilde{\mathrm{Div}}\, X$) be the union of those irreducible components $D_{\alpha}$ of $(\mathrm{Div}\, X)_{\mathrm{red}}$ (resp. $\tilde{D}_{\alpha}$ of $\widetilde{\mathrm{Div}}\, X$) such that $Z_{\alpha}$ (resp. $\tilde{Z}_{\alpha}$) is reduced and irreducible. Then $n$ induces $n'\colon {}'\widetilde{\mathrm{Div}}\, X \to {}'\mathrm{Div}\, X$ which is the normalization of $'\mathrm{Div}\, X$.

Let $Z \subseteqq T \times X$ and $\tau\colon T \to \widetilde{\mathrm{Div}}\, X$ be as above with $Z$ reduced and irreducible.

If $\tilde{D}_\alpha$ is the irreducible component containing $\tau(T)$, then $\tilde{D}_\alpha \subseteq {}'\widetilde{\mathrm{Div}}\, X$. In fact, if $\tilde{Z}_\alpha$ is either nonreduced or irreducible, then $\tilde{Z}_\alpha \times_{\tilde{D}_\alpha} T$ is either nonreduced or irreducible.

We record the following useful result of C. P. Ramanujam.

**Lemma 4.** *Let $X$ be a complex manifold and $S$ a normal complex space. Let $Z \subseteq S \times X$ be a reduced analytic subspace of pure codimension 1. Suppose that $Z$ contains no subspace of the form $\{s\} \times X$, $s \in S$. Then $Z$ is a relative divisor over $S$.*

*Proof.* See [5], 21.14.1.

The next lemma reduces our problem to considering ${}'\widetilde{\mathrm{Div}}\, X$.

**Lemma 5.** *Let $X$ be a compact complex manifold. Let $D_\alpha$ be any irreducible component of $(\mathrm{Div}\, X)_{\mathrm{red}}$. Then there exist irreducible components $D_{\alpha_1}, \dots, D_{\alpha_m}$ of $(\mathrm{Div}\, X)_{\mathrm{red}}$ and an isomorphism $\varphi_\alpha\colon \tilde{D}_{\alpha_1} \times \cdots \times \tilde{D}_{\alpha_m} \to \tilde{D}_\alpha$ such that 1) $\tilde{Z}_{\alpha_i}$ are reduced and irreducible, i.e., $\tilde{D}_{\alpha_i} \subseteq {}'\widetilde{\mathrm{Div}}\, X$ and 2) if $\mu_X(D_\alpha) \subseteq \mathrm{Pic}_\gamma X$ and $\mu_X(D_{\alpha_i}) \subseteq \mathrm{Pic}_{\gamma_i} X$, then $\gamma = \gamma_1 + \cdots + \gamma_m$ and $\tilde{\mu}_\alpha \varphi_\alpha = \psi_\alpha(\tilde{\mu}_{\alpha_1} \times \cdots \times \tilde{\mu}_{\alpha_m})$ where $\tilde{\mu}_\alpha = \mu_X n_\alpha$, $\tilde{\mu}_{\alpha_i} = \mu_X n_{\alpha_i}$ and $\psi_\alpha\colon \mathrm{Pic}_{\gamma_1} X \times \cdots \times \mathrm{Pic}_{\gamma_m} X \to \mathrm{Pic}_\gamma X$ is given by $\psi_\alpha(p_1, \dots, p_m) = p_1 + \cdots + p_m$ (addition in $\mathrm{Pic}\, X$).*

