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Equivariant Stable Homotopy Theory and Idempotents of Burnside Rings

By

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Introduction

Let G be a *finite* group throughout the present work. We denote by $A(G)$ the Burnside ring of G. The stable G-homotopy theory is a G-homology-cohomology theory of $A(G)$-modules and any idempotent of $A(G)$ decomposes it as a direct sum of G-homology-cohomology theories. Such a decomposition for p-localized case was partly investigated by Kosniowski [13] and tom Dieck [7].

Let X and Y be pointed G-CW complexes. We assume X to be finite. The group of stable G-maps from X to Y is denoted by $\tilde{\omega}_G^0(X: Y)$. We put $\tilde{\omega}_G^\alpha(X: Y)=\tilde{\omega}_G^0(\Sigma^V X: \Sigma^U Y)$ for $\alpha=U-V\in RO(G)$. We study $e\tilde{\omega}_G^\alpha(X: Y)$ for each primitive idempotent e of $A(G)$. Denote by P the set of all conjugacy classes of perfect subgroups of G. Primitive idempotents of $A(G)$ correspond bijectively with members of P, Dress [9]. Denote by e_H the primitive idempotent of $A(G)$ corresponding to $(H)\in P$, then

$$\tilde{\omega}_G^\alpha(X: Y)=\coprod_{(H)\in P} e_H\tilde{\omega}_G^\alpha(X: Y).$$

Let H be a perfect subgroup of G. We denote $N=N_G(H)$ and $W=N_G(H)/H$ for simplicity. The main result of the present work is the following.

Theorem A. *There hold the isomorphisms*

$$e_H\tilde{\omega}_G^\alpha(X: Y)\cong\bar{e}_H\tilde{\omega}_N^{\alpha'}(X: Y)\cong\tilde{e}_{\langle 1\rangle}\tilde{\omega}_W^{\alpha''}(X^H: Y^H)$$

which are $e_HA(G)$- and $\bar{e}_HA(N)$-module isomorphisms respecitvely, where $\bar{e}_H$ and $\tilde{e}_{\langle 1\rangle}$ denote the primitive idempotents of $A(N)$ and $A(W)$ corresponding to $(H)_N$ and the trivial perfect subgroup $\{1\}$ of W respecitvely, $\alpha'=\mathrm{res}_N^G\alpha$ and

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$\alpha''=U^H-V^H$ *for* $\alpha'=U-V\in RO(N)$.

Corollary B. *There hold the ring isomorphisms*

$$e_H A(G)\cong \bar{e}_H A(N)\cong \tilde{e}_{\langle 1\rangle} A(W).$$

Direct proof of this corollary is not difficult. T. Miyata and T. Yoshida communicated to the author relatively short direct proofs of this corollary.

Theorem A hold also for any G-homology-cohomology theories defined by G-spectra. The p-localized version of Theorem A is also ture. In fact we prove the more generalized version of Theorem A (Theorem 3.6, (3.7) and Theorem 4.7). We obtain Theorem A by specializing $\pi=\{\text{all primes}\}$ and the p-localized version by $\pi=\{p\}$.

In Section 1 we observe certain relations between primitive idempotnents of $A(G)_{(\pi)}$ and $A(N)_{(\pi)}$, and their behaviors in Mackey double coset formula. The explicit formula (1.2) for primitive idempotents due to Yoshida [17] is essential. In Section 2 we prove an isomorphism theorem (Theorem 2.5) for Mackey functors on the category $\hat{G}$ of finite G-sets. In Section 3 we see briefly that stable G-homotopy theory provides Mackey functors on $\hat{G}$, then we obtain the first isomorphism of Theorem A (Theorem 3.6 and (3.7)) by applying Theorem 2.5. In Section 4 we construct the fixed-point exact sequences for stable G-homotopy theory and prove the second isomorphism of Theorem A (Theorem 4.7).

§1. Idempotents of Burnside Rings

Let $\hat{G}$ be the category of finite G-sets and G-maps. The set of all isomorphism classes in G forms a commutative semi-ring $A^+(G)$ with addition and multiplication defined by disjoint unions and direct products (with diagonal G-actions) respectively. The *Burnside ring* of G, denoted by $A(G)$, is the Grothendieck ring of $A^+(G)$. A finite G-set S represents an element of $A(G)$, denoted by $[S]$. Then every element of $A(G)$ can be expressed in the form $[S]-[T]$. Every finite G-set is expressed uniquely as the disjoint union of ts orbits, which implies that $A(G)$ is additively a free Z-module with basis $\{[G/L];(L)\in C(G)\}$, where $C(G)$ denotes the set of conjugacy classes of subgroups of G. As to the basic properties of $A(G)$ we refer to [8] [9] [10].

Let π be a set of primes and $\boldsymbol{Z}_{(\pi)}$ the subring of Q consisting of all fractions a/b such that $(a, b)=1$ and b is prime to every member of π. Thus, $\boldsymbol{Z}_{(\pi)}=\boldsymbol{Q}$ in case $\pi=\emptyset$; $\boldsymbol{Z}_{(\pi)}=\boldsymbol{Z}$ in case $\pi=\{\text{all primes}\}$; $\boldsymbol{Z}_{(\pi)}=\boldsymbol{Z}_{(p)}$ in case $\pi=\{p\}$, the set consisting of a single prime p. We write $A_{(\pi)}=A\otimes \boldsymbol{Z}_{(\pi)}$ for any abelian group A. Let $G\geqq L$, a subgroup. The assignment "$S\mapsto|S^L|$" defines a semi-ring homomorphism $A^+(G)\to Z$ and induces the ring homomorphism

$$\phi_L\colon A(G)_{(\pi)}\longrightarrow Z_{(\pi)},$$

which is important in studying structure of $A(G)_{(\pi)}$ [8] [9] [17]. E.g., $A(G)_{(\pi)}\ni x=0\Leftrightarrow\phi_L(x)=0$ for all $L\leqq G$.

Primitive idempotents of $A(G)_{(\pi)}$ are discussed in [8] [9] [11] [17]. Following [17] we denote by $S^\pi(G)$ the minimal normal subgroup of G by which the quotient is a solvable π-group. $S^\pi(G)$ is the uniquely determined characteristic subgroup of G [9]. G is called to be *π-perfect* provided $S^\pi(G)=G$. When $\pi=\{\text{all primes}\}$, π-perfect groups are perfect groups.

$S^\pi(G)$ is always π-perfect as $S^\pi(S^\pi(G))=S^\pi(G)$. Let P_π denote the set of all conjugacy classes of π-perfect subgroups of G. Primitive idempotents of $A(G)_{(\pi)}$ correspond bijectively with members of P_π [9] [17].

Let H be a π-perfect subgroup of G and e^π_H the primitive idempotent corresponding to the conjugacy class (H). Put

$$S_\pi(H,G)=\{L\leqq G;\ S^\pi(L)=H\}$$

following [17]. e^π_H is characterized by

$$(1.1)\qquad \phi_L(e^\pi_H)=\begin{cases}1 & \text{if } L\sim S_\pi(H,G)\\ 0 & \text{otherwise,}\end{cases}$$

where "$\sim$" means "conjugate to a member of" [8] [9] [17].

