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On Weakly 1-Complete Surfaces without Non-Constant Holomorphic Functions

By

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Introduction

In this paper, we are interested in a class of two dimensional complex manifolds which are called weakly 1-complete surfaces. Here we call a two dimensional complex manifold X a weakly 1-complete surface if X possesses a C^{∞}-exhausting plurisubharmonic function. This class includes two different extreme objects: compact analytic surfaces and two dimensional Stein manifolds. But at the same time, this class includes some curious examples from the function theoretic point of view i.e. there are weakly 1-complete surfaces without non-constant holomorphic functions (see [3] [6] [8] [12]) and moreover non-compact weakly 1-complete surfaces have an extreme function theoretic property i.e. a non-compact weakly 1-complete surface X is holomorphically convex if and only if X possesses a non-constant holomorphic function (see [9]). Looking back to the case of compact analytic surfaces, roughly speaking, they are classified by the existence or non-existence of meromorphic function. Hence it is natural to suppose that this aspect might give a new standpoint to analyze such a curious example in the class of weakly 1-complete surfaces as far as weakly 1-completeness is expected as a nice intermediate concept between compactness and Stein. This note is an attempt towards the problem of the existence of meromorphic function on non-compact weakly 1-complete surfaces. From now on, all weakly 1-complete surfaces are connected and non-compact and have no exceptional compact curves of the first kind unless otherwise is explicitly stated. Then we shall prove the following theorem.

Main theorem. *Let X be a weakly 1-complete surface without non-constant holomorphic functions and let Φ be a C^{∞}-exhausting plurisubharmonic*

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function on X. If X possesses a non-constant meromorphic function, then each sublevel set $X_c=\{x\in X\mid \Phi(x)<c\}$ is projectively embeddable. Moreover if X contains no exceptional compact curves, then X is projectively embeddable if and only if X possesses a non-constant meromorphic function.

Up to the present, the known examples of weakly 1-complete surfaces without non-constant holomorphic functions contain no exceptional compact curves. But it is still unknown if there is a weakly 1-complete surface without non-constant meromorphic functions (see also [3] [6] [8] [12]).

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1. The following lemma was proved by Ohsawa (see [9] Theorem 1.1).

Lemma 1. *Let X be a weakly 1-complete surface and let $f: X\to \boldsymbol{P}^1$ be a holomorphic map, where $\boldsymbol{P}^1$ is the complex projective line. Then either*

i) *$f^{-1}(p)\cap X_c$ is empty or non-compact for any $p\in \boldsymbol{P}^1$ and $c\in \boldsymbol{R}$*

or

ii) *$f^{-1}(p)\cap X_c$ is compact for any $p\in \boldsymbol{P}^1$ and $c\in \boldsymbol{R}$.*

For a complex manifold Y, we denote by $\mathcal{O}(Y)$ the ring of holomorphic functinos on Y.

Proposition 2. *Let X be a weakly 1-complete surface and assume $\mathcal{O}(X)\cong \boldsymbol{C}$. If there exists a non-constant meromorphic function f on X, then the fibres of f are non-compact.*

Proof. Let Σ be the set of indeterminacy of f. If $\Sigma\neq\emptyset$, by taking a finite number of iterated quadratic transformations with centres at points of Σ, we obtain a complex manifold Y and a proper surjective holomorphic map h: $Y\to X$ such that $\mathcal{O}(Y)\cong \boldsymbol{C}$ and the map $g=h^*f$: $Y\to \boldsymbol{P}^1$ is holomorphic. Then Y is a weakly 1-complete surface because, if we let Φ be a C^∞-exhausting plurisubharmonic function on X, we can take a function $h^*\Phi$ as a C^∞-exhausting plurisubharmonic function on Y. In this case, it sufficies to show that the fibres of g are non-compact. Hence we may assume that f: $X\to \boldsymbol{P}^1$ is holomorphic. Then i) or ii) of Lemma 1 holds. If ii) holds, the connected components of $f^{-1}(p)\cap X_c$ are compact. Hence from Theorem 3 in [1], the equivalence relation R defined by these connected components is proper and the quotient space X/R is an analytic space. Let $p: X \to X/R$ be the natural projection. Then there exists a holomorphic map $q: X/R \to \mathbf{P}^1$ such that q is finite and $f = q \circ p$. Combining this with the fact that X is non-compact and p is proper, we can conclude that X/R is a one dimensional analytic space. Since X/R is holomorphically convex and p is proper, X is holomorphically convex i.e. $\mathcal{O}(X) \not\cong \mathbf{C}$. This is a contradiction. q.e.d.