*Proof.* Let $\tilde{Z}_{\alpha,i} \subseteq \tilde{D}_\alpha \times X$, $i = 1, \dots, m$, be the irreducible components of $\tilde{Z}_{\alpha,\mathrm{red}}$ and $\mathscr{I}_i$ their ideal sheaves. Since $\tilde{Z}_\alpha$ is a relative divisor over $\tilde{D}_\alpha$ and $\tilde{D}_\alpha$ is normal, by Lemma 4 $\tilde{Z}_{\alpha,i}$ are also relative divisors over $\tilde{D}_\alpha$. Moreover $\mathscr{I} = \mathscr{I}_1^{k_1} \cdots \mathscr{I}_m^{k_m}$ is the ideal sheaf of $\tilde{Z}_\alpha$ for unique positive integers $k_i$ (cf. [5] IV, 21.6.9). Let $\tau_i\colon \tilde{D}_\alpha \to \widetilde{\mathrm{Div}}\, X$ be a universal morphism associated to $\tilde{Z}_{\alpha,i} \to \tilde{D}_\alpha$. Let $\tilde{D}_{\alpha_i}$ be the irreducible component of $\widetilde{\mathrm{Div}}\, X$ which contains $\tau_i(\tilde{D}_\alpha)$. Let $\hat{D}_\alpha = \tilde{D}_{\alpha_1} \times \cdots \times \tilde{D}_{\alpha_m}$. Let $\hat{Z}_{\alpha_i} := \tilde{D}_{\alpha_1} \times \cdots \times \tilde{Z}_{\alpha_i} \times \cdots \times \tilde{D}_{\alpha_m}$ ($\tilde{Z}_{\alpha_i}$ on the $i$-th place) naturally considered as a subspace of $\hat{D}_\alpha \times X$. Let $\mathscr{I}'_i$ be the ideal sheaf of $\hat{Z}_{\alpha_i}$. Let $\hat{Z}_\alpha \subseteq \hat{D}_\alpha \times X$ be the relative divisor defined by the ideal sheaf $\mathscr{I}' = \mathscr{I}'^{k_1}_1 \cdots \mathscr{I}'^{k_m}_m$ (cf. Lemma 4). Let $\varphi_\alpha\colon \hat{D}_\alpha \to \widetilde{\mathrm{Div}}\, X$ be an associated universal morphism. From our construction it then follows readily that $\varphi_\alpha(\tau_1 \times \cdots \times \tau_m)$ induces the identity of $\tilde{D}_\alpha$. In particular $\tilde{D}_\alpha \subseteq \varphi_\alpha(\hat{D}_\alpha)$. However, since $\hat{D}_\alpha$ is irreducible and $\tilde{D}_\alpha$ is an irreducible component of $\widetilde{\mathrm{Div}}\, X$ it follows that $\tilde{D}_\alpha = \varphi_\alpha(\hat{D}_\alpha)$. On the other hand, again from our construction it is clear that for any distinct points $d = (d_1, \dots, d_m)$, $d' = (d'_1, \dots, d'_m) \in \hat{D}_\alpha$, $\mathscr{I}'_d \neq \mathscr{I}_{d'}$. Hence $\varphi_\alpha$ is injective. Since both $\tilde{D}_\alpha$ and $\hat{D}_\alpha$ are normal, this implies that $\varphi_\alpha$ is isomorphic. Moreover, since $\tilde{Z}_{\alpha,i}$ are reduced and irreducible, the same is true for $\tilde{Z}_{\alpha_i}$; 1) follows. We show 2).

Let $d=(d_1,\dots, d_m)\in \hat{D}_\alpha$. Then from our construnction $\psi_\alpha(\tilde{\mu}_{\alpha_1}\times\cdots\times\tilde{\mu}_{\alpha_m})(d)$ $=c_1([\tilde{Z}_{\alpha_1,d_1}]^{k_1})+\cdots+c_1([\tilde{Z}_{\alpha_k,d_k}]^{k_m})=c_1([\tilde{Z}_{\alpha_1,d_1}]^{k_1}\otimes\cdots\otimes[\tilde{Z}_{\alpha_k,d_k}]^{k_m}=c_1([\tilde{Z}_{\alpha,\varphi_\alpha(d)}])$ $=\tilde{\mu}_\alpha\varphi_\alpha(d)$. q. e. d.