Recently an explicit formula for the idempotent e^π_H has been given by Yoshida [17]. (The formula for the case $\pi=\emptyset$ is given also by Gluck [11].) Let μ be the Möbius function on the subgroup lattice of G. For $D\leqq G$ he defines

$$\lambda(D,H)=\textstyle\sum_{L\in S_\pi(H,G)}\mu(D,L)$$

and obtains the explicit formula for e^π_H [17], Theorem 3.1, as follows:

$$(1.2)\qquad e^\pi_H=(1/|N_G(H)|)\cdot\textstyle\sum_{D\leqq N_G(H)}|D|\lambda(D,H)[G/D].$$

Let $K\leqq G$. Restricting G-actions to K on each finite G-set S, one obtains the ring homomorphism

$$\mathrm{res}^G_K\colon A(G)_{(\pi)}\longrightarrow A(K)_{(\pi)},$$

called the *restriction* homomorphism. Clearly

$$\phi_L(\mathrm{res}^G_K\,x)=\phi_L(x)$$

for $x \in A(G)_{(\pi)}$ and $L \leqq K$. The assignment "$S \mapsto G \times_K S$" for each finite K-set S induces the linear homomorphism

$$\mathrm{tr}_K^G \colon A(K)_{(\pi)} \longrightarrow A(G)_{(\pi)},$$

called the *transfer* homomorphism. By definition

$$\mathrm{tr}_K^G [K/L] = [G/L].$$

There holds the Frobenius formula

$$\mathrm{tr}_K^G (x \cdot \mathrm{res}_K^G y) = (\mathrm{tr}_K^G x) \cdot y$$

for $x \in A(K)_{(\pi)}$ and $y \in A(G)_{(\pi)}$. res_K^G maps idempotents to idempotents (which may be decomposable), whereas tr_K^G does not in general. Obviously $\mathrm{res}_G^G = \mathrm{tr}_G^G = \mathrm{id}$ for $K = G$.

Let H be a π-perfect subgroup of G and put $N = N_G(H)$, the normalizer of H in G. Let $\bar{e}_H^\pi$ denote the primitive idempotent of $A(N)_{(\pi)}$ corresponding to $(H)_N$, the conjugacy class of H in N, which we call the *central idempotent* of $A(N)_{(\pi)}$. It is characterized by

$$(1.3) \qquad \begin{aligned} \phi_L(\bar{e}_H^\pi) &= 1 \qquad \text{if} \quad L \in S_\pi(H, N) \\ &= 0 \qquad \text{otherwise}, \end{aligned}$$

since $H \lhd N$. (Compare with (1.1).) Remark that $S_\pi(H, G) = S_\pi(H, N)$ and $\lambda(D, H)$, $D \leqq N$, is the same for G and N. Since $N_G(H) = N_N(H)$, we compute by (1.2) as follows:

$$\begin{aligned} \mathrm{tr}_N^G \bar{e}_H^\pi &= (1/|N_N(H)|) \cdot \textstyle\sum_{D \leqq N_N(H)} |D| \lambda(D, H) \cdot \mathrm{tr}_N^G [N/D] \\ &= (1/|N_G(H)|) \cdot \textstyle\sum_{D \leqq N_G(H)} |D| \lambda(D, H) [G/D] \\ &= e_H^\pi, \end{aligned}$$

i.e., we obtain

$$(1.4) \qquad \mathrm{tr}_N^G \bar{e}_H^\pi = e_H^\pi.$$

$\mathrm{res}_N^G e_H^\pi$ is an idempotent of $A(N)_{(\pi)}$ and we see easily by (1.1) that it decomposes as a sum of primitive idempotents which correspond to conjugacy classes $(H')_N$ in N such that $H' \sim H$ in G. Such conjugacy classes correspond bijectively to a part of the double cosets $N\backslash G/N$. Let $\{g_1, \ldots, g_t\}$ be a complete system of representatives of $N\backslash G/N$. Choose a numeration of this system so that $i \leqq s \Leftrightarrow H_i = g_i H g_i^{-1} \leqq N$ (which does not depend on the choice of the representative g_i). Then $\{(H_i)_N, 1 \leqq i \leqq s\}$ forms the complete set of the above mentioned conjugacy classes $(H')_N$ in N. We choose $g_1 = 1$ always, then $H_1 = H$.

Let $\bar{e}_i^\pi$ denote the primitive idempotent of $A(N)_{(\pi)}$ which corresponds to $(H_i)_N$. $\bar{e}_1^\pi=\bar{e}_H^\pi$, the central idempotent of $A(N)_{(\pi)}$. And we obtain

$$\text{(1.5)} \qquad \mathrm{res}_N^G e_H^\pi = \textstyle\sum_{1\leq i\leq s} \bar{e}_i^\pi\,.$$

By (1.4) and (1.5) we see that

$$\mathrm{res}_N^G \circ \mathrm{tr}_G^N \bar{e}_H^\pi = \textstyle\sum_{1\leq i\leq s} \bar{e}_i^\pi\,.$$

Next we apply the Mackey decomposition to $\mathrm{res}_N^G \circ \mathrm{tr}_N^G$. Putting $N_i=N_G(H_i)$, $1\leqq i\leqq t$, we obtain

$$\text{(1.6)} \qquad \mathrm{res}_N^G \circ \mathrm{tr}_N^G = \textstyle\sum_{1\leq i\leq t} \mathrm{tr}_{N\cap N_i}^N \circ \mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*\,,$$

where $c_i^*\colon A(N)_{(\pi)}\to A(N_i)_{(\pi)}$, the isomorphism induced by the conjugation isomorphism $N_i\simeq N$ with respect to g_i^{-1}.

We observe $\mathrm{tr}_{N\cap N_i}^N \circ \mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*(\bar{e}_H^\pi)$ for each i, $1\leqq i\leqq t$. c_i^* maps primitive idempotents to primitive ones. By (1.3) we see that

$$\begin{aligned} \phi_L(c_i^*(\bar{e}_H^\pi)) &= 1 \qquad \text{if} \quad L\in S_\pi(H_i, N_i) \\ &= 0 \qquad \text{otherwise}. \end{aligned}$$

Thus $c_i^*(\bar{e}_H^\pi)$ is the central idempotent of $A(N_i)_{(\pi)}$. Then

$$\begin{aligned} \phi_L(\mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*(\bar{e}_H^\pi)) &= 1 \qquad \text{if} \quad L\in S_\pi(H_i, N\cap N_i) \\ &= 0 \qquad \text{otherwise}, \end{aligned}$$

which shows that $\mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*(\bar{e}_H^\pi)=0$ for $i>s$ and $=$the central idempotent of $A(N_N(H_i))_{(\pi)}$ for $1\leqq i\leqq s$ by (1.3) as $N\cap N_i=N_N(H_i)$. Let $\bar{\bar{e}}_i^\pi$ denote the central idempotent of $A(N_N(H_i))_{(\pi)}$. We have obtained

$$\text{(1.7)} \qquad \begin{aligned} \mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*(\bar{e}_H^\pi) &= \bar{\bar{e}}_i^\pi \qquad \textit{for} \quad 1\leqq i\leqq s \\ &= 0 \qquad \textit{for} \quad s<i\leqq t\,. \end{aligned}$$

Apply (1.4) for the pair (N, H_i) and obtain

$$\text{(1.8)} \qquad \mathrm{tr}_{N\cap N_i}^N (\bar{\bar{e}}_i^\pi) = \bar{e}_i^\pi \qquad \textit{for} \quad 1\leqq i\leqq s\,.$$

We add two remarks. Since $\bar{e}_i^\pi$ is the primitive idempotent of $A(N\cap N_i)_{(\pi)}$ corresponding to $(H_i)_{N\cap N_i}$, we have the decomposition

$$\mathrm{res}_{N\cap N_i}^N \bar{e}_i^\pi = \bar{\bar{e}}_i^\pi + \cdots$$

into primitive ones for $1\leqq i\leqq s$ by (1.5). Thus

$$\text{(1.9)} \qquad (\mathrm{res}_{N\cap N_i}^N \bar{e}_i^\pi)\cdot \bar{\bar{e}}_i^\pi = \bar{\bar{e}}_i^\pi \qquad \text{for} \quad 1\leqq i\leqq s\,.$$

The second remark is that $g_1=1$, $H_1=H$ and $N_1=N$ by our choice. Thus

(1.10) $$\mathrm{tr}^{N}_{N\cap N_1}=\mathrm{res}^{N_1}_{N\cap N_1}=c_1^*=\mathrm{id}\,.$$

