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Infinite Dimensional Lie Algebras Acting on Chiral Fields and the Riemann-Hilbert Problem

By

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§ 0. Introduction

The purpose of this article is to construct the transformation theory for the reduction problem of $SU(2)$ Chiral field and $SU(n)$, $SO(n)$ Chiral field by using the Riemann-Hilbert problem, and to study the structure of the infinite dimensional Lie algebra of the infinitesimal transformations.

The equation of motion of $SU(n)$ ($SO(n)$) Chiral field

$$\partial_x(g^{-1}\partial_y g)+\partial_y(g^{-1}\partial_x g)=0\,, \tag{0.1}$$

where $g=g(x, y)$ is an $SU(n)$-valued (resp. $SO(n)$-valued) matrix function. This equation has been studied by many physicists from the viewpoint of the inverse scattering method [13], [14]. Dolan [17] has recently found, by using the method of variations that the infinitesimal transformation group

$$\mathfrak{su}(n)\otimes \boldsymbol{R}[t] \tag{0.2}$$

acts on the totality of solutions of $SU(n)$ Chiral field. But she did not discuss the reduction problem of $SU(2)$ Chiral field. In this paper we show that the Lie algebras

$$\mathfrak{su}(2)\otimes \boldsymbol{R}[t, t^{-1}]\,,\quad \mathfrak{su}(n)\otimes \boldsymbol{R}[t, t^{-1}]\,,\quad \mathfrak{so}(n)\otimes \boldsymbol{R}[t, t^{-1}] \tag{0.3}$$

infinitesimally act on the solutions of the reduction problem of $SU(2)$ Chiral field and $SU(n)$, $SO(n)$ Chiral field, respectively. Our transformation theory is much indebted to the results of Kinnersley-Chitre et al. [1], [2], [3], [4] and Hauser-Ernst [5], [6], [7] and Zakharov-Mikhailov [9] (the Kinnersley-Chitre theory will be briefly described in the appendix of this paper). The relationship between the results of Dolan [17] and ours will be considered in the forthcoming paper [18].

This paper is planned as follows: First of all, we consider the transformation theory for the reduction problem of $SU(2)$ Chiral field. In Section 1, following the discussion of Kinnersley-Chitre [1], [2], we introduce a potential E, which is an analogue of the Ernst potential, and also show the existence of

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infinite number of potentials $\{N^{(m,n)}\}_{m\geq 0, n\geq 1}$. In Section 2, introducing a generating function $F(t)$, we show that the reduction problem is equivalent to the linear problem for a generation function (Theorem 2.7). In Sections 3 and 4, in accordance with the method exploited by Hauser-Ernst [5], [6], we construct the transformation theory by means of the Riemann-Hilbert problem. The infinite dimensional Lie algebra (0.3) is found to act infinitesimally on the solutions of the problem (Theorem 4.3). In Section 5, we discuss the Lie algebras for $SU(n)$, $SO(n)$ Chiral fields.

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§1. The Reduction Problem of $SU(2)$ Chiral Field

The field equation of $SU(2)$ Chiral field is

$$\partial_x(g^{-1}\partial_y g)+\partial_y(g^{-1}\partial_x g)=0 \tag{1.1}$$

where $g=g(x, y)$ is $SU(2)$-matrix function, and x, y denote the light cone coordinates

$$x=\frac{1}{2}(x^0-x^1), \quad y=\frac{1}{2}(x^0+x^1), \quad (x^0, x^1)\in \boldsymbol{R}^2. \tag{1.2}$$

By the reduction problem we mean that we solve the equation (1.1) with the algebraic constraints for the field g

$$g^2=1, \quad g=g^*, \quad \mathrm{tr}\, g=0. \tag{1.3}$$

Here $*$ denotes the hermitian conjugate. These conditions are consistent with the original equation (1.1). The field equation now reads

$$\partial_y(g\partial_x g)+\partial_x(g\partial_y g)=0. \tag{1.4}$$

The reduction problem of $SU(n)$ or $SO(n)$ Chiral field was considered by Zakharov-Mikhailov [9].

It is well known that the reduction problem of $SU(2)$ Chiral field is equivalent to $O(3)$ non-linear σ-model

$$\partial_x\partial_y\vec{g}+(\partial_x\vec{g}\cdot\partial_y\vec{g})\vec{g}=0 \tag{1.5}$$

where $\vec{g}$ is a vector function of x and y valued on the unit sphere of $\boldsymbol{R}^3$, i.e. $\vec{g}^2=1$. From the equation (1.4) we get the following lemma.

Lemma 1.1. *There exists a twist potential* ϕ *uniquely up to integration constants* $i\gamma$, $\gamma\in\mathfrak{su}(2)$, *such that*

$$i\partial_x\phi=g\partial_x g, \quad i\partial_y\phi=-g\partial_y g, \tag{1.6}$$

$$\phi^*=\phi, \quad \mathrm{tr}\,\phi=0. \tag{1.7}$$

Here $\mathfrak{su}(2)$ *is the Lie algebra of* $SU(2)$.

Proof. Since the equation (1.4) is an integrability condition for (1.6), it is clear that there exists a twist potential ψ. And then $\psi^*-\psi$ and $\mathrm{tr}\,\psi$ are constant because of (1.3), (1.6). Hence we get (1.7) with choosing appropriate integration constants. □

Following the discussion in [1], we introduce a potential E through

$$E=g+i\psi . \tag{1.8}$$

This is an analogue of the Ernst potential in the case of the gravitational field equation.

Proposition 1.2. *The Potential E satisfies the following equations:*

$$\partial_x E=\frac{1}{2}(E+E^*)\partial_x E\,, \quad \partial_y E=-\frac{1}{2}(E+E^*)\partial_y E \tag{1.9}$$

$$\det(1-(E+E^*)t)=1-4t^2\,, \tag{1.10}$$

$$\mathrm{tr}\,E=0\,. \tag{1.11}$$

Proof. From $g^2=1$, it follows that

$$\begin{aligned} g\partial_x E &= g\partial_x g+ig\partial_x\psi \\ &= i\partial_x\psi+g^2\partial_x g=\partial_x E\,. \end{aligned}$$

The second equation of (1.9) is also obtained in the same way. The equations (1.10) and (1.11) follow from (1.3), (1.7). □

The equation (1.9) corresponds to the Ernst equation in the gravitational field equation. Conversely, starting from the equations (1.9), we can get the original field equation (1.4).

Proposition 1.3. *Suppose that E satisfies the equation in Proposition* 1.2. *We set*

$$g=\frac{1}{2}(E+E^*)\,. \tag{1.12}$$

Then g is a solution of the reduction problem.

Proof. From the definition (1.10), (1.12), it follows that g is an hermitian matrix whose eigenvalues are ± 1. Hence we obtain (1.3). Next define a potential ψ by

$$i\psi=\frac{1}{2}(E-E^*)\,. \tag{1.13}$$

Then ψ is a trace-free, hermitian matrix because of (1.11). Next we show

$$i\partial_x\psi=g\partial_x g\,, \quad i\partial_y\psi=-g\partial_y g\,. \tag{1.14}$$

Note that the first equation in (1.9) reads

$$\partial_x g+i\partial_x\psi=g\partial_x g+ig\partial_x\psi\,. \tag{1.15}$$

Taking trace of the above equation, we get

$$\mathrm{tr}\,(g\partial_x\psi)=0\,, \tag{1.16}$$

from which we further obtain $g\partial_x\psi=-\partial_x\psi\cdot g$. Compairing the hermitian part and the anti-hermitian part of (1.15), we get the first equation of (1.14). The second one can be obtained in a similar way. The compatibility condition for (1.14) leads to (1.4). The proposition is proved. □

Thus the original field g corresponds to the potential E, and vice versa.

Next we introduce an infinite number of potentials. For technical reasons, we work in the coordinates

$$z=\frac{1}{2}(x^1+ix^0), \quad \bar{z}=\frac{1}{2}(x^1-ix^0). \tag{1.17}$$

Let ∇ be the gradient, $\tilde{\nabla}$ the formal dual operator

$$\nabla=(\partial, \bar{\partial}), \quad \tilde{\nabla}=(\bar{\partial}, -\partial) \tag{1.18}$$

where $\partial=\partial_z$, $\bar{\partial}=\partial_{\bar{z}}$. The equation (1.9) is now written as

$$\nabla E=\frac{i}{2}(E+E^*)\cdot\tilde{\nabla}E \tag{1.19}$$

or equivalently,

$$\nabla E^*=\frac{i}{2}\tilde{\nabla}E^*\cdot(E+E^*). \tag{1.20}$$

According to the discussion of Kinnersley-Chitre [2], we introduce potentials $\{N^{(m,n)}\}_{m\geq 0, n\geq 1}$ through

$$\nabla N^{(m,n)}=E^{(m)*}\nabla E^{(n)} \tag{1.21}$$

$$E^{(n+1)}=N^{(1,n)}+E^{(1)}E^{(n)} \tag{1.22}$$

$$E^{(0)}=1, \quad E^{(1)}=E, \quad N^{(0,n)}=E^{(n)}. \tag{1.23}$$

Proposition 1.4. *The potentials $\{N^{(m,n)}\}_{m\geq 0, n\geq 1}$ are determined uniqnely up to integration constants.*

This proposition can be obtained by induction. To the end, we need the following lemma.