**Lemma 6.** *Let $f\colon X\to Y$ be a fiber space of compact complex varieties. Let $T$ be a complex variety and $Z\subseteqq T\times X$ a relative divisor over $T$. Then the following conditions are equivalent.* 1) *$f(Z_t)=Y$ for all $t\in T$, and* 2) *$f(Z_t)=Y$ for some $t\in T$.*

*Proof.* Let $\bar{Z}=(id_T\times f)(Z)\subseteqq T\times Y$. By the upper semi-continuity of dimensions of the fibers of $\bar{Z}\to T$ we see that the set $A=\{t\in T;\ \bar{Z}_t=f(Z_t)=Y\}$ is analytic in $T$ where we identify $\{t\}\times X$ with $X$ and $\{t\}\times Y$ with $Y$. Let $r=\dim X-\dim Y$. Let $B=\{(t,y)\in\bar{Z};\ \dim Z_{t,y}\geqq r\}$ where $Z_{t,y}=\{t\}\times f_Z^{-1}(y)$. By the same reason as above $B$ is analytic in $\bar{Z}$. Then for any $t\in A$, $B_t:=B\cap(\{t\}\times Y)\neq\bar{Z}_t=\{t\}\times Y$; otherwise $\dim Z_t=\dim Y+r=\dim X$ so that $Z_t=X$. Hence if $A\neq\emptyset$, by the upper semi-continuity $\dim Z_{t,y}<r$ for general $(t,y)\in\bar{Z}$ and then for general $t\in T-A$, $\dim Z_t<\dim\bar{Z}_t+r<\dim Y+r=\dim X$, i.e., $\dim Z_t\leqq\dim X-2$. This is impossible. Thus either $A=\emptyset$ or $A=T$. This shows the equivalence of the lemma. q. e. d.

**Definition.** Let $f\colon X\to Y$ be as in the above lemma. Let $\tilde{D}_\alpha$ be any irreducible component of $\tilde{\mathrm{D}}\mathrm{iv}\,X$. i) $\tilde{D}_\alpha$ is called *transversal* to $f$ if $f(\tilde{Z}_{\alpha,d})=Y$ for some $d\in\tilde{D}_\alpha$ (and hence for all $d\in\tilde{D}_\alpha$ by the above lemma). ii) $\tilde{D}_\alpha$ is called *isolated* if $\tilde{D}_\alpha$ consists of a single point.

*Remark* 1. $\tilde{D}_\alpha$ is isolated if and only if there exists a proper analytic subset $A\subseteqq X$ such that the supports of $\tilde{Z}_{\alpha,d}$ are contained in $A$ for all $d\in\tilde{D}_\alpha$. In fact, since there exist at most countably many divisors whose supports are contained in $A$, $\tilde{Z}_\alpha\to\tilde{D}_\alpha$ is a trivial family in the sense that $\tilde{Z}_{\alpha,d}=\tilde{Z}_{\alpha,d'}$ for all $d, d'\in D_\alpha$. More generally if $Z\subseteqq T\times X$ is a relative divisor over $T$ where $T$ is any connected complex space, and if $Z_t\subseteqq A$ for any $t\in T$ with $A$ as above then $Z\to T$ is a trivial family in the sense that $Z_t=Z_{t'}$ for any $t, t'\in T$. (The proof is the same.)

**Lemma 7.** *Let $f\colon X\to Y$ be a fiber space of compact complex manifolds. Then there exists a natural bijective correspondence between the set $\mathfrak{E}_X$ of non-isolated and non-transversal irreducible components of $'\tilde{\mathrm{D}}\mathrm{iv}\,X$ and the set $\mathfrak{E}_Y$ of non-isolated irreducible components of $'\tilde{\mathrm{D}}\mathrm{iv}\,Y$ in such a way that if $\tilde{D}_\alpha\in\mathfrak{E}_X$ and $\tilde{D}_\beta\in\mathfrak{E}_Y$ correspond to each other, then there exist an isomorphism $\varphi_{\beta\alpha}\colon\tilde{D}_\beta\to\tilde{D}_\alpha$ and a point $d\in\mathrm{Div}\,X$ such that if $\bar{d}=\mu(d)\in\mathrm{Pic}\,X$, then $\tilde{\mu}_\alpha\varphi_{\alpha\beta}$*

$=\bar{d}^*f^*\tilde{\mu}_\beta$ *as a morphism* $\tilde{D}_\beta \to \mathrm{Pic}_\alpha X$ *where* $\tilde{\mu}_\alpha = \mu_X n_\alpha$, $\tilde{\mu}_\beta = \mu_Y n_\beta$, $f^*$: Pic $Y \to$ Pic $X$ *and* $\bar{d}^*$ *is the translation by* $\bar{d}$.