§2. Idempotents and Mackey Functors

Dress [10], Section 4, defined the Burnside functor on $\hat{G}$. Let T be a finite G-set and $\hat{G}/T$ the category of objects over T. The set of all isomorphism classes of $\hat{G}/T$ forms a commutative semi-ring $A_G^+(T)$ with addition and multiplication defined by disjoint unions and pull-backs. Its Grothendieck ring is denoted by $A_G(T)$. The element of $A_G(T)$ represented by an object $f: S\to T$ of $\hat{G}/T$ is denoted by $[f: S\to T]$. The *Burnside functor* $A_G=(A_{G*}, A_G^*)$ on $\hat{G}$ is a pair of functors $A_{G*}: \hat{G}\to\mathbf{Ab}$ and $A_G^*: \hat{G}^{op}\to\mathbf{Ab}$ such that $A_{G*}(T)=A_G^*(T)=A_G(T)$ on each object T and, for a morphism $f: S\to T$ in $\hat{G}$, $A_{G*}(f)=f_*: A_G(S)\to A_G(T)$ is given by $f_*[g: U\to S]=[f\circ g: U\to T]$ and $A_G^*(f)=f^*: A_G(T)\to A_G(S)$ by $f^*[h: W\to T]=[W\times_T S\to S]$.

As for the definition of a *Mackey functor* $M=(M_*, M^*)$ on $\hat{G}$ we refer to [7], p. 68. The Burnside functor A_G is a Mackey functor on $\hat{G}$. Moreover, f^* is multiplicative (i.e., A_G^* is ring-valued) and there holds the Frobenius property among f_*, f^* and multiplication, i.e., A_G is a Green functor in the sense of [10].

There holds the canonical isomorphism

$$A_G(G/K)\simeq A(K)$$

for $K\leqq G$ such that

$$p_*=\mathrm{tr}_L^K \qquad \text{and} \qquad p^*=\mathrm{res}_L^K$$

for $L\leqq K\leqq G$ and $p: G/L\to G/K$, the canonical projection.

Let $M=(M_*, M^*)$ be any Mackey functor on $\hat{G}$. We write $M_*(f)=f_*$ and $M^*(f)=f^*$ for a morphism $f: S\to T$ in $\hat{G}$. M(T) becomes an $A_G(T)$-module by $[f: S\to T]\cdot x=f_*\circ f^*x$, $x\in M(T)$, [7] [10]. By these module actions M is an A_G-module in the sense that M^* is a module-valued functor ($f^*(xy)=(f^*x)(f^*y)$ for $f: S\to T$, $x\in A_G(T)$ and $y\in M(T)$) and there holds the Frobenius property among f_*, f^* and module action [10], Proposition 4.2. We write $p_*=\mathrm{tr}_L^K$, $p^*=\mathrm{res}_L^K$ for any Mackey functor M, $L\leqq K\leqq G$ and p: $G/L\to G/K$, the canonical projection, in conformity with the above mentioned identities for A_G.

Let π be a set of primes and M a $\boldsymbol{Z}_{(\pi)}$-module-valued Mackey functor. Put $A_{G,\pi}=A_G\otimes Z_{(\pi)}$. The above module action of A_G on M makes M an $A_{G,\pi}$-module.

For each $K \leqq G$, $M(G/K)$ is an $A(K)_{(\pi)}$-module. Hence primitive idempotents of $A(K)_{(\pi)}$ decomposes $M(G/K)$ as a direct sum of submodules. In particular

$$M(pt) = \coprod_{(H)\in P_\pi} e_H^\pi M(pt).$$

We observe $e_H^\pi M(pt)$ as an $e_H^\pi A(G)_{(\pi)}$-module.

Let H be a π-perfect subgroup of G and $N = N_G(H)$. Let $\bar{e}_H^\pi$ be the central idempotent of $A(N)_{(\pi)}$. We want to discuss $\mathrm{res}_N^G \circ \mathrm{tr}_N^G (\bar{e}_H^\pi x)$ for $\bar{e}_H^\pi x \in \bar{e}_H^\pi M(G/N)$. The axiom $(M1)$ for the Mackey functor [7] applied to the pull-back diagram

$$\begin{array}{ccc} G/N \times G/N & \longrightarrow & G/N \\ \downarrow & & \downarrow \\ G/N & \longrightarrow & pt \end{array}$$

implies the Mackey decomposition

$$\mathrm{res}_N^G \circ \mathrm{tr}_N^G = \sum_{1 \leqq i \leqq t} \mathrm{tr}_{N\cap N_i}^N \circ \mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^*$$

for M [10] [12] (the same formula as (1.6)), where we used the same notations as in Section 1, i.e., $\{g_1, \ldots, g_t\}$ $(g_1 = 1)$ is a complete system of representatives of $N\backslash G/N$, $N_i = g_i N g_i^{-1}$, and c_i^*: $M(G/N) \simeq M(G/N_i)$, the isomorphism induced by the right multiplication with g_i: $G/N_i \simeq G/N$, for $1 \leqq i \leqq t$.

Put

$$\bar{x}_i = \mathrm{res}_{N\cap N_i}^{N_i} (c_i^* x) \in M(G/N \cap N_i),\ 1 \leqq i \leqq t.$$

As $\mathrm{res}_{N\cap N_i}^{N_i}$ and c_i^* preserve module actions we see that

$$\begin{aligned} \mathrm{res}_{N\cap N_i}^{N_i} \circ c_i^* (\bar{e}_H^\pi x) &= \bar{e}_i^\pi \bar{x}_i \qquad \text{for} \quad 1 \leqq i \leqq s, \\ &= 0 \qquad \text{for} \quad s < i \leqq t \end{aligned}$$

by (1.7). Next we put

$$x_i = \mathrm{tr}_{N\cap N_i}^N (\bar{e}_i^\pi \bar{x}_i) \in M(G/N), \qquad 1 \leqq i \leqq s.$$

Then

$$\mathrm{tr}_{N\cap N_i}^N (\bar{e}_i^\pi \bar{x}_i) = \mathrm{tr}_{N\cap N_i}^N ((\mathrm{res}_{N\cap N_i}^N \bar{e}_i^\pi) \bar{e}_i^\pi \bar{x}_i) = \bar{e}_i^\pi x_i, \qquad 1 \leqq i \leqq s,$$

by (1.9). For $i = 1$, the remark (1.10) is applicable also for M and we see that

$$\bar{e}_1^\pi x_1 = \bar{e}_H^\pi x,$$

the given element. Thus we obtain

Proposition 2.1. *Using the notations of Section* 1 *we have the direct sum decomposition*

$$(\mathrm{res}_N^G\, e_H^\pi)M(G/N)=\coprod_{1\leq i\leq s}\bar{e}_i^\pi M(G/N),$$

and, for any $\bar{e}_H^\pi\, x\in\bar{e}_H^\pi M(G/N)$, *we have the decomposition*

$$\mathrm{res}_N^G\circ\mathrm{tr}_N^G(\bar{e}_H^\pi x)=\sum_{1\leq i\leq s}\bar{e}_i^\pi x_i$$

such that

$$\bar{e}_i^\pi x_i=\mathrm{tr}_{N\cap N_i}^N\circ\mathrm{res}_{N\cap N_i}^{N_i}\circ c_i^*(\bar{e}_H^\pi x)$$

and

$$\bar{e}_1^\pi x_1=\bar{e}_H^\pi x, \text{ the given element}.$$

Put

$$\mathrm{tr}'^G_N=\mathrm{tr}_N^G\,|\bar{e}_H^\pi M(G/N)\colon\ \bar{e}_H^\pi M(G/N)\longrightarrow e_H^\pi M(pt). \tag{2.2}$$

