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Asymptotic Behaviors of Dynamical Systems with Random Parameters

By

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§1. Introduction

In this paper we will investigate asymptotic behaviors of random orbits of dynamical systems with random parameters. In many biological models (for example, May's model [13]), the dynamical systems have parameters. Asymptotic behaviors of orbits of such dynamical systems depend on the parameters very sensitively. But usually it is not easy to decide values of parameters theoretically. They are decided only experimentally. Hence, it seems to be natural to think that the parameters of the dynamical system are chosen randomly at every time of its iteration.

More precisely we will explain the idea of dynamical systems with random parameters. Let f_λ, $\lambda\in\Lambda$, be a family of maps from a set M into itself. The randomness of the parameter λ is governed by a probability measure γ on the parameter space Λ. Let $\lambda_n(\omega)$, $n=1, 2, \cdots$, be a sequence of independent, identically γ-distributed random variables on Λ. Then we think that the orbits of the dynamical system f_λ may be determined according to the random sequence $\lambda_n(\omega)$. Namely, the state of the system at time n started from the point $x\in M$ is given by

$$X_n^x(\omega)=f_{\lambda_n(\omega)}\circ f_{\lambda_{n-1}(\omega)}\circ\cdots\circ f_{\lambda_1(\omega)}(x)$$

which is, of course, a radom point.

It is easy to see that $X_n^x(\omega)$ becomes a (time homogeneous) Markov chain

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(which we call Π) with the initial point $X_0(\omega)=x$ and the transition probability

$$P(A, x)=\int_{\Lambda} 1_A(f_\lambda(x))d\gamma(\lambda),$$

where 1_A denotes the indicator function of the measurable set A. Hence the theory of Markov chains is applicable to the study of behaviors of dynamical systems with random parameters.

The random sequence $\{\lambda_n(\omega)\}$ mentioned above is defined as follows. Let $(\Omega, \boldsymbol{b})=(\Lambda^N, \gamma^N)$ be the infinite product probability space of the copies of the space $(\Lambda, \gamma)(\boldsymbol{N}=\{1, 2, \cdots\})$, and let $\omega(n)$ denote the n-th coordinate of $\omega\in\Omega$. Then $\lambda_n(\omega)=\omega(n)$ is a random variable and $\{\lambda_n(\omega)\}$ constitutes a sequence of independently identically γ-distributed random variables.

Define the shift operator σ of Ω by

$$(\sigma\omega)(n)=\omega(n+1), \quad \text{for all} \quad n\geqq 1.$$

Then we get another representation of the dynamical system with random parameters

$$T(x, \omega)=(f_{\omega(1)}x, \sigma\omega),$$

which is called a skew product transformation in ergodic theory. Actually, $T^n(x, \omega)=(X_n^x(\omega), \sigma^n\omega)$.

S. Kakutani [8] investigated the relation between the Markov chain Π and the transformation T under the assumption that each f_λ preserves a fixed probability measure ν on M. In this situation the product measure $\nu\times\boldsymbol{b}$ is invariant with respect to T and the measure ν becomes a stationary measure of the Markov chain Π. He proved that the stationary Markov chain Π under the initial measure ν is ergodic if and only if the transformation T is ergodic under $\nu\times\boldsymbol{b}$. In this paper, we assume only that the transformation T preserves the product measure $\nu\times\boldsymbol{b}$ without the assumption of the measure-preserveness of each transformation f_λ, and we prove Kakutani's result mentioned above.

Our investigation of dynamical systems with random parameters is suggested by the studies of deterministic dynamical systems, such as one-dimensional dynamical systems (see [1] [2] [3] [4] [5] [6] [7]) and dynamical systems with hyperbolic structures (see [10] and [11]). We will see that there are very similar phenomena in dynamical systems with random parameters as in the deterministic dynamical systems metioned above.

