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On a Certain Class of *-Algebras of Unbounded Operators

By

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Abstract

A *-algebra $\mathfrak{A}$ of linear operators with a common invariant dense domain $\mathscr{D}$ in a Hilbert space is studied relative to the order structure given by the cone $\mathfrak{A}^+$ of positive elements of $\mathfrak{A}$ (in the sense of positive sesquilinear form on $\mathscr{D}$) and the ρ-topology defined as an inductive limit of the order norm ρ_A (of the subspace $\mathfrak{A}_A$ with A as its order unit) with $A \in \mathfrak{A}^+$. In particular, for those $\mathfrak{A}$ with a countable cofinal sequence A_i in $\mathfrak{A}^+$ such that $A_i^{-1} \in \mathfrak{A}$, the ρ-topology is proved to be order convex, any positive elements in the predual is shown to be a countable sum of vector states, and the bicommutant within the set $B(\mathscr{D}, \mathscr{D})$ of continuous sesquilinear forms on $\mathscr{D}$ is shown to be the ultraweak closure of $\mathfrak{A}$. The structure of the commutant and the bicommutant are explicitly given in terms of their bounded operator elements which are von Neumann algebras and the commutant of each other.

§ 1. Introduction

Our aim in this paper is to develop a theory of a certain class of *-algebras $\mathfrak{A}$ of linear operators with a common invariant dense domain $\mathscr{D}$ in a Hilbert space $\mathscr{H}$ in parallel with theory of von Neumann algebras as a continuation of [7]. (Also see [14].) The set $B(\mathscr{D}, \mathscr{D})$ of all continuous sesquilinear forms on $\mathscr{D}$ (the continuity relative to the collection of norms $\mathscr{D} \ni x \mapsto \|Ax\|$, $A \in \mathfrak{A}$) plays the rôle of the set $L(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$ in theory of von Neumann algebras. For those $\mathfrak{A}$ satisfying Condition I described below, we can give the decomposition of continuous linear forms into positive components, i.e. the strong normality of the positive cone $\mathfrak{A}^+$, the description of positive elements in the predual of $\mathfrak{A}$ and the notion of commutant, for which the bicommutant coincides with the ultraweak closure.

Under weaker Condition I_0 or I'_0, also described below, structure of the

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commutant and bicommutant (both of which actually satisfy Condition I) can be analyzed.

The cone $B^+(\mathscr{D}, \mathscr{D})$ of all positive forms β in $B(\mathscr{D}, \mathscr{D})$ (i.e. $\beta(x, x) \geqq 0$ for all $x \in \mathscr{D}$), gives a natural order structure in $B(\mathscr{D}, \mathscr{D})$. Any $T \in \mathfrak{A}$ can be viewed as an element $T(x, y) \equiv (Tx, y)$ of $B(\mathscr{D}, \mathscr{D})$, and this defines an order structure in $\mathfrak{A}$. For $A \in \mathfrak{A}^+ (\equiv \mathfrak{A} \cap B^+(\mathscr{D}, \mathscr{D}))$, the set of all $T \in \mathfrak{A}$ satisfying $|(Tx, x)| \leqq \lambda(Ax, x)$ for some $\lambda \geqq 0$ and all $x \in \mathscr{D}$ is denoted by $\mathfrak{A}_A$ and the infimum of λ is denoted as $\rho_A(T)$ or $\|T\|_A$. The locally convex inductive limit topology (Chapter II §6 [13]) for the system of normed spaces $(\mathfrak{A}_A, \rho_A)$ is the ρ-topology of $\mathfrak{A}$ introduced and studied in [10]. (Note that $\mathfrak{A} = \bigcup_{A \in \mathfrak{A}^+} \mathfrak{A}_A$.)

In the present paper, we consider only countably dominated $\mathfrak{A}$, i.e. we assume the existence of a sequence $A_n \in \mathfrak{A}^+$, $A_n \geqq \mathbf{1}$ which is cofinal in $\mathfrak{A}^+$, i.e. we may choose a monotone increasing sequence $A_n \in \mathfrak{A}$ satisfying $\mathfrak{A} = \bigcup_n \mathfrak{A}_{A_n}$ except that the countability is not essential in the result on structure of $\mathfrak{A}'$ and $\mathfrak{A}''$. As already noticed in [10], all positive linear maps of $\mathfrak{A}$ into another space of the same type decreases the ρ-norms and hence is automatically continuous; in particular, the ρ-norms are preserved under an isomorphism (preserving linear and order structure) in analogy with C^*-norms. We also note that $(\mathfrak{A}, \rho)$ is a separated, bornologic *DF* space*, for which the strong dual $\mathfrak{A}^\rho$ of $(\mathfrak{A}, \rho)$ is a Fréchet space.

A further condition on $\mathfrak{A}$ on which we focus our attention in this paper is the following:

Condition I. There exists a cofinal sequence A_n in $\mathfrak{A}^+$ such that $A_n \geqq \mathbf{1}$ and $A_n^{-1} \in \mathfrak{A}$.

Again, we may choose a monotone increasing sequence A_n without loss of generality.

Condition I together with our definition of *-algebras imply $A_n \mathscr{D} = \mathscr{D}$. On the other hand if $A_n \mathscr{D} = \mathscr{D}$ is satisfied, then the algebra $\tilde{\mathfrak{A}}$ generated by $\mathfrak{A}$ and all A_n^{-1} (restricted to $\mathscr{D}$) is a *-algebra on $\mathscr{D}$ satisfying Condition I. In fact, our structure result will be formulated in terms of the following:

Condition $\mathbf{I_0}$. There exists a cofinal sequence A_n in $\mathfrak{A}^+$ such that $A_n \geqq \mathbf{1}$ and $A_n \mathscr{D} = \mathscr{D}$.

* $\mathfrak{A}$ is separated because $\mathscr{D} \times \mathscr{D}$ which gives ρ-continuous forms already separate $\mathfrak{A}$; bornologic as an inductive limit of bornologic spaces by Corollary 1, Chapter 2, §8, [13]; DF as an inductive limit of DF spaces by Proposition 5, Chapter 4, Part 3, §3 in [7].

Actually our structure result holds under the following weaker condition:

Condition I_0'. There exists a cofinal sequence A_n in $\mathfrak{A}^+$ such that $A_n \geqq \mathbb{1}$ and A_n^2 is essentially selfadjoint (on $\mathscr{D}$).

Condition I_0 implies I_0' (Lemma 4.5).

The ultraweak closure of $\mathfrak{A}$ (and the bicommutant) contains, in general, elements $\beta \in B(\mathscr{D}, \mathscr{D})$ which cannot be expressed as $\beta(x, y) = (Bx, y)$ for operators B satisfying $B\mathscr{D} \subset \mathscr{D}$. (Either there is no operator B or $B\mathscr{D}$ is not contained in $\mathscr{D}$). It is therefore convenient to consider a condition similar to Condition I for a subset of $B(\mathscr{D}, \mathscr{D})$. It is formulated as follows:

Condition II. Let A_n be a cofinal sequence in $B^+(\mathscr{D}, \mathscr{D})$ satisfying $A_n \geqq \mathbb{1}$ and $A_n\mathscr{D} = \mathscr{D}$. $\mathfrak{B}$ is a subspace (by which we always mean a linear subset) of $B(\mathscr{D}, \mathscr{D})$ satisfying the following:

(1) $\mathfrak{B}$ is symmetric, i.e. $\beta \in \mathfrak{B}$ implies $\beta^* \in \mathfrak{B}$ where $\beta^*(x, y) = \overline{\beta(y, x)}$, the bar denoting the complex conjugate.

(2) $\beta \in \mathfrak{B}$ implies $A_n \beta A_n \in \mathfrak{B}$ and $A_n^{-1} \beta A_n^{-1} \in \mathfrak{B}$ where $(C\beta D)(x, y) = \beta(Dx, C^* y)$.

(3) The set $\mathfrak{B}_{\mathrm{id}}$ of all bounded operators B in $\mathfrak{B}$(i.e. $\beta(x, y) \equiv (Bx, y) \in \mathfrak{B}$) is an algebra containing $\mathbb{1}$.

In Section 2, we describe definitions of basic notion in our study. The terminology is then used to describe main results in Section 3 in the form of Theorems and Corollaries. Their proof will be given in subsequent sections.

§ 2. Basic Notation

In the present paper we shall be concerned with linear operators (not necessarily bounded) A defined on a dense linear subset $\mathscr{D}$ of a Hilbert space $\mathscr{H}$. The closure of A will be denoted by $\bar{A}$, the adjoint of A by $A^\dagger$ and, if Dom $A^\dagger$ (the domain of $A^\dagger$) contains $\mathscr{D}$, the restriction of $A^\dagger$ to $\mathscr{D}$ by A^*.

A set $\mathfrak{A}$ of linear operators will be called a *$*$-algebra on a domain $\mathscr{D}$* if the following conditions are satisfied.

1. $\mathfrak{A}$ is symmetric, i.e. Dom $A^\dagger \supset \mathscr{D}$ and $A^* \in \mathfrak{A}$ for any $A \in \mathfrak{A}$.
2. $\mathscr{D}$ is invariant, i.e. $A\mathscr{D} \subset \mathscr{D}$ for any $A \in \mathfrak{A}$.
3. $\mathfrak{A}$ is an algebra, i.e. $A + B \in \mathfrak{A}$, $AB \in \mathfrak{A}$, $cA \in \mathfrak{A}$ for any $A, B \in \mathfrak{A}$ and any complex number c.

4. $\mathfrak{A}$ is unital, i.e. $\mathbf{1} \in \mathfrak{A}$.

We shall assume below that $\mathfrak{A}$ is countably dominated (see Section 1). We shall always equip $\mathscr{D}$ with the topology given by the collection of semi-norms $x \to \|Ax\|$, $A \in \mathfrak{A}$, where we may take a monotone increasing sequence of semi-norms $\|A_n x\|$ (Lemma 4.2).

The completion $\hat{\mathscr{D}}$ of $\mathscr{D}$ is (Lemma 3.6, [12])

$$\hat{\mathscr{D}} = \bigcap_{A \in \mathfrak{A}} \mathrm{Dom}\, \bar{A}. \tag{2.1}$$

Semi-norms in $\hat{\mathscr{D}}$ are given by $\hat{\mathscr{D}} \in x \mapsto \|\bar{A}x\|$ for $A \in \mathfrak{A}$. Without loss of generality (Lemma 4.3) we may and do assume the completeness of $\mathscr{D}$:

$$\mathscr{D} = \hat{\mathscr{D}}. \tag{2.2}$$

The set of all continuous sesquilinear forms on $\mathscr{D}$ will be denoted by $B(\mathscr{D}, \mathscr{D})$. For $\beta \in B(\mathscr{D}, \mathscr{D})$, the continuity implies the existence of $A \in \mathfrak{A}$ such that for some constant M and all $x, y \in \mathscr{D}$,

$$|\beta(x, y)| \leqq M \|Ax\| \, \|Ay\|. \tag{2.3}$$

It is equivalent to the existence of some $M' (\leqq M)$ satisfying

$$|\beta(x, x)| \leqq M'(A^*Ax, x) \tag{2.4}$$

for all $x \in \mathscr{D}$. Therefore a confinal sequence in $\mathfrak{A}^+$ is also cofinal in $B(\mathscr{D}, \mathscr{D})$. We define

$$\rho_A(\beta) = \inf\{\lambda \geqq 0;\ |\beta(x, x)| \leqq \lambda(Ax, x) \quad \text{for all} \quad x \in \mathscr{D}\} \tag{2.5}$$

for $A \in \mathfrak{A}^+$.

For a symmetric subspace $\mathfrak{B}$ of $B(\mathscr{D}, \mathscr{D})$, $\mathfrak{B} = \mathfrak{B}_{\mathrm{R}} + i\mathfrak{B}_{\mathrm{R}}$ where $\mathfrak{B}_{\mathrm{R}}$ denotes the set of all hermitian β (i.e. $\beta^* = \beta$) in $\mathfrak{B}$. Let $\mathfrak{B}^+$ denote $\mathfrak{B} \cap B^+(\mathscr{D}, \mathscr{D})$ and $A \in \mathfrak{B}^+$. The set of all $D \in \mathfrak{B}_{\mathrm{R}}$ with $\rho_A(D) \leqq 1$ (i.e. $-A \leqq D \leqq A$) will be denoted as $[-A, A]$ or $[-A, A]_{\mathfrak{B}}$.

A subset S is called order convex if $x, y \in S$ and $x \leqq z \leqq y$ implies $z \in S$. If a topology can be generated by order convex sets, the topology is called order convex. This is equivalent in a topological vector space the so-called normality of the positive cone $\mathfrak{A}^+$. (For example, Chapter 5, §3, 3.1 [13].) This property is shown in Theorem 1 for a *-algebra $\mathfrak{A}$ satisfying Condition I, its ultraweak closure and some other cases.

The linear form

$$\omega_{x,y}(\beta) = \beta(x, y), \quad \beta \in B(\mathscr{D}, \mathscr{D}), \tag{2.6}$$

for $x,\ y\in\mathscr{D}$ is in the strong dual $\mathfrak{A}^\rho$ of $(\mathfrak{A},\rho)$ (and in the strong dual $B(\mathscr{D},\mathscr{D})^\rho$ of $(B(\mathscr{D},\mathscr{D}),\rho)$) and ω extends to a continuous linear map from the projective completion $\mathscr{D}\hat{\otimes}\mathscr{D}^-$ of $\mathscr{D}\otimes\mathscr{D}^-$ into $\mathfrak{A}^\rho$ (or $B(\mathscr{D},\mathscr{D})^\rho$) (Chapter 1, §1, Proposition 2, [6]). Here $\mathscr{D}^-$ is the complex conjugate of $\mathscr{D}$ (i.e. multiplication of a complex number z in $\mathscr{D}^-$ is that of $\bar{z}$ in $\mathscr{D}$).

A general element u in $\mathscr{D}\hat{\otimes}\mathscr{D}^-$, which is a Fréchet space under our assumption, is of the form

$$u=\sum_{i=1}^{\infty}\lambda_i x_i\otimes y_i,\quad \sum_{i=1}^{\infty}|\lambda_i|<\infty \tag{2.7}$$

where x_i and y_i are sequences converging to $\mathbf{0}$ in $\mathscr{D}$ and $\mathscr{D}^-$ (Chapter 1, §2, Theorem 1, [6]). The above-mentioned extension of ω is given by

$$\omega_u=\sum_{i=1}^{\infty}\lambda_i\omega_{x_i,y_i}\,. \tag{2.8}$$

The topology on $\mathfrak{A}$ (and on $B(\mathscr{D},\mathscr{D})$) induced by $\sigma(B(\mathscr{D},\mathscr{D}),\mathscr{D}\otimes\mathscr{D}^-)$ and $\sigma(B(\mathscr{D},\mathscr{D}),\mathscr{D}\hat{\otimes}\mathscr{D}^-)$ are called weak and ultraweak topology.