*Proof.* Let $U \subseteq Y$ be a Zariski open subset over which $f$ is smooth. First we specify the correspondence. Let $\tilde{D}_\beta$ be any non-isolated irreducible component of $'\tilde{\mathrm{D}}\mathrm{iv}\, Y$. Let $F_\beta := id_{\tilde{D}_\beta} \times f$: $\tilde{D}_\beta \times X \to \tilde{D}_\beta \times Y$. Let $\tilde{E}_\beta := F_\beta^* \tilde{Z}_\beta \subseteq \tilde{D}_\beta \times X$ be the pull-back of $\tilde{Z}_\beta$ to $\tilde{D}_\beta \times X$ as a divisor. Since $\tilde{Z}_\beta$ is reduced and irreducible and $F_\beta|_{\tilde{D}_\beta \times X_U} = id_{\tilde{D}_\beta} \times f_U$: $\tilde{D}_\beta \times X_U \to \tilde{D}_\beta \times U$ is a smooth fiber space, $\tilde{E}_\beta \cap (\tilde{D}_\beta \times X_U) = F_\beta^{-1}(\tilde{Z}_\beta \cap (\tilde{D}_\beta \times U))$ also is reduced and irreducible. Hence there exists a unique irreducible component $\tilde{E}_{\beta 1}$ of $\tilde{E}_{\beta,\mathrm{red}}$ such that $F_\beta(\tilde{E}_{\beta 1}) = \tilde{Z}_\beta$. (Note that since $\tilde{D}_\beta$ is non-isolated $\tilde{E}_{\beta 1} \cap (\tilde{D}_\beta \times X_U) \neq \emptyset$ by Remark 1.) Since $f$ is surjective, $\tilde{E}_{\beta 1}$ contains no subspace of the form $\{d\} \times X$, $d \in \tilde{D}_\beta$. Hence $\tilde{E}_{\beta 1}$ is a relative divisor over $\tilde{D}_\beta$ by Lemma 4. Let $\tau_\beta$: $\tilde{D}_\beta \to \tilde{\mathrm{D}}\mathrm{iv}\, X$ be an associated universal morphism. Let $\tilde{D}_\alpha$ be the irreducible component of $\tilde{\mathrm{D}}\mathrm{iv}\, X$ which contains $\tau_\beta(\tilde{D}_\beta)$. As we have already remarked, actually we have $\tilde{D}_\alpha \subseteq {}'\tilde{\mathrm{D}}\mathrm{iv}\, X$. Moreover, $\tilde{D}_\alpha$ is non-isolated since $\tilde{E}_{\beta 1,d}$ moves as well as $\tilde{Z}_{\beta,d}$ when $d$ moves in $\tilde{D}_\beta$, and it is non-transversal to $f$ since for any $d \in \tilde{D}_\beta$, $\tilde{Z}_{\alpha,\tau_\beta(d)} = \tilde{E}_{\beta 1,d}$ and $f(\tilde{E}_{\beta 1,d}) = \tilde{Z}_{\beta,d} \neq Y$. We set $a(\tilde{D}_\beta) = \tilde{D}_\alpha$.