Lemma 1.5. *The potentials $\{E^{(n)}\}_{n\geq 0}$ exists uniquely up to integration constants, and satisfy*

$$\nabla E^{(n)}=\frac{i}{2}(E+E^*)\tilde{\nabla}E^{(n)} \qquad \text{for} \quad n\geqq 0. \tag{1.24}$$

Proof. The proof is done by induction. When $n=1$, the claim of the lemma is nothing but the equation (1.9). Suppose that we have proved the n-th induction step. We show that there exists $N^{(1,n)}$ given by (1.19). Noting that $\tilde{\nabla}\circ\nabla=0$, and (1.19), (1.20), we have

$$\begin{aligned}\tilde{\nabla}\circ(E^*\nabla E^{(n)})&=\tilde{\nabla}E^*\circ\nabla E^{(n)}\\ &=-\frac{i}{2}\nabla E^*\circ(E+E^*)\nabla E^{(n)}\\ &=-\tilde{\nabla}E^*\circ\nabla E^{(n)}\\ &=-\tilde{\nabla}\circ(E^*\nabla E^{(n)}),\end{aligned} \tag{1.25}$$

from which we obtain $\tilde{\nabla}\circ(E^*\nabla E^{(n)})=0$. This guarantees the existence of $N^{(1,n)}$ and $E^{(n+1)}$. Next we show that $\nabla E^{(n+1)}=\frac{i}{2}(E+E^*)\tilde{\nabla}E^{(n+1)}$. From (1.23), it follows that

$$\begin{aligned}\nabla E^{(n+1)}&=\nabla N^{(1,n)}+\nabla E\cdot E^{(n)}+E\nabla E^{(n)}\\&=E^*\nabla E^{(n)}+\frac{i}{2}(E+E^*)\tilde{\nabla}E\cdot E^{(n)}+\frac{i}{2}E(E+E^*)\tilde{\nabla}E^{(n)}\\&=\frac{i}{2}(E+E^*)^2\tilde{\nabla}E^{(n)}+\frac{i}{2}(E+E^*)\tilde{\nabla}E\cdot E^{(n)}\\&=2i\tilde{\nabla}E^{(n)}+\frac{i}{2}(E+E^*)\tilde{\nabla}E\cdot E^{(n)}\,.\end{aligned}$$

In the last step of the above equations, we have used the fact that $(E+E^*)^2=4$. On the other hand, $\tilde{\nabla}E^{(n+1)}=(E+E^*)\tilde{\nabla}E^{(n)}+\tilde{\nabla}E\cdot E^{(n)}$. Hence we obtain the desired result. This completes the induction step. □

The proposition 1.4 can be proved in the same way so that we omit it.

There are remarkable relations among the potentials.

Proposition 1.6. *When choosing appropriate integration constants, the following recursive relations hold;*

(1.26) $$N^{(m,n)}+N^{(n,m)*}=E^{(m)*}E^{(n)}\,,$$

(1.27) $$N^{(m,n+1)}-N^{(m+1,n)}=N^{(m,1)}E^{(n)}\,,\qquad \textit{for}\quad m\geqq 0,\ n\geqq 1\,.$$

Proof. Since $N^{(m,n)}=E^{(m)*}\nabla E^{(n)}$, and $N^{(n,m)*}=\nabla E^{(m)*}\cdot E^{(n)}$, we have $\nabla(N^{(m,n)}+N^{(n,m)*})=\nabla(E^{(m)*}E^{(n)})$, which implies (1.26). Next we shall prove (1.27). From (1.21), (1.22), and (1.26), it follows that

$$\begin{aligned}&\nabla(N^{(m,n+1)}-N^{(m+1,n)})\\&\quad=E^{(m)*}\nabla E^{(n+1)}-E^{(m+1)*}\nabla E^{(n)}\\&\quad=E^{(m)*}(E^*\nabla E^{(n)}+\nabla E\cdot E^{(n)}+E\nabla E^{(n)})-E^{(m+1)*}\nabla E^{(n)}\\&\quad=\nabla N^{(m,1)}\cdot E^{(n)}+(E^{(m)*}E-N^{(1,m)*})\nabla E^{(n)}\\&\quad=\nabla N^{(m,1)}\cdot E^{(n)}+N^{(m,1)}\nabla E^{(n)}\\&\quad=\nabla(N^{(m,1)}E^{(n)})\,.\end{aligned}$$

This completes the proof.

§ 2. Generating Function

The concept of a generating function of the Ernst potential was originated in the gravitational field theory [2]. In our case it is defined by

(2.1) $$F(t)=\sum_{n=0}^{\infty}E^{(n)}t^n$$

where $\{E^{(n)}\}_{n\geqq 0}$ are the potentials given by (1.23).

In this section we shall show that the reduction problem is equivalent to a system of linear differential equations with several algebraic constraints satisfied by a generating function. Such a method was presented by Hauser-Ernst [6].

First we obtain the follwing proposition.

Proposition 2.1. *A generating function* $F(t)$ (2.1) *solves the following system of linear differential equations;*

$$\nabla F(t)=\frac{i}{2}(E+E^{*})\tilde{\nabla}F(t)\,, \tag{2.2}$$

$$\nabla F(t)=\frac{t}{1-4t^{2}}(\nabla E+2it\tilde{\nabla}E)F(t)\,. \tag{2.3}$$

Proof. By virtue of (1.24), one can easily obtain (2.2). To show (2.3), we note that we obtain from (1.22)

$$\nabla E^{(n+1)}=(E+E^{*})\nabla E^{(n)}+\nabla E\cdot E^{(n)}\,.$$

Multiplying t^{n+1} to the both sides, and summing over n, we have

$$\nabla F(t)=t\{(E+E^{*}\nabla F(t)+\nabla E\cdot F(t))\}\,.$$

By making use of (2.2), this is rewritten as

$$\partial F-2it\bar{\partial}F=t\partial E\cdot F,\quad 2it\partial F+\bar{\partial}F=t\bar{\partial}E\cdot F\,,$$

which leads to (2.3). □

The system (2.3) reads

$$dF(t)=\Omega(t)F(t)\,, \tag{2.4}$$

where d denotes the exterior differentiation with respect to x, y, and $\Omega(t)$ is a 1-form given by

$$\Omega(t)=\frac{t}{1-2t}\partial_{x}E\,dx+\frac{t}{1+2t}\partial_{y}E\,dy\,. \tag{2.5}$$

As a corollary of Proposition 2.1, we have

Corollary 2.2.

$$d(\det F(t))=0\,. \tag{2.6}$$

To rewrite the equation (1.9), we introduce $A(t)$ through

$$A(t)=1-(E+E^{*})t \tag{2.7}$$

Lemma 2.3. *The equations* (1.9) *are equivalent to*

$$tdE=A(t)\Omega(t)\,. \tag{2.8}$$

This lemma follows from

$$\begin{aligned} &tdE-A(t)\Omega(t)\\ &=\frac{t^{2}}{1-2t}\{2\partial_{x}E+(E+E^{*})\partial_{x}E\}\,dx+\frac{t^{2}}{1+2t}\{2\partial_{y}E+(E+E^{*})\partial_{y}E\}\,dy\,. \end{aligned}$$

Next we show that the integrability condition for (2.4)

(2.9)
$$\partial_x\partial_y E+\frac{1}{4}[\partial_x E,\ \partial_y E]=0$$

can be derived from (2.8) (Proposition 2.5). For the purpose, we need the following lemma.

Lemma 2.4. *We have*

(21.0)
$$\partial_y E^*\cdot\partial_x E=0\,,\quad \partial_x E^*\cdot\partial_y E=0\,.$$

Proof. By virtue of (1.9), we obtain

$$\begin{aligned}\partial_y E^*\cdot\partial_x E&=-\frac{1}{2}\partial_y E^*\cdot(E+E^*)\partial_x E\\&=-\partial_y E^*\cdot\partial_x E\,,\end{aligned}$$

which leads to the first equation of (2.10). The second one of (2.9) can be obtained in the same way. □

Proposition 2.5. *The equations* (2.8) *yield the integrability condition for* (2.4)

(2.11)
$$d\Omega(t)=\Omega(t)^2\,.$$

Proof. Form the previous lemma, it follows that

$$\begin{aligned}dA(t)\cdot\Omega(t)&=\frac{t^2}{1-2t}(\partial_y E\cdot\partial_x E+\partial_y E^*\cdot\partial_x E)dxdy\\&\quad-\frac{t^2}{1+2t}(\partial_x E\cdot\partial_y E+\partial_x E^*\cdot\partial_y E)dxdy\\&=\Big(\frac{t^2}{1-2t}\partial_y E\cdot\partial_x E-\frac{t^2}{1+2t}\partial_x E\cdot\partial_y E\Big)dxdy\\&=-tdE\cdot\Omega(t)\,.\end{aligned}$$

Noting that $d(A(t)\Omega(t))=0$ which follows from (2.8), we have $A(t)d\Omega(t)=A(t)\Omega(t)^2$. Since $A(t)$ is invertible when t is small, (2.11) is proved. □

It should be noted that the equation (1.9) cannot be derived from the integrability condition (2.9) since (2.9) is an equation of second order. Proposition 2.5 means that the equation (1.9) is a special class of (2.9).

The equation (2.8) is rewritten as

(2.12)
$$A(t)dF(t)=tdE\cdot F(t)\,.$$

By virtue of this we obtain

Lemma 2.6. *We have*

(2.13)
$$d(F(t)^\dagger A(t)F(t))=0$$

where $F(t)^\dagger=F(\bar{t})^*$, $\bar{t}$ *denotes the complex conjugate of* t.