They are given by semi-norms $\beta\in\mathfrak{A}\to|\beta(x,y)|$ with x and y varying in $\mathscr{D}$ for the weak topology and by semi-norms $\beta\in\mathfrak{A}\to|\langle\beta,\omega_u\rangle|=|\sum_{i=1}^{\infty}\lambda_i\beta(x_i,y_i)|$ for the ultraweak topology. The set of all ultraweakly continuous linear functionals on $(\mathfrak{A},\rho)$ is called the predual of $\mathfrak{A}$ and denoted $\mathscr{P}$. The predual of $B(\mathscr{D},\mathscr{D})$ is $\mathscr{D}\hat{\otimes}\mathscr{D}$ and $B(\mathscr{D},\mathscr{D})$ is the dual space of $\mathscr{D}\hat{\otimes}\mathscr{D}^-$ (Chapter 1, §1, Proposition 2, [6]. However the strong topology on $B(\mathscr{D},\mathscr{D})$ induced by $\mathscr{D}\hat{\otimes}\mathscr{D}$ has been shown to coincide with ρ so far only under the condition that $\mathscr{D}$ is quasi-normable. See Proposition 7 (1°) in [9].) The map ω induces a topological isomorphism of the predual of $\mathfrak{A}$ onto the Fréchet space $\mathscr{D}\hat{\otimes}\mathscr{D}^-/\mathfrak{A}^0$ (Proposition 7.2) where $\mathfrak{A}^0$ is the polar of $\mathfrak{A}$ in the duality $\langle\mathscr{D}\hat{\otimes}\mathscr{D},B(\mathscr{D},\mathscr{D})\rangle$.

For any selfadjoint ultraweakly closed subspace $\mathfrak{B}$ of $B(\mathscr{D},\mathscr{D})$, a positive linear form φ on $\mathfrak{B}$ is called normal if

$$\varphi(\sup_\alpha T_\alpha)=\sup_\alpha\varphi(T_\alpha) \tag{2.9}$$

for any bounded increasing net T_α in $\mathfrak{B}^+$. We shall determine in Theorem 2 a form of such φ on the ultraweak closure of a *-algebra satisfying Condition I. Together with Corollary 1a, it gives a concrete description of elements in the predual.

The commutant $\mathfrak{A}'$ of a *-algebra $\mathfrak{A}$ is the subspace of $B(\mathscr{D},\mathscr{D})$ consisting of all $\beta\in B(\mathscr{D},\mathscr{D})$ satisfying $\beta(Ax,y)=\beta(x,A^*y)$ for all $A\in\mathfrak{A}$ and $x,\ y\in\mathscr{D}$.

If an $\mathfrak{A}$ satisfies Condition I, then $\mathfrak{A}'$ is a *-algebra over $\mathscr{D}$ in the sense that any $\beta \in \mathfrak{A}'$ can be written as $\beta(x, y)=(Bx, y)$ with $B\mathscr{D} \subset \mathscr{D}$ and B's form a *-algebra over $\mathscr{D}$ as defined eariler, except that $\mathscr{D}$ is not complete relative to the topology induced by $\mathfrak{A}'$. The same conclusion can be achieved under Condition I_0 or I_0'. In either case $\mathfrak{A}'$ satisfies Condition I.

We shall define the bicommutant $\mathfrak{A}''$ as the commutant of $\mathfrak{A}'$ within $B(\mathscr{D}, \mathscr{D})$. We may consider the commutant $(\mathfrak{A}', D)'$ of $\mathfrak{A}'$ in the earlier sense, namely within $B(D, D)$ for the completion D of $\mathscr{D}$ relative to the topology on $\mathscr{D}$ induced by $\mathfrak{A}'$. It is smaller than $\mathfrak{A}''$ in general. We shall clarify in Theorem 3, the structure of $\mathfrak{A}'$, $\mathfrak{A}''$ and $(\mathfrak{A}', D)'$.

Another topology called the λ-topology on a *-algebra $\mathfrak{A}$ on $\mathscr{D}$ has been introduced in [10]. We define for $T \in \mathfrak{A}$ and $0 \neq A \in \mathfrak{A}$,

$$\lambda_A(T) = \sup\{\|Tx\|/\|Ax\| : x \in \mathscr{D},\ \|Tx\| + \|Ax\| \neq 0\}$$

where we define $a/0 = +\infty > b$ for any $a > 0$ and real b, and $\mathfrak{M}_A$ denotes the set of all $T \in \mathfrak{A}$ with $\lambda_A(T) < \infty$. Then the λ-topology is the inductive limit topology of $\mathfrak{A} = \cup\{\mathfrak{M}_A : 0 \neq A \in \mathfrak{A}\}$ for the system of normed spaces $(\mathfrak{M}_A, \lambda_A)$. In general, the λ-topology is known to be different from the ρ-topology, a typical example being the set $L(\mathscr{D})$ of all operators A such that $\mathrm{Dom}\, A = \mathscr{D}$, $A\mathscr{D} \subset \mathscr{D}$, $\mathrm{Dom} \cdot A^\dagger \supset \mathscr{D}$ and $A^\dagger \mathscr{D} \subset \mathscr{D}$. The coincidence of the ρ and λ topologies for $\mathfrak{A}$ has a significance because it is equivalent to the ρ-continuity of the product ST as a map from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$. We shall prove the coincidence of the two topologies for some $\mathfrak{A}$ (Theorem 3 (1)).

§ 3. Main Results

We describe our main results as theorems in this section, their proof being given in later sections. If $\mathfrak{A}$ is countably dominated, $B(\mathscr{D}, \mathscr{D})$ satisfies Condition I and hence all the following results hold for $B(\mathscr{D}, \mathscr{D})$.

This first group of results are about order-convexity of ρ-topology.

Theorem 1. *Let $\mathfrak{B}$ be a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ containing the cofinal sequence A_n of $B^+(\mathscr{D}, \mathscr{D})$. The ρ-topology on $\mathfrak{B}_{\mathrm{R}}$ (or on $\mathfrak{B}$) (defined as the inductive limit topology of a system of normed spaces $((\mathfrak{B}_{\mathrm{R}})_{A_n}, \rho_{A_n})$ where $\rho_{A_n}(T) = \inf\{\lambda \geqq 0;\ |(Tx, x)| \leqq \lambda(A_n x, x)$ for all $x \in \mathscr{D}\}$) is order convex if one of the following conditions is satisfied.*

(i) *$\mathfrak{B}$ is a *-algebra over $\mathscr{D}$ satisfying Condition I.*

(ii) $\mathfrak{B}$ *is the completion (relative to the ρ-topology) of a *-algebra over* $\mathscr{D}$ *satisfying Condition I.*

(iii) $\mathfrak{B}$ *is the ultraweak closure of a *-algebra over* $\mathscr{D}$ *satisfying Condition I.*

(iv) *Condition II is satisfied.*

Remark. (ii) corresponds to C^*-closure and (iii) corresponds to W^*-colsure in the case of *-algebras of bounded operators. (i), (ii) and (iii) are actually special cases of (iv) and hence it suffices to prove theorem 1 under (iv).

The order-convexity of ρ-topology immediately implies the following. (Chapter 5, 3.3, Corollary 1, [13].)

Corollary 1a. *Any continuous linear form on* $(\mathfrak{B}, \rho)$ *decomposes as a linear combination of positive continuous linear forms if* $\mathfrak{A}$ *satisfies one of conditions of the preceding Theorem.*

Corollary 1b. *Let* $\mathfrak{B}$ *be a symmetric subspace of* $B(\mathscr{D}, \mathscr{D})$ *containing a cofinal sequence in* $B^+(\mathscr{D}, \mathscr{D})$. *The positive cone* $\mathfrak{B}^+$ *is normal for* ρ *if and only if* $(\mathfrak{C}, \rho)$ *induces the topology* $(\mathfrak{B}, \rho)$ *for any symmetric subspace* $\mathfrak{C}$ *of* $B(\mathscr{D}, \mathscr{D})$ *containing* $\mathfrak{B}$.

This is the converse of Proposition 6 in [9].

The second group of results are about description of the predual.

Theorem 2. (1) *Any ultraweakly continuous linear functional* f *on a symmetric subspace of* $B(\mathscr{D}, \mathscr{D})$ *is of the form* $f=f_1-f_2+i(f_3-f_4)$ *where* f_j $(j=1,\dots,4)$ *is an ultraweakly continuous positive linear functional.*

(2) *Any ultraweakly continuous positive linear functional on a symmtric ultraweakly closed subspace of* $B(\mathscr{D}, \mathscr{D})$ *is normal.*

(3) *Any normal positive linear functional* f *on the ultraweak closure* $\mathfrak{B}^\sigma$ *of* $\mathfrak{B}$ *can be written as*

$$f(T)=\sum_{i=1}^{\infty} T(x_i, x_i),\ T\in\mathfrak{B}^\sigma, \tag{3.1}$$

for a sequence $x_i\in\mathscr{D}$ *if* $\mathfrak{B}$ *is a subspace of* $B(\mathscr{D}, \mathscr{D})$ *satisfying Condition II. In particular,* f *is ultraweakly continous on* $\mathfrak{B}^\sigma$. *A *-algebra* $\mathfrak{A}$ *on* $\mathscr{D}$ *satisfying Condition I is a special case of such* $\mathfrak{B}$.

(4) *Any normal positive linear functional* f *on* $B(\mathscr{D}, \mathscr{D})$ *is of the form* (3.1) *(without any condition on* $\mathfrak{A}$).

The third group of results are about structure of $\mathfrak{A}'$ and $\mathfrak{A}''$.

Theorem 3. *Let $\mathfrak{A}$ be a *-algebra on a domain $\mathscr{D}$ satisfying Condition I_0.*

(1) *The commutant $\mathfrak{A}'$ is an ultraweakly closed *-algebra on $\mathscr{D}$ satisfying Condition I and the topologies ρ and λ coincide on $\mathfrak{A}'$.*

(2) *The bicommutant $\mathfrak{A}''$ is ultraweakly closed and satisfies Condition II.*

(3) *The bicommutant $\mathfrak{A}''$ is the ultraweak closure of *-algebra $\tilde{\mathfrak{A}}$ generated by $\mathfrak{A}$ and all A_n^{-1}.*

(4) *Let $\mathfrak{M}$ be the von Neumann algebra generated by all bounded operators in $\tilde{\mathfrak{A}}$ and $\mathfrak{Z}$ be its center. The set of all bounded operators in $\mathfrak{A}'$ is $\mathfrak{M}'$ (the commutant of $\mathfrak{M}$ in the sense of von Neumann algebras) and $\mathfrak{A}'$ is algebraically generated by $\mathfrak{M}'$ and $[\bar{A}_n]_{\mathfrak{Z}}$ ($n=1, 2,\dots$) where $\bar{A}_n$, the closure of A_n, is a selfadjoint positive operator affiliated with $\mathfrak{M}$ and $[A_n]_{\mathfrak{Z}}$* is *the greatest lower bound of A_n affiliated with $\mathfrak{Z}$ (i.e. the largest element of the set of all selfadjoint positive operators C affiliated with $\mathfrak{Z}$ satisfying Dom $C \supset \mathscr{D}$ and $(A_n x, x) \geqq (Cx, x)$* for all *$x \in \mathscr{D}$) which exists and satisfies $[A_n]_{\mathfrak{Z}} \geqq \mathbf{1}$, $[A_n]_{\mathfrak{Z}}^{-1} \in \mathfrak{Z} \subset \mathfrak{M}' \subset \mathfrak{A}'$. (See Appendix.)*

(5) *A closable operator defined on $\mathscr{D}$ is in $\mathfrak{A}'$ if and only if it is affiliated with $\mathfrak{M}'$.*

(6) *All $B \in (\mathfrak{A}')^+$ is essentially selfadjoint on $\mathscr{D}$, $B^\alpha \in (\mathfrak{A}')^+$ for $\alpha > 0$ and, if $B \geqq \mathbf{1}$, $B^\alpha \mathscr{D} = \mathscr{D}$.*

(7) *Let D be the completion of $\mathscr{D}$ relative to the topology induced by a countable set of norms $\|[A_n]_{\mathfrak{Z}} x\|$ for $x \in \mathscr{D}$ and $(\mathfrak{A}', D)'$ be the commutant of $\mathfrak{A}'$ in $B(D, D)$. Then $\mathfrak{A}'$ is ultraweakly closed relative to $D \hat{\otimes} D$, $(\mathfrak{A}', D)'$ is an ultraweakly closed *-algebra on D satisfying Condition I, generated algebraically by $\mathfrak{M}$ and $[A_n]_{\mathfrak{Z}}$ ($n=1, 2,\dots$), and D is complete relative to $(\mathfrak{A}', D)'$. $(\mathfrak{A}', D)'' = \mathfrak{A}'$.*

Theorem 3'. *Under Condition I_0' for $\mathfrak{A}$, the conclusion* (1), (4), (6) *and* (7) *of Theorem* 3 *hold except $\mathfrak{M}$ is the von Neumann algebra generated by bounded operators in $\mathfrak{A} A_n^{-1}$ for all n. In addition,* (2) *takes the following form: $\mathfrak{A}''$ is ultraweakly closed and the set of all bounded operators in $\mathfrak{A}''$ is $\mathfrak{M}$.*

The reason for failure of (3) and Condition II in (2) in Theorem 3' is because $A_n^{-1}\mathscr{D}$ need not be contained in $\mathscr{D}$.

§4. Preliminary Lemmas

Before plunging into proof of main results, we settle minor problems in this section.

Lemma 4.1. *If* $A \in \mathfrak{A}^+$ *and* $A \geqq \mathbf{1}$, *then* $A^2 \geqq A$.

Proof. By Schwarz inequality and the assumption $A \geqq \mathbb{1}$,

$$\|Ax\| \, \|x\| \geqq (Ax, x) \geqq \|x\|^2 . \tag{4.1}$$

Therefore $\|Ax\| \geqq \|x\|$, which implies

$$(A^2x, x) = \|Ax\|^2 \geqq \|Ax\| \, \|x\| \geqq (Ax, x) . \tag{4.2}$$

Lemma 4.2. *If* $A_n \in \mathfrak{A}$ $(n=1, 2,\dots)$ *is cofinal in* $\mathfrak{A}^+$, *then for any* $A \in \mathfrak{A}$, *there exists an integer n and* $\lambda > 0$ *such that* $\lambda \|A_n x\| \geqq \|Ax\|$ *for all* $x \in \mathscr{D}$.

Proof. For given $A \in \mathfrak{A}$, $A^*A \in \mathfrak{A}^+$ and hence there exists (n, λ) such that $\lambda A_n \geqq A^*A$ and $\lambda A_n \geqq \mathbf{1}$. By Lemma 4.1, $\lambda^2 A_n^2 \geqq \lambda A_n \geqq A^*A$ and hence $\lambda^2 \|A_n x\|^2 = (\lambda^2 A_n^2 x, x) \geqq (A^*Ax, x) = \|Ax\|^2$ for all $x \in \mathscr{D}$.

Lemma 4.3. *Let* $\hat{\mathscr{D}}$ *be the completion of* $\mathscr{D}$ *given by* (2.1). *If* $\beta \in B(\mathscr{D}, \mathscr{D})$, *then there is a unique extension* $\bar{\beta} \in B(\hat{\mathscr{D}}, \hat{\mathscr{D}})$ *of* β. *For any given* $A \in \mathfrak{A}$ *and* $\lambda > 0$, $|\beta(x, x)| \leqq \lambda (Ax, x)$ *for all* $x \in \mathscr{D}$ *if and only if* $|\bar{\beta}(x, x)| \leqq \lambda (\bar{A}x, x)$ *for all* $x \in \hat{\mathscr{D}}$.

Proof. The existence of the extension and its unqiueness is straightforward because $\beta \in B(\mathscr{D}, \mathscr{D})$ is continuous in $\mathscr{D} \times \mathscr{D}$. For $x \in \hat{\mathscr{D}}$, there exists a net $x_\alpha \in \mathscr{D}$ such that $(A(x_\alpha - x), x_\alpha - x) \to 0$ for all $A \in \mathfrak{A}^+$. Since $|\beta(x_\alpha, x_\alpha)| \leqq (Ax_\alpha, x_\alpha)$ for some $A \in \mathfrak{A}^+$, $\beta(x_\alpha, x_\alpha)$ has a limit which must be $\bar{\beta}(x, x)$. Hence the first inequality implies the second. The second inequality implies the first as its restriction.