As a consequence of Proposition 2, we obtain the following

Theorem 3. *Let X be a weakly 1-complete surface and assume $\mathcal{O}(X) \cong \mathbf{C}$. Let f be a non-constant meromorphic function on X. Then $X_c \backslash P(f)$ is 1-convex for every $c \in \mathbf{R}$, where $P(f)$ is the pole divisor of f.*

Proof. Let Φ be a C^∞-exhausting plurisubharmonic function on X and let Σ be the set of indeterminacy of f. By taking a finite number of quadratic transformations with centres at points of Σ, we obtain a complex manifold Y and a proper surjective holomorphic map $h: Y \to X$ such that Y is a weakly 1-complete surface with respect to $h^*\Phi$, $\mathcal{O}(Y) \cong \mathbf{C}$ and the map $g = h^*f: Y \to \mathbf{P}^1$ is holomorphic. Let Ξ_1 (resp. Ξ_2) be the total (resp. strict) transformation of the pole divisor $P(f)$ of f. Then it holds that $g^{-1}(\infty)$ ($\mathbf{P}^1 = \{\infty\} \cup \mathbf{C}$) coincides with Ξ_2. Let $\{U_i\}_{i=0,1}$ be the standard covering of $\mathbf{P}^1$ with the local coodinate (z) in U_0 and (w) in U_1 respectively and $Y_c \backslash \Xi_2$ is weakly 1-complete with respect to $(\exp(c) - \exp(h^*\Phi))^{-1} + g^*(|z|^2)$, where $Y_c = \{y \in Y \mid h^*\Phi(y) < c\}$. Since $Y_c \backslash \Xi_2$ is a weakly 1-complete surface and $g|_{Y_c \backslash \Xi_2}$ is a holomorphic map whose fibres are non-compact (Proposition 2), $Y_c \backslash \Xi_2$ is holomorphically convex by [9] Proposition 1.4. On the other hand, the critical values of $g|_{Y_c}$ are finitely many and $g|_{Y_c \backslash \Xi_2}$ is constant on every connected curve contained in $Y_c \backslash \Xi_2$. Hence $Y_c \backslash \Xi_2$ contains only finitely many connected compact curves. This fact implies that $Y_c \backslash \Xi_2$ is obtained from a two dimensional Stein space by blowing up a finite number of points to compact curves. Hence $Y_c \backslash \Xi_2$ is 1-convex. Next we assert that $Y_c \backslash \Xi_1$ is 1-convex. To prove this, we use the following assertion (see [11] lemma 2):

*) *The complement of a Stein divisor of a two dimensional 1-convex manifold is 1-convex.*

The set of indeterminacy Σ of f in $X_c = \{x \in X \mid \Phi(x) < c\}$ is a finite points set of X_c and Ξ_2 meets each connected component of $\Xi_3 = h^{-1}(\Sigma \cap X_c)$. Let D_1 be the union of the irreducible components of Ξ_3 which meet Ξ_2 and $E_1 = D_1 \backslash \Xi_2$.