Conversely, let $\tilde{D}_\alpha$ be any irreducible component of $'\tilde{\mathrm{D}}\mathrm{iv}\, X$ which is non-isolated and non-transversal to $f$. Let $F_\alpha := (id_{\tilde{D}_\alpha} \times f)$: $\tilde{D}_\alpha \times X \to \tilde{D}_\alpha \times Y$. We set $\bar{Z}_\alpha := F_\alpha(\tilde{Z}_\alpha) \subseteq \tilde{D}_\alpha \times Y$. Then by Lemma 4, $\bar{Z}_\alpha$ is a relative divisor over $Y$ since $\tilde{D}_\alpha$ is not transversal to $f$. Let $\tau_\alpha$: $\tilde{D}_\alpha \to \tilde{\mathrm{D}}\mathrm{iv}\, Y$ be an associated universal morphism. Let $\tilde{D}_\beta$ be the irreducible component which contains $\tau_\alpha(\tilde{D}_\alpha)$. Then by the same argument as above $\tilde{D}_\beta \subseteq {}'\tilde{\mathrm{D}}\mathrm{iv}\, Y$ and it is non-isolated (cf. Remark 1). Then we set $b(\tilde{D}_\alpha) = \tilde{D}_\beta$.

We now show that the above correspondences $a$ and $b$ are in fact bijective, inverse to each other, and have the property of the proposition. First, we note that from our construction it follows readily that $\tau_\beta$ is generically injective and moreover that each fiber of $\tau_\alpha$ is discrete; for any $d \in \tau_\alpha(\tilde{D}_\alpha)$, the support of $\tilde{Z}_{\alpha,d'}$ is contained in $f^{-1}(\bar{Z}_{\alpha,d})$ for each $d' \in \tau_\alpha^{-1}(d)$ and hence by Remark 1 dim $\tau_\alpha^{-1}(d) = 0$. We further show that $\bar{Z}_{\alpha,d}$ is reduced if $\tilde{Z}_{\alpha,d}$ is reduced and if $\tilde{Z}_{\alpha,d} \cap X_U$ is dense in $\tilde{Z}_{\alpha,d}$. In fact, since $f_U$ is a smooth fiber space, we have $\tilde{F}_\alpha^{-1}(\bar{Z}_\alpha \cap (\tilde{D}_\alpha \times U)) = \tilde{Z}_\alpha \cap (\tilde{D}_\alpha \times X_U)$, both sides being reduced. Hence $f^{-1}(\bar{Z}_{\alpha,d} \cap U) = \tilde{Z}_{\alpha,d} \cap X_U$, so that $\bar{Z}_{\alpha,d} \cap U$ is reduced if so is $\tilde{Z}_{\alpha,d}$. If further, $Z_{\alpha,d} \cap X_U$ is dense in $Z_{\alpha,d}$ then $\bar{Z}_{\alpha,d} \cap U$ also is dense in $\bar{Z}_{\alpha,d}$ and hence $\bar{Z}_{\alpha,d}$ also is reduced.

Now we fix $\tilde{D}_\beta \subseteq {}'\tilde{\mathrm{D}}\mathrm{iv}\, X$. We consider the corresponding $\tilde{D}_\alpha = a(\tilde{D}_\beta)$, $\tau_\beta$:

$\tilde{D}_\beta \to \tilde{D}_\alpha$ and $\tau_\alpha$: $\tilde{D}_\alpha \to {}'\tilde{\mathrm{D}}\mathrm{iv}\, Y$. Suppose that we have shown that $\tau_\alpha \cdot \tau_\beta = id_{\tilde{D}_\beta}$ so that in particular $\tau_\alpha(\tilde{D}_\alpha) \supseteqq D_\beta$. Then, since $\tilde{D}_\alpha$ and $\tilde{D}_\beta$ are normal and irreducible, $\tau_\alpha(\tilde{D}_\alpha) = \tilde{D}_\beta$ and, in view of the generic injectivity of $\tau_\beta$ and the fact that $\dim \tau_\alpha^{-1}(d) = 0$, this would imply that $\tau_\alpha$ and $\tau_\beta$ give isomorphisms of $\tilde{D}_\alpha$ and $\tilde{D}_\beta$ and that $ba$ $=$identity. So we show that $\tau_\alpha \cdot \tau_\beta = id_{\tilde{D}_\beta}$. Let $V \subset \tilde{D}_\beta$ be a Zariski open subset such that $\tilde{Z}_{\beta,d}$ are reduced and $\tilde{Z}_{\beta,d} \cap U$ is dense in $\tilde{Z}_{\beta,d}$ for all $d \in V$. Then we have only to show that $\tau_\alpha(\tau_\beta|_V) = id_V$. This follows if we show that $\tilde{Z}_{\beta,d} = \bar{Z}_{\alpha,d'}$ for any $d \in V$, with $d' = \tau_\beta(d)$, as a subspace of $Y$. By our construction it is clear that $\tilde{Z}_{\beta,d} = (\bar{Z}_{\alpha,d'})_{\mathrm{red}}$, while by what we have proved above $\bar{Z}_{\alpha,d'}$ is reduced since $\tilde{Z}_{\alpha,d'} = \tilde{E}_{\beta 1,d}$ is reduced and $(\tilde{E}_{\beta 1,d} \cap X_U)$ is dense in $\tilde{E}_{\beta 1,d}$. Hence the assertion is proved.

