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A Stein Domain with Smooth Boundary Which Has a Product Structure

By

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It is well known that the unit ball in $\mathbb{C}^2$ is not biholomorphically equivalent to the bidisc. Its first proof is due to H. Cartan [1], although it is customally called Poincaré's theorem. H. Rischel [3] extended this theorem by proving that there exists no surjective proper holomorphic map from a strictly pseudoconvex domain $\boldsymbol{B}$ to a product domain $\boldsymbol{E}$. Recently, A. Huckleberry and E. Ormsby [2] generalized it to the case where $\boldsymbol{B}$ is a bounded domain in $\mathbb{C}^n$ with smooth boundary and $\boldsymbol{E}$ is the total space of a holomorphic fiber bundle. It will be natural to ask whether we can generalize it to the case where $\boldsymbol{B}$ is a relatively compact Stein domain with smooth boundary in a complex manifold and $\boldsymbol{E}$ is the total space of a fiber bundle. The purpose of the present note is to show an example of a Stein domain $\boldsymbol{B}$ in a compact complex manifold which has the following properties.

1) The boundary of $\boldsymbol{B}$ is smooth.

2) $\boldsymbol{B}$ is biholomorphically equivalent to the product of two Stein manifolds.

Let $\mathcal{A}$ be an elliptic curve which is isomorphic to $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$. We denote the points of $\mathcal{A}$ by $[z]$, where $z\in\mathbb{C}$. Let $\boldsymbol{B}$ be a domain in $\mathcal{A}\times\mathbb{P}^1$ defined by

$$\boldsymbol{B}:=\{([z],\zeta)\in\mathcal{A}\times\mathbb{P}^1 \mid \mathrm{Re}\,(\zeta\exp(2\pi i\,\mathrm{Re}\,z))<0\}\,.$$

Here ζ denotes an inhomogeneous coordinate of $\mathbb{P}^1$. Clearly, $\boldsymbol{B}$ is well-defined and the boundary of $\boldsymbol{B}$ is smooth. Note that $\boldsymbol{B}$ is contained in $\mathcal{A}\times\mathbb{C}^*$, where $\mathbb{C}^*:=\mathbb{P}^1\backslash\{\zeta=0,\,\infty\}$, and that we have a biholomorphic map

$$\begin{array}{ccc} \mathcal{A}\times\mathbb{C}^* & \longrightarrow & \mathcal{A}\times\mathbb{C}^* \\ \Downarrow & & \Downarrow \\ ([z],\zeta) & \longmapsto & \left(\left[z+\frac{i}{2\pi}\cdot\ln\frac{i}{\zeta}\right],\,\zeta\right). \end{array}$$

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Clearly, $\mathrm{Re}\left(\zeta \exp\left(2\pi i \,\mathrm{Re}\left(z - \frac{i}{2\pi}\cdot \ln \frac{i}{\zeta}\right)\right)\right) = 0$ if and only if $\mathrm{Re}\, z$ is an integral multiple of 1/2. Hence,

$$\boldsymbol{B} \simeq \{[z] \in \boldsymbol{A} \mid 0 < \mathrm{Re}\, z < 1/2\} \times \mathbf{C}^*,$$

so that $\boldsymbol{B}$ is the product of an annulus and the punctured plane.

Remark. Note that the boundary of $\boldsymbol{B}$ is everywhere pseudoflat.

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References

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Added in proof: Recently, Diederich and Fornaess constructed in an analoguous way a domain with smooth pseudoconvex boundary D in $\mathbf{P}^1 \times$Hopf surface, and showed that D cannot be exhausted by pseudoconvex subdomains. In fact, D is biholomorphic to $(\mathbf{C}^2 \setminus \{0\}) \times$ an annulus!