Lemma 4.4. *If* $A_n \geqq \mathbf{1}$ *is a cofinal sequence in* $\mathfrak{A}^+$, *then* A_n^2 *is also a cofinal sequence in* $\mathfrak{A}^+$. *Furthermore, there is a subsequence* $n(m)$, $m=1, 2,\dots$ *and a monotone increasing sequence* $\lambda_m \geqq 1$, $\lambda_m \to \infty$ *such that both* $B_m = \lambda_m A_{n(m)}$ *and* B_m^2 *are monotone increasing and cofinal in* $\mathfrak{A}^+$, *and* $B_{m+1} \geqq B_m^2$.

Proof. For $x \in \mathscr{D}$, $A_n \geqq \mathbf{1}$ implies $(A_n x, x) \leqq \|A_n x\|^2 = (A_n^2 x, x)$ by Lemma 4.1. Because A_n is cofinal, there exists $N(n)$ and $g(n) \geqq 1$ such that

$$g(n) A_{N(n)} \geqq \sum_{k=1}^{n} (A_k + A_k^2) , \tag{4.3}$$

which implies $g(n)A_{N(n)} \geqq A_k$ and, due to Lemma 4.3 for $A_{N(n)} \geqq \mathbf{1}$, $g(n)A_{N(n)}^2 \geqq g(n)A_{N(n)} \geqq A_k^2$. Starting with $n(1)=1$ and $B_1=A_1$, we choose inductively $n(m)=N(l)$ for $l=\max(n(m-1), m)$ and $\lambda_m=(\lambda_{m-1})^2 g(l)$. Then the desired properties are satisfied.

Lemma 4.5. *If A and B are linear operators with a common dense domain $\mathscr{D}$ satisfying $A\mathscr{D} \subset \mathscr{D}$, $B\mathscr{D} \subset \mathscr{D}$, $A \geqq \mathbf{1}$ (as a form) and $AB=BA=\mathbf{1}$ on $\mathscr{D}$, then the closures of A and B are positive self-adjoint operators, $B=A^{-1} \leqq \mathbf{1}$ and $\bar{A}^2$ is the closure of A^2.*

Proof. We have $A\mathscr{D} \subset \mathscr{D}$ on one hand and $B\mathscr{D} \subset \mathscr{D}$ implies $A\mathscr{D} \supset AB\mathscr{D} = \mathscr{D}$ on the other hand. Therefore $A\mathscr{D}=\mathscr{D}$. Similarly $B\mathscr{D}=\mathscr{D}$. For $x \in \mathscr{D}$,

$$(4.4) \qquad (Bx, x)=(Bx, ABx)=(ABx, Bx) \qquad (\geqq 0)$$
$$=(x, Bx)$$

and hence $B \in \mathfrak{A}^+$. Further, Schwarz inequality and $A \geqq \mathbf{1}$ imply

$$(4.5) \qquad \|Bx\|\,\|x\| \geqq (Bx, x)=(ABx, Bx) \geqq (Bx, Bx)=\|Bx\|^2$$

for all $x \in \mathscr{D}$ and hence $\|B\| \leqq 1$. Namely $\bar{B}$ is a positive bounded self-adjoint operator. For the selfadjoint operator $\bar{B}^{-1}$, $\bar{B}\mathscr{D}$ for any dense set $\mathscr{D}$ is a core due to $\|B\|<\infty$ and hence $\mathscr{D}=B\mathscr{D}$ is the core of $\bar{B}^{-1}$ in the present case. Furthermore, for $x \in \mathscr{D}$, $\bar{B}^{-1}x=\bar{B}^{-1}BAx=Ax$ and hence $\bar{A}=\bar{B}^{-1}$. Since $A^2 \in \mathfrak{A}$, $B^2 \in \mathfrak{A}$, $A^2 \geqq \mathbf{1}$ and $A^2B^2=B^2A^2=\mathbf{1}$, we have $\overline{A^2}=(\bar{B}^2)^{-1}=\bar{B}^{-2}=\bar{A}^2$.

In the following, we shall denote $B=A^{-1}$ in the above situation.

Lemma 4.6. *If a closable operator T defined on $\mathscr{D}$ has its range in $\mathscr{D}$, then T is continuous as a map from the Fréchet space $\mathscr{D}$ into itself, where the countable domination and completeness $\mathscr{D}=\hat{\mathscr{D}}$ are assumed. For any $\beta \in B(\mathscr{D}, \mathscr{D})$ and such operators T_1 and T_2, $T_2^*\beta T_1(x, y)=\beta(T_1x, T_2y)$ is again in $B(\mathscr{D}, \mathscr{D})$.*

Proof. Since T is closable on $\mathscr{H}$, it is closed on the Fréchet space $\mathscr{D}$. (Note that $x_\alpha \in \mathscr{D}$ and $x_\alpha \to \mathbf{0}$ in $\mathscr{D}$ implies $x_\alpha \to \mathbf{0}$ in $\mathscr{H}$.) By closed graph theorem, T is continuous.

§5. A Representation of $\mathfrak{A}_A$ by Bounded Linear Operators

In this section we work with subspaces $\mathfrak{B}$ in $B(\mathscr{D}, \mathscr{D})$. A *-algebra $\mathfrak{A}$ over $\mathscr{D}$ can be viewed as $\mathfrak{B}$ and, if $\mathfrak{A}$ satisfies Condition I, then it satisfies Condition II.

Proposition 5.1. *Let $\mathfrak{B}$ be a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$, $A \in \mathfrak{A}^+$, $A \geqq \mathbb{1}$. Then $(\mathfrak{B}_A, \rho_A)$ is canonically isometrically isomorphic to a subspace of $(L(\mathscr{H}), \rho_1)$ where $\mathfrak{B}_A = \mathfrak{B} \cap B(\mathscr{D}, \mathscr{D})_A$, $\rho_1(A)$ is the infimum of $\lambda \geqq 0$ satisfying $|(Ax, x)| \leqq \lambda \|x\|^2$ for all $x \in \mathscr{H}$ and coincides with $\|A\|$ for any normal A.*

Proof. For $A \in B^+(\mathscr{D}, \mathscr{D})$, there exists (for example, see [9]) a unique positive selfadjoint opertor Δ_A such that $\mathscr{D}$ is the core of $\Delta_A^{1/2}$ (hence $\Delta_A^{1/2}\mathscr{D}$ is dense in $\mathscr{H}$) and

$$A(x, y) = (\Delta_A^{1/2}x, \Delta_A^{1/2}y),$$

because A is closable. Since $A \geqq \mathbb{1}$, $\Delta_A \geqq \mathbb{1}$ and in particular the kernel of Δ_A is $\mathbb{0}$.

Let $\beta \in B(\mathscr{D}, \mathscr{D})$, $\lambda \geqq 0$ and $|\beta(x, x)| \leqq \lambda A(x, x)$ for all $x \in \mathscr{D}$.

$$\beta(x, y) = T_\beta(\Delta_A^{1/2}x, \Delta_A^{1/2}y) \tag{5.1}$$

defines a hermitian sesquilinear form on $\Delta_A^{1/2}\mathscr{D}$ which satisfies $|T_\beta(x, x)| \leqq \lambda \|x\|^2$. This implies the existence of unique $T_\beta \in L(\mathscr{H})$ satisfying

$$\beta(x, y) = (T_\beta \Delta_A^{1/2}x, \Delta_A^{1/2}y). \tag{5.2}$$

The map $\beta \to T_\beta$ is obviously linear. Since $\rho_1(T_\beta)$ is the infimum of $\lambda \geqq 0$ satisfying $|(T_\beta x, x)| \leqq \lambda \|x\|^2$, which is equivalent to $|\beta(x, x)| \leqq \lambda(Ax, x)$ for all $x \in \mathscr{D}$, it is $\rho_A(\beta)$. Hence the map is isometric.

Remark 5.2. If $A\mathscr{D} = \mathscr{D}$ in Proposition 5.1, then $\Delta_{A^2} = \bar{A}^2$ by Lemma 4.5 and hence $(\Delta_{A^2})^{1/2} = \bar{A}$.

Proposition 5.3. *Let $\mathfrak{B}$ be a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ containing a cofinal seqeunce of operators A_n in $B^+(\mathscr{D}, \mathscr{D})$ such that $A_n \geqq \mathbb{1}$, $A_n\mathscr{D} = \mathscr{D}$, $\beta \in \mathfrak{B}$ implies $A_n\beta A_n \in \mathfrak{B}$ and $A_n^{-1}\beta A_n^{-1} \in \mathfrak{B}$. Then $\mathfrak{B} = \bigcup_n \mathfrak{B}_{B_n}$ for $B_n = A_n^2$, and $\mathfrak{B}_{B_n}$ are isometrically isomorphic to $\mathfrak{B}_{\mathrm{id}}$ (the set of all bounded operators in $\mathfrak{B}$), where $\mathfrak{B}_A \equiv \mathfrak{B} \cap B(\mathscr{D}, \mathscr{D})_A$ for $A = B_n$ or $A = \mathrm{id}$.*

Proof. By Lemma 4.4, $\mathfrak{B} = \bigcup_n \mathfrak{B}_{B_n}$. By Remark 5.2, $\Delta_{B_n}^{1/2} = \bar{A}_n$. For $\beta \in \mathfrak{B}$, we have

$$A_n^{-1}\beta A_n^{-1}(x, y) = \beta(A_n^{-1}x, A_n^{-1}y) = (T_\beta^{(n)}x, y) \tag{5.3}$$

where $T^{(n)}$ is T for $A = B_n$ in Proposition 5.1. Thus $T_\beta^{(n)}$ (as a sesquilinear form on $\mathscr{D}$) is in $\mathfrak{B}_{\mathrm{id}}$. Conversely if a bounded linear operator B_1 is in $\mathfrak{B}_{\mathrm{id}}$, then

$B \equiv A_n B_1 A_n \in \mathfrak{B}$ and $T_B = B_1$. Namely $T^{(n)}$ is an isometric isomorphism of $\mathfrak{B}_{B_n}$ onto $\mathfrak{B}_{\text{id}}$.

The following is an immediate consequence of the definition.

Lemma 5.4. *The map $T^{(n)}$ is bicontinuous from $\mathfrak{A}_{A_n}$ with $\sigma(\mathfrak{A}_{A_n}, \mathscr{D}\otimes\mathscr{D})$ onto N_n with $\sigma(N_n, \mathscr{D}_n\otimes\mathscr{D}_n)$ where $N_n = T^{(n)}\mathfrak{A}_{A_n}$ and $\mathscr{D}_n = \Delta_{A_n}^{1/2}\mathscr{D}$.*

Proposition 5.5. *Let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_{A_n}$ be a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ containing a cofinal sequence A_n of $B^+(\mathscr{D}, \mathscr{D})$ satisfying $A_n \geqq 1$ and let $T^{(n)}$ be the map $\beta \in \mathfrak{B}_{A_n} \to T_\beta \in L(\mathscr{H})$ defined in Proposition 5.1 for $A = A_n$. Then $\mathfrak{B}$ is ultraweakly closed if and only if the image N_n of $T^{(n)}$ is ultraweakly closed in $L(\mathscr{H})$ for all n.*

Proof. Obviously $\mathfrak{B}$ is ultraweakly closed if and only if $\mathfrak{B}_{\mathrm{R}}$ is. The circled closed convex hull of $\{\lambda^{-1}(A_n x, x)^{-1}(x\otimes x); x \in \mathscr{D}\}$ forms a fundamental system of neighbourhood of 0 in $\mathscr{D}\hat{\otimes}\mathscr{D}$ when λ varies over positive reals and n varies over natural numbers. (Chapter 1, §1, Proposition 2(2), [6]. The circled closed convex hull is actually circled σ-convex hull, Exercise 2, §13, Chapter 2, [6].) Its polar in $B(\mathscr{D}, \mathscr{D})$ is the set of $\beta \in B(\mathscr{D}, \mathscr{D})$ satisfying $|\beta(x, x)| \leqq \lambda(A_n x, x)$. Therefore, by the Banach-Dieudonné theorem (Chapter 4, §6, Theorem 6.4, [13]), $\mathfrak{B}_{\mathrm{R}}$ is ultraweakly closed if and only if $[-A_n, A_n]_{\mathfrak{B}}$ is ultraweakly compact for all n, and by the Ascoli theorem which asserts equivalence of simple convergence on $\mathscr{D}\hat{\otimes}\mathscr{D}$ and simple convergence on its dense subset $\mathscr{D}\otimes\mathscr{D}$ for the equicontinuous set $[-A_n, A_n]_{\mathfrak{B}}$ (Theorem 4, Chapter 0 in [7]), if and only if $[-A_n, A_n]_{\mathfrak{B}}$ is $\sigma(\mathfrak{B}_{A_n}, \mathscr{D}\otimes\mathscr{D})$ compact for all n. By isomorphism $T^{(n)}$, this is the same as the $\sigma(\mathfrak{N}_n, \mathscr{D}_n\otimes\mathscr{D}_n)$ compactness of $[-\text{id}, \text{id}]_{\mathfrak{N}_n}$ where $\mathfrak{N}_n$ is the image of $\mathfrak{B}_{A_n}$ by $T^{(n)}$ and $\mathscr{D}_n = \Delta_{A_n}^{1/2}\mathscr{D}$, which is dense in $\mathscr{H}$. By the Ascoli theorem, this is equivalent to the $\sigma(\mathfrak{N}_n, \mathscr{H}\hat{\otimes}\mathscr{H})$ compactness of $[-\text{id}, \text{id}]_{\mathfrak{N}_n}$ and, by the Banach-Dieudonné theorem, to the condition that $(\mathfrak{N}_n)_{\mathrm{R}}$ and hence $\mathfrak{N}_n$ is ultraweakly closed in $L(\mathscr{H})$ for all n. (Note that $T^{(n)}$ preserves the adjoint.)

Corollary 5.6. *Assume that $A_n \geqq 1$ is a cofinal sequence of $B(\mathscr{D}, \mathscr{D})^+$ and $A_n\mathscr{D} = \mathscr{D}$. Let $\mathfrak{B}$ a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ such that*

1° $\text{id} \in \mathfrak{B}$,

2° *$A_n \beta A_n$ and $A_n^{-1}\beta A_n^{-1}$ are in $\mathfrak{B}$ if β is in $\mathfrak{B}$. Then $\mathfrak{B}$ is ultraweakly closed if and only if $[\text{id}, \text{id}]_{\mathfrak{B}}$ is compact relative to the σ-weak topology of bounded operators, equivalently if and only if the set of all bounded operators* *in $\mathfrak{B}$ is σ-weakly closed.*

Proof. The two conditions imply that $\mathfrak{B}$ contains the cofinal seqeunce A_n^2 of $B(\mathscr{D}, \mathscr{D})$. By the proof of Proposition 5.5, $\mathfrak{B}$ is ultraweakly closed if and only if $[-A_n^2, A_n^2]_{\mathfrak{B}}$ is $\sigma(\mathfrak{B}, \mathscr{D}\otimes\mathscr{D})$ compact for all n. By Lemma 5.4, this is the case if and only if $T^{(n)}[-A_n^2, A_n^2]_{\mathfrak{B}}$ is $\sigma(\cdot, \mathscr{D}_n\otimes\mathscr{D}_n)$ compact, where $\Delta_{B_n}^{1/2}=\bar{A}_n$ for $B_n=A_n^2$ by Remark 5.2 and hence $\mathscr{D}_n=A_n\mathscr{D}=\mathscr{D}$.