Section 2 is the preliminary part, where we define the Markov chain Π and the skew product transformation T precisely and study the relations between a stationary probability measure of the Markov chain Π (a Π-invariant measure) and an invariant probability measure of the transformation T (a T-invariant measure). In Section 3, we prove that a Π-invariant probability measure ν is Π-ergodic if and only if $\nu\times\boldsymbol{b}$ is T-ergodic.

In the rest of the paper, we make some smoothness assumptions on the phase space M, the parameter space Λ and the family of transformations $\{f_\lambda\}$, which is described in Section 4. We will show under these assumptions, that any Π-invariant measure is absolutely continuous with respect to the Riemannian volume on M. In Section 5, we will study the exactness and the cluster property of Π and T. We will get some topological condition for exactness and show that any ergodic invariant measure can be represented by an exact invariant measure. In Section 6 we will investigate the condition for the uniqueness of Π-invariant measure and asymptotic behavior of random orbits of such random systems. We will also see that the randomized system of May's model is uniquely ergodic. In Section 7, we will study the relations between a deterministic dynamical system and systems with random parameters. We consider a deterministic dynamical system as a limit of random systems when the parameter spaces tend to one point. Especially we are interested in the dynamical system which is given as a limit of uniquely ergodic systems. Lastly we will give a simple two dimensonal example in such case.

§ 2. Skew Product Transformation and Markov Chain

In this section we set up our objects. Let (M, d_M) and (Λ, d_Λ) be compact metric spaces and f a continuous map from $M\times\Lambda$ to M. We denote $f^x(\lambda)=f_\lambda(x)=f(x, \lambda)$ for $x\in M$ and $\lambda\in\Lambda$.

Given a Borel probability measure γ on Λ, we call the system (f, M, Λ, γ) a random dynamical system with phase space M and random parameter space (Λ, γ). Let $\Omega=\Lambda^{\boldsymbol{N}}$ and $\boldsymbol{b}=\gamma^{\boldsymbol{N}}$ the product measure on Ω where $\boldsymbol{N}=\{1, 2, \cdots\}$, and let σ be the shift operator defined on Ω by

$$(\sigma\omega)(n)=\omega(n+1)\,, \qquad n\in\boldsymbol{N},\ \omega\in\Omega\,,$$

where $\omega(n)$ denotes the n-th coordinate of $\omega\in\Omega$.

Define a skew product transformation T on $M\times\Omega$ by

$$T(x, \omega)=(f_{\omega(1)}(x), \sigma(\omega))\,, \qquad x\in M\,, \quad \omega\in\Omega\,.$$

The x-coordinate of the n-th iteration $T^n(x, \omega)$ is given by

$$X_n^x(\omega)=f^{n(\omega)}(x)\equiv f_{\omega(n)}\circ f_{\omega(n-1)}\circ\cdots\circ f_{\omega(1)}(x)\,. \tag{2.1}$$

For each fixed (x, ω), the sequence $\{X_n^x(\omega)\}_{n\geqq 1}$ is an orbit of our random system (f, M, Λ, γ). It is easy to see that the random sequence $\{X_n^x(\omega)\}$ governed by the probability measure $\boldsymbol{b}$ forms a time homogeneous Markov chain with the initial point $X_0^x(\omega)=x$, which we call Π. The transition probability is given by the image measure $f_*^x(\gamma)$:

$$P(A, x)=f_*^x(\gamma)(A)=\int_\Lambda 1_A(f^x(\lambda))d\gamma(\lambda)\,,$$

where $1_A(x)$ is the indicator function of the measurable set A in M. Let P be the transition operator of Π:

$$P(g)(x)=\int_M g(y)P(dy, x)=\int_\Lambda g(f_\lambda(x))d\gamma(\lambda), \qquad g\in C(M).$$

Let P^* be the dual operator of P defined by $P^*(\nu)(g)=\nu(P(g))$ for a probability measure ν on M.