Suppose $\bar{e}_H^\pi x\in\mathrm{Ker}\,\mathrm{tr}'^G_N$. Then $\mathrm{res}_N^G\circ\mathrm{tr}_N^G(\bar{e}_H^\pi x)=\sum_{1\leq i\leq s}\bar{e}_i^\pi x_i=0$. Hence $\bar{e}_i^\pi x_i=0$ for all i, $1\leq i\leq s$. In particular $\bar{e}_H^\pi x=\bar{e}_1^\pi x_1=0$. Thus we obtain

$$\mathrm{tr}'^G_N\colon\ \bar{e}_H^\pi M(G/N)\longrightarrow e_H^\pi M(pt) \text{ is monomorphic}. \tag{2.3}$$

Let

$$\mathrm{res}'^G_N\colon\ e_H^\pi M(pt)\longrightarrow\bar{e}_H^\pi M(G/N) \tag{2.4}$$

be the $e_H^\pi A(G)_{(\pi)}$-module map defined by

$$\mathrm{res}'^G_N(x)=\bar{e}_H^\pi\cdot\mathrm{res}_N^G\,x,\ x\in e_H^\pi M(pt).$$

By Frobenius property and (1.4) we see that

$$\mathrm{tr}'^G_N\circ\mathrm{res}'^G_N(x)=\mathrm{tr}_N^G(\bar{e}_H^\pi\cdot\mathrm{res}_N^G\,x)=e_H^\pi\,x=x$$

for $x\in e_H^\pi M(pt)$. Thus

$$\mathrm{tr}'^G_N\circ\mathrm{res}'^G_N=\mathrm{id},$$

which shows that tr'^G_N is epimorphic and hence isomorphic by (2.3). Clearly res'^G_N is the inverse to tr'^G_N and we obtain

Theorem 2.5. *Let* π *be a set of primes,* M *a* $Z_{(\pi)}$*-module-valued Mackey functor on* G, H *a* π*-perfect subgroup of* G *and* $N=N_G(H)$. *Let* e_H^π *be the primitive idempotent of* $A(G)_{(\pi)}$ *corresponding to* $(H)\in P_\pi$ *and* $\bar{e}_H^\pi$ *the central idempotent of* $A(N)_{(\pi)}$. *Then there holds the* $e_H^\pi A(G)_{(\pi)}$*-module isomorphism*

$$\mathrm{res}'^G_N\colon\ e_H^\pi M(pt)\cong\bar{e}_H^\pi M(G/N).$$

§3. Stable G-Homotopy Theory

By a *G-module* V we mean a finite dimensional real or complex G-module equipped with an invariant metric for simplicity. By S^V and B^V we denote the unit sphere and unit ball of V respectively. We put $\Sigma^V = B^V/S^V$, which is G-homeomorphic to the one-point compactification of V.

Let X and Y be pointed G-CW complexes. We assume X to be finite. By the group of *stable-G-maps* from X to Y we understand

$$\tilde{\omega}_G^0(X\colon Y) = \operatorname{colim}\, [\Sigma^V X, \Sigma^V Y]^G$$

[8], Section 7, where $[\ ,\ ]^G$ denotes the set of G-homotopy classes of pointed G-maps, $\Sigma^V X = \Sigma^V \wedge X$, V runs over the system of complex G-modules which is directed by G-embeddings as G-submodules, and the colimit is taken with respect to suspensions

$$\Sigma_*^W\colon [\Sigma^V X, \Sigma^V Y]^G \longrightarrow [\Sigma^{W\oplus V} X, \Sigma^{W\oplus V} Y]^G .$$

$\tilde{\omega}_G^0(X\colon Y)$ is a well-defined abelian group.

We use complex G-modules by the following two reasons: i) the directed system of complex G-modules may be regarded as a cofinal subsystem of that of real G-modules so that we loose nothing by this restriction; ii) the group of complex automorphisms of a complex G-module V is connected so that G-maps $\Sigma^V \to \Sigma^V$ induced by complex automorphisms of V are all G-homotopic to the identity, which makes several identifications among G-homotopy sets coming from isomorphisms of G-modules unique.

Let $f\colon S \to T$ be a map in $\hat{G}$. Endowing discrete topology to S and T respectively, a G-embedding $i\colon S \subset T \times V$ such that V is a complex G-module and $\mathrm{pr}_1 \circ i = f$ is called an *admissible embedding* for f. The existence of an admissible embedding is easily shown by making use of the complex permutation representation V_S of S. Let $i\colon S \subset T \times V$ be an admissible embedding for f. We may assume that $i(S) \subset T \times \operatorname{Int} B^V$. Regard S and T as 0-dimensional G-manifolds and let νi be the normal G-bundle of the embedding i. Then $\nu i \simeq_G S \times V$. Choose the normal disk G-bundle $D\nu i$ so that $D\nu i \subset T \times B^V$. Since $D\nu i \simeq_G S \times B^V$, the Thom construction gives a pointed G-map

$$\operatorname{tr} f\colon T^+ \wedge \Sigma^V \longrightarrow S^+ \wedge \Sigma^V .$$

This construction is of course a very special case of the equivariant Becker-

Gottlieb transfer [15]. (Compare also with [8], §7, in which the case of compact Lie group actions is discussed.) The following properties of $\mathrm{tr}\, f$ are easily shown by standard techniques and left to readers.

(3.1) *The stable class* $\{\mathrm{tr}\, f\} \in \tilde{\omega}_G^0(T^+ : S^+)$ *is uniquely determined by* f.

(3.2) *Let* $f\colon S_1 \to S_2$ *and* $g\colon S_2 \to S_3$ *be morphisms in* G. *Then*

$$\{\mathrm{tr}\,(g \circ f)\} = \{\mathrm{tr}\, f\} \circ \{\mathrm{tr}\, g\}$$

as stable G-maps.

(3.3) *Let*

$$\begin{array}{ccc} S' & \xrightarrow{g'} & S \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

be a pull-back diagram in $\hat{G}$. *Then*

$$\{g'^+\} \circ \{\mathrm{tr}\, f\} = \{\mathrm{tr}\, f\} \circ \{g^+\}$$

as stable G-maps.

We define a bifunctor

$$\omega_G[X : Y] \colon \hat{G} \longrightarrow \mathbf{Ab}$$

as follows:

$$\omega_G[X : Y](S) = \tilde{\omega}_G^0(S^+ \wedge X : Y)$$

on objects; for a morphism $f\colon S \to T$ in $\hat{G}$ we put

$$f_* = (\mathrm{tr}\, f \wedge 1)^* \colon \tilde{\omega}_G^0(S^+ \wedge X : Y) \longrightarrow \tilde{\omega}_G^0(T^+ \wedge X : Y)$$

which gives a covariant functor by (3.2), and

$$f^* = (f^+ \wedge 1)^* \colon \tilde{\omega}_G^0(T^+ \wedge X : Y) \longrightarrow \tilde{\omega}_G^0(S^+ \wedge X : Y)$$

which gives obviously a contravariant functor.