Next we fix $\tilde{D}_\alpha$. We consider the corresponding $\tilde{D}_\beta = b(\tilde{D}_\alpha)$, $\tau_\alpha$: $\tilde{D}_\alpha \to \tilde{D}_\beta$, and $\tau_\beta$: $\tilde{D}_\beta \to \tilde{\mathrm{D}}\mathrm{iv}\, X$. Then just as above we show that $\tau_\beta \cdot \tau_\alpha = id_{\tilde{D}_\alpha}$ and then that $ab$ $=$identity. We set $\varphi_{\beta\alpha} = \tau_\beta$: $\tilde{D}_\beta \cong \tilde{D}_\alpha$.

It remains to show the existence of $d \in \mathrm{Div}\, X$ satisfying $\tilde{\mu}_\alpha \varphi_{\alpha\beta} = \bar{d}^* f^* \tilde{\mu}_\beta$. Write $\tilde{E} = \tilde{E}_{\beta 1} \cup \tilde{E}_{\beta 2}$ for a unique relative divisor $\tilde{E}_{\beta 2}$ over $\tilde{D}_\beta$ with $\tilde{E}_{\beta 1} \not\supseteqq \tilde{E}_{\beta 2}$. Then by our definition of $\tilde{E}_{\beta 1}$ we have $\tilde{E}_{\beta 1} \cap (\tilde{D}_\beta \times X_U) = \tilde{E}_\beta \times X_U$. Hence if $A := X - X_U$, then $\tilde{E}_{\beta 2} \subseteqq \tilde{D}_\beta \times A$. Hence by Remark 1 $\tilde{E}_{\beta 2} \to \tilde{D}_\beta$ is a constant family, so that the image of an associated universal morphism $\tau_{\beta 2}$: $\tilde{D}_\beta \to \tilde{\mathrm{D}}\mathrm{iv}\, X$ is a unique point. Then it suffices to take this point as $d$. (When $\tilde{E}_{\beta 2} = \emptyset$, we set $d = 0$.) q.e.d.

*Proof of Porposition.* Let $D_\alpha = D_{\gamma,\alpha}$ be any irreducible component of $D_{\gamma,\mathrm{red}}$ and $\bar{D}_\alpha = \mu_\gamma(D_\alpha) \subseteqq \mathrm{Pic}_\gamma X$. Then it suffices to show that $\bar{D}_\alpha$ is contained in some orbit $P_\gamma^a = P_\gamma^a(\alpha)$. In fact, if $\bar{D}_\gamma^i$ is any connected component of $\bar{D}_{\gamma,\mathrm{red}}$ and $\bar{D}_\gamma^i = \bigcup_{\alpha \in \mathfrak{A}_i} \bar{D}_\alpha$, then $\bigcup_{\alpha \in \mathfrak{A}_i} P_\gamma(\alpha)$ also is connected and hence $P_\gamma(\alpha) = P_\gamma(\alpha')$ for any $\alpha$, $\alpha' \in \mathfrak{A}_i$ since the orbits are mutually disjoint. Hence $\bar{D}_\gamma^i \subseteqq P_\gamma^a$ for a unique orbit $P_\gamma^a$. Now we show that $\bar{D}_\alpha \subseteqq P_\gamma^a$ for some $P_\gamma^a$. First, by Lemma 5 we infer that we may assume that $\tilde{D}_\alpha \subseteqq {}'\tilde{\mathrm{D}}\mathrm{iv}\, X$. If $\tilde{D}_\alpha$ is isolated, then the assertion is clearly true. So we may assume that $\tilde{D}_\alpha$ is not isolated. We take a holomorphic model $(*)$ of algebraic reduction of $X$. Then by Lemma 7 applied to $\varphi$, we can replace $X$ by $X^*$ so that we may assume from the beginning that $X = X^*$ and $f$ is defined on $X$. Now by Fischer-Forster [2] if $\tilde{D}_\alpha$ is transversal to $f$, then $D_\alpha$ is isolated (cf. Remark 1). Hence we may further assume that $\tilde{D}_\alpha$ is not transversal. Then applying Lemma 7 this time to $f$, Proposition follows immediately.