A probability measure ν on M is called *Π-invariant* if $P^*(\nu)=\nu$. The set of all Π-invariant probability measures is denoted by $\mathcal{I}_\Pi$. Let $\mathcal{I}_T^b$ denote the set of all T-invariant probability measures of the form $\nu\times\boldsymbol{b}$, where ν are probability measures on M. Then we have

Lemma 2.1. *A probability measure ν on M belongs to $\mathcal{I}_\Pi$ if and only if $m=\nu\times\boldsymbol{b}$ belongs to $\mathcal{I}_T^b$.*

Proof. For any $g\in C(M\times\Omega)$, put

$$\hat{g}(x)=\int_\Omega g(x, \omega)d\boldsymbol{b}(\omega),$$

then we have

$$\begin{aligned}(P^*\nu)(\hat{g})=\nu(P(\hat{g}))&=\iiint g(f_\lambda(x), \omega)d\boldsymbol{b}(\omega)d\nu(x)d\gamma(\lambda)\\ &=\iint g(f_{\omega(1)}(x), \sigma(\omega))dm(x, \omega)\\ &=\iint g(T(x, \omega))dm(x, \omega).\end{aligned}$$

Hence $P^*\nu=\nu$ if and only if m is T-invariant.

Lemma 2.2. *$\mathcal{I}_\Pi\neq\emptyset$ and hence $\mathcal{I}_T^b\neq\emptyset$.*

Proof. This follows from Tihkonov's fixed point theorem.

§ 3. Ergodicity

In this section we will prove the equivalence of the ergodicity of the Markov chain Π and that of the transformation T defined in the previous section. Let $\nu\in\mathcal{I}_\Pi$ be given. A measurable set $E\subset M$ is called *(Π, ν)-invariant* if $P(1_E)=1_E$ (a.e. ν). The measure ν is called *Π-ergodic* if every (Π, ν)-invariant set has ν-measure 0 or 1. We denote by $\mathcal{E}_\Pi$ the set of all Π-ergodic probability measures. We also denote by $\mathcal{E}_T$ the set of all T-invariant ergodic probability measures, and we put $\mathcal{E}_T^b=\mathcal{E}_T\cap\mathcal{I}_T^b$. Then we have

Theorem 3.1. *A probability measure ν on M belongs to $\mathcal{E}_\Pi$ if and only if the product measure $m=\nu\times\boldsymbol{b}$ belongs to $\mathcal{E}_T^b$.*

Remark 3.1. S. Kakutani [8] gave an elegant proof of this theorem under the assumption that each f_λ preserves the probability measure ν. But it can be seen that in his proof it does not need to use the above assumption. Nevertheless we give a proof of the theorem for the completeness.

Proof of the "if" part. Let $m=\nu\times\boldsymbol{b}\in\mathcal{E}_T^b$. Suppose that there is a (Π, ν)-invariant measurable set B with $0<\nu(B)<1$. Then by Lemma 2.1, $\nu|_B\times\boldsymbol{b}$ is T-invariant. This contradicts to the T-ergodicity of the measure m.

In order to prove the only if part, we use the following lemma. We denote $E^x=\{\omega;(x,\omega)\in E\}$ for $E\subset M\times\Omega$ and $x\in M$.

Lemma 3.1. *Let ν be a probability measure on M and $m=\nu\times\boldsymbol{b}$. For any measurable set $E\subset M\times\Omega$ with $m(E)>0$ and any $\varepsilon>0$, there exists an $n(\varepsilon)>0$ such that*

$$\nu(\{x\in M;\, b((T^nE)^x)>1-\varepsilon\})\geqq(1-\varepsilon)m(E)$$

for all $n\geqq n(\varepsilon)$.