Proposition 3.4. $\omega_G[X : Y]$ *is a Mackey functor.*

Proof. (3.3) implies the axiom ($M1$) of [7], p. 68. As to the axiom ($M2$), let $S \amalg T$ be a disjoint union of finite G-sets, then $(S \amalg T)^+ = S^+ \vee T^+$ and

$$\begin{aligned} \tilde{\omega}_G^0((S \amalg T)^+ \wedge X : Y) &= \tilde{\omega}_G^0((S^+ \wedge X) \vee (T^+ \wedge X) : Y) \\ &\simeq \tilde{\omega}_G^0(S^+ \wedge X : Y) \oplus \tilde{\omega}_G^0(T^+ \wedge X : Y). \end{aligned}$$

□

Let $L \leqq G$. Since the directed system of L-modules which are obtained from G-modules by restriction of actions is a cofinal subsystem of that of arbitrary L-modules, we get the homomorphism

$$\psi_L^G = \mathrm{res}_L^G : \tilde{\omega}_G^0(X: Y) \longrightarrow \tilde{\omega}_L^0(X: Y)$$

by restricting G-actions to L-actions. On the other hand we get the isomorphism

$$\kappa : \tilde{\omega}_G^0((G/L)^+ \wedge X: Y) \simeq \tilde{\omega}_L^0(X: Y)$$

by restricting stable G-maps to $\{L\}^+ \wedge X \simeq_L X$, which we regard as the canonical isomorphism. Let

$$p: G/L \longrightarrow pt$$

be the unique G-map. We can easily identify

$$p^* = \mathrm{res}_L^G$$

via the canonical isomorphism κ. We define

$$\mathrm{tr}_L^G = p_* \circ \kappa^{-1} : \tilde{\omega}_L^0(X: Y) \longrightarrow \tilde{\omega}_G^0(X: Y).$$

With these setting we apply Theorem 2.5 to the Mackey functor $\omega_G[X: Y]$ and obtain

Theorem 3.5. *Let X and Y be pointed G-CW complexes. Assume X to be finite. Let π be a set of primes. Using the same notations as in Theorem 2.5 there holds the $e_H^\pi A(G)_{(\pi)}$-module isomorphism*

$$\mathrm{res}'^G_N : e_H^\pi \tilde{\omega}_G^0(X: Y)_{(\pi)} \cong \bar{e}_H^\pi \tilde{\omega}_N^0(X: Y)_{(\pi)}.$$

The above theorem applies also to G-homology and G-cohomology theories. Any G-cohomology theory defined on the category of (finite) G-CW complexes satisfying suitable axioms is representable by a G-spectrum [2] [14]. So we discuss here only G-homology and G-cohomology theories defined by G-spectra [2] [13]. We use G-spectra indexed by complex (virtual) G-modules in the same reason as the definition of the group of stable G-maps. Practically we may restrict our G-spectra to those indexed by a cofinal subsystem of that of complex G-modules and will do so in the sequel.

Let $\rho = \rho_G$ be the complex regular representation of G. $\{n\rho : n \in \mathbb{Z}\}$ is one of such cofinal subsystems. We use this system particularly. A *G-spectrum* $E_G = \{E_n, \varepsilon_n : \Sigma^\rho E_n \to E_{n+1}; n \in \mathbb{Z}\}$ consists of a pointed G-CW complex E_n and a pointed G-map (*structure map*) $\varepsilon_n : \Sigma^\rho E_n \to E_{n+1}$ for each $n \in \mathbb{Z}$. When $E_n = \Sigma^{n\rho}$ and $\varepsilon_n = \mathrm{id} : \Sigma^\rho \Sigma^{n\rho} = \Sigma^{(n+1)\rho}$ for $n \geqq 0$ ($E_n = pt$ for $n < 0$), the G-spectrum is called the *G-sphere* spectrum and denoted by Σ_G.

Let $E_G=\{E_n, \varepsilon_n;\ n\in \boldsymbol{Z}\}$ be a G-spectrum and $L\leqq G$. As $\operatorname{res}^G_L \rho_G=|G/L|\cdot\rho_L$, where $\rho'=\rho_L$ is the complex regular representation of L, putting

$$\begin{aligned} E'_{|G/L|n+k}&=\Sigma^{k\rho'} E_n && \text{for}\quad 0\leqq k<|G/L| \\ \varepsilon'_{|G/L|n+k}&=\mathrm{id} && \text{for}\quad 0\leqq k<|G/L|-1 \\ &=\varepsilon_n && \text{for}\quad k=|G/L|-1\,, \end{aligned}$$

we get an L-spectrum

$$\psi_L E_G=\{E'_n,\ \varepsilon'_n;\ n\in \boldsymbol{Z}\}$$

by restricting G-actions to L-actions. Clearly

$$\psi_L\Sigma_G=\Sigma_L\,.$$

The *E_G-homology-cohomology group in degree* 0 (homology with respect to Y and cohomology with respect to X) is defined by

$$E^0_G(X;\ Y)=\operatorname{colim}\,[\Sigma^{n\rho}X,\ E_n\wedge Y]^G\,,$$

where the colimit is taken with respect to the compositions $\varepsilon_{n*}\circ\Sigma^{\rho}_{*}$ as usual. $E^0_G(X\colon Y)$ is a well-defined abelian group. Obviously

$$\Sigma^0_G(X\colon Y)=\tilde{\omega}^0_G(X\colon Y)\,.$$

Again we obtain a Mackey functor $\hat{G}\to\mathbf{Ab}$ by the assignment: $S\mapsto E^0_G(S^+\wedge X\colon Y)$ and "$f\colon S\to T$"$\mapsto f_*=(\operatorname{tr}f\wedge 1)^*$ and $f^*=(f^+\wedge 1)^*$. Also we have the *restriction* homomorphism

$$\psi^G_L=\operatorname{res}^G_L\colon E^0_G(X\colon Y)\longrightarrow(\psi_L E_G)^0(X\colon Y)$$

and the *transfer* homomorphism

$$\operatorname{tr}^G_L\colon(\psi_L E_G)^0(X\colon Y)\longrightarrow E^0_G(X\colon Y)$$

together with the canonical isomorphism

$$\kappa\colon E^0_G((G/L)^+\wedge X\colon Y)\cong(\psi_L E_G)^0(X\colon Y)$$

in the parallel way to the case of $\tilde{\omega}^0_G$.

Now apply Theorem 2.5 to the above Mackey functor and obtain

Theorem 3.6. *Under the same assumptions and notations as in Theorem 3.5 there holds the $e^{\pi}_H A(G)_{(\pi)}$-module isomorphism*

$$\operatorname{res}'^G_N\colon e^{\pi}_H E^0_G(X\colon Y)_{(\pi)}\cong\bar{e}^{\pi}_H(\psi_L E_G)^0(X\colon Y)_{(\pi)}$$

for any G-spectrum E_G.

Let $\alpha \in RO(G)$ and express $\alpha = U - V$ as a difference of real G-modules. The E_G-homology-cohomology group *in degree* α is defined by

$$E_G^{\alpha}(X: Y) = E_G^{0}(\Sigma^V X: \Sigma^U Y).$$

Let $\alpha = U' - V'$ be another expression. We can certainly find an additive isomorphism

$$E_G^0(\Sigma^V X: \Sigma^U Y) \simeq E_G^0(\Sigma^{V'} X: \Sigma^{U'} Y),$$

but it is no more canonical and there are many choices of this isomorphism. So, as far as we are interested in additive structures we may use the $RO(G)$-grading; but, when we are interested in mulitplicative structure based on ring-G-spectra, we will meet with serious troubles in $RO(G)$-grading as to commutativity etc., and we need some other device which will be discussed in another occasion.

Anyway we get the restriction homomorphism

$$\psi_L^G = \mathrm{res}_L^G: E_G^{\alpha}(X: Y) \longrightarrow (\psi_L E_G)^{\psi_L \alpha}(X: Y)$$

and the transfer homomorphism

$$\mathrm{tr}_L^G: (\psi_L E_G)^{\psi_L \alpha}(X: Y) \longrightarrow E_G^{\alpha}(X: Y)$$

in degree $\alpha \in RO(G)$, where $\psi_L \alpha = \mathrm{res}_L^G U - \mathrm{res}_L^G V \in RO(L)$ for $\alpha = U - V \in RO(G)$.