These preparations enables us to convert the equations (1.9), (1.10) and (1.11) into those satisfied by a generating function.

Theorem 2.7. *Suppose that the potential* E *is a solution of the equations* (1.9), (1.10) *and* (1.11). *Then there exists a fundamental solution matrix* $F(t)$ *subject to the five equations below*:

$$dF(t)=\Omega(t)F(t) \tag{2.14}$$

$$A(t)dF(t)=tdE\cdot F(t)\,, \tag{2.15}$$

$$F(t)=1+Et+\cdots\,, \quad as\ t\to 0\,, \tag{2.16}$$

$$\det F(t)=(1-4t^2)^{-1/2}\,, \tag{2.17}$$

$$F(t)^{\dagger}A(t)F(t)=1\,. \tag{2.18}$$

Here $\Omega(t)$ *and* $A(t)$ *are given by* (2.5) *and* (2.7), *respectively. Conversely* E *satisfies the equations* (1.9), (1.10) *and* (1.11) *provided that there is a fundamental solution matrix* $F(t)$ *subject to the above equations* (2.14)—(2.18).

Proof. First suppose that E satisfies the equations (1.9), (1.10) and (1.11). From Lemma 2.3 and Proposition 2.5, (2.10), it follows that there exists a fundamental solution matrix $F(t)$ of (2.14) and (2.15) satisfying (2.16). Moreover Lemma 2.6 shows that $F(t)$ satisfies (2.18) at the same time under a suitable choice of a gauge of $F(t)$. Since $\det A(t)=1-4t^2$, one has $|\det F(t)|^2=(1-4t^2)^{-1}$. Use of Corollary 2.2 leads to $\det F(t)=(1-4t^2)^{-1/2}e^{2ia(t)}$ where $a(t)$ is real valued when t is real. Since (2.16) gives $\det F(t)=1+O(t^2)$ as $t\to 0$, one can find $a(t)$ such that $a(0)=\dot{a}(0)=0$. Thus $F(t)e^{-ia(t)}$ satisfies all of the requirements. Next we show the converse. The equations (2.14), (2.15) yield (2.8) which is equivalent to (1.9). The equation (1.12) follows from (1.15) and (1.16). Finally we verify (1.11). The expansion (2.16) gives

$$\det F(t)=1+(\mathrm{tr}\,E)t+\cdots \qquad \text{as} \quad t\to 0\,.$$

On the other hand, (2.17) yields

$$\det F(t)=1+O(t^2) \qquad \text{as} \quad t\to 0\,.$$

Compairing these expression, one obtains (1.11). □

From now on, a function $F(t)$ satisfying the conditions of Theorem 2.7 will be simply called a generating function.

Now we shall express the potentials $\{N^{(m,n)}\}$ introduced in the previous section in the language of a generating function. Set

$$G(s,\,t)=\frac{1}{s-t}\{s-tF(s)^{-1}F(t)\}\,. \tag{2.19}$$

The function of this type was originally considered in [2] and [16]. Expand $G(s,\,t)$ into a power series of s, t

$$G(s,\,t)=\sum_{m,n=0}^{\infty}G^{(m,n)}s^m t^n\,. \tag{2.20}$$

Note that $G^{(m,0)}$ for $m\geqq 1$. First we obtain a proposition correspoding to Proposition 1.7.

Proposition 2.8. *Suppose that $F(t)$ is a generating function. Then the recursive relations*

$$G^{(m,n+1)}-G^{(m+1,n)}=G^{(m,1)}F^{(n)}, \tag{2.21}$$

$$G^{(m,n)}+G^{(n,m)*}=F^{(m)*}F^{(n)} \tag{2.22}$$

are valid for $m\geqq 0$, $n\geqq 1$. Here we set $G^{(0,n)}=F^{(n)}$.

Proof. From the definition of $G(s, t)$, it follows that

$$t^{-1}\{G(s, t)-1\}-s^{-1}\{G(s, t)-F(t)\}=\partial_w G(s, w)|_{w=0}F(t). \tag{2.23}$$

Substituting the expansions of $F(t)$ and $G(s, t)$ into the both sides of (2.23), and collecting all terms of the same order in s, t, equating the resulting coefficients, we have (2.21). Setting $m=0$ in (2.21), we obtain

$$F^{(n+1)}=G^{(1,n)}+F^{(1)}F^{(n)}. \tag{2.24}$$

To prove (2.22), it is sufficient to show

$$G(s, t)+G(s, t)^{\dagger}=1+F(s)^{\dagger}F(t). \tag{2.25}$$

The definition of $G(s, t)$ leads to

$$G(s, t)+G(t, s)^{\dagger}=1+\frac{1}{s-t}\{sF(s)^{\dagger}[F(t)^{\dagger}]^{-1}-tF(s)^{-1}F(t)\}.$$

Since $F(s)^{-1}=F(s)^{\dagger}A(s)$, we get (2.25). □

The equations (2.21), (2.22) and (2.24) correspond to (1.27), (1.26) and (1.22), respectively. In closing this section, we shall show the relations corresponding to (1.21).

Proposition 2.9. *The differential recursive relations*

$$dG^{(m,n)}=F^{(m)*}dF^{(n)} \tag{2.26}$$

hold for $m\geqq 0$, $n\geqq 1$.

Proof. By means of (2.18), $G(s, t)$ is rewritten as

$$G(s, t)=\frac{1}{s-t}\{s-tF(s)^{\dagger}A(s)F(t)\}.$$

Differentiating the both sides of the above identity, we have

$$\begin{aligned} dG(s, t)&=-\frac{t}{s-t}\{sF(s)^{\dagger}dE^{*}\cdot F(t)+F(s)^{\dagger}dA(s)\cdot F(t)+F(s)^{\dagger}A(s)dF(t) \\ &=-\frac{t}{s-t}\{-sF(s)^{\dagger}dE\cdot F(t)+F(s)^{\dagger}A(s)dF(t)\} \\ &=\frac{t}{s-t}\{t^{-1}sF(s)^{\dagger}A(t)dF(t)-F(s)^{\dagger}A(s)dF(t)\}. \end{aligned}$$

Therefore

$$dG(s, t)=F(s)^{\dagger}dF(t). \tag{2.27}$$

Expanding the both sides into a power series in s, t, we obtain (2.26).

Thus $\{G^{(m,n)}\}$ can be identified as the potentials $\{N^{(m,n)}\}$. Among the above equations (2.21), (2.22) and (2.26), (2.22) is essential to construct the transformation theory. We emphasize that this equation is an immediate conclusion of the definition of $G(s, t)$.

§ 3. Riemann-Hilbert Problem

In this section, following Hauser-Ernst [6], and Zakharov and his coworkers [9], [10], we shall present an algebraic approach to construct transformations for solutions of the reduction problem of $SU(2)$ Chiral field. The transformations are achieved by use of the Riemann-Hilbert problem. We call them the Riemann-Hilbert (RH) transformations. Our aim is to investigate the algebraic structure of the Lie algebra of the infinitesimal RH transformations, so that the details of analytic aspects of the transformation theory are not considered here.

Begin with a generating function $F_0(t)$ such that

$$dF_0(t)=\Omega_0(t)F_0(t)\,, \tag{3.1}$$

$$A_0(t)dF_0(t)=t\,dE_0\cdot F_0(t)\,, \tag{3.2}$$

$$F_0(0)=1\,,\quad \dot{F}_0(0)=E_0\,, \tag{3.3}$$

$$\det F_0(t)=(1-4t^2)^{-1/2}\,, \tag{3.4}$$

$$F_0(t)^\dagger A_0(t)F_0(t)=1\,. \tag{3.5}$$

Here the dot in (3.3) denotes the differentiation with respect to t, and $A_0(t)$, $\Omega_0(t)$ are respectively defined by (2.5), (2.8) for the potential E_0. Let C be a small circle in the complex t-plane whose center is the origin, such that $F_0(t)$ is holomorphic in $C\cup C_+$. Here $C_+(C_-)$ denotes the inside (outside) of C.

Note that $F_0(t)$ is not uniquely determined by the above conditions. In fact, if $v(t)$ is a 2×2 matrix depending on only t and holomorphic in $C\cup C_+$ such that

$$v(t)^\dagger v(t)=1\,,\quad \det v(t)=1\,, \tag{3.6}$$

$$v(0)=1\,,\quad \dot{v}(0)=0\,, \tag{3.7}$$

then $F_0(t)v(t)$ is also a generating function for the potential E_0. Let $u(t)$ be an 2×2 matrix depending on only t, analytic on C, such that

$$u(t)^\dagger u(t)=1\,, \tag{3.8}$$

$$\det u(t)=1\,. \tag{3.9}$$

Consider the Riemann-Hilbert problem

$$X_-(s)=X_+(s)H(s)\quad (s\in C) \tag{3.10}$$

$$H(t)=F_0(t)u(t)F_0(t)^{-1}\,, \tag{3.11}$$

with the normalization condition

$$X_+(0)=1\,, \tag{3.12}$$

where $X_\pm(t)$ is holomorphic in $C_\pm$ and continuous on C, and invertible in $C_\pm\cup C$, respectively. We assume that one can uniquely solve this problem.