Then E_1 is a Stein divisor in $Y_c \backslash \Xi_2$, hence from $*$) $(Y_c \backslash \Xi_2) \backslash E_1$ is 1-convex. Let D_2 be the union of the irreducible components of Ξ_3 which meet D_1 and $E_2 = D_2 \backslash D_1$. Then E_2 is a Stein divisor in $(Y_c \backslash \Xi_2) \backslash E_1$. Hence $(Y_c \backslash \Xi_2) \backslash (E_1 \cup E_2)$ is 1-convex. We continue this process. Since Ξ_2 meets each connected component of Ξ_3, there exists an integer k_0 such that $D_{k_0} = \Xi_3$. Hence $(Y_c \backslash \Xi_2) \backslash \cup_{i=1}^{k_0} E_i = (Y_c \backslash \Xi_2) \backslash (D_{k_0} \backslash \Xi_2) = Y_c \backslash \Xi_1$ is 1-convex. So our assertion holds. Since $Y_c \backslash \Xi_1$ is isomorphic to $X_c \backslash P(f)$, $X_c \backslash P(f)$ is 1-convex. q. e. d.

2. Let M be a complex manifold or space and $\pi: \boldsymbol{B} \to M$ be a holomorphic line bundle over M with trivializing covering $\{U_i\}$ and transition functions $\{b_{ij}\}$. $\boldsymbol{B}$ is said to be positive (resp. semi-positive) on a subset Y of M if there exists a metric $\{a_i\}$ along the fibres of $\boldsymbol{B}$ i.e. a system of positive C^∞-functions a_i on U_i satisfying $a_i |b_{ij}|^2 = a_j$ on $U_i \cap U_j$, such that $-\log a_i$ is strictly plurisubharmonic (resp. plurisubharmonic) on every $U_i \cap Y$.

Proposition 4. *Let X be a weakly 1-complete surface and assume $\mathcal{O}(X) \cong \boldsymbol{C}$. Let f be a non-constant meromorphic function on X and let $\boldsymbol{F}$ be the line bundle on X determined by the pole divisor $P(f)$ of f. Then, for any $c \in \boldsymbol{R}$, $\boldsymbol{F}$ is semi-positive on X_c and positive outside a compact subset K_c of X_c.*

Proof. We may assume that the pole divisor $P(f)$ of f contains no compact components. Take a real number c' with $c' > c$. Then $D' = X_{c'} \cap P(f)$ is a union of one dimensional analytic spaces $\{D'_1, \ldots, D'_m\}$. By the theorem of Richberg (see [10] Satz 3.3), there exist a neighborhood W'_k of D'_k in $X_{c'}$ and a C^∞-strictly plurisubharmonic function μ'_k on W'_k such that the restriction of μ'_k onto D'_k coincides with a C^∞-strictly plurisubharmonic function on D'_k $(1 \leqq k \leqq m)$. For some real number d with $c < d < c'$ and k, we take a neighborhood W_k of $D'_k \cap \bar{X}_d$ and a C^∞-function χ_k on X such that $W_k \Subset W'_k$, $0 \leqq \chi_k \leqq 1$, $\operatorname{supp} \chi_k \Subset W'_k$ and $\chi_k = 1$ on $\bar{W}_k \cap \bar{X}_d$. Put $W = \cup_{k=1}^m W_k$, then W is a neighborhood of the closure of $D = D' \cap X_c$. On the other hand, there exist a finite covering $\{V_i\}_{1 \leqq i \leqq n}$ of $\bar{D}$ and a family of holomorphic functions $\{\sigma_i\}_{1 \leqq i \leqq n}$ such that 1) if $V_i \subset W_{k(i)}$, $V_i \Subset W_{k(i)}$ 2) $V_i \cap \bar{D} = \{\sigma_i = 0\}$. Put $V = \cup_{i=1}^n V_i$, then $D \Subset V \Subset W$. If $V_i \cap V_j \neq \emptyset$, we set $F_{ij} = \sigma_i / \sigma_j$, then F_{ij} is a nowhere vanishing holomorphic function on $V_i \cap V_j$. Then there exists a family of positive C^∞-functions $\{a_i\}_{1 \leqq i \leqq n}$ such that $|F_{ij}|^2 = a_i \cdot a_j^{-1}$ on $V_i \cap V_j$. We set $A'_i = a_i \cdot \exp \{C \cdot (\sum_{k=1}^m \chi_k \cdot \mu'_k)\}$ on V_i, where C is a positive constant. Since $V \Subset W$, if C is large enough, $\log A'_i$ is a C^∞-strictly plurisubharmonic function on V_i. Put