Proof. Let $\mathcal{A}_n$ be the σ-field generated by the first n coordinates $\omega(1),\cdots,\omega(n)$ of Ω. Let $\varphi_n^B(\omega)=\boldsymbol{b}_n(B\cap\omega_n)$, where $\omega_n=\{\omega'\in\Omega;\,\omega'(1)=\omega(1),\cdots,\omega'(n)=\omega(n)\}$ and $\boldsymbol{b}_n=\prod_{k=n+1}^{\infty}\gamma(d\omega'(k))$. Then it is easy to see that φ_n^B is a version of $\boldsymbol{b}(B|\mathcal{A}_n)$, and so $\varphi_n^B\to 1_B$ for a.e. ω. Let $E_n^x=\{\omega\in\Omega;\,\varphi_x^{E^x}(\omega)>1-\varepsilon\}$ and $F_n=\{(x,\omega)\in E;\,\omega\in E_n^x\}$. Then we have

$$\begin{aligned}\lim_{n\to\infty}m(F_n)&=\lim_{n\to\infty}\int_M\boldsymbol{b}(F_n^x)d\nu(x)\\&=\lim_{n\to\infty}\int_M\int_{E_n^x}\boldsymbol{b}(E^x|\mathcal{A}_n)(\omega)d\boldsymbol{b}(\omega)d\nu(x)\\&=\int_M\boldsymbol{b}(E^x)d\nu(x)=m(E)\,.\end{aligned}$$

From the relation $\boldsymbol{b}(\sigma^n(B\cap\omega_n))=\varphi_n^B(\omega)$ we obtain that for any $y\in M$ such that $(T^n(F_n))^y\neq\emptyset$, there exist $x\in M$ and an atom ω_n of $\mathcal{A}_n$ with $\omega_n\subset E_n^x$ such that $(T^n(F_n))^y\supset\sigma^n(F_n^x\cap\omega_n)$. Hence we have

$$\begin{aligned}\boldsymbol{b}(\{T^n(F_n)\}^y)&\geqq\boldsymbol{b}(\sigma^n(F_n^x\cap\omega_n))\\&=\varphi_n^{F_n^x}(\omega)\\&=1_{E_n^x}(\omega)\boldsymbol{b}(E^x|A_n)(\omega)\quad\text{a.e. }\omega\in\omega_n\\&=\boldsymbol{b}(E^x|A_n)(\omega)>1-\varepsilon\,.\end{aligned}$$

Let G_n be the projection of F_n to M i.e. $G_n=\{x\in M;\,(x,\omega)\in F_n \text{ for some } \omega\in\Omega\}$, then $G_n\subset\{x\in M;\,\boldsymbol{b}((T^nE)^x)>1-\varepsilon\}$. Since $\nu(G_n)\geqq m(F_n)\geqq(1-\varepsilon)m(E)$ for sufficiently large n, we obtain the lemma.

Proof of the "only if" part of Theorem 3.1. Take a measurable set E in $M\times\Omega$ with $T^{-1}(E)=E$ and $m(E)>0$. Let $F=\{x\in M;\,\boldsymbol{b}(E^x)=1\}$. Then Lemma 3.1 implies $\nu(F)=m(E)$. Then the relation

$$m(E)=\int\boldsymbol{b}(E^x)d\nu(x)=\nu(F)+\int_{F^c}\boldsymbol{b}(E^x)d\nu(x)$$

shows that $x\notin F$ implies $\boldsymbol{b}(E^x)=0$ for ν-a.e. x. On the other hand, since

$$\boldsymbol{b}(E^x)=\boldsymbol{b}((T^{-1}E)^x)=\int_\Lambda \boldsymbol{b}(E^{f_\lambda(x)})d\gamma(\lambda),$$

$x\in F$ if and only if $f_\lambda(x)\in F$ for a. e. λ, and $x\notin F$ if and only if $f_\lambda(x)\notin F$ for a. e. λ, which means

$$P(1_F)(x)=\int_\Lambda 1_F(f_\lambda(x))d\gamma(\lambda)=1_F(x).$$

Since (Π, ν) is ergodic, we have $m(E)=\nu(F)=1$.

§ 4. Assumptions for Further Investigations

We assumed only the continuity of the system so far. From this section on, we assume the smoothness of the random dynamical system (f, M, Λ, γ) described as follows. Assume that M and $\Lambda=\Lambda^0+\Lambda_b$ are compact connected Riemannian manifolds where Λ_b is the boundary of Λ^0. We denote the metrics of M and Λ by d_M and d_Λ respectively. The map f is assumed to be a C^1-map.