We shall prove

$$T^{(n)}[-A_n^2, A_n^2]_{\mathfrak{B}}=[-\mathrm{id}, \mathrm{id}]_{\mathfrak{B}}. \tag{5.4}$$

If this is shown, then this set is bounded in the set $B(\mathscr{H})$ of all bounded operators, it is $\sigma(\cdot, \mathscr{D}_n\otimes\mathscr{D}_n)$ compact if and only if it is $\sigma(\cdot, \mathscr{H}\hat{\otimes}\mathscr{H})$ compact (i.e. σ-weakly compact), by the Ascoli theorem for example, and this is the case if and only if $\mathfrak{B}_{\mathrm{id}}$ is σ-weakly closed (Chapter 1, §3, Theorem 1, (iv) [3]), proving Corollary.

To show (5.4), the definition (5.3) of $T^{(n)}$ and our assumption 2° imply

$$T^{(n)}[-A_n^2, A_n^2]_{\mathfrak{B}}\subset[-\mathrm{id}, \mathrm{id}]_{\mathfrak{B}}. \tag{5.5}$$

On the other hand, for any $T\in[-\mathrm{id}, \mathrm{id}]_{\mathfrak{B}}$,

$$\beta(x, y)=(TA_nx, A_ny) \qquad (x, y\in\mathscr{D}) \tag{5.6}$$

belongs to $\mathfrak{B}$ by our assumption 2°, is in $[-A_n^2, A_n^2]$ and satisfies $T_\beta^{(n)}=T$. Therefore we (5.6).

Proposition 5.7. *Let $\mathfrak{B}$ be a subspace of $B(\mathscr{D}, \mathscr{D})$ satisfying Condition II. The ultraweak closure $\mathfrak{B}^\sigma$ of $\mathfrak{B}$ in $B(\mathscr{D}, \mathscr{D})$ also satisfies Condition II with the same cofinal sequence $A_n\geqq\mathbb{1}$ and $(\mathfrak{B}^\sigma)_{\mathrm{id}}$ is the von Neumann algebra generated by $\mathfrak{B}_{\mathrm{id}}$.*

Proof. Since the multiplication of A_n and A_n^{-1} is continuous on $\mathscr{D}$ by Lemma 4.6, $\beta\to A_n\beta A_n$ and $\beta\to A_n^{-1}\beta A_n^{-1}$ is weakly and ultraweakly continuous. Therefore the first half is immediate except for (3) of Condition II. By Corollary 5.6 $(\mathfrak{A}^\sigma)_{\mathrm{id}}$ contains the ultraweak closure $\mathfrak{M}$ of $\mathfrak{A}_{\mathrm{id}}$ which is a von Neumann algebra. Thus it remains to show that $(\mathfrak{A}^\sigma)_{\mathrm{id}}=\mathfrak{M}$.

Let $B\in\mathfrak{M}'$. Since $\bar{A}_n$ is selfadjoint (Lemma 4.5) and $A_n^{-2}\in(\mathfrak{B})_{\mathrm{id}}$, $\bar{A}_n^{-1}$ is in $\mathfrak{M}$ and commutes with B. This implies $B\,\mathrm{Dom}\,\bar{A}_n\subset\mathrm{Dom}\,\bar{A}_n$ and hence $B\mathscr{D}\subset\mathscr{D}$, namely $x\otimes y\to Bx\otimes y$ and $x\otimes y\to x\otimes By$ induces a continuous linear map on $\mathscr{D}\hat{\otimes}\mathscr{D}$.

By Proposition 5.3, any $A\in\mathfrak{B}$ can be written as $A=A_nTA_n$ with $T\in\mathfrak{B}_{\mathrm{id}}$.

Therefore A commutes with B on $\mathscr{D}$, i.e.

$$\beta(Bx, y)=\beta(x, B^*y)$$

holds for $\beta\in\mathfrak{B}$ and hence $\beta\in\mathfrak{B}^\sigma$. This implies

$$(\mathfrak{B}^\sigma)_{\mathrm{id}}\subset(\mathfrak{M}')'=\mathfrak{M}.$$

Therefore $(\mathfrak{B}^\sigma)_{\mathrm{id}}=\mathfrak{M}=(\mathfrak{B}_{\mathrm{id}})''$.

§ 6. Normality of the Positive Cone

Lemma 6.1. *Let $\mathfrak{R}$ be a *-algebra of bounded linear operators on a Hilbert space H, $B\in\mathfrak{R}$, $B\geqq 0$ and $0<\alpha<1$. If $T\in\mathfrak{R}$ and*

$$|(Tx, x)|\leqq((B+\alpha\mathbf{1})x, x) \tag{6.1}$$

for all $x\in\mathscr{H}$, there exists T_1 and T_2 in $\mathfrak{R}$ such that $T=T_1+T_2$,

$$|(T_1x, x)|\leqq(Bx, x), \tag{6.2}$$

$$|(T_2x, x)|\leqq G(\|B\|)\alpha^{1/2}\|x\|^2 \tag{6.3}$$

for all $x\in\mathscr{H}$ where $G(t)=2(1+t)^{1/2}+1$.

Proof. On the interval $I\equiv[0, \|B\|]$, consider a real function

$$f(t)=\min\{[t(t+\alpha)]^{-1/2}-\varepsilon,\ 1+\alpha^{-1}\} \tag{6.4}$$

where $\varepsilon>0$ is sufficiently small. (For example $4^{-1}\alpha^{1/2}[1+\|B\|]^{-3/2}>\varepsilon$.) Let $g(t)$ be a polynomial satisfying $|f(t)-g(t)|\leqq\varepsilon$ for all $t\in I$. Let $h(t)=g(t)t$. We have

$$1\geqq(t/(t+\alpha))^{1/2}\geqq h(t)\geqq 0, \tag{6.5}$$

$$0\leqq(1-h(t))(t+\alpha)^{1/2}\leqq\alpha^{1/2}. \tag{6.6}$$

For $T\in\mathfrak{R}$ satisfying (6.1), let

$$T_1=h(B)Th(B), \quad T_2=T-T_1. \tag{6.7}$$

Then

$$\begin{aligned}|(T_1x, x)|&=|(Th(B)x, h(B)x)|\\&\leqq((B+\alpha\mathbf{1})h(B)x, h(B)x)\\&\leqq(Bx, x)\end{aligned} \tag{6.8}$$

where the last inequality is due to (6.5). Furthermore (6.1) implies $\|T'\|\leqq 1$ for $T'=(B+\alpha\mathbf{1})^{-1/2}T(B+\alpha\mathbf{1})^{-1/2}$ and hence

$$(6.9). \quad \|T_2\| \leqq \|(\mathbb{1}-h(B))Th(B)\| + \|T(\mathbb{1}-h(B))\|$$
$$\leqq \|(\mathbb{1}-h(B))(B+\alpha\mathbb{1})^{1/2}\|(\|(B+\alpha\mathbb{1})^{1/2}h(B)\| + \|(B+\alpha\mathbb{1})^{1/2}\|)$$
$$\leqq \alpha^{1/2}(2(\|B\|+\alpha)^{1/2}+\alpha^{1/2}) \leqq \alpha^{1/2}(2(\|B\|+1)^{1/2}+1).$$

Lemma 6.2. *Let $\mathfrak{B}$ be a subspace of $B(\mathscr{D}, \mathscr{D})$ containing a cofinal sequence $A_n \geqq \mathbb{1}$ in $B^+(\mathscr{D}, \mathscr{D})$ and satisfying Condition II. For any given $B \geqq \mathbb{0}$ in $\mathfrak{B}_{B_n}$ for $B_n = A_n^2$, there exists a number G depending only on $\rho_{B_n}(B)$ such that for any $0<\alpha<1$ and any $\beta \in \mathfrak{B}$ satisfying $|\beta(x, x)| \leqq ((B+\alpha B_n)x, x)$, there exists β_1 and β_2 in $\mathfrak{B}$ satisfying*

$$(6.10) \qquad \beta = \beta_1 + \beta_2, \ |\beta_1(x, x)| \leqq (Bx, x), \ |\beta_2(x, x)| \leqq G\alpha^{1/2}(B_n x, x),$$

for all $x \in \mathscr{D}$.

Proof. By Proposition 5.3, there is an order preserving isometric isomorphism of $\mathfrak{B}_{B_n}$ onto the *-algebra $\mathfrak{B}_{\mathrm{id}}$ of bounded operators where B_n is mapped to the identity operator. The decomposition of Lemma 6.1 in $\mathfrak{B}_{\mathrm{id}}$ then yields the desired decomposition in $\mathfrak{B}_{B_n}$.

Lemma 6.3. *Let $\mathfrak{B}$ be a symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ containing a cofinal monotone sequence B_n $(n=1, 2,\dots)$ in $B^+(\mathscr{D}, \mathscr{D})$ and $\mathfrak{V}$ be a neighbourhood of $\mathbb{0}$ in $(\mathfrak{B}, \rho)$. Let λ_n $(n=1, 2,\dots)$ be a seqeunce of real numbers satisfying $0<\lambda_n<1$ for all n and $\prod_{n=1}^{\infty} \lambda_n > 0$. The following 3 conditions are equivalent.*

1° *$\mathfrak{V}$ contains an absolutely convex and order convex neighbourhood of $\mathbb{0}$ in $(\mathfrak{B}, \rho)$.*

2° *There exist $\alpha_n > 0$ $(n=1, 2,\dots)$ such that*

$$(6.11) \qquad [-\sum_{j=1}^{n} \alpha_j B_j, \quad \prod_{j=1}^{n} \alpha_j B_j]_{\mathfrak{B}} \subset \mathfrak{V}.$$

3° *There exist $\alpha_n > 0$ $(n=1, 2,\dots)$ such that*

$$(6.12) \qquad C_n = \sum_{k=1}^{n-1} (\prod_{j=k}^{n-1} \lambda_j)\alpha_k B_k + \alpha_n B_n$$

satisfies (for all n) $[-C_n, C_n]_{\mathfrak{B}} \subset \mathfrak{V}$.

Proof. 1°→2°: Let $\mathfrak{W}$ be an absolutely convex and order convex neighbourhood of $\mathbb{0}$ in $\mathfrak{B}$ and $\mathfrak{U}$ be an absolutely convex open set in $\mathfrak{W}$. If $C \in \mathfrak{U} \cap \mathfrak{B}^+$, $[-C, C]_{\mathfrak{B}}$ is in $\mathfrak{W}$. For $B_j \in B^+$ and $C \in \mathfrak{U} \cap \mathfrak{B}^+$, there exists $\lambda > 0$ such that $C + \lambda B_j$ is in $\mathfrak{U}$, for which $[-(C+\lambda B_j), C+\lambda B_j]_{\mathfrak{B}} \subset \mathfrak{W}$. By repetition, 2° holds.

2°→3°: Due to $C_n \leqq \sum_{j=1}^{n} \alpha_j B_j$.

$3°\to 1°$: Let $\lambda=\prod_{j=1}^{\infty}\lambda_j>0$. Then $\lambda<\lambda_1\lambda_2\cdots\lambda_n$ and hence

$$C'_n=\lambda\sum_{j=1}^{n}\alpha_j B_j \tag{6.13}$$

satisfies (for all n) $C'_n\leqq C'_{n+1}$ and $[-C'_n, C'_n]\subset\mathfrak{V}$. Since $\mathfrak{B}_{C'_n}=\mathfrak{B}_{C_n}$, the set

$$\bigcup_{n=1}^{\infty}[-C'_n, C'_n] \tag{6.14}$$

is absolutely convex, order convex and is a neighbourhood of $\mathbf{0}$ in $(\mathfrak{B}, \rho)$ contained in $\mathfrak{V}$.

Proof of Theorem 1 (i)–(iii). In all cases (i)–(iii), $\mathfrak{B}$ satisfies Condition II and hence these cases follow from (iv).

Proof of Theorem 1 (iv). The condition 1° of Lemma 6.3 for arbitrary neighbourhood $\mathfrak{V}$ of $\mathbf{0}$ in $(\mathfrak{B}, \rho)$ is the order convexity of the topology ρ by definition. We shall apply Lemma 6.2 to find α_n and show 3° in Lemma 6.3.

Let $\mathfrak{V}$ be a neighbourhood of $\mathbf{0}$ in $(\mathfrak{B}, \rho)$ and $\lambda_1, \lambda_2,\dots, \lambda_n,\dots$ be as in Lemma 6.3. By Lemma 4.4, we may assume that $B_n=A_n^2\geqq\mathbf{1}$ is monotone increasing. Set $\mathfrak{V}_n=\mathfrak{V}\cap\mathfrak{B}_{B_n}$. Then $\mathfrak{V}_1\subset\mathfrak{V}_2\subset\cdots$. By induction on n, we shall find $\alpha_n>0$ satisfying

$$[-\alpha_{n+1}B_{n+1}-\lambda_n C_n, \alpha_{n+1}B_{n+1}+\lambda_n C_n]_{\mathfrak{B}}=[-C_{n+1}, C_{n+1}]_{\mathfrak{B}}\subset\mathfrak{V}_{n+1} \tag{6.15}$$

with C_n defined by (6.12). Since $\mathfrak{V}_1$ is a neighbourhood of $\mathbf{0}$ in the normed space $(\mathfrak{B}_{B_1}, \rho_{B_1})$, there is α_1 such that $[-C_1, C_1]\subset\mathfrak{V}_1$ for $C_1=\alpha_1 B_1$. Suppose we have already found C_n with $[-C_n, C_n]\subset\mathfrak{V}_n$. Choose α_{n+1} such that

$$[-G\alpha_{n+1}^{1/3}B_{n+1}, G\alpha_{n+1}^{1/3}B_{n+1}]\subset(1-\lambda_n)\mathfrak{V}_{n+1} \tag{6.16}$$

where G is given by Lemma 6.2 for $B=C_n$. Note that $C_n\in\mathfrak{B}_{Bn}\subset\mathfrak{B}_{Bn+1}$. Let

$$T\in[-\lambda_n C_n-\alpha_{n+1}B_{n+1}, \lambda_n C_n+\alpha_{n+1}B_{n+1}]_{\mathfrak{B}}. \tag{6.17}$$

By Lemma 6.2, there exists T_1 and T_2 in $\mathfrak{B}$ satisfying

$$T_1\in[-\lambda_n C_n, \lambda_n C_n]\subset\lambda_n\mathfrak{V}_n\subset\lambda_n\mathfrak{V}_{n+1} \tag{6.18}$$

and

$$T_2\in[-G\alpha_{n+1}^{1/2}B_{n+1}, G\alpha_{n+1}^{1/2}B_{n+1}]\subset(1-\lambda_n)\mathfrak{V}_{n+1}. \tag{6.19}$$

Therefore $T\in\mathfrak{V}_{n+1}$, which shows $[-C_{n+1}, C_{n+1}]_{\mathfrak{B}}\subset\mathfrak{V}_{n+1}$.

Proof of Corollary 1b. If $\mathfrak{B}^+$ is normal, then the topology induced by the restriction of $(\mathfrak{C}, \rho)$ to $\mathfrak{B}$ is the ρ-topology of $\mathfrak{B}$ by Proposition 6 (1°) of [9]. Conversely, if $(\mathfrak{C}, \rho)$ induces the ρ-topology of $\mathfrak{B}$ for any $\mathfrak{C} \supset \mathfrak{B}$, then take $\mathfrak{C} = B(\mathscr{D}, \mathscr{D})$. We know by Theorem 1 that the ρ-topology for $B(\mathscr{D}, \mathscr{D})$ is order-convex. The restriction of an order-convex topology to a subspace is again order-convex and hence the ρ-topology of $\mathfrak{B}$ must be order-convex.