$A_0'=A_i'/|\sigma_i|^2$ on V_i, then A_0' is a positive function on V. Moreover $\log A_0'$ is C^∞-strictly plurisubharmonic on $V\backslash D'$ and tends to infinity on $V\cap D'$. We set $V_0=X_{c'}\backslash D'$ and define $F_{i0}=\sigma_i$ on $V_0\cap V_i$. Then $\{F_{ij}\}_{0\leqq i,j\leqq n}$ define a system of transition functions for $\boldsymbol{F}$ on $V\cup V_0$. We take real constants b_1, b_2 and b_3 so that $b_1>b_2>b_3>0$ and $\{x\in V|\, b_3/2\leqq\log A_0'(x)\leqq\infty\}\cap\bar{X}_c\Subset V$. We choose a C^∞-function $\lambda(t)\colon(-\infty,\infty)\to(-\infty,\infty)$ such that $\lambda(t)=b_2$ if $t\leqq b_3$, $\lambda'(t)>0$, $\lambda''(t)>0$ if $t\in(b_3,b_1)$ and $\lambda(t)=t$ if $t\geqq b_1$. We put $\mu(x)=\lambda(\log A_0'(x))$ for $x\in V$ and extend $\mu(x)=b_2$ for $x\in X_{c+\delta}\backslash V\ (0<\delta\ll 1)$. Then the function μ is plurisubharmonic on $X_{c+\delta}\backslash D'$ and strictly plurisubharmonic on $\{x\in V|\, b_1<\log A_0'(x)<+\infty\}\cap X_{c+\delta}$. From Theorem 3, $X_{c'}\backslash D'$ is 1-convex. Hence there exists a C^∞-plurisubharmonic function θ on $X_{c'}\backslash D'$ which is strictly plurisubharmonic on $X_{c'}\backslash(D'\cup M_{c'})$, where $M_{c'}$ is the maximal compact subvariety of $X_{c'}\backslash D'$. We take a C^∞-function τ on X such that $0\leqq\tau\leqq 1$, $\tau=1$ on $\bar{X}_c\backslash\{x\in V|\, 2b_1<\log A_0'(x)\leqq+\infty\}$ and $\operatorname{supp}\tau\cap\bar{X}_c\cap\{x\in V|\log A_0'(x)=3b_1\}=\emptyset$. Since μ is strictly plurisubharmonic on $\{x\in V|\, b_3<\log A_0'(x)<3b_1\}\cap X_{c+\delta}$, if $\varepsilon>0$ is small enough, $\varepsilon\cdot\tau\cdot\theta+\mu$ is plurisubharmonic on $X_c\backslash D$ $(D=X_c\cap D')$ and strictly plurisubharmonic on $X_c\backslash(D\cup M_{c'})$ (we may assume that $V\cap M_{c'}=\emptyset$). We put $A_0=\exp(\varepsilon\cdot\tau\cdot\theta+\mu)$ on $X_c\backslash D$. Then A_0 coincides with the original A_0' near D. We set $A_i=A_0\cdot|\sigma_i|^2$ on $V_i\cap X_c$. Then it is easily verified that $\{A_i^{-1}\}$ is a metric of $\boldsymbol{F}=\{F_{ij}\}$ on X_c and $-\log A_i^{-1}$ is strictly plurisubharmonic on $V_i\cap X_c$ if $i\in\{1,\dots,n\}$, plurisubharmonic on $V_i\cap(X_c\backslash D)$ and strictly plurisubharmonic on $V_0\cap(X_c\backslash(D\cup M_{c'}))$. This implies that $\boldsymbol{F}$ is semi-positive on X_c and positive outside the maximal compact subvariety M_c of $X_c\backslash P(f)$.

q.e.d.