We put the following assumptions.

Assumption 4.1. We assume that $\dim(M)=\dim(\Lambda)>\dim(\Lambda_b)$.

Define

$$S^x=\left\{\lambda\in\Lambda\,;\,\det\left(\frac{\partial}{\partial\lambda}f(x, \lambda)\right)=0\right\}\cup\Lambda_b,$$

$$S=\bigcup_{x\in M}(\{x\}\times S^x).$$

The second assumption is

Assumption 4.2. The sets S^x and S are finite sums of connected submanifolds whose co-dimensions are positive.

Thirdly we put the following natural

Assumption 4.3. The every probability measure γ on Λ, which we consider, is absolutely continuous with respect to the Riemannian volume $d\lambda$ of Λ and its density (Radon-Nikodym derivative) $\theta(\lambda)=d\gamma(\lambda)/d\lambda$ is positive everywhere and continuous.

Under the above assumptions we have

Lemma 4.1. *The transition probability $P(\cdot, x)$ of the Markov chain Π is absolutely continuous with respect to the Riemannian volume μ on M, and it has a density $p(y|x)$ which is lower semicontinuous with respect to (x, y).*

Proof. For each $x\in M$, we have a finite partition $\xi^x=\{U_i^x\}$ of $\Lambda\backslash S^x$, such that each U_i^x is open and connected, f^x is a diffeomorphism on each U_i^x and ξ^x moves smoothly with respect to x. Let f_i^x denote the restriction of f^x to U_i^x. Then it is easy to see that a version of the transition density is given by

$$p(y \mid x)=\begin{cases} 0, & \text{if } (f^x)^{-1}(y) \cap S^x \neq \emptyset, \\ \sum_i |D(f_i^x)^{-1}(y)| \cdot \theta((f_i^x)^{-1}(y)), & \text{if } (f^x)^{-1}(y) \cap S^x = \emptyset, \end{cases}$$

where the summation is taken over all i such that $(f^x)^{-1}(y) \cap U_i^x \neq \emptyset$. By the implicit function theorem, $p(y \mid x)$ is continuous on $\{(x, y); p(y \mid x)>0\}$. This completes the proof of the lemma.

Remark 4.1. In the sequel of this paper, we use the transition density $p(y \mid x)$ given above.

Theorem 4.1. *Every* $\nu \in \mathcal{I}_\Pi$ *is absolutely continuous with respect to the Riemannian volume* μ *on* M *and a version of the density function of* ν *is given by*

$$\phi_\nu(y)=\int_M p(y \mid x) d\nu(x), \tag{4.1}$$

which is lower semicontinuous and satisfies

$$\int_M \phi_\nu(x) p(y \mid x) d\mu(x)=\phi_\nu(y).$$

Proof. For any measurable set $A \subset M$, we have

$$\nu(A)=\int_M P(A, x) d\nu(x)=\int_A \int_M p(y \mid x) d\nu(x) d\mu(y),$$

which implies the theorem.

We introduce

Definition 4.1. a) We define $p^{(n)}(y \mid x)$ inductively by

$$p^{(1)}(y \mid x)=p(y \mid x),$$

$$p^{(n+1)}(y \mid x)=\int_M p(y \mid z) p^{(n)}(z \mid x) d\mu(z), \qquad n=1, 2, \cdots,$$

and denote by $\Pi^{(n)}$ the Markov chain with the transition probability density $p^{(n)}(y \mid x)$.

b) We denote $x \xrightarrow{n} y$ if $p^{(n)}(y \mid x)>0$. For a subset B of M, we also denote $B \xrightarrow{n} y$ if $x \xrightarrow{n} y$ for some $x \in B$ and $x \xrightarrow{n} B$ if $x \xrightarrow{n} y$ for some $y \in B$. We put $R_B^n=\{y; B \xrightarrow{n} y\}$ and $L_B^n=\{x; x \xrightarrow{n} B\}$. If $B=\{x\}$ we denote $R_{\{x\}}^n$ by R_x^n and $L_{\{x\}}^n$ by L_x^n. In the above notations, if $n=1$ we don't write 1 (e.g. $x \longrightarrow y$ instead of $x \xrightarrow{1} y$, $R_B=R_B^1$ and so on).

c) For $\nu \in \mathcal{I}_\Pi$, define $S_\nu=\{x; \phi_\nu(x)>0\}$.