§7. Predual

Proposition 7.1. *Let $\mathfrak{B}$ be a symmetric ultraweakly closed subset of $B(\mathscr{D}, \mathscr{D})$ with a monotone increasing cofinal sequence $A_n \geq \mathbb{1}$ in $B^+(\mathscr{D}, \mathscr{D})$. For a linear functional φ on $\mathfrak{B}$, φ is ultraweakly continuous if and only if the restriction of φ to $[-A_n, A_n]_{\mathfrak{B}}$ is ultraweakly continuous for all n. Further, this is the case if and only if the restriction of φ to $[-A_n, A_n]_{\mathfrak{B}}$ is weakly continuous for all n.*

Proof. Since weak and ultraweak topology coincide on an equicontinuous set $[-A_n, A_n]_{\mathfrak{B}}$, the last two conditions are equivalent. The second condition follows from the first condition by restriction. Thus we have only to show that the last condition implies the first.

Let φ be a linear functional on $\mathfrak{B}$, with its restriction $[-A_n, A_n]_{\mathfrak{B}}$ being weakly continuous for all n. Let $\mathfrak{K} = \ker \varphi$. The set $\mathfrak{K} \cap [-A_n, A_n]_{\mathfrak{B}}$ is weakly closed and hence, by Banach-Diendonné theorem, $\mathfrak{K}$ is ultraweakly closed, which is the ultraweak continuity of φ.

Proposition 7.2. *Let $\mathfrak{B}$ be as in Proposition 6.1. Let $\mathfrak{B}^\rho$ be the strong dual of $(\mathfrak{B}, \rho)$, which is a Fréchet space. In $\mathfrak{B}^\rho$, the closure of the set of all weakly continuous linear functionals is exactly the predual $\mathscr{P}$ of $\mathfrak{B}$, i.e. the set of all ultraweakly continuous linear functional. The map ω from the Fréchet space $\mathscr{D} \hat{\otimes} \mathscr{D}$ into $\mathfrak{A}^\rho$ given by (2.8) induces a topological homomorphism onto the Fréchet space $\mathscr{P}$.*

Proof. By the Banach theorem on the dual $B(\mathscr{D}, \mathscr{D})$ of the Fréchet space, any ultraweakly bounded set is simply bounded on $\mathscr{D} \otimes \mathscr{D}$ and, by Corollary 2, Section 5, Chapter 3, [13], it is equicontinuous and hence is in the polar of a neighbourhoods of $\mathbb{0}$. Hence $[-\lambda A_n, \lambda A_n]_{\mathfrak{B}}$, $\lambda > 0$ and $n = 1, 2, \ldots$, as polars of a fundamental system of neighbourhoods of $\mathbb{0}$ in $\mathscr{D} \hat{\otimes} \mathscr{D}$, is a fundamental system of bounded sets of $\mathfrak{B}$ in ultraweak topology. Thus the strong topology on the predual $\mathscr{P}$ of $\mathfrak{A}$ is the topology induced on $\mathscr{P}$ by $\mathfrak{A}^\rho$. By a theorem of

Grothendieck (Chapter 4, 6.2, Corollary 1, [13]), the completion of $\mathscr{P}$ is the set of all linear functionals f on $\mathfrak{A}$, whose restriction to each $[-A_n, A_n]$ is ultraweakly continuous. By Proposition 7.1, f must be ultraweakly continuous and hence $\mathscr{P}$ is closed in $\mathfrak{A}^\rho$.

Since bounded sets of $\mathfrak{A}$ relative to weak topology is the same as above, the same argument shows that the completion of the set of all weakly continuous linear functionals on $\mathfrak{A}$ is the set of all linear functionals f on $\mathfrak{A}$, whose restriction to each $[-A_n, A_n]$ is weakly continuous. The coincidence of weak and ultra-weak topology on the equicontinuous set $[-A_n, A_n]$ of $\mathfrak{A}$ then proves the first part of Proposition.

$B(\mathscr{D}, \mathscr{D})$ is the dual of $\mathscr{D}\hat{\otimes}\mathscr{D}$ ([6]) as a vector space and $\mathfrak{B}$ is its ultraweakly closed subspace. Hence $\mathfrak{B}$ is, as a vector space, the dual of $(\mathscr{D}\hat{\otimes}\mathscr{D}/\mathfrak{B}^0)$ by the bipolar theorem where $\mathfrak{B}^0$ is the polar of $\mathfrak{B}$ in $\mathscr{D}\hat{\otimes}\mathscr{D}$. By the above proof, $\mathscr{P}$ is a Fréchet space as a closed subset of the Fréchet space $(\mathfrak{B}, \rho)^\rho$ (the dual of $(\mathfrak{B}, \rho)$ and $(\mathfrak{B}, \rho)$ is a DF-space as an inductive limit of normed spaces by Proposition 5, Chapter 4, Part 3, §3 in [7]). By Hahn-Banach extension theorem, ω is surjective and hence ω induces an isomorphism from a Fréchet space $\mathscr{D}\hat{\otimes}\mathscr{D}/\mathfrak{B}^0$ onto another Fréchet space $\mathscr{P}$. Furthermore ω as a homomorphism from $\mathscr{D}\hat{\otimes}\mathscr{D}$ into $(\mathfrak{B}, \rho)^\rho$ is separately continuous by the proof of Proposition 8 in [9] and hence is continuous. Therefore ω induces a topological isomorphism of $\mathscr{D}\hat{\otimes}\mathscr{D}/\mathfrak{B}^0$ onto $\mathscr{P}$.

Lemma 7.3. *Let $\mathfrak{A}$ be a *-algebra satisfying Condition I and π be an amplification $T\in\mathfrak{A}\to\Phi(T)\in\mathfrak{A}\otimes\mathbf{1}$ acting on the Hilbert space*

$$\tilde{\mathscr{H}}\equiv\mathscr{H}\otimes l_2(N)=\sum{}^{\oplus}\mathscr{H}. \tag{7.1}$$

Let the set of all $x=\sum_{i=1}^{\infty}{}^{\oplus}x_i$ such that $x_i\in\mathscr{D}$ and $\sum_{i=1}^{\infty}\|A_nx_i\|^2<\infty$ for all n (called σ-convergent in [10]) be denoted by $\tilde{\mathscr{D}}$. Then

$$\tilde{\mathscr{D}}=\bigcap_{n=1}^{\infty}\mathrm{Dom}\,(\overline{\pi(A_n)}). \tag{7.2}$$

Proof. By Lemma 4.4, we may assume that A_n is monotone increasing, $A_n\geqq\mathbf{1}$, $A_n^{-1}\in\mathfrak{A}$ and A_n^2 is also monotone increasing. Since $\|A_nx\|^2=\sum_{i=1}^{\infty}(A_n^2x_i, x_i)<\infty$, $\tilde{\mathscr{D}}$ is included in the right hand side. Let $x=\sum_{i=1}^{\infty}{}^{\oplus}x_i$ be in $\mathrm{Dom}\,(\overline{\pi(A_n)})$ for all n. Since the projection P on i-th space $\mathscr{H}$ commutes with $\pi(A_n)$, $Px\in\mathrm{Dom}\,(\overline{\pi(A_n)})$ and hence $x_i\in\bigcap_{n=1}^{\infty}D(\bar{A}_n)$.

By Lemma 4.5, $\mathscr{D}$ is the core of the selfadjoint operator $\overline{A_n}$ for all n. Therefore there exists $x_{in} \in \mathscr{D}$ such that

$$\|A_n(x_{in} - x_i)\| < n^{-1}. \tag{7.3}$$

Since A_n^2 is monotone increasing, we have

$$\|A_k(x_{in} - x_{im})\| \leqq n^{-1} + m^{-1} \tag{7.4}$$

for all n, $m \geqq k$ including $A_0 = \mathbf{1}$, x_{in} converges to x_i relative to the topology given by seminorms $\|A_n x\|$ $(n = 1, 2, \dots)$. Since $\mathscr{D}$ is assumed to be complete, we have $x_i \in \mathscr{D}$.

From $x \in \mathrm{Dom}\,(\pi(A_n))$, it follows that $\sum_{i=1}^{\infty} \|A_n x_i\|^2 = \|\pi(A_n)x\|^2 < \infty$. Therefore $x \in \tilde{\mathscr{D}}$.

Proof of Theorem 2 (1). By Proposition 7.2, any ultraweakly continuous functional f can be written as

$$f(\beta) = w_u(\beta) = \sum \lambda_j \beta(x_j, y_j) \tag{7.5}$$

where $\sum |\lambda_j| = 1$, $\lim_j (A_n x_j, x_j) = 0$ and $\lim_j (A_n y_j, y_j) = 0$ for all n. By a change of x_j, we may assume $\lambda_j > 0$. Then $f = g_0 - g_2 + i(g_1 - g_3)$ where

$$g_k(\beta) \equiv 4^{-1} \sum_j \lambda_j \beta(x_j + i^k y_j, x_j + i^k y_j) \tag{7.6}$$

is positive and ultraweakly continuous.

Proof of Theorem 2 (2). Let T_α be a monotone increasing net in $\mathfrak{B}$ bounded by some $\beta \in B(\mathscr{D}, \mathscr{D})$.

Let $T(x, x) \equiv \sup_\alpha T_\alpha(x, x)$. Then $T(x, x) < \infty$ and hence $T(x, x) = \lim \cdot T_\alpha(x, x)$. Therefore T satisfies the parallelogram law and hence is uniquely extended to a sesquilinear form which we write T again. Since $\beta(x, x) \geqq T(x, x) \geqq 0$, $T \in B(\mathscr{D}, \mathscr{D})$ and $T(x, y) \equiv \lim T_\alpha(x, y)$. Since T_α and T are in a bounded set $[0, \beta]$, where weak and ultraweak topologies coincide, T is the ultraweak limit of T_α and hence is in $\mathfrak{B}$. Since $\beta \geqq T_\alpha$ implies $\beta \geqq T$, $T = \sup_\alpha T_\alpha$. If f is ultra-weakly continuous, we have $f = \lim_\alpha f(T_\alpha)$.

Proof of Theorem 2 (3). As in Proposition 5.3, consider the order preserving isometric isomorphisms $T^{(n)}$ from $(\mathfrak{B}^\sigma)_{A_n^2}$ to $(\mathfrak{B}^\sigma)_{\mathrm{id}}$ which is a von Neumann algebra by Corollary 5.7. Let f be a normal positive linear functional on $\mathfrak{B}$ and $\Psi_n = f \cdot (T^{(n)})^{-1}$. Then Ψ_n is a normal positive linear functional on the von Neumann algebra $(\mathfrak{B}^\sigma)_{\mathrm{id}} \equiv \mathfrak{M}$. Hence there exists a sequence of vectors $x_j \in \mathscr{H}$ such that $\Psi_n = \sum_{j=1}^{\infty} \omega_{x_j, x_j}$. Since $\mathscr{D}$ is dense in $\mathscr{H}$, we can find $y_j \in \mathscr{D}$

approximating x_j, so that $\Psi'_n = \sum_{j=1}^{N} \omega_{y_j, y_j}$ satisfies

$$\|\Psi_n - \Psi'_n\|_{\mathfrak{M}} < \varepsilon . \tag{7.7}$$

Setting $z_j = A_n^{-1} y_j \in \mathscr{D}$ and $f_n = \sum_{j=1}^{N} \omega_{z_j, z_j}$, we have

$$\sup_{|T| \leqq A_n^2} |f(T) - f_n(T)| = \sup_{|B| \leqq 1} |\Psi_n(B) - \Psi'_n(B)| \leqq \varepsilon \tag{7.8}$$

and hence f is ultraweakly continuous by Proposition 7.2.

On the amplified space $\tilde{\mathscr{H}}$ defined by (7.1), we define amplification $\pi(\beta)$ of $\beta \in B(\mathscr{D}, \mathscr{D})$ as a continuous sesquilinear form on $\tilde{\mathscr{D}}$ (see (7.2)) by

$$\pi(\beta)(x, y) = \sum \beta(x_i, y_i), \tag{7.9}$$

$$x = \sum^{\oplus} x_i \in \tilde{\mathscr{D}}, \; y = \sum^{\oplus} y_i \in \tilde{\mathscr{D}}, \tag{7.10}$$

where we equip $\tilde{\mathscr{D}}$ with a countable system of norms $\|\pi(A_n)x\|$. An ultraweakly continuous linear functional f can be written as $f(\beta) = \sum_{i=1}^{\infty} \beta(x_i, y_i)$ with σ-convergent sequence x_i and y_i. (If $f(\beta) = \sum_{i=1}^{\infty} \lambda_i \beta(x'_i, y'_i)$ with $\lambda_i \geqq 0$, $x'_i \to \mathbf{0}$ and $y'_i \to \mathbf{0}$ in $\mathscr{D}$, then set $x_i = \lambda_i^{1/2} x'_i$, $y_i = \lambda_i^{1/2} y'_i$.) Setting $x = \sum^{\oplus} x_i$ and $y = \sum^{\oplus} y_i$, both x and y are $\tilde{\mathscr{D}}$ and

$$f(\beta) = \pi(\beta)(x, y) . \tag{7.11}$$

Since any $\beta \in \mathfrak{B}^{\sigma}$ can be written as $A_n T A_n$ for some n and $T \in (\mathfrak{B}^{\sigma})_{\mathrm{id}} \equiv \mathfrak{M}$ and since T can be decomposed as a linear combination of positive operators in the von Neumann algebra $\mathfrak{M}$, β is a linear combination of positive elements in $\mathfrak{B}^{\sigma}$. Since f is positive, we have $f(\beta^*) = \overline{f(\beta)}$. This implies

$$\pi(\beta)(y, x) = \overline{\pi(\beta^*)(x, y)} = \overline{f(\beta^*)} = f(\beta) = \pi(\beta)(x, y) \tag{7.12}$$

for all $\beta \in \mathfrak{B}^{\sigma}$. Therefore $g(\beta) \equiv \beta(u, u)$ for $u = x + y$ satisfies

$$g(\beta) = \beta(x, x) + \beta(y, y) + 2f(\beta) \geqq 2f(\beta) . \tag{7.13}$$

First we restrict our attention to $\beta \in (\mathfrak{B}^{\sigma})_{\mathrm{id}} = \mathfrak{M}$. Then $\tilde{f}(\pi(T)) \equiv f(T)$, $T \in \mathfrak{M}$, is a positive linear functional on $\pi(\mathfrak{M})$ majorized by $(\pi(T)u, u)$. Hence there exists a (unique) positive operator $T_0 \in \mathfrak{M}'$ such that its range is the closure of $\pi(\mathfrak{M})u$ and

$$f(S^*T) = (T_0 \pi(T)u, \pi(S)u) = (\pi(S^*T) T_0^{1/2} u, T_0^{1/2} u) \tag{7.14}$$

for all S and T in $\mathfrak{M}$. Since (1) and $\mathbf{1} \in \mathfrak{B}_{\mathrm{id}}$ in (3) of Condition II imply $\bar{A}_n^{-2} \in \mathfrak{B}_{\mathrm{id}} \subset \mathfrak{M}$, $T_0^{1/2}$ commutes with spectral projections of $\pi(\bar{A}_n) = \overline{\pi(A_n)}$ for all

n, which implies $T_0^{1/2}\,\mathrm{Dom}\,\overline{\pi(A_n)} \subset \mathrm{Dom}\,\overline{\pi(A_n)}$ and hence $T_0^{1/2}\tilde{\mathscr{D}} \subset \tilde{\mathscr{D}}$. Therefore $x = T_0^{1/2}u \in \tilde{\mathscr{D}}$ and, for $x = \sum^{\oplus} x_i$,

$$f(\beta) = \pi(\beta)(x, x) = \sum_{i=1}^{\infty} \beta(x_i, x_i) \tag{7.15}$$

holds for $\beta \in (\mathfrak{B}^{\sigma})_{\mathrm{id}}$. Since both ends of (7.15) are ultraweakly continuous (positive) linear functionals, (7.15) holds for all $\beta \in \mathfrak{B}^{\sigma}$ by the following Lemma. ($\mathfrak{B}^{\sigma}$ is an ultraweakly closed symmetric subspace of $B(\mathscr{D}, \mathscr{D})$ satisfying Condition II.)