This hypothesis is not so strong, since there is a fundamental solution matrix if $u(t)$ is very close to the unit matrix. It is also noted that $X_{\pm}(t)$ consequently is analytic on C from the analyticity of $H(t)$. Set

$$F(t)=\begin{cases} X_+(t)F_0(t) & \text{in } C_+, \\ X_-(t)F_0(t)u(t)^{-1} & \text{in } C_-, \end{cases} \tag{3.13}$$

$$X(t)=X_+(t) \text{ in } C_+, \quad =X_-(t) \text{ in } C_-, \tag{3.14}$$

$$E=E_0+\dot{X}_+(0), \tag{3.15}$$

$$A(t)=1-(E+E^*)t, \tag{3.16}$$

$$\Omega(t)=\frac{t}{1-2t}\partial_x E dx+\frac{t}{1+2t}\partial_y E dy. \tag{3.17}$$

By a similar method as in [6], one can show that E is a solution of the reduction problem and that $F(t)$ is a generating function for E.

Proposition 3.1. *The following equation hold*:

$$\det X(t)=1, \tag{3.18}$$

$$X(t)^\dagger A(t)X(t)=A_0(t), \tag{3.19}$$

$$\det A(t)=\det A_0(t)=1-4t^2, \tag{3.20}$$

$$dX(t)=\Omega(t)X(t)-X(t)\Omega_0(t), \tag{3.21}$$

$$A(t)dX(t)+t[X(t)^\dagger]^{-1}dE_0=t[dE\cdot X(t)]. \tag{3.22}$$

From this proposition, we obtain

Theorem 3.2. *$F(t)$ is a generating function; that is, $F(t)$ satisfies the defining relations*

$$dF(t)=\Omega(t)F(t), \tag{3.23}$$

$$F(0)=1, \quad \dot{F}(0)=E, \tag{3.24}$$

$$\det F(t)=(1-4t^2)^{-1/2}, \tag{3.25}$$

$$F(t)^\dagger A(t)F(t)=1, \tag{3.26}$$

$$A(t)dF(t)=t\, dE\cdot F(t). \tag{3.27}$$

Proof. The above equations except the last one are derived, respectively, from (3.1) and (3.21), (3.3) and (3.14), (3.20) and (3.18), (3.5) and (3.19). Finally we show (3.27). The equations (3.22) and(3.13) give

$$\begin{aligned} A(t)dF(t)&=A(t)\{-t[X(t)^\dagger]^{-1}dE_0+t\, dE\cdot X_+(t)\}F_0(t)+A(t)X_+(t)\Omega_0(t)F_0(t) \\ &=t\, dE\cdot F(t) \end{aligned}$$

since $A(t)X_+(t)\Omega_0(t)=t[X_+(t)^\dagger]^{-1}dE_0\cdot F_0(t)$ from (3.2) and (3.19). This completes the proof. □

Next we show Proposition 3.1.

Proof of (3.24). First we show

$$[X_+(t)^\dagger]^{-1}A_0(t)X_+(t)^{-1}=\text{same} + \text{replacing} - . \tag{3.28}$$

From (3.5) and (3.11), it follows that

the left-hand side of (3.28)

$$=[X_-(t)^\dagger]^{-1}[F_0(t)^\dagger]^{-1}u(t)^\dagger F_0(t)^\dagger A_0(t)F_0(t)u(t)F_0(t)^{-1}X_-(t)^{-1}$$

$$=[X_-(t)^\dagger]^{-1}[F_0(t)^\dagger]^{-1}F_0(t)^{-1}X_-(t)^{-1}$$

$$=[X_-(t)^\dagger]^{-1}A_0(t)X_-(t)^{-1}.$$

Hence $[X(t)^\dagger]^{-1}A_0(t)X(t)^{-1}$ is an entire function of t. Since $A_0(t)$ is a linear function of t, this function is also linear in t,

$$[X(t)^\dagger]^{-1}A_0(t)X(t)^{-1}=B+Ct. \tag{3.29}$$

Considering the expansion of the left-hand side at $t=0$, we have $B=1$ and $C=-(E+E^*)$. □

Proof of (3.20). It is obvious from (3.18), (3.19). □

Proof of (3.21). We can easily show

$$dX_-(t)\cdot X_-(t)^{-1}+X_-(t)\Omega_0(t)X_-(t)^{-1}=\text{same} + \text{replacing} - . \tag{3.30}$$

Hence, taking the simple poles of $\Omega_0(t)$ into account, we can set

$$dX(t)\cdot X(t)^{-1}+X(t)\Omega_0(t)X(t)^{-1}=\frac{t}{1-2t}B\,dx+\frac{t}{1+2t}C\,dy.$$

By the same argument as in the proof of (3.19), we obtain $B=\partial_x E$, $C=\partial_y E$. □

Proof of (3.22). First we show

$$A(t)dX_-(t)\cdot X_-(t)^{-1}+t[X_-(t)^\dagger]^{-1}dE_0\cdot X_-(t)=\text{same} + \text{replacing} - . \tag{3.31}$$

Using (3.1), (3.2), (3.5) and (3.10), (3.11), we have

$$\begin{aligned} A(t)dX_-(t)\cdot X_-(t)^{-1}=&A(t)dX_+(t)\cdot X_+(t)^{-1} \\ &+A(t)X_+(t)\{\Omega_0(t)-H(t)\Omega_0(t)H(t)^{-1}\}X_+(t)^{-1}, \end{aligned} \tag{3.32}$$

$$A(t)X_+(t)\Omega_0(t)X_+(t)^{-1}=t[X_+(t)^\dagger]^{-1}dE_0\cdot X_+(t)^{-1}, \tag{3.33}$$

$$A_0(t)H(t)\Omega_0(t)=t[H(t)^*]^{-1}dE_0. \tag{3.34}$$

Substituting these into the left-hand side of (3.31) we obtain (3.31). The rest of the proof can be done in a similar way as above, so that we omit it. □

In [6], it has been shown that the totality of the RH transformations is equipped with group structure under natural composition of matrices. Precisely we have

Proposition 3.3 (cf. [6]). *Suppose that $u_1(t)$, $u_2(t)$, which satisfy the conditions* (3.8), (3.9), *transform $F_0(t)$ to $F_1(t)$, and $F_1(t)$ to $F_2(t)$. Then $u_2(t)u_1(t)$ transforms $F_0(t)$ to $F_2(t)$.*

§ 4. The Infinitesimal Riemann-Hilbert Transformations

It is known that the Riemann-Hilbert problem (3.10) is rewritten to the integral equation

$$F\cdot(1-K)=F_0 \tag{4.1}$$

(see [6], [11]). The integral operator K is defined by

$$(\Phi\cdot K)(t)=\frac{1}{2\pi i}\int_C ds\ \Phi(s)K(s, t), \quad s, t\in C \tag{4.2}$$

where the kernel function is

$$K(s, t)=\frac{t}{s(s-t)}\{F_0(s)^{-1}F_0(t)-u(s)F_0(s)^{-1}F(t)s(t)^{-1}\}. \tag{4.3}$$

We remark that (4.1) actually gives the solution of the Riemann-Hilbert problem (3.10) if $u(t)$ is very close to the unit matrix (when $u(t)=1$, K is a null operator).

In what follows, the infinitesimal form of (4.1) will be called the infinitesimal Riemann-Hilbert (RH) transformation (the precise definition will be given later). The discussion in this section corresponds to the converse of the procedure exploited by Hauser-Ernst [5].

Assume that

$$u(t)=\exp v(t) \tag{4.4}$$

where $v(t)$ satisfies the conditions that $v(t)^{\dagger}+v(t)=0$, and $\operatorname{tr} v(t)=0$. Substituting (4.4) into (4.1), and neglecting the higher order terms with respect to $v(t)$, we have the infinitesimal form of (4.1)

$$F_0(t)\to F_0(t)+\frac{1}{2\pi i}\int_C ds\frac{t}{s(s-t)}\{F_0(t)-F_0(s)(1+v(t))F_0(s)^{-1}F_0(t)(1-v(t))\}$$

$$=F_0(t)+\frac{1}{2\pi i}\int_C ds\frac{t}{s(s-t)}\{-F_0(s)v(s)F_0(s)^{-1}F_0(t)+F_0(t)v(t)\}.$$

Since the integrand of the last equation is analytic at $s=t$, we may analytically continue t into C_+. Then we have

$$F_0(t)\to F_0(t)-\frac{1}{2\pi i}\int_C ds\frac{t}{s(s-t)}F_0(s)v(s)F_0(s)^{-1}F_0(t) \tag{4.5}$$

$$=F_0(t)+\frac{1}{2\pi i}\int_C ds\{s^{-1}F_0(s)v(s)(G_0(s, t)-1)-\frac{t}{s(s-t)}F_0(s)v(s)\}$$

where $G_0(s, t)$ is defined by (2.19) where $F(t)$ is replaced with $F_0(t)$.

Expand $G_0(s, t)$ and $F_0(t)$ into power series of s and t

$$G_0(s, t)=\sum_{m,n=0}^{\infty}G^{(m,n)}s^m t^n, \quad F_0(t)=\sum_{n=0}^{\infty}F^{(n)}t^n. \tag{4.6}$$

Also define

$$v^{(p)}=\frac{1}{2\pi i}\int_C ds\ s^{p-1}v(s). \tag{4.7}$$

Noting that $t/s(s-t)=\sum_{n=1}^{\infty}t^{n}/s^{n+1}$, and inserting (4.6), (4.7) into (4.5), we obtain *the infinitesimal RH transformation* (associated with $v(t)$)

$$F^{(n)}\to F^{(n)}+\sum_{p,q=0}^{\infty}F^{(p)}v^{(p+q)}G^{(q,n)}-\sum_{p=0}^{\infty}F^{(p)}v^{(p-n)}\,. \tag{4.8}$$

We remark that (4.8) actually expresses an infinitesimal form of the RH transformation associated with $u(t)$ when $v(t)$ is very close to the null matrix.