Remark 4.2. 1) $x \longrightarrow y$ if and only if $y=f_\lambda(x)$ for some $\lambda \in \Lambda \backslash S^x$. 2) The sets S_ν, R_B^n and L_B^n are open because $\phi_\lambda(x)$ and $p^{(n)}(y \mid x)$ are lower semicontinuous.

Lemma 4.2. *For any $\nu\in\mathcal{I}_{\Pi}$, we have $R_{S_{\nu}}\subset S_{\nu}$.*

Proof. If $x\notin S_{\nu}$ then

$$\int_{M}\phi_{\nu}(y)p(x\mid y)d\mu(y)=\phi_{\nu}(x)=0,$$

and so $\mu(L_{x}\cap S_{\nu})=0$. Since $L_{x}\cap S_{\nu}$ is open, $L_{x}\cap S_{\nu}=\emptyset$ and so $x\notin R_{S_{\nu}}$. This proves the lemma.

§ 5. Exactness and Cluster Property

In this section, we will investigate the properties of π-ergodic probability measures. The notations and the assumptions are the same as in the preceding sections.

The limiting measures defined in the following play an important role in the study of relations between the asymptotic behavior of ω-orbit $\{f^{n(\omega)}(x)\,;\,n\geqq 0\}$ and Π-invariant measures.

Definition 5.1. For $(x,\omega)\in M\times\Omega$, we denote by $\Pi_{x}^{\omega}(\Pi_{x})$ the set of all limit points of the family of probability measures $\{n^{-1}\sum_{k=1}^{n}\delta_{f^{k(\omega)}(x)}\,;\,n\geqq 1\}$ $(\{n^{-1}\sum_{k=1}^{n}P^{*k}(\delta_{x})\,;\,n\geqq 1\}$ respectively) on M in the weak topology, where δ_{a} denotes the point mass at a.

Remark 5.1. Since M is compact, every Π_{x}^{ω} and Π_{x} are not empty. Moreover, if $\nu\in\mathcal{E}_{\Pi}$ then $\Pi_{x}=\{\nu\}$ for any $x\in S_{\nu}$.

Let us define a bounded linear operator Q on $L^{1}(\mu)$ by

$$Q(g)(x)=\int g(y)p(x\mid y)d\mu(y)\,,\qquad g\in L^{1}(\mu).$$

We denote by $\|g\|_{1}$ the norm of $g\in L^{1}(\mu)$.

Theorem 5.1. *If $\nu\in\mathcal{I}_{\Pi}$ is $\Pi^{(n)}$-ergodic for all $n\geqq 1$, then there exist $C>0$ and $0<\rho<1$ such that*

$$|P^{*n}(\hat{\nu})(g)-\nu(g)|<C\rho^{n}$$

for any $n>0$, any measurable g with $\|g\|_{\infty}=\sup_{x}|g(x)|\leqq 1$ and any probability measure $\hat{\nu}$ on S_{ν}.

In order to prove the theorem we prepare the following

Lemma 5.1. *Under the same assumption as in Theorem* 5.1, *there exist $0<\alpha<1$ and $m>0$ such that*

$$\|Q^{m}(h)-\phi_{\nu}\|_{1}<\alpha\|h-\phi_{\nu}\|_{1}$$

for any non-negative function h with $\{x\,;\,h(x)>0\}\subset S_{\nu}$ and $\|h\|_{1}=1$.