Lemma 7.4. *Let $\mathfrak{B}$ be an ultraweakly closed symmetric subspace satisfying Condition II. Any ultraweakly continuous linear functional f on $\mathfrak{B}$ is positive if it is positive on $\mathfrak{B}_{\mathrm{id}}$. Any two ultraweakly continuous linear functionals f_1 and f_2 coincide on $\mathfrak{B}$ if they coincide on $\mathfrak{B}_{\mathrm{id}}$.*

Proof. The second assertion is obtained by applying the first assertion to $f_1 - f_2$ and $-(f_1 - f_2)$. To prove the first assertion, we note that $(\bar{A}_n)^{-2} \in \mathfrak{M} = \mathfrak{B}_{\mathrm{id}}$ and hence $g(\bar{A}_n)$ for any bounded measurable function g is in $\mathfrak{M}$. If $f(\beta) = \sum_{i=1}^{\infty} \beta(x_i, y_i)$ and $T \geqq 0$ is in $\mathfrak{B}_{\mathrm{id}}$, then

$$\sum_i (Tg(\bar{A}_n)x_i, g(\bar{A}_n)y_i) \geqq 0. \tag{7.16}$$

Taking a monotone increasing sequence of bounded positive $g_k(x)$ converging to x, on $[0, \infty)$, we have the estimate

$$\begin{aligned} \sum_{i>N} |(Tg(\bar{A}_n)x_i, g(\bar{A}_n)y_j)| &\leqq \sum_{i>N} \|g_k(\bar{A}_n)x_i\| \, \|g_k(\bar{A}_n)y_i\| \, \|T\| \\ &\leqq \|T\| \{\sum_{i>N} (g_k(A_n)^2 x_i)\}^{1/2} \{\sum_{i>N} (g_k(A_n)^2 y_i, y_i)\}^{1/2} \\ &\leqq \|T\| \{\sum_{i>N} (A_n^2 x_i, x_i)\}^{1/2} \{\sum_{i>N} (A_n^2 y_i, y_i)\}^{1/2}, \end{aligned} \tag{7.17}$$

which can be made smaller than any given $\varepsilon > 0$ for some N due to $\sum (A_n^2 x_i, x_i) < \infty$ and $\sum (A_n^2 y_i, y_i) < \infty$. Then

$$\sum_{i=1}^{M} (TA_n x_i, A_n y_i) = \lim_k \sum_{i=1}^{M} (Tg_k(A_n)x_i, g_k(A_n)y_i) \geqq -\varepsilon, \tag{7.18}$$

for $M \geqq N$. Taking the limit of $M \to \infty$ and then $\varepsilon \to 0$, we obtain

$$\sum_{i=1}^{\infty} (TA_n x_i, A_n y_i) \geqq 0. \tag{7.19}$$

Any positive β in $\mathfrak{B}_{B_n}$ for $B_n = A_n^2$ is of the form $\beta(x, y) = (TA_n x, A_n y)$ with $T \in \mathfrak{M}^+$ and hence (7.19) implies $f(\beta) \geqq 0$ for such β. Since B_n is cofinal in $B^+(\mathscr{D}, \mathscr{D})$, we have $f(\beta) \geqq 0$ for all $\beta \in \mathfrak{B}^+$.

Proof of Theorem 2 (4). Let $\{A_i\}$ be a monotone increasing confinal sequence in $B^+(\mathscr{D}, \mathscr{D})$ with $A_1 = \mathbf{1}$ and $B_i = A_i^2$ monotone increasing. $\mathfrak{B}_i = B^+(\mathscr{D}, \mathscr{D})_{B_i}$, $\mathfrak{B}_1$ is the von Neumann algebra $L(\mathscr{H})$ of all bounded linear operators. Proposition 5.3 implies that $T \in \mathfrak{B}_i \to A_i^{-1} T A_i^{-1} \in \mathfrak{B}_1$ is a bijective isometric isomorphism (isometry relative to $\|T\|_{B_i}$ in $\mathfrak{B}_i$).

If f is a normal positive linear form on $B(\mathscr{D}, \mathscr{D})$, then $f_i(S) = f(A_i S A_i)$ for $S \in \mathfrak{B}_1$ is a normal positive linear form on $L(\mathscr{H})$ and hence $f_i(S) = \operatorname{Tr} S\rho_i$ for a unique positive trace class operator ρ_i. Since $f_i(A_i^{-1} S A_i^{-1}) = f(S)$, we have $\operatorname{Tr}(S A_i^{-1} \rho_i A_i^{-1}) = \operatorname{Tr} S\rho_1$ for all $S \in L(\mathscr{H})$ and hence $A_i^{-1} \rho_i A_i^{-1} = \rho_1$. This implies that $\rho_1 \mathscr{H} \subset \operatorname{range} A_i^{-1} = \operatorname{Dom} A_i$ for all i. Hence $\rho_1 \mathscr{H} \subset \mathscr{D}$ and in particular, all eigenvectors of ρ_1 (with non-zero eignevalues) are in $\mathscr{D}$. This implies

$$f(T) = \sum T(x_i, x_i) \tag{7.20}$$

for $x_i \in \mathscr{D}$ for all $T \in L(\mathscr{H})$. By normality of f, we have (7.20) for all $T \in B^+(\mathscr{D}, \mathscr{D})$ and, by linearity, for all $T \in \mathfrak{B}(\mathscr{D}, \mathscr{D})$.

§ 8. Commutants and Bicommutants

In this section, the *-algebra $\mathfrak{A}$ on $\mathscr{D}$ is assumed to have a cofinal sequence A_n in $B^+(\mathscr{D}, \mathscr{D})$ satisfying $A_n \geqq \mathbf{1}$ and $A_n \mathscr{D} = \mathscr{D}$. (Condition I_0) By Lemma 4.4, we may and do assume that A_n and $B_n = A_n^2$ are both monotone increasing.

Lemma 8.1. *The set of all bounded operators in the commutant $\mathfrak{A}'$ is the commutant (in the sense of von Neumann algebras) of all bounded operators in the *-algebra $\tilde{\mathfrak{A}}$ generated by $\mathfrak{A}$ and A_n^{-1}.*

Proof. Let $T \in L(\mathscr{H})$ be in $\mathfrak{A}'$. By definition,

$$(Tx, A_n y) = (A_n x, T^* y) \tag{8.1}$$

for all $x, y \in \mathscr{D}$. By taking limit, the same equation holds for $\bar{A}_n$ and for all x, y in the domain of $\bar{A}_n$. Since $A_n \mathscr{D} = \mathscr{D}$ implies the existence of B for $A = A_n$ in Lemma 4.5 ($BA_n x = x$ for $x \in \mathscr{D}$) and hence A_n is a selfadjoint positive operator with $\|A_n^{-1}\| \leqq 1$ (due to $A_n \geqq \mathbf{1}$). Substituting $x = \bar{A}_n x'$ and $y = \bar{A}_n y'$ with arbitrary x' and y' in $\mathscr{H}$, we obtain the commutativity of T with $\bar{A}_n^{-1}$. It now follows that T commutes with all bounded elements of $\tilde{\mathfrak{A}}$ because T is in $\mathfrak{A}'$.

Conversely, $T \in L(\mathscr{H})$ be in the commutant of all bounded elements in $\tilde{\mathfrak{A}}$. Obviously $T(x, y) \equiv (Tx, y) \in \mathfrak{B}(\mathscr{D}, \mathscr{D})$ due to $T(x, x) \leqq \|T\| \, \|x\|^2$. Furthermore, for $A \in \mathfrak{A}$, there eixsts an n such that $A^2 \leqq A_n^2$, which implies that AA_n^{-1} is bounded and hence

$$(8.2) \qquad T(x, Ay) = (Tx, (AA_n^{-1})A_n y) = (\bar{A}_n^{-1}A^*x, T^*A_n y) \\ = (\bar{A}_n^{-1}A^*x, \bar{A}_n T^*y) = (A^*x, T^*y) = T(A^*x, y).$$

Lemma 8.2 *$\mathfrak{A}'$ is an ultraweakly closed *-algebra on $\mathscr{D}$.*

Proof. Let $\beta \in \mathfrak{A}'$. By definition, β is continuous, i.e. there exists an n such that

$$(8.3) \qquad |\beta(x, y)| \leqq \|A_n x\| \, \|A_n y\|.$$

Therefore, there exists a bounded operator B_m for every $m \geqq n$ such that

$$(8.4) \qquad \beta(x, y) = (B_m A_m x, A_m y).$$

By $\beta \in \mathfrak{A}'$, we have $\beta(Ax, y) = \beta(x, A^*y)$ for all $A \in \mathfrak{A}$. For $A = A_m$, we have

$$(8.5) \qquad (B_m A_m^2 x, A_m y) = (B_m A_m x, A_m^2 y).$$

Setting $x = A_m^{-1}x'$, $y = A_m^{-1}y'$ with arbitrary x' and y' in $\mathscr{D}$, we have

$$(8.6) \qquad (B_m A_m x', y') = (B_m x', A_m y').$$

Therefore $B_m \mathscr{D} \subset \mathrm{Dom}\,(\bar{A}_m)$ and $A_m B_m = B_m A_m$ on $\mathscr{D}$. Let $B = B_m A_m^2$. Then $B\mathscr{D} \subset \mathrm{Dom}\,(\bar{A}_m)$ due to $A_m^2 \mathscr{D} \subset \mathscr{D}$ and $B_m \mathscr{D} \subset \mathrm{Dom}\,\bar{A}_m$. Furthermore

$$(8.7) \qquad \beta(x, y) = (\bar{A}_m B_m A_m x, y) = ((B_m A_m) A_m x, y) = (Bx, y)$$

for all $x, y \in \mathscr{D}$. Therefore B is actually independent of m. Hence

$$(8.8) \qquad B\mathscr{D} \subset \bigcap_{m \geqq n} \mathrm{Dom}\,(\bar{A}_m) = \bigcap_{m \geqq 1} \mathrm{Dom}\,(\bar{A}_m) \\ = \bigcap_{A \in \mathfrak{A}} \mathrm{Dom}\,(\bar{A}) = \mathscr{D}.$$

We also note that $\beta \in \mathfrak{A}'$ implies

$$(8.9) \qquad BA = AB \quad \text{on } \mathscr{D} \quad \text{for any } A \in \mathfrak{A}.$$

Let $\beta_i \in \mathfrak{A}'$ and $\beta_i(x, y) = (\beta_i x, y)$ $(i = 1, 2)$. Then it is immediate that $\beta_1 + \beta_2 \in \mathfrak{A}'$, $\beta_i^* \in \mathfrak{A}'$ and $c\beta_i \in \mathfrak{A}'$ for a complex number c. For sufficiently large n, $A_n^{-1}B_i A_n^{-1}$ is bounded by 1 and hence $A_n^{-2}B_1 B_2 A_n^{-2} = (A_n^{-1}B_1 A_n^{-1})(A_n^{-1}B_2 A_n^{-1})$ is bounded and

$$(8.10) \qquad |(B_1 B_2 x, y)| \leqq \|A_n^2 x\| \, \|A_n^2 y\|$$

for $x, y \in \mathscr{D}$, which shows that $\beta_1\beta_2(x, y) \equiv (B_1 B_2 x, y)$ is in $B(\mathscr{D}, \mathscr{D})$. Furthermore

(8.11) $$\beta_1\beta_2(Ax, y)=(B_1B_2Ax, y)=(B_1B_2x, A^*y)=\beta_1\beta_2(x, A^*y)$$

due to the commutativity of A with B_i. Hence $\beta_1\beta_2 \in \mathfrak{A}'$ and $\mathfrak{A}'$ is a *-algebra over $\mathscr{D}$.

The relation $\beta(Ax, y)=\beta(x, A^*y)$ for $A \in \mathfrak{A}$ is stable under ultraweak limit (of β) and hence $\mathfrak{A}'$ is ultraweakly closed.

For any selfadjoint operator A affiliated with a von Neumann algebra $\mathfrak{M}$, there exists a unique operator $[A]_{\mathfrak{Z}}$ affiliated with $\mathfrak{Z}$. (See Appendix.) If $A=A^*$ and $B=B^*$ are affiliated with $\mathfrak{M}$ and $\mathfrak{M}'$, then $A \geqq B$ (in the sence of $(Ax, x) \geqq (Bx, x)$ for all x in (Dom A) $\cap$ (Dom B)) if and only if $[A]_{\mathfrak{Z}} \geqq B$. In particular, if $A \geqq \alpha$, then $[A]_{\mathfrak{Z}} \geqq \alpha$ for a real number α. (Similarly we can define the lowest upper bound $[A]^{\mathfrak{Z}}$ of A affiliated with $\mathfrak{Z}$. Then $A \geqq B$ is equivalent to $[A]_{\mathfrak{Z}} \geqq [B]^{\mathfrak{Z}}$ in the above situation. We also have $[A^{-1}]_{\mathfrak{Z}}=([A]^{\mathfrak{Z}})^{-1}$ for positive A.)

Lemma 8.3 *$\mathfrak{A}'$ is algebraically generated by its bounded part and $[\bar{A}_n]_{\mathfrak{Z}}$. In particular $[\bar{A}_n]_{\mathfrak{Z}}$ is a central cofinal sequence in $(\mathfrak{A}')^+$. $\mathfrak{A}'$ satisfies Condition I.*

Proof. Let $B \in \mathfrak{A}'$. By the proof of Lemma 7.2,

(8.12) $$A_n^2 \geqq B^*B$$

for some n and hence $\|\bar{A}_n x\|^2 \geqq \|\bar{B}x\|^2$ for all x in Dom $\bar{A}_n$. Therefore

(8.13) $$B^*\bar{B} \leqq [\bar{A}_n^2]_{\mathfrak{Z}}=[\bar{A}_n]_{\mathfrak{Z}}^2$$

where $\mathfrak{Z}$ is the center of von Neumann algebra generated by the bounded part of $\tilde{\mathfrak{A}}$. Because $[\bar{A}_m]$ is affiliated with $\mathfrak{Z}$,

(8.14) $$([\bar{A}_n]_{\mathfrak{Z}}^2 x, Ay)=(A^*x, [\bar{A}_n]_{\mathfrak{Z}}^2 y)$$

for $A \in \mathfrak{A}$ and $x, y \in \mathscr{D}$. Furthermore $A_n^2 \geqq [\bar{A}_n]_{\mathfrak{Z}}^2$. Therefore $B_n'(x, y) \equiv ([\bar{A}_n]_{\mathfrak{Z}}^2 x, y)$ belongs to $\mathfrak{A}'$ and the same holds for $[A_n]_{\mathfrak{Z}}^{-1}$. (Note that $A_n \geqq \mathbf{1}$ implies $[A_n]_{\mathfrak{Z}} \geqq \mathbf{1}$ and hence $[\bar{A}_n]_{\mathfrak{Z}}^{-1}$ is bounded.) Then

(8.15) $$B'=B[\bar{A}_n]_{\mathfrak{Z}}^{-1}$$

is bounded due to (8.12), and is in $\mathfrak{A}'$. Thus

(8.16) $$B=(B[A_n]_{\mathfrak{Z}}^{-1}) \quad [\bar{A}_n]_{\mathfrak{Z}}$$

where $[\bar{A}_n]_{\mathfrak{Z}}$ (restricted to $\mathscr{D}$ or considered as a sesquilinear form on $\mathscr{D} \times \mathscr{D}$) is in $\mathfrak{A}'$. This also shows that $[\bar{A}_n]_{\mathfrak{Z}}^2$ is a cofinal sequence in $(\mathfrak{A}')^+$. Since $A_{n+1} \geqq A_n^2$ implies $[\bar{A}_{n+1}]_{\mathfrak{Z}} \geqq [\bar{A}_n]_{\mathfrak{Z}}^2$, $[\bar{A}_n]_{\mathfrak{Z}}$ is also a cofinal sequence. Since $[\bar{A}_n]_{\mathfrak{Z}}^{-1} \in \mathfrak{A}'$, $\mathfrak{A}'$ satisfies Condition I.