## §3. Proof of the Theorems

Let $X$ be a normal compact complex space. Let $r\colon \tilde{X}\to X$ be a resolution. Since $X$ is normal so that $r_*\mathcal{O}_{\tilde{X}}\cong\mathcal{O}_X$, the natural morphism $r^*\colon \operatorname{Pic} X\to\operatorname{Pic}\tilde{X}$ is injective.

**Lemma 8.** *For each $\gamma\in NS(X)$, $r^*(\operatorname{Pic}_\gamma X)$ is a closed submanifold in* $\operatorname{Pic} X$. *In particular $r^*$ is a closed embedding.*

*Proof.* It suffices to show that $r^*(\operatorname{Pic}_0 X)$ is closed in $\operatorname{Pic}_0\tilde{X}$. Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & \operatorname{Pic}_0 X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} & & \\ 0 & \longrightarrow & H^1(\tilde{X}, \mathbb{Z}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & \operatorname{Pic}_0\tilde{X} & \longrightarrow & 0 \end{array}$$

where the vertical maps are injective and the horizontal sequences come from the exponential sequences on $X$ and $\tilde{X}$. Then it is enough to show that the subgroup in $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ generated by $H^1(\tilde{X}, \mathbb{Z})$ and $H^1(X, \mathcal{O}_X)$ is closed. Fisrt, recall that we have the natural inclusions $H^1(X, \mathbb{R})\to H^1(X, \mathcal{O}_X)$ and $H^1(\tilde{X}, \mathbb{R})\to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ of real vector spaces (cf. [6] IX, Prop. 3.2). Then, clearly the subgroup in $H^1(\tilde{X}, \mathbb{R})$ generated by $H^1(X, \mathbb{R})$ and $H^1(\tilde{X}, \mathbb{Z})$ is closed. Since $H^1(X, \mathcal{O}_X)$ is a vector subspace of $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, from this the lemma follows immediately. q.e.d.

*Proof of Theorem* 1. Clearly we may assume that $X$ is connected. Suppose that $X$ is nonsingular. Then, as we have already noted, Theorem follows immediately from Lemma 3 and Proposition. So suppose that $X$ is not nonsingular. Let $\gamma\in NS(X)$ be arbitrary. Then we have only to show that any connected component $\bar{D}_{\gamma,\alpha}$ of $\bar{D}_\gamma$ is projective (cf. §1). Let $r\colon \tilde{X}\to X$ be a resolution of $X$. Then there exists a unique $\tilde{\gamma}\in NS(\tilde{X})$ such that $r^*(\operatorname{Pic}_\gamma X)\subseteq\operatorname{Pic}_{\tilde{\gamma}}\tilde{X}$. Then by Lemma 1 and the definition of $r^*$ we have $r^*(\bar{D}_{\gamma,\mathrm{red}})=\bar{D}_{\tilde{\gamma},\mathrm{red}}\cap r^*(\operatorname{Pic}_\gamma X)$. On the other hand, every connected component of $\bar{D}_{\tilde{\gamma},\mathrm{red}}$ is contained in some orbit $P^a_{\tilde{\gamma}}$ of $P^a=P^a(\tilde{X})$ in $\operatorname{Pic}_{\tilde{\gamma}}\tilde{X}$ by Proposition. Hence by Lemma 8, $r^*(\bar{D}_{\gamma,\alpha,\mathrm{red}})$ is a closed analytic subspace of $P^a_{\tilde{\gamma}}$. Therefore noting that $r^*$ is an embedding, by Lemma 3, $\bar{D}_{\gamma,\alpha}$ is projective as was desired.