Proof. Let $x\in S_{\nu}$ and $y\in R_{x}$. By Remark 5.1, $\bigcup_{n}R_{y}^{n}\supset S_{\nu}$ a.e. and since they are open we have $\bigcup_{n}R_{y}^{n}\supset S_{\nu}$. On the other hand, we have $L_{y}\cap S_{\nu}\neq\emptyset$

and hence $\bigcup_n R_y^n \cap L_y \neq \emptyset$. Thus there exists an $n>0$ such that $y \in L_y^n$. Hence $\{L_y^{kn}\}_{k\geqq 1}$ are increasing open sets. We claim that $\bigcup_k L_y^{kn+1} \supset \bar{S}_\nu$. Indeed, if $z \in \bar{S}_\nu$ we have $R_z \cap S_\nu \neq \emptyset$. By the assumption of the lemma and Remark 5.1, we have $\bigcup_k R_z^{kn} \supset S_\nu$ a.e. and hence there exists a k such that $R_z^{kn} \cap L_y \neq \emptyset$, which implies that $z \in L_y^{kn+1}$.

Now, by the compactness of $\bar{S}_\nu$, we have $L_y^m \supset \bar{S}_\nu$ for some $m>0$. This means that $p^{(m)}(y|z)>0$ for any $z \in \bar{S}_\nu$. By the lower semicontinuity of $p^{(m)}(y|z)$, there exists $\beta>0$ and $\delta>0$ such that $p^{(m)}(y'|z)>\beta$ for $z \in \bar{S}_\nu$ and $y' \in B_\delta(y)=\{y' ; d_M(y, y')<\delta\}$. Thus

$$Q^m(h)(y')=\int h(z)p^{(m)}(y'|z)d\mu(z) \geqq \beta \int_{S_\nu} h \, d\mu ,$$

for $y' \in B_\delta(y)$. Let $\hat{h}=h-h\wedge\phi_\nu$ and $\hat{\phi}_\nu=\phi_\nu-h\wedge\phi_\nu$, then we have

$$\begin{aligned}
\|Q^m(h)-\phi_\nu\|_1 &= \|Q^m(h-\phi_\nu)\|_1=\|Q^m(\hat{h}-\hat{\phi}_\nu)\|_1 \\
&= \|Q^m(\hat{h})-Q^m(\hat{h})\wedge Q^m(\hat{\phi}_\nu)\|_1+\|Q^m(\hat{\phi}_\nu)-Q^m(\hat{h})\wedge Q^m(\hat{\phi}_\nu)\|_1 \\
&= \|Q^m(\hat{h})\|_1+\|Q^m(\hat{\phi}_\nu)\|_1-2\|Q^m(\hat{h})\wedge Q^m(\hat{\phi}_\nu)\|_1 \\
&\leqq \|\hat{h}\|_1+\|\hat{\phi}_\nu\|_1-\beta\mu(B_\delta(y))(\|\hat{h}\|_1+\|\hat{\phi}_\nu\|_1) \\
&= (1-\beta\mu(B_\delta(y)))\|h-\phi_\nu\|_1 .
\end{aligned}$$

This implies the lemma.

Proof of Theorem 5.1. First we assume that $\hat{\nu}$ is absolutely continuous with respect to μ and has the density $\phi_{\hat{\nu}}$. Then

$$\begin{aligned}
P^{*n}(\hat{\nu})(g) &= \hat{\nu}(P^n(g)) \\
&= \iint g(y)p^{(n)}(y|x)d\mu(y)\phi_{\hat{\nu}}(x)d\mu(x) \\
&= \int g(y)Q^n(\phi_{\hat{\nu}})(y)d\mu(y) .
\end{aligned}$$

Put $\rho=\alpha^{1/m}$ where α and m are given in Lemma 5.1. Noting that $\|Q^n(g-\phi_\nu)\|_1$ are decreasing, we have

$$\begin{aligned}
&\left|\int g(y)Q^n(\phi_{\hat{\nu}})(y)d\mu(y)-\int g(y)\phi_\nu(y)d\mu(y)\right| \\
&\leqq \int |Q^n(\phi_{\hat{\nu}})-\phi_\nu|\, d\mu < C\rho^n
\end{aligned}$$

for some positive constant C which is independent of $\hat{\nu}$ and g. When $\hat{\nu}$ is not absolutely continuous, we may consider $P^*(\hat{\nu})$ in place of $\hat{\nu}$.