Lemma 8.4 *The ρ and λ topologies coincide on $\mathfrak{A}'$.*

Proof. The ρ topology is given as an iductive limit of the norm topology on $\mathfrak{A}_{A'_n}$ given by $\rho_{A'_n}(\cdot)$, where $A'_n=[\bar{A}_n]_{\mathfrak{Z}}$, for example.

If $A\in\mathfrak{A}'$, there exists n such that $A^*A\leqq B'_n$. Then $\|Ax\|^2\leqq\|[\bar{A}_n]_{\mathfrak{Z}}x\|^2$. Thus the λ-topology is given as an inductive limit of the norm topology on $\mathfrak{A}'_n$ given by

$$\|A\|_{\lambda n}\equiv\sup_x\{\|Ax\|/\|[\bar{A}_n]_{\mathfrak{Z}}x\|;\ x\in\mathscr{D}\}=\|A[\bar{A}_n]_{\mathfrak{Z}}^{-1}\| \tag{8.17}$$

where $\mathfrak{A}'_n$ is the set of $A\in\mathfrak{A}'$ such that the above supremum is finite.

To compare the two, we first note that for $x\in\mathscr{D}$ and $A\in\mathfrak{A}'$,

$$\begin{aligned}|(Ax,\ x)|&\leqq\|Ax\|\ \|x\|\leqq\|A\|_{\lambda n}\|[\bar{A}_n]_{\mathfrak{Z}}x\|\ \|x\|\\&\leqq\|A\|_{\lambda n}([\bar{A}_n]_{\mathfrak{Z}}^2x,\ x)\leqq\|A\|_{\lambda n}([\bar{A}_{n+1}]_{\mathfrak{Z}}x,\ x)\end{aligned} \tag{8.18}$$

due to $[\bar{A}_m]_{\mathfrak{Z}}^2\geqq\mathbf{1}$ and $[\bar{A}_{n+1}]_{\mathfrak{Z}}\geqq[\bar{A}_n]_{\mathfrak{Z}}^2$ by our convention. Therefore

$$\|A\|_{\lambda n}\geqq\rho_{A'_{n-1}}(A). \tag{8.19}$$

On the other hand, $\rho_{B'_n}(A)=k<\infty$ for $B'_n=[\bar{A}_n]_{\mathfrak{Z}}^2$ implies

$$\rho_{B'_n}(\mathrm{Re}\,A)\leqq k,\quad \rho_{B'_n}(\mathrm{Im}\,A)\leqq k \tag{8.20}$$

where $2\,\mathrm{Re}\,A=A+A^*$ and $2\mathrm{i}\,\mathrm{Im}\,A=A-A^*$. For selfadjoint A, $\rho_{B'_n}(A)=k$ implies $\|A_n'^{-1}AA_n'^{-1}\|=\|AB_n'^{-1}\|\leqq k$ and hence

$$\|Ax\|\leqq k^2\|B'_nx\|=k^2\|A_n'^2x\|\leqq k^2\|A'_{n+1}x\| \tag{8.21}$$

where the last inequality is due to $A_n'^2\leqq A'_{n+1}$ for selfadjoint operators A'_n and A'_{n+1} associated with an abelian von Neumann algebra $\mathfrak{Z}$. Therefore

$$\|A\|_{\lambda(n+1)}\leqq 2\rho_{B'_n}(A)\leqq 2\rho_{A'_n}(A). \tag{8.22}$$

Lemma 8.5 *The bicommutant $\mathfrak{A}''$ satisfies Condition II, it is the ultraweak closure of $\tilde{\mathfrak{A}}$ in $B(\mathscr{D},\ \mathscr{D})$ and the set of bounded operators in $\mathfrak{A}''$ is the von Neumann algebra generated by bounded operators in $\tilde{\mathfrak{A}}$.*

Proof. By definition, A_n is cofinal in $(\mathfrak{A}'')^+\subset B^+(\mathscr{D},\ \mathscr{D})$. By Lemma 8.1 and 8.3, $\bar{A}_n^{-1}$ is in $\mathfrak{A}''$. By assumption, $\bar{A}_n^{-1}\mathscr{D}\subset\mathscr{D}$ and $\tilde{\mathfrak{A}}$ is a *-algebra over $\mathscr{D}$. Any $T\in\tilde{\mathfrak{A}}$ is a closed map from the F-space $\mathscr{D}$ into $\mathscr{D}$ and hence is continuous, i.e. $\|T\|_{A_n}<\infty$ for some n. Thus $\tilde{\mathfrak{A}}\subset B(\mathscr{D},\ \mathscr{D})$ and hence $\tilde{\mathfrak{A}}\subset\mathfrak{A}''$.

From Lemma 8.1 and 8.3, it also follows that bounded elements of $\mathfrak{A}''$ is the von Neumann algebra $\mathfrak{M}$ generated by all bounded elements of $\tilde{\mathfrak{A}}$ and hence is the ultraweak closure of $\tilde{\mathfrak{A}}$. If $A\in\mathfrak{A}''$, then there exists n such that $A_n^{-1}AA_n^{-1}$

is bounded. Therefore there exists a net $C_\alpha \in \tilde{\mathfrak{A}}$ such that $\lim C_\alpha = A_n^{-1} A A_n^{-1}$ in the ultraweak topology of the von Neumann algebra $\mathfrak{M}$. Then

$$(Ax,\ y) = \lim_\alpha (C_\alpha A_n x,\ A_n y) \tag{8.23}$$

for all $x,\ y \in \mathscr{D}$. Namely A is in the ultraweak closure of $\tilde{\mathfrak{A}}$. The fact that $\mathfrak{A}''$ is ultraweakly closed is immediate.

Since $\tilde{\mathfrak{A}}$ is a *-algebra over $\mathscr{D}$ (contained in $B(\mathscr{D},\ \mathscr{D})$) and $A_n,\ A_n^{-1} \in \tilde{\mathfrak{A}} \subset \mathfrak{A}''$, $\mathfrak{A}''$ clearly satisfies the Condition II.

By Lemma 8.1, any bounded operator in $\mathfrak{A}''$ must be in the double commutant (in the sense of von Neumann algebras) of the set of all bounded operator in $\tilde{\mathfrak{A}}$, (which is a *-algebra by definition). On the other hand, the preceeding proof shows the opposite inclusion and hence equality. (Note that the ultraweak closure in the sense of bounded operators is included in the ultraweak closure in $B(\mathscr{D},\ \mathscr{D})$.)

Proof of Theorem 3 (1) *and* (4). By Lemmas 8.1, 8.2, 8.3 and 8.4.

Proof of Theorem 3 (2) *and* (3). By Lemma 8.5.

Proof of Theorem 3 (5). Let T be a closable operator defined on $\mathscr{D}$. Then it is a closable map from the F-space $\mathscr{D}$ into the Hilbert space $\mathscr{H}$ and hence $\|Tx\|^2 \leqq c\|A_n x\|^2 = (cA_n^* A_n x,\ x)$ for some $c > 0$. Hence $T(x,\ y) \equiv (Tx,\ y)$ is in $B(\mathscr{D},\ \mathscr{D})$. As we have seen in (3), $\mathfrak{M}$ is in $\mathfrak{A}''$ and hence, if $T \in \mathfrak{A}'$, then $T(x,\ S^* y) = T(Sx,\ y)$ for all $x,\ y \in \mathscr{D}$ and $S \in \mathfrak{M}$, which implies $ST \subset TS$. Therefore T is affiliated with $\mathfrak{M}'$. Conversely, if T is affiliated with $\mathfrak{M}'$, then $ST \subset TS$ for any $S \in \mathfrak{M}$. Taking $S = \bar{A}_n^{-1}$, we obtain $\bar{A}_n T \bar{A}_n^{-1} \subset T$. If we restrict this equation on $A_n \mathscr{D}$, we obtain $A_n T = T A_n$ on $\mathscr{D}$. For any $S \in \mathfrak{A}$, there exists n such that $A_n^{-1} S^* S A_n^{-1}$ is bounded, i.e. $B \equiv S A_n^{-1} \in \mathfrak{M}$. Then $S = B A_n \mid \mathscr{D}$, $S^* = \bar{A}_n B^* \mid \mathscr{D}$ and

$$\begin{aligned} T(Sx,\ y) &= (TBA_n x,\ y) = (BTA_n x,\ y) = (A_n Tx,\ B^* y) \\ &= (Tx,\ S^* y) = T(x,\ S^* y). \end{aligned} \tag{8.24}$$

Therefore $T \in \mathfrak{A}'$.

Proof of Theorem 3 (6). For any given $B \in (\mathfrak{A}')^+$, there exists an n and $c > 0$ such that $B \leqq cA_n$. Let E_k be the spectral projection of $[A_n]_{\mathfrak{Z}}$ for the interval $[0,\ k]$. Then $E_k \in \mathfrak{Z}$, $\mathscr{H} = \bigoplus_{k=1}^{\infty} \mathscr{H}_k$, $\mathscr{H}_k \equiv (E_k - E_{k-1})\mathscr{H}$ and B is reduced by this decomposition (because $E_k \in \mathfrak{Z}$ leaves $\mathscr{D}$ invariant and commutes with B on $\mathscr{D}$). Furthermore B on each $\mathscr{H}_k$ is bounded (by ck). Therefore B is essentially selfadjoint.

Proof of Theorem 3 (7). Since $\mathfrak{A}'$ satisfies Condition I by (1) and the set of all bounded operators in $\mathfrak{A}'$ is a von Neumann algebra $\mathfrak{M}'$, $\mathfrak{A}'$ is ultraweakly closed in $B(D, D)$ due to the last half of Corollary 5.6. If we use $(\mathfrak{A}', D)$ for $(\mathfrak{A}, \mathscr{D})$ and $[A_n]_3$ for A_n and apply (1) and (4), then we see that $(\mathfrak{A}', D)'$ is an ultraweakly closed *-algebra on D satisfying Condition I and generated argebraically by $\mathfrak{M} = (\mathfrak{M}')'$ and $[A_n]_3 = [[A_n]_3]_3$. Repeated appications of this argument yield the last assertion.

§ 9. Proof of Theorem 3′

Lemma 9.1 *Let A be defined on a dense domain $\mathscr{D}$, $A\mathscr{D} \subset \mathscr{D}$ and $(Ax, x) \geqq \|x\|^2$ for all $x \in \mathscr{D}$. If A^2 is essentially selfadjoint on $\mathscr{D}$, then A is also essentially selfadjoint, $(\bar{A})^2 = \bar{A}^2$ and $A\mathscr{D}$ is a core for $\bar{A}$.*

Proof. By Lemma 4.1, $\|A^2x\|^2 \geqq (A^2x, x) \geqq (Ax, x) \geqq \|x\|^2$ for $x \in \mathscr{D}$. Therefore $\mathrm{Dom}\,\bar{A}^2 \subset \mathrm{Dom}\,\bar{A}$, $\bar{A}^2 \subset (\bar{A})^2$ (because $A^2x_n \to y$ implies x_n and Ax_n converging), the relation holds for $x \in \mathrm{Dom}\,\bar{A}^2$ and $\bar{A}^2\mathscr{D}$ is the whole Hilbert space $\mathscr{H}$. Since $A\mathscr{D} \subset \mathscr{D}$ implies $A^2\mathscr{D} \subset A\mathscr{D}$ and since $A^2\mathscr{D}$ is dense because $\mathscr{D}$ is a core of the strictly positive selfadjoint operator $\bar{A}^2$, $A\mathscr{D}$ is also dense. Hence $(Ax, x) \geqq \|x\|^2$ for $x \in \mathscr{D}$ implies that any $y \in \mathscr{H}$ is a limit of Ax_n with x_n also converging and hence $\bar{A}(\mathrm{Dom}\,\bar{A}) = \mathscr{H}$. Furthermore $(\bar{A}x, x) \geqq \|x\|^2$ and hence $\|\bar{A}x\|^2 \geqq (\bar{A}x, x) \geqq \|x\|^2$ for all $x \in \mathrm{Dom}\,\bar{A}$. This means that $\bar{A}^{-1}$ is defined on $\mathscr{H}$ with $\|\bar{A}^{-1}\| \leqq 1$ and $(\bar{A}^{-1}\bar{A}x, \bar{A}x) = (x, Ax) \geqq 0$ for all $x \in \mathrm{Dom}\,\bar{A}$. Hence $\bar{A}^{-1} \geqq 0$. Hence $A = (\bar{A}^{-1})^{-1}$ is positive selfadjoint. Since $(\bar{A})^2$ is selfadjoint, $\bar{A}^2 \subset (\bar{A})^2$ above implies $A^2 = (\bar{A})^2$. For any $x \in \mathrm{Dom}\,\bar{A}^2$, there exists $x_n \in \mathscr{D}$ satisfying $x_n \to x$ and $A^2x_n \to \bar{A}^2x$, which implies $Ax_n \to Ax$. Therefore the restriction of A to $A\mathscr{D}$ has the closure with a domain containing $\bar{A}\,\mathrm{Dom}\,\bar{A}^2$, which is $\mathrm{Dom}\,\bar{A}$ and hence $A\mathscr{D}$ is a core of $\bar{A}$.

Proof of (1) *and* (4). Since A_n is essentially selfadjoint on $\mathscr{D}$ by Condition I_0' and Lemma 9.1, Lemma 8.1 holds. Note that $\mathfrak{A}A_n^{-1}$ is defined on $A_n\mathscr{D}$ and for any $B \in \mathfrak{A}$, there exists an n such that $cA_n^2 \geqq B^*B$, i.e. BA_n^{-1} is bounded.

In the proof of Lemma 8.2, we any take x' and y' in $A_m\mathscr{D}$ which is a core for $\bar{A}_m$ and hence (8.6) holds for all x' and y' in the domain of $\bar{A}_m$ when A_m is replaced by $\bar{A}_m$. Therefore $B_m(\mathrm{Dom}\,\bar{A}_m) \subset \mathrm{Dom}\,\bar{A}_m$ and $B_m\bar{A}_m = \bar{A}_mB_m$ on $\mathrm{Dom}\,\bar{A}_m$. This implies that $B_m\bar{A}_m^3 = \bar{A}_mB_m\bar{A}_m^2$ on $\mathrm{Dom}\,A_m^3 \supset \mathrm{Dom}\,A_m^3 \supset \mathscr{D}$ and hence, in particular $B_m\bar{A}_m^2\mathscr{D} \subset \mathscr{D}(\bar{A}_m)$. Setting $B \equiv B_mA_m^2 \mid \mathscr{D}$, we obtain (8.7) and hence B is independent of m. Furthermore $B\mathscr{D}\subset\bigcap_m \mathrm{Dom}(\bar{A}_m)=\mathscr{D}$. The rest of the proof of Lemma 8.2 holds as it is and hence $\mathfrak{A}'$ is again an ultraweakly closed *-algebra on $\mathscr{D}$.

The proof of Lemma 8.3 and 8.4 is unchanged.

Proof of modified (2), (6) *and* (7). The same as proof of Theorem 3.

§10. Abelian Case

We shall discuss the Gelfand transform for abelian case.