Next consider the generators of the infinitesimal RH transformations. Set

$$v(t)=\gamma^{(k)}t^{-k}\,,\quad \gamma^{(k)}\in\mathfrak{su}(2)\,. \tag{4.9}$$

Then we have the generators

$$G^{(0,n)}\to G^{(0,n)}+\gamma^{(k)}G^{(k,n)}-G^{(0,n+k)}\gamma^{(k)}+\sum_{j=0}^{k}G^{(0,j)}\gamma^{(k)}G^{(k-j,n)} \quad \text{for } k\geqq 0,\ n\geqq 1\,, \tag{4.10$_k$}$$

and

$$G^{(0,n)}\to G^{(0,n)}-G^{(0,n-k)}\gamma^{(-k)} \qquad \text{for } k\geqq 1,\ n\geqq 1\,. \tag{4.10$_{-k}$}$$

We denote these generators by $\gamma^{(k)}t^{-k}$. We set the totality of the infinitesimal RH transformations as $\mathcal{G}$. That is,

$$\mathcal{G}=\text{span of }\{\gamma^{(k)}t^{-k}\,|\,k\in\boldsymbol{Z},\ \gamma^{(k)}\in\mathfrak{su}(2)\}\,. \tag{4.11}$$

Next we consider the action of $\mathcal{G}$ on the totality of the potentials $G^{(m,n)}$. We have the following proposition.

Proposition 4.1. *For any $k\geqq 0$, $\gamma^{(k)}t^{-k}$ infinitesimally acts on the totality of potentials $\{G^{(m,n)}\}$ as follows:*

$$\gamma^{(k)}t^{-k}:\ G^{(m,n)}\mapsto G^{(m,n)}+\gamma^{(k)}G^{(m+k,n)}-G^{(m,n+k)}\gamma^{(k)}+\sum_{j=1}^{k}G^{(m,j)}\gamma^{(k)}G^{(k-j,n)} \quad \textit{for } m\geqq 0,\ n\geqq 1\,. \tag{4.12}$$

Proof. The proof is carried out by induction. The claim of the proposition is true for $n\geqq 1$ when $m=0$. Assume that (4.12) holds for $n\geqq 1$ when $m-1$. The recursive formula (2.21) and the assumption of induction give

$$\begin{aligned} G^{(m,n)}&=G^{(m-1,n+1)}-G^{(m-1,1)}G^{(0,n)}\\ &\overset{\gamma^{(k)}t^{-k}}{\longmapsto} G^{(m,n)}+\gamma^{(k)}G^{(m+k,n)}-G^{(m,n+k)}\gamma^{(k)}\\ &\quad -G^{(m-1,1)}\gamma^{(k)}G^{(k,n)}+G^{(m-1,1+k)}\gamma^{(k)}G^{(0,n)}\\ &\quad +\sum_{j=1}^{k}G^{(m-1,j)}\gamma^{(k)}G^{(k-j,n+1)}-\sum_{j=1}^{k}G^{(m-1,1)}G^{(0,j)}\gamma^{(k)}G^{(k-j,n)}\\ &\quad -\sum_{j=1}^{k}G^{(m-1,j)}\gamma^{(k)}G^{(k-j,1)}G^{(0,n)}\,. \end{aligned} \tag{4.13}$$

Here we neglect the higher order terms with respect to $\gamma^{(k)}$. Note that the last three terms of the above equation read

$$\sum_{j=0}^{k-1} G^{(m-1,j+1)}\gamma^{(k)}G^{(k-j,n)}-\sum_{j=1}^{k} G^{(m-1,1)}G^{(0,j)}\gamma^{(k)}G^{(k-j,n)}$$

$$=\sum_{j=0}^{k-1} G^{(m,j)}\gamma^{(k)}G^{(k-j,n)}+G^{(m-1,1)}\gamma^{(k)}G^{(k,n)}-G^{(m-1,1)}G^{(0,k)}\gamma^{(k)}G^{(0,n)},$$

and that

$$G^{(m-1,k+1)}-G^{(m-1,1)}G^{(0,k)}=G^{(m,k)}.$$

Substituting these into (4.13), we complete the induction step. □

Before proceeding to the next proposition, we note that $(4.10)_{-k}$ is rewritten as

$$G^{(0,n)}\longmapsto G^{(0,n)}+\sum_{j=0}^{k-1}\delta_{0,j}\gamma^{(-k)}\delta_{k-j,n}+\gamma^{(-k)}G^{(-k,n)}-G^{(0,n-k)}\gamma^{(-k)}. \tag{4.14}$$

Here we have set

$$G^{(p,-p)}=-G^{(-p,p)}=1 \qquad \text{for any } p\geqq 1, \tag{4.15}$$

and $G^{(m,n)}=0$ for other negatives indices. Then the recursive formula (2.21) is valid for $m\geqq 0$ or $n\geqq 1$.

Proposition 4.2. *For any* $k\geqq 1$, $\gamma^{(-k)}t^{-k}$ *infinitesimally acts on the totality of potentials* $\{G^{(m,n)}\}$ *as follows*:

$$\gamma^{(-k)}t^{k}: G^{(m,n)}\longmapsto G^{(m,n)}+\sum_{j=0}^{k-1}\delta_{m,j}\gamma^{(-k)}\delta_{k-j,n} +\gamma^{(-k)}G^{(m-k,n)}-G^{(m,n-k)}\gamma^{(-k)} \qquad \textit{for } m\geqq 0,\ n\geqq 1. \tag{4.16}$$

Proof. The proof is done by induction. Assume that the $(m-1)$-th induction step holds. Then, by a similar way as in Proposition 4.1, we obtain

$$\begin{aligned} G^{(m,n)}\xrightarrow{\gamma^{(-k)}t^{k}} & G^{(m,n)}+\sum_{j=0}^{k-1}\delta_{m-1,j}\gamma^{(-k)}\delta_{k-j,n+1} \\ &+\gamma^{(-k)}G^{(m-k,n)}-G^{(m,n-k)}\gamma^{(-k)} \\ &-G^{(m-1,1)}\sum_{j=0}^{k-1}\delta_{0,j}\gamma^{(-k)}\delta_{k-j,n}-G^{(m-1,1)}\gamma^{(-k)}G^{(-k,n)} \\ &+G^{(m-1,1-k)}\gamma^{(-k)}G^{(0,n)}-\sum_{j=0}^{k-1}\delta_{m-1,j}\gamma^{(-k)}\delta_{k-j,1}G^{(0,n)}. \end{aligned} \tag{4.17}$$

Note that the following identities hold:

$$\sum_{j=0}^{k-1}\delta_{0,j}\delta_{k-j,n}+G^{(k,n)}=\delta_{k,n}-\delta_{k,n}=0,$$

$$\sum_{j=0}^{k-1}\delta_{m-1,j}\delta_{k-j,1}-G^{(m-1,1-k)}=\delta_{m-1,k-1}-\delta_{m-1,k-1}=0,$$

$$\sum_{j=0}^{k-1}\delta_{m-1,j}\gamma^{(-k)}\delta_{k-j,n+1}=\sum_{j=1}^{k-1}\delta_{m,j}\gamma^{(-k)}\delta_{k-j,n}, \qquad \text{for } m\geqq 1,\ n\geqq 1.$$

Substituting these into (4.17), we complete the m-th induction step. □

We obtain the main theorem in this section.

Theorem 4.3. *The Lie algebra* $\mathfrak{G}$ *is isomorphic to*

(4.18) $$\mathfrak{su}(2)\underset{\boldsymbol{R}}{\otimes}\boldsymbol{R}[t, t^{-1}].$$

Namely the commutation relation

(4.19) $$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}]=[\gamma^{(k)}, \gamma^{(l)}]t^{-k-l}$$

holds for any k, l. *The bracket of the right hand side is the usual one of* $\mathfrak{su}(2)$, *and that of left hand side is the bracket among the infinitesimal transformations.*

Kinnersley-Chitre [2] showed that the infinite dimensional Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})\otimes\boldsymbol{R}[t, t^{-1}]$ acts on the totality of solutions of the stationary axially-symmetric gravitational equation (refer to the appendix). The proof of Theorem 4.3 can be done in the same manner as in [2]. However Kinnersley-Chitre give no proofs there. Hence we would like to give the proof in detail.