By the above definition we see that we may apply Theorem 3.6 to E_G^{α} and obtain the $e_H^{\pi} A(G)_{(\pi)}$-module isomorphism

$$\text{(3.7)} \qquad \mathrm{res}'^G_N: e_H^{\pi} E_G^{\alpha}(X: Y)_{(\pi)} \cong \bar{e}_H^{\pi} (\psi_L E_G)^{\psi_L \alpha}(X: Y)_{(\pi)}.$$

§4. Fixed-Point Exact Sequences

Let $G \rhd K$, a normal subgroup; then $(\rho_G)^K = \rho_{G/K}$, the complex regular representation of G/K. Let $E_G = \{E_n, \varepsilon_n; n \in \boldsymbol{Z}\}$ be a G-spectrum. Putting

$$E''_n = E_n^K,$$
$$\varepsilon''_n = \varepsilon_n^K: \Sigma^{\rho''} E''_n \longrightarrow E''_{n+1}, \quad \rho'' = \rho_{G/K},$$

for $n \in \boldsymbol{Z}$, we get a G/K-spectrum

$$\phi_K E_G = \{E''_n, \varepsilon''_n; n \in \boldsymbol{Z}\}$$

which is called the *K-fixed-point spectrum* of E_G. Clearly

$$\phi_K \Sigma_G = \Sigma_{G/K}.$$

By restriction to K-fixed-points we get a homomorphism

$$\phi_K^G\colon E_G^\alpha(X\colon Y) \longrightarrow (\phi_K E_G)^{\phi_K\alpha}(X^K\colon Y^K)$$

called the *K-fixed-point homomorphism*, where $\phi_K\alpha = U^K - V^K \in RO(G)$ for $\alpha = U - V \in RO(G)$.

We construct an exact sequence involving ϕ_K^G which generalizes the fixed-point exact sequence for $G = \boldsymbol{Z}/2$, [3], Section 1.

Decompose

$$\rho_G = \rho_1 \oplus \rho_2,\ \rho_2 = \rho_G^K \simeq \rho_{G/K} \qquad \text{and} \quad \rho_1^K = \{0\}.$$

For each integer $n > 0$ we get a G-homotopy commutative diagram of pointed G-cofibrations

$$\begin{array}{ccccc}
S_+^{(n+1)\rho_1} & \longrightarrow & B_+^{(n+1)\rho_1} & \longrightarrow & \Sigma^{(n+1)\rho_1} \\
\downarrow & & \downarrow & & \| \\
S^{(n+1)\rho_1}/S^{\rho_1}\times B^{n\rho_1} & \longrightarrow & B^{(n+1)\rho_1}/S^{\rho_1}\times B^{n\rho_1} & \longrightarrow & \Sigma^{(n+1)\rho_1}, \\
\wr\wr & & \wr\wr & \nearrow & \\
\Sigma^{\rho_1}(S_+^{n\rho_1}) & \longrightarrow & \Sigma^{\rho_1}(B_+^{n\rho_1}) & &
\end{array}$$

where we identify $B^{(n+1)\rho_1} = B^{\rho_1}\times B^{n\rho_1}$, $S^{(n+1)\rho_1} = \partial(B^{\rho_1}\times B^{n\rho_1}) = S^{\rho_1}\times B^{n\rho_1} \cup B^{\rho_1}\times S^{n\rho_1}$, which implies the following commutative diagram with two horizontal exact sequences:

$$\begin{array}{ccccccc}
\cdots\longrightarrow E_G^{\alpha+n\rho_1-1}(S_+^{n\rho_1}\wedge X\colon Y) & \xrightarrow{\delta_n} & E_G^\alpha(X;Y) & \xrightarrow{\chi^n} & E_G^{\alpha+n\rho_1}(X\colon Y) & \longrightarrow \cdots \\
\downarrow{\scriptstyle \xi_n} & & \| & & \downarrow{\scriptstyle \chi} & \\
\longrightarrow E_G^{\alpha+(n+1)\rho_1-1}(S_+^{(n+1)\rho_1}\wedge X\colon Y) & \xrightarrow{\delta_{n+1}} & E_G^\alpha(X\colon Y) & \xrightarrow{\chi^{n+1}} & E_G^{\alpha+(n+1)\rho_1}(X\colon Y) & \longrightarrow \cdots
\end{array} \tag{4.1}$$

for each $\alpha \in RO(G)$ by fixing the same expression $\alpha = U - V$, where the homomorphism χ is induced by the inclusion $\chi = \chi_{\rho_1}\colon \Sigma^0 \subset \Sigma^{\rho_1}$ and ξ_n is induced by the collapsing map $S_+^{(n+1)\rho_1} \to \Sigma^{\rho_1}(S_+^{n\rho_1})$. (Compare with the commutative diagram of [3], p. 5.) Take the colimit in vertical direction of this diagram and obtain an exact seqeunce which is an S-dual version of the localization exact sequence of tom Dieck [5] under a specified situation. We identify this exact sequence with our desired exact sequence.

Define

$$(\lambda_K E_G)^\alpha(X\colon Y) = \operatorname*{colim}_n\,[E_G^{\alpha+n\rho_1-1}(S_+^{n\rho_1}\wedge X\colon Y),\ \xi_n], \tag{4.2}$$

and we prove the isomorphism

(4.3) $$\operatorname*{colim}_n\,[E_G^{\alpha+n\rho_1}(X\colon Y),\,\chi]\cong(\phi_K E_G)^{\phi_K\alpha}(X^K\colon Y^K).$$

First we prove

Lemma 4.4. $\operatorname*{colim}_n\,[E_G^{n\rho_1}(X/X^K\colon Y),\,\chi]=0.$

Proof. Take $x=\{f\}\in\operatorname{colim}\,[E_G^{n\rho_1}(X/X^K;\,Y),\,\chi]$. x is represented by a G-map $f\colon\Sigma^{m\rho}(X/X^K)\to\Sigma^{n\rho_1}E_m\wedge Y$. We want to show that replacing f by another representative g of x, $g^L\simeq 0$ for all $L\leqq G$; then $g\simeq_G 0$ by [4], Chapter II, Lemma 5.2, and hence $x=0$. Suppose $L\geqq K$, then $pt=(X/X^K)^K\supset(X/X^K)^L$; thus $(X/X^K)^L=pt$, $(\Sigma^{m\rho}(X/X^K))^L=pt$ and $f^L=0$. Next, suppose $L\ngeqq K$. Since ρ_G is the complex regular representation of G, there exists a non-zero $v\in\rho_G$ such that $G_v=L$. Let $v=(v_1,\,v_2)\in\rho_1\oplus\rho_2=\rho_G$, then $v_1\neq 0$ and $\rho_1^L\neq\{0\}$. Thus $(\Sigma^{k\rho_1})^L$ is a sphere of dimension $\geqq 2k$ for any integer $k>0$. In the present colimit f and $(\chi^k\wedge 1)\circ f\colon\Sigma^{m\rho}(X/X^K)\to\Sigma^{(k+n)\rho_1}E_m\wedge Y$ represent the same element x for any integer $k>0$. Since X is finite by our assumption, we may choose k large enough so that $\dim\Sigma^m(X/X^K)<2(k+n)-1$. Now, put $g=(\chi^k\wedge 1)\circ f$; $\dim(\Sigma^{m\rho}(X/X^K))^L<2(k+n)-1$ and $(\Sigma^{(k+n)\rho_1}E_m\wedge Y)^L$ is at least $(2(k+n)-1)$-connected for any $L\ngeqq K$; thus $g^L\simeq 0$ for all $L\leqq G$ and $g\simeq_G 0$. □

Proof of (4.3). We prove the case $\alpha=0$. General case follows from this special case by replacing X by $\Sigma^V X$ and Y by $\Sigma^U Y$ for $\alpha=U-V\in RO(G)$.