*Proof of Theorem* 2. Since $D_\alpha(A)=\rho(Z(X)\cap(D_\alpha\times A))$, $D_\alpha(A)$ is analytic.

On the other hand, we have $D_\alpha(A)=\bigcup_{x\in A} D_\alpha(x)$ where $D_\alpha(x)=D_\alpha(\{x\})$. Now $D_\alpha(x)=Z(X)\cap(D_\alpha\times\{x\})\subseteqq D_\alpha$ is regarded naturally as a divisor on $D_\alpha$ (not necessarily reduced). It then follows that there exist a finite number of points $x_1,\dots,x_m\in X$ such that $D_\alpha(A)=\bigcup_{i=1}^{m} D_\alpha(x_i)$ as a set. (Recall that $D_\alpha$ is compact by Theorem 1.) Thus it suffices to show that $D_\alpha(x_i)$ is ample for any $i$. We first note that under our assumption there exists a Zariski open subset $U$ of $X$ containing $x_\iota$ such that $D_\alpha(x)$ is a divisor on $D_\alpha$ for any $x\in U$. Further since the divisors $D_\alpha(x)$, $x\in U$, are all mutually algebraically equivalent, it suffices to show that $D_\alpha(x_0)$ is ample for some fixed $x_0\in U$.

*Claim*. For any nowhere discrete reduced analytic subspace $B$ of $D_\alpha$ we can find $x\in U$ such that 1) $D_\alpha(x)$ intersect each irreducible component of $B$ and 2) $B\cap D_\alpha(x)$ is nowhere dense in $B$.

In fact, let $B_i$, $i=1,\dots,r$, be the irreducible components of $B$. Fix a point $b_i\in B_i-\bigcup_{j\neq i} B_j$ for each $i$. Let $Z_{b_i}\subseteqq X$ be the corresponding divisor and then take any $x\in U-\bigcup_{i=1}^{r} Z_{b_i}$. Then obviously $b_i\notin D_\alpha(x)$. Hence 2) is satisfied. Moreover since the natural map $Z_{B_i}\to X$ is surjective by our assumption that $\dim B_i>0$, $D_\alpha(x)\ni b'_i$ for some $b'_i\in B_i$. Hence 1) also is true. The claim is proved.

Now using this claim inductively we see that for any $p\geqq 0$ and any complex subvariety $C$ of dimension $p$ of $D_\alpha$, we can always find $x_1,\dots,x_p\in U$ such that $D_\alpha(x_1)\cap\cdots\cap D_\alpha(x_p)\cap C$ is a nonempty finite set of points. This implies that the intersection number $D_\alpha(x_0)^p\cdot C=D_\alpha(x_1)\cdot D_\alpha(x_2)\cdot\cdots\cdot D_\alpha(x_p)\cdot C>0$ (cf. [7]). Hence by Nakai crieterion (cf. [7]) $D_\alpha(x_0)$ is ample. q. e. d.

*Remark* 2. The above proof shows in fact the following: For any complex variety $X$ and any compact subspace $B$ of $\mathrm{Div}\,X$, $B(A)=\{d\in B;\ Z_{X,d}\cap A\neq\emptyset\}$ is a support of an ample divisor if it is a divisor on $B$ at all. (Note that $B(A)=\bigcup_{x\in A} B(x)$ and $B(x)$ has the natural structure of a locally principal analytic subspace of $B$.) See Banica, C., and Ueno, K., *J. Math. Kyoto Univ.*, 20 (1980), 381–389, for the intersection theory on a general compact complex space generalizing that of [7]. Further $B$–$B(A)$ is affine if $B(A)$ contains no irreducible component of $B_{\mathrm{red}}$. The result of Ohsawa [8] mentioned in the introduction implies that $B$–$B(A)$ is Stein.

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