Proof of Theorem 4.3. The proof is subdevided into four cases. The first case is

(4.20) $$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}]=[\gamma^{(k)}, \gamma^{(l)}]t^{-k-l} \qquad \text{for any } k, l\geqq 0.$$

By making use of Proposition 4.1, for any non-negative integer k, l, we have

$$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}]:$$

$$\begin{aligned} G^{(m,n)}\longmapsto &[\gamma^{(k)}, \gamma^{(l)}]G^{(m+l-k,n)}-G^{(m,n+l+k)}[\gamma^{(k)}, \gamma^{(l)}] \\ &-\sum_{j=1}^{k}G^{(m,j+l)}\gamma^{(l)}\gamma^{(k)}+\sum_{j=0}^{k}G^{(m,j)}\gamma^{(k)}\gamma^{(l)}G^{(k+l-j,n)} \\ &+\sum_{j=1}^{l}G^{(m,j+k)}\gamma^{(k)}\gamma^{(l)}G^{(l-j,n)}-\sum_{j=1}^{l}G^{(m,j)}\gamma^{(l)}\gamma^{(k)}G^{(l+k-j,n)} \\ =&[\gamma^{(k)}, \gamma^{(l)}]G^{(m+l+k,n)}-G^{(m,n+l+k)}[\gamma^{(k)}, \gamma^{(l)}] \\ &+\sum_{j=1}^{l+k}G^{(m,j)}\gamma^{(k)}\gamma^{(l)}G^{(l+k-j,n)}-\sum_{j=1}^{l+k}G^{(m,j)}\gamma^{(l)}\gamma^{(k)}G^{(l+k-j,n)} \\ =&[\gamma^{(k)}, \gamma^{(l)}]G^{(m+l+k,n)}-G^{(m,n+l+k)}[\gamma^{(k)}, \gamma^{(l)}] \\ &+\sum_{j=1}^{l+k}G^{(m,j)}[\gamma^{(k)}, \gamma^{(l)}]G^{(l+k-j,n)}. \end{aligned}$$

This implies (4.20). The second case

(4.21) $$[\gamma^{(-k)}t^{k}, \gamma^{(-l)}t^{l}]=[\gamma^{(-k)}, \gamma^{(-l)}]t^{k+l} \qquad \text{for any } k, l\geqq 0,$$

can be proved in a similar way. In fact we obtain

$$[\gamma^{(-k)}t^{k}, \gamma^{(-l)}t^{l}]:$$

$$\begin{aligned} G^{(m,n)}\longmapsto &\left(\sum_{j=0}^{k-1}\delta_{m,j}\delta_{k-j,n-l}+\sum_{j=0}^{l-1}\delta_{m-k,j}\delta_{l-j,n}\right)\gamma^{(-k)}\gamma^{(-l)} \\ &-\left(\sum_{j=0}^{k-1}\delta_{m-l,j}\delta_{k-j,n}+\sum_{j=0}^{l-1}\delta_{m,j}\delta_{l-j,n-k}\right)\gamma^{(-l)}\gamma^{(-k)} \end{aligned}$$

$$+[\gamma^{(-k)}, \gamma^{(-l)}]G^{(m-l-k,n)}-G^{(m,n-k-l)}[\gamma^{(-k)}, \gamma^{(-l)}]$$
$$=\sum_{j=0}^{l+k-1}\delta_{m,j}[\gamma^{(-k)}, \gamma^{(-l)}]\delta_{l+k-j,n}+[\gamma^{(-k)}, \gamma^{(-l)}]G^{(m-l-k,n)}$$
$$-G^{(m,n-l-k)}[\gamma^{(-k)}, \gamma^{(-l)}].$$

Next we proceed to the third case

$$[\gamma^{(k)}t^{-k}, \gamma^{(-l)}\gamma^{l}]=[\gamma^{(k)}, \gamma^{(-l)}]t^{-k+l} \qquad \text{for any } 0<l\leqq k. \tag{4.22}$$

The proof of this case is rather complicated than those of the previous two cases. Neglecting higher order terms with respect to $\gamma^{(k)}$ and $\gamma^{(-l)}$, we have

$$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}]:$$
$$G^{(m,n)}\longmapsto[\gamma^{(k)}, \gamma^{(-l)}]G^{(m-l+k,n)}-G^{(m,n-l-k)}[\gamma^{(k)}, \gamma^{(-l)}]$$
$$+(*)+(*\cdot *)+(*\cdot *\cdot *).$$

Here the last three terms are computed as follows:

$$(*)=\sum_{j=0}^{l-1}\delta_{m+k,j}\gamma^{(k)}\gamma^{(-l)}\delta_{l-j,n}-\sum_{j=0}^{l-1}\delta_{m,j}\gamma^{(-l)}\gamma^{(k)}\delta_{l-j,n+k}=0$$

because $\delta_{m+k,j}=\delta_{l-j,n+k}=0$ for any $0<l\leqq k$, and $m\geqq 0$, $n\geqq 1$, $0\leqq j\leqq l-1$. The rest of the terms are

$$(*\cdot *)=\sum_{j=0}^{k}\sum_{i=0}^{l-1}G^{(m,j)}\gamma^{(k)}\delta_{k-j,i}\gamma^{(-l)}\delta_{l-i,n}+\sum_{j=1}^{k}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}G^{(k-l-j,n)}$$
$$=\sum_{i=0}^{l-1}G^{(m,k-i)}\gamma^{(k)}\gamma^{(-l)}\delta_{l-i,n}+\sum_{j=1}^{k-l}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}G^{(k-l-j,n)}$$
$$-\sum_{j=k-l+1}^{k}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}\delta_{k-l-j+n,0}$$
$$=\sum_{j=0}^{l}G^{(m,k-l+j)}\gamma^{(k)}\gamma^{(-l)}\delta_{j,n}-\sum_{j=k-l+1}^{k}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}\delta_{k-l-j+n\ 0}$$
$$+\sum_{j=1}^{k-l}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}G^{(k-l-j,n)}$$
$$=\sum_{j=1}^{k-l}G^{(m,j)}\gamma^{(k)}\gamma^{(-l)}G^{(k-l-j,n)}.$$

$$(***)=\sum_{j=1}^{k}\sum_{i=0}^{l-1}\delta_{m,i}\gamma^{(-l)}\delta_{l-i,j}\gamma^{(k)}G^{(k-j,n)}-\sum_{j=1}^{k}G^{(m,j-l)}\gamma^{(-l)}\gamma^{(k)}G^{(k-j,n)}$$
$$=-\sum_{j=1}^{k-l}G^{(m,l)}\gamma^{(-l)}\gamma^{(k)}G^{(k-l-j,n)}.$$

Hence we obtain

$$[\gamma^{(k)}t^{-k}, \gamma^{(-l)}t^{l}]:$$
$$G^{(m,n)}\longmapsto[\gamma^{(k)}, \gamma^{(-l)}]G^{(m-l+k,n)}-G^{(m,n-l+k)}[\gamma^{(k)}, \gamma^{(-l)}]$$
$$+\sum_{j=1}^{k-l}G^{(m,j)}[\gamma^{(k)}, \gamma^{(-l)}]G^{(k-l-j,n)}.$$

The last case

(4.31) $$[\gamma^{(k)}t^{-k}, \gamma^{(-l)}t^{l}]=[\gamma^{(k)}, \gamma^{(-l)}]t^{-k+l} \quad \text{for } 0<k\leqq l$$

can be proved in the same way. □

§ 5. The Riemann-Hilbert Transformations for $SU(n)$ and $SO(n)$ Chiral Field

In this final section we consider the RH transformations for $SU(n)$, $SO(n)$ Chiral field. Zakharov-Mikhailov [9] found that $SU(n)$, $(SO(n))$ Chiral field equation

(5.1) $$\partial_y(g^{-1}\partial_x g)+\partial_x(g^{-1}\partial_y g)=0$$

is equivalent to the compatibility condition for the following linear problem;

(5.2) $$dY(t)=\Omega(t)Y(t)$$

where d denotes the exterior differentiation with respect to x and y, and $\Omega(t)$ is a 1-form given by

(5.3) $$\Omega(t)=\frac{tA}{1-t}dx-\frac{tB}{1+t}dy$$

(5.4) $$A=A(x, y), \quad B=B(x, y)\in\mathfrak{su}(n) \text{ (resp. } \mathfrak{so}(n)).$$

Since, for any fundamental solution matrix $Y(t)$ of (5.2),

(5.5) $$d(Y(t)^{\dagger}Y(t))=0, \qquad d(\det Y(t))=0$$

hold, we see that there exists a fundamental solution matrix $Y(t)$ such that

(5.6) $$\det Y(t)=1, \qquad Y(t)^{\dagger}Y(t)=1, \qquad Y(0)=1.$$

Here † has benn defined in Section 2. In the case of $SO(n)$ Chiral field, the first equation in (5.5) and the second equation in (5.6) must be replaced with $d({}^tY(t)Y(t))=0$, and ${}^tY(t)Y(t)=1$, respectively. Here ${}^tY(t)$ denotes the transposed matrix of $Y(t)$. Conversely, if there exists a fundamental solution matrix $Y(t)$ of (5.2) subject to the condition (5.6) (we call such a solution a generating function for the 1-form $\Omega(t)$), the coefficients A and B in (5.4) are $\mathfrak{su}(n)$ (resp. $\mathfrak{so}(n)$) matrices.