Lemma 10.1 *Let $\mathfrak{A}$ be an abelian *-algebra over $\mathscr{D}$ with the a cofinal sequence A_n in $\mathfrak{A}^+$ satisfying Condition I. Let $\mathscr{A}$ be the C^*-algebra which is the norm closure of the * algebra $\mathfrak{A}_{\mathrm{id}}$ of bounded operators in $\mathfrak{A}$. Let X be the space of all characters of $\mathscr{A}$ with the weak * topology. Then*

(1) *Any character $\chi\in X$ such that $\chi(A_n^{-1})\neq 0$ for all n has a unique extension of its restriction on $\mathfrak{A}_1$ to a character (again denoted by χ) of $\mathfrak{A}$.*

(2) *For any non-zero T in $\mathfrak{A}$, there exists a character $\chi\in X$ such that $\chi(T)\neq 0$.*

Proof. (1) Let χ be a character on $\mathscr{A}$. For any $A\in\mathfrak{A}$, there exists an n and $c>0$ such that $A^*A\leqq cA_n^2$. Then $AA_n^{-1}\in\mathfrak{A}_1$. We define

$$\chi(A)\equiv\chi(\bar{A}A_n^{-1})\chi(A_n^{-1})^{-1}.$$

Suppose $\bar{A}A_n^{-1}\in\mathfrak{A}_1$ and $\bar{A}A_m^{-1}\in\mathfrak{A}_1$ with $n>m$. Then

$$\begin{aligned}\chi(\bar{A}A_n^{-1})\chi(A_n^{-1})^{-1}&=\chi(\bar{A}A_m^{-1}A_mA_n^{-1})\chi(A_n^{-1})^{-1}\\&=\chi(\bar{A}A_m^{-1})\chi(A_mA_n^{-1})\chi(A_n^{-1})^{-1}\\&=\{\chi(\bar{A}A_m^{-1})\chi(A_m^{-1})^{-1}\}\{\chi(A_m^{-1})\chi(A_mA_n^{-1})\}\chi(A_n^{-1})^{-1}\\&=\chi(\bar{A}A_m^{-1})\chi(A_m^{-1})^{-1}\{\chi(A_n^{-1})\}\chi(A_n^{-1})^{-1}\\&=\chi(\bar{A}A_m^{-1})\chi(A_m^{-1})^{-1}\end{aligned}$$

which shows the independence of the definition. The linear dependence on A is then immediate. If $AA_n^{-1}\in\mathfrak{A}_1$ and $BA_m^{-1}\in\mathfrak{A}_1$, there exists a k such that $A_nA_mA_k^{-1}\in\mathfrak{A}_1$ and hence AA_k^{-1}, BA_k^{-1}, ABA_k^{-1} are all in $\mathfrak{A}_1$. Then

$$\begin{aligned}\chi(\overline{AB})&=\chi(\overline{AB}A_k^{-1})\chi(A_k^{-1})^{-1}\\&=\chi(\overline{AB}A_k^{-1}A_k^{-1})\chi(A^{-1})^{-2}\\&=\chi(\bar{A}A_k^{-1}\bar{B}A_k^{-1})\chi(A_k^{-1})^{-2}\\&=\{\chi(\bar{A}A_k^{-1})\chi(A_k^{-1})^{-1}\}\{\chi(\bar{B}A_k^{-1})\chi(A_k^{-1})^{-1}\},\\&=\chi(\bar{A})\chi(\bar{B}).\end{aligned}$$

(2) Let $T \in \mathfrak{A}$ and n be such that $\bar{T}A_n^{-1} \in \mathfrak{A}_1$. The norm closure $\overline{\mathfrak{A}}_1$ of $\mathfrak{A}_1$ is a commutative C^*-algebra. Hence the set of character χ on $\overline{\mathfrak{A}}_1$ which is non-zero on any specific elements of $\overline{\mathfrak{A}}_1$ is a dense open set in the set of all characters of $\overline{\mathfrak{A}}_1$. In particular the set of characters χ on $\overline{\mathfrak{A}}_1$ such that $\chi(\bar{T}A_n^{-1}) \neq 0$ for an n and $\chi(A_j^{-1}) \neq 0$ for all j is a G_δ and hence non-empty. Such a character induces a character $\mathfrak{A}$ such that $\chi(T) \neq 0$.

Proposition 10.2. *Let $\mathfrak{A}$ be an ablian *-algebra on $\mathscr{D}$ satisfying Condition I. Let $\mathscr{A}$ and X be as in Lemma* 10.1. *Let S be the set of all $\chi \in X$ such that $\chi(A_n^{-1}) \neq 0$ for all $n \neq 0$. Let Φ be the Gelfand transform from $\mathfrak{A}$ to $C(S)$ defined by $\Phi(T)(\chi) = \chi(T)$. Then Φ is a positive isomorphism of $\mathfrak{A}$ into $C(S)$.*

Proof. By Lemma 10.1, Φ is an isomorphism. If $T \in \mathfrak{A}$, $T \geqq 0$ and $TA_n^{-1} \in \mathfrak{A}_1$, then $A_n^{-1}TA_n^{-1} \geqq 0$ as an element of $\mathscr{A}$ and hence

$$\chi(T) = \chi(A_n^{-1}TA_n^{-1})\chi(A_n)^{-2} \geqq 0$$

because χ is a character on a C^*-algebra $\mathscr{A}$.

Remark. Since Φ preserves order, it preserves ρ-norms defined by order.

Appendix

Proposition A1. *Let A be an essentially selfadjoint operator affiliated with a von Neumann algebra $\mathfrak{M}$ and satisfying*

$$(Ax, x) \geqq \|x\|^2 \tag{A.1}$$

for all $x \in \operatorname{Dom} A$. Then there exists the largest lower bound $[A]_{\mathfrak{Z}}$ of A among positive selfadjoint operators affiliated with the center $\mathfrak{Z}$ of $\mathfrak{M}$.

Proof. Let $\mathscr{F}$ be the family of all positive selfadjoint operators B affiliated with $\mathfrak{Z}$, with Dom A included in Dom B and staisfying

$$(Ax, x) \geqq (Bx, x) \qquad (\geqq 0) \tag{A.2}$$

for all $x \in \operatorname{Dom} A$. Since $1 \in \mathscr{F}$, $\mathscr{F}$ is non-empty. By taking closure, Dom $\bar{A}$ $\subset$ Dom B for all $B \in \mathscr{F}$ and (A.2) holds with A replaced by its closure $\bar{A}$ for all $x \in \operatorname{Dom} \bar{A}$.

Let B_1 and B_2 be in $\mathscr{F}$. Since $\mathfrak{Z}$ is a commutative von Neumann algebra, there exists a projection $E \in \mathfrak{Z}$ satisfying

$$B_1E \geqq B_2E,\ B_1(1-E) \leqq B_2(1-E). \tag{A.3}$$

Setting $B=B_1E+B_2(1-E)$, it immediately follows that $B \geqq B_1$, $B \geqq B_2$ and $B \in \mathscr{F}$. Thus $\mathscr{F}$ with the ordering as sesquilinear forms on $\operatorname{Dom} \bar{A}$ is a net. (All $B \in \mathscr{F}$ is essentially selfadjoint on $\operatorname{Dom} \bar{A}$, which is $\mathfrak{Z}$-invariant.)

Let $E(L)$ be the spectral projection of A for $(-\infty, L)$. Then $BE(L)$ for all L are bounded operators in $\mathfrak{Z}E(L)$ and have the supremum

$$B(L) = \sup\{BE(L);\ B \in \mathscr{F}\}. \tag{A.4}$$

If $L_1 \geqq L_2$, then $B(L_1)E(L_2)=B(L_2)$ and $(AE(L))^2 \geqq B(L)^2$. (For a pure state φ of $E(L)\mathfrak{M}E(L)$, which is a character for its center $\mathfrak{Z}E(L)$, $\varphi((AE(L))^2) \geqq \varphi(AE_\alpha(L))^2 \geqq \varphi(B(L))^2 = \varphi(B(L)^2)$ and hence the same holds for any state φ by convex combination and weak limit.) Therefore $B(n)$ is a monotone increasing sequence as a sesquilinear form on $\operatorname{Dom} \bar{A}$ with $\|B(n)x\|^2$ monotone increasing and bounded by $\|Ax\|^2$ for all $x \in \operatorname{Dom} \bar{A}$. Hence there is a limit $B = \lim B(n)$ defined on $\operatorname{Dom} \bar{A}$. On each $E(L) \operatorname{Dom} \bar{A}$, it coincides with the bounded positive selfadjoint operator $B(L)$. Therefore its closure is positive selfadjoint. Since $B(L)$ commutes with $\mathfrak{M}'$ and $E(L)\mathfrak{M}E(L)$ (both having $\operatorname{Dom} \bar{A}$ invariant), $\bar{B}$ commutes with both $\mathfrak{M}'$ and $E(L)\mathfrak{M}E(L)$ for all L. Therefore it is affiliated with $\mathfrak{Z}$. Furthermore $A \geqq B$ and B is the largest element of $\mathscr{F}$.

Proposition A2. *Let A be as in Proposition A1. Let B be a positive operator on* $\operatorname{Dom} A$ *such that the closure $\bar{B}$ is affiliated with $\mathfrak{M}'$ and* (A.2) *is satisfied for all $x \in \operatorname{Dom} A$. Then*

$$([A]_{\mathfrak{Z}}x, x) \geqq (Bx, x) \tag{A.5}$$

for all $x \in \operatorname{Dom} \bar{A}$.

Proof. Without loss of generality, we may assume that $A=\bar{A}$, because (A.2) implies the same for $\bar{A}$ and $x \in \operatorname{Dom} \bar{A}$, and $[\bar{A}]_{\mathfrak{Z}} = [A]_{\mathfrak{Z}}$. We assume the existence of $x \in \operatorname{Dom} A$ satisfing $\|x\|=1$ and

$$\varepsilon \equiv (Bx, x) - ([A]_{\mathfrak{Z}}x, x) > 0, \tag{A.6}$$

and derive a contradiction.

Let E_L be the spectral projection of A for $(-\infty, L)$. Then E_L leaves $\operatorname{Dom} A$ invariant, commutes with B and $[A]_{\mathfrak{Z}} (E_L \in \mathfrak{M})$,

$$(B(1-E_L)x, x) - ([A]_{\mathfrak{Z}}(1-E_L)x, x) \tag{A.7}$$

is monotone decreasing tending to 0 as $L \to \infty$ and hence there exists an L such that $x_L = E_Lx/\|E_Lx\|$ satisfies

(A.8) $$(Bx_L, x_L) - ([A]_{\mathfrak{Z}} x_L, x_L) \geqq 3\varepsilon/4 .$$

Since AE_L is bounded (by L),

(A.9) $$(Ax, x) \geqq (Bx, x) \geqq 0$$

(A.10) $$(Ax, x) \geqq ([A]_{\mathfrak{Z}} x, x) \geqq \|x\|^2$$

implies that A, $[A]_{\mathfrak{Z}}$, and B restricted to $E_L H$ are bounded and belong to $\mathfrak{M}_{E_L}$, the center $\mathfrak{Z}_{E_L}$ of $\mathfrak{M}_{E_L}$ and $(\mathfrak{M}')_{E_L} = (\mathfrak{M}_{E_L})'$, respectively. Thus we are reduced to the case where A and B are bounded, which we shall now assume and again start from (A.6).

Let E be the spectral projection of $A - [A]_{\mathfrak{Z}}$ ($\geqq 0$) for $(-\infty, \varepsilon/2)$. By definition of $[A]_{\mathfrak{Z}}$, the central support $s^{\mathfrak{Z}}(E)$ must be $\mathbb{1}$. (Otherwise $[A]_{\mathfrak{Z}} + (\varepsilon/2)(1 - s^{\mathfrak{Z}}(E))$ will be a lower bounded of A in $\mathfrak{Z}$ larger than $[A]_{\mathfrak{Z}}$.) It is then possible to find a partition of the unity $\mathbb{1}$ by projections E_α of $\mathfrak{M}$ such that each E_α is equivalent to subprojection of E in $\mathfrak{M}$. Since

(A.11) $$\sum_\alpha \{(BE_\alpha x, E_\alpha x) - ([A]_{\mathfrak{Z}} E_\alpha x, E_\alpha x)\} = (Bx, x) - ([A]_{\mathfrak{Z}} x, x) = \varepsilon \sum \|E_\alpha x\|^2 ,$$

there exists α such that $E_\alpha x \neq 0$ and $x_\alpha \equiv E_\alpha x / \|E_\alpha x\|$ satisfies

(A.12) $$(Bx_\alpha, x_\alpha) - ([A]_{\mathfrak{Z}} x_\alpha, x_\alpha) \geqq \varepsilon .$$

Let u be a partial isometry in $\mathfrak{M}$ such that $u^*u = E_\alpha$ and $uu^* \leqq E$. Let $y = ux_\alpha$. Then

(A.13) $$\|y\|^2 = (ux_\alpha, ux_\alpha) = (u^*ux_\alpha, x_\alpha) = (x_\alpha, x_\alpha) = 1 .$$

(A.14) $$(By, y) = (u^*Bux_\alpha, x_\alpha) = (Bu^*ux_\alpha, x_\alpha) = (Bx_\alpha, x_\alpha) ,$$

(A.15) $$([A]_{\mathfrak{Z}} y, y) = (u^*[A]_{\mathfrak{Z}} ux_\alpha, x_\alpha) = ([A]_{\mathfrak{Z}} x_\alpha, x_\alpha) .$$

Hence, (A.12), (A.14) and (A.15) imply

(A.16) $$(By, y) \geqq ([A]_{\mathfrak{Z}} y, y) + \varepsilon .$$

Since $\|(A - [A]_{\mathfrak{Z}})E\| \leqq \varepsilon/2$ by definition of E and $Eu = u$, (A.13) implies

(A.17) $$(Ay, y) - ([A]_{\mathfrak{Z}} y, y) \leqq \varepsilon/2 .$$

This implies

(A.18) $$(By, y) - (Ay, y) \geqq \varepsilon/2 > 0$$

which contradicts $A \geqq B$.

Lemma A3. *If $f(\cdot)$ is a polynomial with positive coefficients, then*

(A.19) $$[f(A)]_{\mathfrak{Z}} = f([A]_{\mathfrak{Z}}) .$$

Proof. The von Neumann algebra $\mathfrak{N}$ generated by spectral projections of A and $\mathfrak{Z}$ is commutative. Since A and $[A]_{\mathfrak{Z}}$ can be viewed as continuous functions on the spectrum of $\mathfrak{N}$, $A \geqq [A]_{\mathfrak{Z}}$ implies

$$f(A) \geqq f([A]_{\mathfrak{Z}}). \tag{A.20}$$

Let $B = [f(A)]_{\mathfrak{Z}}$. Then $B \geqq f([A]_{\mathfrak{Z}})$ by (A.20). If $B \neq f([A]_{\mathfrak{Z}})$, there exists $\delta > 0$ such that the spectral projection F on $B - f([A]_{\mathfrak{Z}})$ for (∞, δ) is non-zero. There also exists a spectral projection F_1 of $[A]_{\mathfrak{Z}} F$ such that $F_1 \leqq F$, $F_1 \neq 0$ and $[A]_{\mathfrak{Z}} F_1$ is bounded. There exists $\varepsilon > 0$ such that $f(x + \varepsilon) \leqq f(x) + (\delta/2)$ for $x \in [0, \|[A]_{\mathfrak{Z}} F_1\|]$. We then have $f(A) \geqq B \geqq f([A]_{\mathfrak{Z}} + \varepsilon F_1)$ and by monotonicity of f on $[0, \infty]$, we obtain $A \geqq [A]_{\mathfrak{Z}} + \varepsilon F_1$ with $F_1 \in \mathfrak{Z}$ which is a contradiction.

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