Theorem 5. *Let X be a weakly 1-complete surface and assume $\mathcal{O}(X)\cong\mathbf{C}$. If X possesses a non-constant meromorphic function f, then there exists a positive line bundle on each sublevel set X_c and so X_c is realized as a locally closed subspace of a complex projective space.*

Proof. We have only to prove the former assertion since the latter one follows from [2] Lemma 3. From Theorem 3, $X_c\backslash P(f)$ is 1-convex. Let M_c be the maximal compact subvariety of $X_c\backslash P(f)$. First we assume that M_c is connected. Let $\{M_{c,i}\}_{1\leqq i\leqq n}$ be the irreducible components of M_c. Since M_c is exceptional in X_c, the intersection matrix $(M_{c,i}\cdot M_{c,j})$ is negative definite and $M_{c,i}\cdot M_{c,j}\geqq 0$ if $i\neq j$. Hence there exist natural numbers $r_1,\dots,r_n$ such that $\sum_{i=1}^n r_iM_{c,i}\cdot M_{c,j}<0$ for $1\leqq j\leqq n$. Let p_i be the ideal sheaf of $M_{c,i}$ and set

$\mathscr{I}=p_1^{r_1}\cdots p_n^{r_n}$. Let $\boldsymbol{L}$ be the line bundle over X_c corresponding to the invertible sheaf $\mathscr{I}$ and let μ_i: $\hat{M}_{c,i}\to M_{c,i}$ be the normalization of $M_{c,i}$. By the choice of natural numbers $r_1,\dots, r_n$, $\mu_i^*(\boldsymbol{L}|_{M_{c,i}})$ is positive over $\hat{M}_{c,i}$. Since each μ_i is a finite map, $\boldsymbol{L}|_{M_{c,i}}$ and so $\boldsymbol{L}|_{M_c}$ is positive over M_c. In a suitable manner, we extend the metric $\{a_i\}$ which gives the positivity of $\boldsymbol{L}|_{M_c}$ over M_c to a metric $\{A'_i\}$ of $\boldsymbol{L}|_W$, where W is a neighborhood of M_c. Let θ be a C^∞-plurisubharmonic function on $X_c\backslash P(f)$ which is strictly plurisubharmonic on $X_c\backslash(P(f)\cup M_c)$, then $\{A_i=A'_i\cdot\exp(-C\theta)\}$ becomes a new metric of $\boldsymbol{L}|_W$. If C is large enough, $\boldsymbol{L}|_W$ is positive over W. Hence by the same way as in the proof of Proposition 4, we can conclude that $\boldsymbol{L}$ is semipositive outside a compact neighborhood K of M_c and positive on a neighborhood W of M_c with $W\subset \mathrm{Int}\ K$. If M_c is not connected, it has a finite number of connected components, and by applying the above argument to each connected component and tensoring the line bundles obtained as the consequence, we see that there exists a line bundle $\boldsymbol{L}$ over X_c such that $\boldsymbol{L}$ is semi-positive outside a compact neighborhood K of M_c and positive on a neighborhood W of M_c with $W\subset \mathrm{Int}\ K$. Since from Proposition 4 the line bundle $\boldsymbol{F}$ corresponding to the pole divisor of f is semi-positive on X_c and positive on $X_c\backslash M_c$, $\boldsymbol{F}^{\otimes m}\otimes\boldsymbol{L}$ is a positive line bundle on X_c if m is large enough. Hence our assertion holds. q. e. d.

In case X contains no exceptional compact curves, we can prove the following theorem which has been suggested by T. Ohsawa.

Theorem 6. *Let X be a weakly 1-complete surface without non-constant holomorphic functions and assume that X contains no exceptional compact curves. Then the following three conditions are equivalent:*

1) *X possesses a non-constant meromorphic function,*
2) *X possesses a holomorphic line bundle $\boldsymbol{F}$ on X such that $\boldsymbol{F}|_{X_c}$ is positive for every $c\in\boldsymbol{R}$,*
3) *X is projectively embeddable i.e. X is realized as a locally closed subspace of a complex projective space.*

Proof. 1)→2) follows from Proposition 4 and 3)→1) is clear. Hence we have only to prove 2)→3). First we prove the following assertion:

a) *There exists a positive integer m_0 such that $\dim_{\boldsymbol{C}}\Gamma(X, \mathcal{O}(\boldsymbol{F}^{\otimes m}\otimes\boldsymbol{K}_X))\geqq 2$ for every $m\geqq m_0$, where $\boldsymbol{K}_X$ is the canonical line bundle of X.*