Next we consider the RH transformations (Zakharov-Mikhailov constructed the transformation of this type in [9]). Let $Y_0(t)$ be a generating function for the 1-form $\Omega_0(t)$ with the coefficients A_0 and B_0, and C a small circle whose center is the origin such that $Y_0(t)$ is holomorphic in $C\cup C_+$ (as for the notations, see Section 3). And let $u(t)$ be an $n\times n$ matrix function of t, analytic on C such that

(5.7) $$u(t)^{\dagger}u(t)=1, \qquad \det u(t)=1.$$

For $SO(n)$ Chiral field, the first equation in (5.7) must be replaced with ${}^tu(t)u(t)=1$. As in Section 3, we consider the Riemann-Hilbert problem

(5.8) $$X_-(s)=X_+(s)H(s), \qquad s\in C$$

(5.9) $$H(t)=Y_0(t)u(t)Y_0(t)^{-1},$$

with normalization condition $X_+(0)=1$. For the solution of the problem, we define $Y(t)$ and $\Omega(t)$ as follows;

$$Y(t)=\begin{cases} X_+(t)Y_0(t) & \text{in } C_+, \\ X_-(t)Y_0(t)u(t)^{-1} & \text{in } C_-, \end{cases} \tag{5.10}$$

$$\Omega(t)=\frac{tA}{1-t}dx-\frac{tB}{1+t}dy, \tag{5.11}$$

$$A=A_0+\partial_x\dot{X}_+(0), \qquad B=B_0-\partial_y\dot{X}_+(0).$$

Then we have

Proposition 5.1 (cf. [9]). *$Y(t)$ is a generating function for the 1-form $\Omega(t)$* (5.11).

The infinitesimal RH transformation for $SU(n)$, $SO(n)$ Chiral fields can be obtained in the same way as in Section 4. We only show the results. As in Section 2, define the potentials $\{G^{(m,n)}\}_{m\geq 0, n\geq 1}$ by

$$G(s, t)=\frac{1}{s-t}\{s-tY_0(s)^{-1}Y_0(t)\}=\sum_{m,n=0}^{\infty}G^{(m,n)}s^m t^n. \tag{5.12}$$

In Section 4, we have only used the relation (2.21) and the integral equation (4.1) to show the propositions in Section 4. Notice that (2.21) directly follows from the definition of $G(s, t)$, and that (4.1) also represents the Riemann-Hilbert problem in this case. Hence we can define the infinitesimal RH transformations just as in Section 4. The generators $\gamma^{(k)}t^{-k}$'s of the infinitesimal transformations are defined by (4.10) where $\gamma^{(k)}$'s belong to $\mathfrak{su}(n)$ (resp. $\mathfrak{so}(n)$). Let $\mathcal{G}$ be the totality of the infinitesimal RH transformations

$$\mathcal{G}=\text{span of } \{\gamma^{(k)}t^{-k}\,|\,k\in \boldsymbol{Z}, \gamma^{(k)}\in\mathfrak{su}(n) \text{ (resp. } \mathfrak{so}(n))\}. \tag{5.13}$$

Then we have

Theorem 5.2. *The Lie algebra $\mathcal{G}$ is isomorphic to the graded Lie algebra*

$$\mathfrak{su}(n)\otimes\boldsymbol{R}[t, t^{-1}] \text{ (resp. } \mathfrak{so}(n)\otimes\boldsymbol{R}[t, t^{-1}]). \tag{5.14}$$

Namely the bracket relations

$$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}]=[\gamma^{(k)}, \gamma^{(l)}]t^{-k-l} \tag{5.15}$$

hold for any generators $\gamma^{(k)}t^{-k}$, $\gamma^{(l)}t^{-l}$ of $\mathcal{G}$.

The reason why the algebra of $SU(2)$ Chiral fields is isomorphic to that of the reduction problem is that the Riemann-Hilbert problems for these equations are formulated in the same manner (compare (3.8), (3.9) and (5.7)).

Appendix. Kinnersley-Chitre Theory

In the references [1], [2], W. Kinnersley and D. M. Chitre constructed the so-called Geroch group. This is an infinite dimensional transformation group acting on the manifold of solutions of stationary axially symmetric vacuum gravitational field equations (ASVG). In this short note, we shall review the essence of their theory.

First of all we define ASVG. Consider a 4-dimentional metric form expressed by

$$-ds^2=e^{2\Gamma}(d\rho^2+dz^2)-g_{ab}dx^a dx^b \quad (a, b=0, 1) \tag{A.1}$$

where $(x^0, x^1)=(t, \phi)$, and Γ, g_{ab} are functions depending on only ρ and z. Furthermore we assume that

$$g=(g_{ab}) \text{ is real, symmetric and } \det g=-\rho^2. \tag{A.2}$$

We demand that the metric form (A.1) satisfies the Einstein equations $R_{ij}=0$. The essential part of these equations is

$$\nabla\circ(\rho^{-1}g\sigma\nabla g)=0 \tag{A.3}$$

where $\nabla=(\partial_\rho, \partial_z)$ is the 2-dimensional gradient, and $\tilde{\nabla}=(\partial_z, -\partial_\rho)$ is the dual operator of ∇, and where $\sigma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. This is ASVG.

Kinnersley-Chitre observed two internal symmetries hidden behind the equations (A.3), which do not commute to each other, and composed them. The Geroch group is generated by these symmetries.

The first symmetry is immediately found:

$$g\longmapsto {}^t\xi g\xi, \qquad \xi\in SL(2, \boldsymbol{R}), \text{ constant}. \tag{A.4}$$

The equations (A.3) are obviously invariant under the above transformations. We denote by $\mathcal{G}^{(0)}$ the Lie algebra of this transformation group, which is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$.

In order to find another internal symmetry, we must introduce the so-called Ernst potential. First we note that there exists a twist potential ψ defined by

$$\nabla\psi=\rho^{-1}g\sigma\tilde{\nabla}g. \tag{A.5}$$

The Ernst potential is given by

$$E=g+i\psi, \tag{A.6}$$

and satisfies the equation

$$\nabla E=i\rho^{-1}g\tilde{\nabla}E. \tag{A.7}$$

Let E_{11} be the (1, 1) component of E. Then E_{11} satisfies

$$(\mathrm{Re}\, E_{11})\Delta E_{11}=(\nabla E_{11})^2, \tag{A.8}$$

where Δ is the 3-dimensional Laplacian. (A.8) is called the Ernst equation. Since other components are regained from E_{11}, the symmetry of the Ernst equation becomes important.

Proposition A.1. (*The second symmetry*). *The Ernst equations has the three symmetries below*:

(A.9) $$E_{11} \longmapsto E_{11} - i\alpha \quad (\textit{a gauge transformation}),$$

(A.10) $$E_{11} \longmapsto \beta E_{11} \quad (\textit{rescaling}),$$

(A.11) $$E_{11} \longmapsto \frac{E_{11}}{i\gamma E_{11}+1}.$$

Here α, β, γ are real constants. The last transformation (A.11) *is not trivial, and is called the Ehlers transformation. These second symmetries do not commute with* (A.4).

First we mix the symmetry (A.4) with Ehlers transformations (A.11). Denote by $\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}$ the infinitesimal Ehlers transformation.

Theorem A.2. *The infinitesimal transformation* $\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}$ *acts on the field g and the potential E as follows*:

(A.12) $$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \longmapsto \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} + 2\begin{pmatrix} \gamma_3 g_{11}\psi_{11} & \gamma_3 g_{11}\psi_{21} \\ \gamma_3 g_{11}\psi_{21} & \gamma_3(-g_{22}\psi_{11}+2\psi_{21}g_{12}) \end{pmatrix}$$

(A.13) $$E \longmapsto E + iE\sigma\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix}\sigma E + i\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix}\sigma N - i(N+E\sigma E)\sigma\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix}$$

where N is defined by

(A.14) $$\nabla N = E^* \sigma \nabla E.$$

Sketch of the proof. We only show that $\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}$ transforms E_{21} to

(A.15) $$E_{21} - i\gamma_3 E_{21} E_{11} - i\gamma_3 N_{11}.$$

From the definition of twist potential ψ (A.5) and (A.12), we see that $\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}$ transforms $\nabla\psi_{21}$ as follows:

$$\nabla\psi_{21} = -g_{22}\tilde{\nabla} g_{11} + g_{21}\tilde{\nabla} g_{21} \longmapsto \nabla\psi_{21} + 2\gamma_3(\psi_{21}\nabla\psi_{11} - g_{11}\nabla g_{21}).$$

Since the right hand side of the above equation is $\nabla(\psi_{21} - \gamma_3 \operatorname{Re}(E_{21}E_{11}+N_{11})\}$, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1} : \psi_{21} \longmapsto \psi_{21} - \gamma_3 \operatorname{Re}(E_{21}E_{11}+N_{11}).$$

By the same way, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1} : g_{21} \longmapsto g_{21} + \gamma_3 \operatorname{Im}(E_{21}E_{11}+N_{11}).$$

This completes the proof.

It is noted that $\mathfrak{sl}(2, \boldsymbol{R})$ is isomorphic to $\mathfrak{sym}(2, \boldsymbol{R})$ (algebra of 2×2 real symmetric matrices) as Lie algebra by

$$\mathfrak{sym}(2, \boldsymbol{R}) \ni \gamma \longmapsto \sigma\gamma \in \mathfrak{sl}(2, \boldsymbol{R}). \tag{A.16}$$

The $\mathfrak{sym}(2, \boldsymbol{R})$ bracket is given by $[\gamma, \gamma'] = \gamma\sigma\gamma' - \gamma'\sigma\gamma$.