Consider the exact sequences associated with the G-cofibration $X^K\to X\to X/X^K$ and take the colimit of these sequences with respect to χ. We get an exact sequence

$$\begin{aligned}&\operatorname*{colim}_n\,[E_G^{n\rho_1}(X/X^K\colon Y),\,\chi]\longrightarrow\operatorname*{colim}_n\,[E_G^{n\rho_1}(X\colon Y),\,\chi]\\&\longrightarrow\operatorname*{colim}_n\,[E_G^{n\rho_1}(X^K\colon Y),\,\chi]\longrightarrow\operatorname*{colim}_n\,[E_G^{n\rho_1+1}(X/X^K\colon Y),\,\chi].\end{aligned}$$

By the above lemma $\operatorname{colim}\,[E_G^{n\rho_1}(X/X^K\colon Y),\,\chi]=0$ and also $\operatorname{colim}\,[E_G^{n\rho_1+1}(X/X^K\colon Y),\,\chi]=0$ replacing Y by ΣY. Thus we get the isomorphism

(♯) $$\operatorname*{colim}_n\,[E_G^{n\rho_1}(X\colon Y),\,\chi]\cong\operatorname*{colim}_n\,[E_G^{n\rho_1}(X^K\colon Y),\,\chi].$$

Consider the following sequence

$$\begin{aligned}&[\Sigma^{m\rho}X^K,\Sigma^{n\rho_1}E_m\wedge Y]^G\xrightarrow{\Sigma_*^{n\rho_2}}[\Sigma^{m\rho+n\rho_2}X^K,\,\Sigma^{n\rho}E_m\wedge Y]^G\\&\xrightarrow{\varepsilon_*^n}[\Sigma^{m\rho+n\rho_2}X^K,\,E_{n+m}\wedge Y]^G\xrightarrow{\Sigma_*^{n\rho_1}}[\Sigma^{(n+m)\rho}X^K,\,\Sigma^{n\rho_1}E_{m+n}\wedge Y]^G\end{aligned}$$

and observe that the composition $=(\varepsilon_*\circ\Sigma_*^{\rho})^n$, which proves the isomorphism

$$E_G^{n\rho_1}(X^K\colon Y)\cong\operatorname*{colim}_m\,[[\Sigma^{m\rho+n\rho_2}X^K,\,E_{n+m}\wedge Y]^G,\ \varepsilon_*\circ\Sigma_*^\rho]\,.$$

And we get the isomorphism

$$(\sharp\sharp)\qquad \operatorname*{colim}_n E_G^{n\rho_1}(X^K\colon Y)\cong\operatorname*{colim}_{n,m}\,[\Sigma^{m\rho+n\rho_2}X^K,\,E_{n+m}\wedge Y]^G\,.$$

Observe the commutative diagram:

$$\begin{array}{ccc} [X,\,Y]^G & \xrightarrow{\Sigma_*^{\rho_1}} & [\Sigma^{\rho_1}X,\,\Sigma^{\rho_1}Y]^G \\ & {\scriptstyle(\chi\wedge1)^*}\searrow \quad \swarrow{\scriptstyle(\chi\wedge1)^*} & \\ & [X,\,\Sigma^{\rho_1}Y]^G, & \end{array}$$

which shows that the homomorphism χ may be used as $\chi=(\chi\wedge1)^*$ as well as $\chi=(\chi\wedge1)_*$. In the right hand side of the isomorphism ($\sharp\sharp$) we may understand $\chi=(\chi\wedge1)_*$. Then we see that the directed system of this double colimit contains the sequence $\{[\Sigma^{n\rho_2}\wedge X^K,\,E_n\wedge Y]^G,\ \chi\circ\varepsilon_*\circ\Sigma_*^\rho\}$ as a cofinal subsequence. Thus

$$\operatorname*{colim}_n E_G^{n\rho_1}(X^K\colon Y)\cong\operatorname*{colim}_n\,[[\Sigma^{n\rho_2}X^K,\,E_n\wedge Y]^G,\ \chi\circ\varepsilon_*\Sigma_*^\rho]\,.$$

Now, K acts trivially on $\Sigma^{n\rho_2}X^K$. Hence

$$[\Sigma^{n\rho_2}X^K,\,E_n\wedge Y]^G=[\Sigma^{n\rho_2}X^K,E_n{}^K\wedge Y^K]^G=[\Sigma^{n\rho_2}X^K,\,E_n{}^K\wedge Y^K]^{G/K}\,,$$

and we get the isomorphism

$$\operatorname*{colim}_n\ E_G^{n\rho_1}(X^K\colon Y)\cong(\phi_K E_G)^0(X^K\colon Y^K)\,,$$

which, together with ($\sharp$), completes the proof of (4.3). □

In the exact seqeunce obtained by taking the colimit of (4.1) in the vertical direction, identify one term with $(\phi_K E_G)^{\phi_K\alpha}(X^K\colon Y^K)$ by (4.3). It is easy to identify $\operatorname{colim}\chi^n$ with the fixed-point homomorphism ϕ_K^G, and we obtain the desired exact sequence

$$(4.5)\qquad \begin{aligned}\cdots\longrightarrow(\lambda_K E_G)^\alpha(X\colon Y)\longrightarrow E_G^\alpha\,(X\colon Y)\xrightarrow{\phi_K^G}(\phi_K E_G)^{\phi_K\alpha}(X^K\colon Y^K)\\ \longrightarrow(\lambda_K E_G)^{\alpha+1}(X\colon Y)\longrightarrow\cdots\end{aligned}$$

for $\alpha\in RO(G)$, which we call the *K-fixed-point exact sequence.*

Let π be a set of primes and H a π-perfect subgroup of G. Denote $N=N_G(H)$, $W=N_G(H)/H$, $E_N=\psi_N E_G$, $E_W=\phi_H E_N$, $\alpha'=\psi_N\alpha$ and $\alpha''=\phi_H\alpha'$ for $\alpha\in RO(G)$. Consider the following H-fixed-point exact sequence (tensored with $\boldsymbol{Z}_{(\pi)}$):

$$\cdots\longrightarrow(\lambda_H E_N)^{\alpha'}(X\colon Y)_{(\pi)}\longrightarrow E_N^{\alpha'}(X\colon Y)_{(\pi)}\xrightarrow{\phi_H^N}E_W^{\alpha''}(X^H\colon Y^H)_{(\pi)}\longrightarrow\cdots$$

Since actions of $A(N)$ on $E_N^{\alpha'}(X: Y)$ are natural with respect to X and Y, the central idempotent $\bar{e}_H^\pi$ of $A(N)_{(\pi)}$ acts on this seqeunce as an idempotent. Remark that $\bar{e}_H^\pi$ acts on $E_W^{\alpha''}(X^H: Y^H)_{(\pi)}$ through the homomorphism

$$\phi_H^N: A(N)_{(\pi)} \longrightarrow A(W)_{(\pi)}$$

defined by $\phi_H^N[S]=[S^H]$ for finite N-sets S. By (1.3) we see easily that

$$\phi_H^N \bar{e}_H^\pi = \tilde{e}_{\langle 1\rangle}^\pi ,$$

the primitive idempotent of $A(W)_{(\pi)}$ corresponding to the trivial π-perfect subgroup $\{1\}$ of W. Thus we get the following exact seqeunce

$$\cdots \longrightarrow \bar{e}_H^\pi(\lambda_H E_N)^{\alpha'}(X: Y)_{(\pi)}$$
$$\longrightarrow \bar{e}_H^\pi E_N^{\alpha'}(X: Y)_{(\pi)} \xrightarrow{\phi_H^N} \tilde{e}_{\langle 1\rangle}^\pi E_W^{\alpha''}(X^H: Y^H)_{(\pi)} \longrightarrow \cdots .$$

We will prove

Proposition 4.6. $\bar{e}_H^\pi(\lambda_H E_N)^{\alpha'}(X: Y)=0$.