For a real number c, we take a point $x_0\in X_c$. Let h: $\hat{X}_c\to X_c$ be the quadratic transformation at x_0 and let $\boldsymbol{L}$ be the line bundle on $\hat{X}_c$ corresponding to the divisor $h^{-1}(x_0)$. Then $\hat{X}_c$ is weakly 1-complete and there exists a positive integer m_0 such that $h^*\boldsymbol{F}^{\otimes m}\otimes\boldsymbol{L}^{*\otimes 3}$ is positive on $\hat{X}_c$ for every $m\geqq m_0$. When we denote the canonical line bundle of $\hat{X}_c$ by $\boldsymbol{K}_{\hat{X}_c}$, from Nakano's vanishing theorem (see [7] Theorem 1), we have $H^1(\hat{X}_c, \mathcal{O}(h^*\boldsymbol{F}^{\otimes m}\otimes\boldsymbol{L}^{*\otimes 3}\otimes\boldsymbol{K}_{\hat{X}_c}))=0$ for every $m\geqq m_0$. By using the adjunction formula $\boldsymbol{K}_{\hat{X}_c}=h^*\boldsymbol{K}_X\otimes\boldsymbol{L}$, we have $H^1(\hat{X}_c, \mathcal{O}(h^*\boldsymbol{E}_m\otimes\boldsymbol{L}^{*\otimes 2}))=0$ for $\boldsymbol{E}_m=\boldsymbol{F}^{\otimes m}\otimes\boldsymbol{K}_X$ and every $m\geqq m_0$. Finally we obtain $H^1(X_c, \boldsymbol{I}_{x_0}^2\otimes\mathcal{O}(\boldsymbol{E}_m))=0$ for every $m\geqq m_0$, where $\boldsymbol{I}_{x_0}$ is the maximal ideal sheaf associated to $\{x_0\}$. Hence we obtain that the restriction homomorphism $\rho\colon \Gamma(X_c, \mathcal{O}(\boldsymbol{E}_m))\to\Gamma(\{x_0\}, \mathcal{O}/\boldsymbol{I}_{x_0}^2\otimes\mathcal{O}(\boldsymbol{E}_m))$ is surjective for every $m\geqq m_0$. Since $\dim_{\boldsymbol{C}}\Gamma(\{x_0\}, \mathcal{O}/\boldsymbol{I}_{x_0}^2\otimes\mathcal{O}(\boldsymbol{E}_m))=3$, we have $\dim_{\boldsymbol{C}}\Gamma(X_c, \mathcal{O}(\boldsymbol{E}_m))\geqq 2$ for every $m\geqq m_0$. On the other hand, from [13] Lemma 5.4, we have that the restriction homomorphism $r\colon \Gamma(X_d, \mathcal{O}(\boldsymbol{E}_m))\to\Gamma(X_e, \mathcal{O}(\boldsymbol{E}_m))$ has a dense image with respect to the topology of uniform convergence on compact subsets for every $m\geqq 1$ and real numbers d and e with $d>e$. Using this, we have $\dim_{\boldsymbol{C}}\Gamma(X, \mathcal{O}(\boldsymbol{E}_m))\geqq 2$ for every $m\geqq m_0$.

Since X contains no exceptional compact curves, from a), Theorem 3 and Proposition 4, we obtain that $\boldsymbol{E}_m|_{X_c}$ is positive for every $c\in\boldsymbol{R}$ and $m\geqq m_0$. Hence combining Nakano's vanishing theorem with [13] Lemma 5.4, we obtain the following global vanishing theorem:

b) $$H^1(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))=0 \quad \textit{for every } m\geqq m_0 \quad \textit{and} \quad n\geqq 1.$$

Secondly we prove the following assertion.

c) For every $m\geqq m_0$ and $n\geqq 2$, there exist elements φ_0 and φ_1 of $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))$ such that the map $f\colon X\to\boldsymbol{P}^1$ defined by the quotient of φ_0 and φ_1 is holomorphic.