Lemma A.3. *Under the isomorphism* (A.16), *we identify* $\gamma^{(0)} \in \mathfrak{sym}(2, \boldsymbol{R})$ *with an element of* $\mathcal{G}^{(0)}$. *Then* $\gamma^{(0)}$ *gives an infinitesimal transformation*

$$\gamma^{(0)} : E \longmapsto E + \gamma^{(0)}\sigma E - E\sigma\gamma^{(0)}. \tag{A.17}$$

We define $\mathcal{G}^{(1)}$ as a class of infinitesimal transformations given by

$$\mathcal{G}^{(1)} = \left[\begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}, \mathcal{G}^{(0)}\right] \oplus \begin{pmatrix} 0 & 0 \\ 0 & \gamma_3 \end{pmatrix} t^{-1}. \tag{A.18}$$

It is noted that $\mathcal{G}^{(1)}$ is canonically isomorphic to $\mathfrak{sym}(2, \boldsymbol{R}) \otimes t^{-1}$ as a vector space:

$$\mathcal{G}^{(1)} \cong \{\gamma^{(1)} t^{-1} ; \gamma^{(1)} \in \mathfrak{sym}(2, \boldsymbol{R})\}. \tag{A.19}$$

Lemma A.4. *Under the isomorphism* (A.19), $\gamma^{(1)} t^{-1}$ *gives an infinitesimal transformation*

$$\gamma^{(1)} t^{-1} : E \longmapsto E + iE\sigma\gamma^{(1)}\sigma E + i\gamma^{(1)}\sigma N - i(N + E\sigma E)\sigma\gamma^{(1)}. \tag{A.20}$$

It should be noted that $\mathcal{G}^{(1)}$ is not closed as a Lie algebra. However this is an important point. In fact we extend the group on the ground of this point. We define

$$\mathcal{G}^{(k+1)} = [\mathcal{G}^{(k)}, \mathcal{G}^{(1)}] \qquad \text{for } k \geqq 1. \tag{A.21}$$

In order to represent $\mathcal{G}^{(k)}$, we must introduce new potentials.

Proposition A.5. *There exist an infinite number of potentials* $\{N^{(m,n)}\}$ *defined by*

$$\begin{aligned} & E^{(n+1)} = i(N^{(1,n)} + E^{(1)}\sigma E^{(n)}), \qquad n \geqq 1, \\ & E^{(0)} = i\sigma, \qquad E^{(1)} = E, \\ & \nabla N^{(m,n)} = E^{(m)*}\sigma\nabla E^{(n)}, \qquad N^{(0,n)} = iE^{(n)}, \qquad m \geqq 0,\ n \geqq 1. \end{aligned} \tag{A.22}$$

Lemma A.6. *Under a suitable choice of integration constants, the following recursive relations hold;*

$$N^{(m,n)} - N^{(n,m)*} = E^{(m)*}\sigma E^{(n)} \qquad \textit{for } m \geqq 0,\ n \geqq 1, \tag{A.23}$$

$$N^{(m,n+1)} - N^{(m+1,n)} = iN^{(m,1)}\sigma E^{(n)} \qquad \textit{for } m \geqq 0,\ n \geqq 1 \tag{A.24}$$

We observe that $\mathcal{G}^{(k)}$ is canonically isomorphic to $\mathfrak{sym}(2, \boldsymbol{R}) \otimes t^{-k}$ as a vector space.

$$\mathcal{G}^{(k)} \cong \{\gamma^{(k)} t^{-k} : \gamma^{(k)} \in \mathfrak{sym}(2, \boldsymbol{R})\}. \tag{A.25}$$

Theorem A.7. *Under the isomorphism* (A.25), $\gamma^{(k)} t^{-k}$ *gives an infinitesimal transformation*

$$\gamma^{(k)} t^{-k} : N^{(m,n)} \longmapsto N^{(m,n)} + \gamma^{(k)}\sigma N^{(m+k,n)} - N^{(m,n+k)}\sigma\gamma^{(k)} \tag{A.26}$$

$$-\sum_{j=1}^{k} N^{(m,j)}\sigma\gamma^{(k)}\sigma N^{(k-j,n)} \qquad \textit{for } m\geqq 0,\ n\geqq 1.$$

And the commutation relations are given by

(A.27) $$[\gamma^{(k)}t^{-k},\ \gamma^{(l)}t^{-l}]=[\gamma^{(k)},\ \gamma^{(l)}]t^{-k-l} \qquad \textit{for } k\geqq 0,\ l\geqq 0$$

Sketch of the proof. In order to verify the theorem, we need the following lemma.

Lemma A.8. *An infinitesimal transformation* $\gamma^{(1)}t^{-1}\in\mathcal{G}^{(1)}$ *acts on* $\{N^{(m,n)}\}_{m\geqq 0,n\geqq 1}$ *as follows*:

(A.28) $$\gamma^{(1)}t^{-1}: N^{(m,n)} \longmapsto N^{(m,n)}+\gamma^{(1)}\sigma N^{(m+1,n)}-N^{(m,n+1)}\sigma\gamma^{(1)} -N^{(m,1)}\sigma\gamma^{(1)}\sigma N^{(0,n)} \qquad \textit{for } m\geqq 0,\ n\geqq 1.$$

This lemma is proved by induction with respect to m and n. Let us assume that we have verified the n-th induction step for $m=0$ or 1. Then, by using (A.22), (A.24) and the hypothesis of induction, we can prove

$$\gamma^{(1)}t^{-1}: N^{(0,n+1)} \longmapsto N^{(0,n+1)}+\gamma^{(1)}\sigma N^{(1,n+1)} -N^{(0,n+2)}\sigma\gamma^{(1)}-N^{(0,1)}\sigma\gamma^{(1)}\sigma N^{(0,n+1)},$$

and

$$\gamma^{(1)}t^{-1}: \nabla N^{(1,n+1)} \longmapsto \nabla(N^{(1,n+1)}+\gamma^{(1)}\sigma N^{(2,n+1)} -N^{(1,n+2)}\sigma\gamma^{(1)}-N^{(1,1)}\sigma\gamma^{(1)}\sigma N^{(0,n+1)}.$$

Thus we see that (A.28) is true for any $n\geqq 0$, and $m=0$ or 1. That the assertion of the lemma is true for any $m\geqq 0$ can be also verified by induction. Theorem A.7 is shown by using Lemma A.8 in the same way as in Theorem 4.3.

Next we define $\mathcal{G}^{(-1)}$ as a class of infinitesimal gauge transformations. If we denote by $\gamma^{(-1)}t$ an infinitesimal gauge transformation

(A.29) $$E \longmapsto E+i\gamma^{(-1)}, \qquad \gamma^{(-1)}\in\mathfrak{sym}(2, \boldsymbol{R}),$$

$\mathcal{G}^{(-1)}$ is canonically isomorphic to $\mathfrak{sym}(2, \boldsymbol{R})\otimes t$ as a vector space.

Lemma A.9. *An infinitesimal transformation* $\gamma^{(-1)}t\in\mathcal{G}^{(-1)}$ *acts on the potentials* $\{N^{(m,n)}\}_{m\geqq 0,n\geqq 1}$ *as follows*:

(A.30) $$\gamma^{(-1)}t: N^{(m,n)} \longmapsto N^{(m,n)}-\delta_{m,0}\gamma^{(-1)}\delta_{n,1} +\gamma^{(-1)}\sigma N^{(m-1,n)}-N^{(m,n-1)}\sigma\gamma^{(-1)}.$$

Here the potentials with negative indices are defined as

(A.31) $$N^{(p,-p)}=-N^{(-p,p)}=\sigma \qquad \textit{for } p\geqq 1,\ N^{(m,n)}=0 \textit{ for other indices.}$$

Let us define

(A.32) $$\mathcal{G}^{(-k)}=[\mathcal{G}^{(-k+1)},\ \mathcal{G}^{(-1)}] \qquad \text{for } k\geqq 1.$$

This vector space is isomorphic to $\mathfrak{sym}(2, \boldsymbol{R})\otimes t^{-k}$,

(A.33) $$\mathcal{G}^{(-k)}\cong\{\gamma^{(-k)}t^{k}: \gamma^{(-k)}\in\mathfrak{sym}(2, \boldsymbol{R})\}.$$

We have

Theorem A.10. *Under the isomorphism* (A.31), $\gamma^{(-k)}t^k$ *gives an infinitesimal transformation*

(A.35) $$\gamma^{(-k)}t^k : N^{(m,n)} \longmapsto N^{(m,n)} - \sum_{j=0}^{k-1} \delta_{m,j}\gamma^{(-k)}\delta_{k-j,n} + \gamma^{(-k)}\sigma N^{(m-k,n)} - N^{(m,n-k)}\sigma\gamma^{(-k)} \qquad \text{for } m \geqq 0,\ n \geqq 1.$$

And the commutation relations are given by

(A.36) $$[\gamma^{(-k)}t^k, \gamma^{(-l)}t^l] = [\gamma^{(-k)}, \gamma^{(-l)}]t^{-k-l} \qquad \text{for } k, l \geqq 0.$$

We set Lie algebra $\mathcal{G}$ as

(A.37) $$\mathcal{G} = \bigoplus_{k=-\infty}^{\infty} \mathcal{G}^{(k)}.$$

The structure of $\mathcal{G}$ is stated in our main theorem.

Theorem A.11. *As a graded Lie algebra,* $\mathcal{G}$ *is isomorphic to* $\mathfrak{sym}(2, \mathbf{R}) \underset{\mathbf{R}}{\otimes} \mathbf{R}[t, t^{-1}]$. *Namely the commutation relations*

(A.38) $$[\gamma^{(k)}t^{-k}, \gamma^{(l)}t^{-l}] = [\gamma^{(k)}, \gamma^{(l)}]t^{-k-l}$$

hold for any integers k *and* l.

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