As a corollary of this proposition we obtain

Theorem 4.7. *Under the same setting as in Theorem* 3.5 *there holds the* $\bar{e}_H^\pi A(N)_{(\pi)}$*-module isomorphism*

$$\phi_H^N: \bar{e}_H^\pi E_N^{\alpha'}(X: Y)_{(\pi)} \cong \tilde{e}_{\langle 1\rangle}^\pi E_W^{\phi_H \alpha'}(X^H: Y^H)_{(\pi)} ,$$

where $W=N_G(H)/H$, E_N *is an* N*-spectrum*, $E_W=\phi_H E_N$, $\alpha' \in RO(N)$ *and* $\tilde{e}_{\langle 1\rangle}$ *is the primitive idempotent of* $A(W)_{(\pi)}$ *corresponding to the trivial* π*-perfect subgroup* $\{1\}$ *of* W.

Proof of Proposition 4.6. Again it is sufficient to prove the special case $\alpha'=0$. Decompose $\rho_N=\rho_1\oplus\rho_2$, $\rho_2=\rho_N^H\cong\rho_W$, $\rho_1^H=\{0\}$. By (4.2)

$$\bar{e}_H^\pi(\lambda_H E_N)^0(X: Y)_{(\pi)} = \operatorname{colim} \bar{e}_H^\pi E_N^{n\rho_1-1}(S_+^{n\rho_1}\wedge X: Y)_{(\pi)} .$$

Therefore it is sufficient to prove

$$\bar{e}_H^\pi E_N^0(S_+^{n\rho_1}\wedge \Sigma X: \Sigma^{n\rho_1} Y)_{(\pi)}=0 .$$

Since $\rho_1^H=\{0\}$, we see that $(S^{n\rho_1})^H=\emptyset$. $S^{n\rho_1}$ is an N-CW complex. Let $\sigma^k \times N/L$ be an N-cell of $S^{n\rho_1}$. Then $(\sigma^k\times N/L)^H=\sigma^k\times(N/L)^H=\emptyset$, whence $H\nleq L$. The standard argument by induction on N-cell of $S_+^{n\rho_1}$ reduces the problem to show that

$$\bar{e}_H^\pi E_N^0(((\sigma^k\times N/L)/(\partial\sigma^k\times N/L))\wedge\Sigma X: \Sigma^{n\rho_1} Y)_{(\pi)}=0 .$$

The left hand side

$$= \bar{e}^{\pi}_{H} E^0_N((N/L)^+ \wedge \Sigma^{k+1} X \colon \Sigma^{n\rho_1} Y)_{(\pi)}$$
$$\cong (\mathrm{res}^N_L \bar{e}^{\pi}_H) \cdot (\psi_L E_N)^0 (\Sigma^{k+1} X \colon \Sigma^{n\rho_1} Y)_{(\pi)}$$

and

$$\mathrm{res}^N_L\, \bar{e}^{\pi}_H = 0$$

by (1.3) as $H \not\leq L$, which completes the proof. □

Apply (3.7) and Theorem 4.7 to the G-sphere spectrum. We get the isomorphisms

$$(4.8) \qquad e^{\pi}_H \tilde{\omega}^{\alpha}_G (X \colon Y)_{(\pi)} \cong \bar{e}^{\pi}_H \tilde{\omega}^{\alpha'}_N (X \colon Y)_{(\pi)} \cong \tilde{e}^{\pi}_{\langle 1 \rangle} \tilde{\omega}^{\alpha''}_W (X^H \colon Y^H)_{(\pi)}$$

for each $(H) \in P_{\pi}$, where $\alpha \in RO(G)$, $\alpha' = \psi_N \alpha$ and $\alpha'' = \phi_H \alpha'$. Specialize $\pi = \{$all primes$\}$, then we get Theorem A (Introduction).

Put

$$\omega^{\alpha}_G(pt) = \tilde{\omega}^{\alpha}_G\, (\Sigma^0 \colon \Sigma^0)\,.$$

Segal [16] showed the isomorphism

$$\omega^0_G(pt) \cong A(G)\,.$$

Then, by (4.8) we obtain

Corollary 4.9. *There hold the ring isomorphisms*

$$e^{\pi}_H A(G)_{(\pi)} \cong \bar{e}^{\pi}_H A(N)_{(\pi)} \cong \tilde{e}^{\pi}_{\langle 1 \rangle} A(W)_{(\pi)}\,.$$

Specialize Corollary 4.9 to $\pi = \{$all primes$\}$, then we get Corollary B (Introduction).

Finally we may apply the classifying spaces of families of subgroups due to tom Dieck [6], Let $F_{\pi-\mathrm{solv}}$ denote the family of all solvable π-subgroups of W and $EF_{\pi-\mathrm{solv}}$ its classifying space. There holds the isomorphism

$$(4.10) \qquad \tilde{e}^{\pi}_{\langle 1 \rangle} \tilde{\omega}^{\alpha''}_W\, (X^H \colon Y^H)_{(\pi)} \cong \tilde{\omega}^{\alpha''}_W (X^H \colon Y^H \wedge EF^+_{\pi-\mathrm{solv}})_{(\pi)}$$

by arguments of [6] [7].

Let $H_0, H_1, \ldots, H_k$ $(H_0 = \{1\})$ be a complete system of representatives of P_{π}. Then, from the direct sum decomposition

$$\tilde{\omega}^{\alpha}_G (X \colon Y)_{(\pi)} = \coprod_{0 \leq i \leq k} e^{\pi}_{H_i} \tilde{\omega}^{\alpha}_G (X \colon Y)_{(\pi)}$$

and (4.8) we get the direct sum decomposition

$$(4.11) \qquad \tilde{\omega}^{\alpha}_G\, (X \colon Y)_{(\pi)} = e^{\pi}_{\langle 1 \rangle} \tilde{\omega}^{\alpha}_G (X \colon Y)_{(\pi)} \oplus \coprod_{1 \leq i \leq k} \tilde{e}^{\pi}_{\langle 1 \rangle} \tilde{\omega}^{\alpha_i}_{W_i} (X^{H_i} \colon Y^{H_i})_{(\pi)}$$

where $N_i=N_G(H_i)$, $W_i=N_i/H_i$ and $\alpha_i=\phi_{H_i}(\psi_{N_i}\alpha))$ for $1\leqq i\leqq k$.

Example 1. $G=\mathbb{S}_5$, $\pi=\{\text{all primes}\}$. Conjugacy classes of perfect subgroups are $(\mathbb{A}_5)$ and $(\{1\})$, and we obtain the direct sum decomposition

$$\tilde{\omega}_G^{\alpha}(X:Y)=e_{\langle 1\rangle}\tilde{\omega}_G^{\alpha}(X:Y)+\tilde{\omega}_{Z/2}^{\alpha''}(X^{A_5}:Y^{A_5}),$$

where $\alpha''=\phi_{A_5}\alpha$.

Example 2. $G=\mathbb{S}_6$, $\pi=\{\text{all primes}\}$. There are 4 conjugacy classes of perfect subgroups: (H_1), (H_2), (H_3) and (H_4), where $H_1=\mathbb{A}_6$, $H_2=\mathbb{A}_5$, $H_3\cong\mathbb{A}_5$ and $H_4=\{1\}$. H_2 and H_3 are isomorphic but not conjugate. There is an outer automorphism a of $\mathbb{S}_6$ such that $a(H_2)=H_3$. Thus: $N_G(H_1)=\mathbb{S}_6$, $N_G(H_2)=\mathbb{S}_5$ and $N_G(H_3)\cong\mathbb{S}_5$. We obtain the direct sum decomposition

$$\tilde{\omega}_G^{\alpha}(X:Y)=e_{\langle 1\rangle}\tilde{\omega}_G^{\alpha}(X:Y)\oplus\coprod_{1\leqq i\leqq 3}\tilde{\omega}_{Z/2}^{\alpha_i}(X^{H_i}:Y^{H_i}),$$

where $\alpha_i=\phi_{H_i}(\psi_{N_i}\alpha)$ for $1\leqq i\leqq 3$.

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