From a), we have $\dim_{\boldsymbol{C}}\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))\geqq 2$ for every $m\geqq m_0$ and $n\geqq 1$. We fix two integers m and n with $m\geqq m_0$ and $n\geqq 2$ and take elements ψ_0 and ψ_1 of $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))$. From Proposition 2, there exist complex numbers a_0 and a_1 such that the divisor D defined by $a_0\psi_1+a_1\psi_0$ is Stein. We set $\varphi_0=a_0\psi_1+a_1\psi_0$. We consider an exact sequence $0\to\mathcal{O}(\boldsymbol{E}_m^{\otimes n-1})\to\mathcal{O}(\boldsymbol{E}_m^{\otimes n})\to\mathcal{O}_D(\boldsymbol{E}_m^{\otimes n})\to 0$. From this and b), we obtain that the restriction homomorphism $\rho\colon \Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))\to\Gamma(D, \mathcal{O}_D(\boldsymbol{E}_m^{\otimes n}))$ is surjective. Since D is a one dimensional Stein space, we obtain that the second singular cohomology group $H^2(D, Z)$ of D and $H^1(D, \mathcal{O})$ vanish. Hence any holomorphic line bundle on D is analytically trivial. Combining this fact with the surjectivity of ρ, we have that there exists an element

φ_1 of $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes n}))$ such that φ_1 nowhere vanishes on D. Hence φ_0 and φ_1 are the desired elements.

Thirdly we prove the following assertion.

d) $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}))$ *separates points of* X *and gives local coordinates at each point of* X *for every* $m \geqq m_0$.

We fix an integer m with $m \geqq m_0$. Let φ_0 and φ_1 be the elements of $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}))$ which realize the situation of c). We consider the set $L_m = \{(\varphi) \mid \varphi = a_0\varphi_1 + a_1\varphi_0$ and $a_0, a_1 \in \boldsymbol{C}\}$, where (φ) is the divisor defined by φ. Since $f = \varphi_1/\varphi_0 \colon X \to \boldsymbol{P}^1$ is holomorphic, if an element of L_m contains compact curves, each connected component of them is exceptional in X. This contradicts the assumption. Hence each element of L_m contains no compact curves and so Stein. For every point x of X, there exists an element D_x of L_m passing through x. Let $\mathscr{I}_{D_x}$ be the ideal sheaf associated to the divisor D_x. Let y be a point of X which is different from x. Then if $y \notin D_x$, from the choice of φ_0 and φ_1, $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}))$ separates points x and y. Hence concerning the property of separating points, we have only to prove the case $y \in D_x$. We consider the following two exact sequences: $0 \to \mathcal{O}(\boldsymbol{E}_m^{\otimes 2}) \to \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}) \to \mathcal{O}/\mathscr{I}_{D_x} \otimes \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}) \to 0$ and $0 \to \mathcal{O}(\boldsymbol{E}_m) \to \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}) \to \mathcal{O}/\mathscr{I}_{D_x}^2 \otimes \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}) \to 0$. From b), restriction homomorphisms $\rho_i \colon \Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes 3})) \to \Gamma(D_x, \mathcal{O}/\mathscr{I}_{D_x}^i \otimes \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}))$ $(i=1, 2)$ are surjective. Since D_x is Stein, the surjectivity of ρ_i $(i=1, 2)$ implies the assertion d).

From the assertion d), we can apply the standard argument of Whitney type (see for example, Hörmander [4] Chap. V, §3). Hence for $N \geqq 5$, we can choose $N+1$ elements $\{\varphi_0, \varphi_1, \dots, \varphi_N\}$ of $\Gamma(X, \mathcal{O}(\boldsymbol{E}_m^{\otimes 3}))$ in such a way that the map $X \ni x \mapsto (\varphi_0(x) \colon \varphi_1(x) \colon \cdots \colon \varphi_N(x)) \in \boldsymbol{P}^N$ is holomorphic, one-to-one and of maximal Jacobian rank on X. q.e.d.

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