Theorem 5.2 (*Cluster property*). *Under the same assumption as in Theorem* 5.1, *we have*

$$|\nu((P^n g)h)-\nu(g)\nu(h)| \leqq C\rho^n \|g\|_\infty \cdot \|h\|_\infty$$

for any non-negative continuous functions g and h on M, where $C>0$ and $0<\rho<1$

are the same constants as in Theorem 5.1.

Proof. It is easy to see that

$$\nu((P^n g)h)=\iint g(y)p^{(n)}(y\mid x)d\mu(y)h(x)\phi_\nu(x)d\mu(x)$$
$$=\int g(y)Q^n(h\phi_\nu)(y)d\mu(y)\,.$$

Hence, using an analogous argument to the one in the proof of Theorem 5.1, we have

$$|\nu((P^n g)h)-\nu(g)\nu(h)|$$
$$=\nu(h)\left|\int (g(y)Q^n(h\phi_\nu)(y)/\nu(h))d\mu(y)-\nu(g)\right|$$
$$\leqq C\rho^n\nu(h)\|g\|_\infty\leqq C\rho^n\|g\|_\infty\cdot\|h\|_\infty\,.$$

q. e. d.

It seems natural to make the following

Definition 5.2. a) If $\nu\in\mathcal{J}_\Pi$ satisfies the conclusion of Lemma 5.1, then ν is called *Π-exact.*

b) If $\nu\in\mathcal{J}_\Pi$ satisfies the conclusion of Theorem 5.2, then ν is said to have the *cluster property.*

c) A probability measure $m\in\mathcal{J}_T$ is called *T-exact,* if $\lim_{n\to\infty}m(T^nE)=1$ for any measurable set E with $m(E)>0$.

Lemma 5.2. *Let $\nu\in\mathcal{J}_\Pi$. If S_ν is connected then ν satisfies the assumption of Theorem* 5.1 *and hence it is Π-exact and has the cluster property.*

Proof. Let ν_1, $\nu_2\in\mathcal{E}_{\Pi^{(n)}}$. If there exists a point $x\in\bar{S}_{\nu_1}\cap\bar{S}_{\nu_2}$, then $R_x^n\subset S_{\nu_1}\cap S_{\nu_2}$ and so $\nu_1=\nu_2$ by Remark 5.1. This implies that if $\nu_1\neq\nu_2$ then $\bar{S}_{\nu_1}\cap\bar{S}_{\nu_2}=\emptyset$. Assume that $\nu=\sum_{k=1}^{m_n}\alpha_k\nu_k^{(n)}$, $\nu_k^{(n)}\in\mathcal{E}_{\Pi^{(n)}}$. Then $\bar{S}_\nu=\bigcup_{k=1}^{m_n}\bar{S}_{\nu_k^{(n)}}$ (disjoint sum). Since $\bar{S}_\nu$ is connected, $\nu=\nu_k^{(n)}$ for some k. Thus $\nu\in\mathcal{E}_{\Pi^{(n)}}$ for any $n\geqq 1$.

Lemma 5.3. *If $\nu\in\mathcal{J}_\Pi$ is Π-exact then $m=\nu\times\boldsymbol{b}$ is T-exact.*

Proof. Take a measurable set E with $m(E)>0$ and any $\varepsilon>0$. Define $E_\varepsilon^n=\{x\in M;\ \boldsymbol{b}((T^nE)^x)>1-\varepsilon\}$. Then by Lemma 3.1, there exists an $n(\varepsilon)$ such that $\nu(E_\varepsilon^{n(\varepsilon)})>(1-\varepsilon)m(E)$. Take a sequence of positive numbers $\{\varepsilon_k\}$ such that $\sum_k\varepsilon_k<\infty$. Then writing $n_k=n(\varepsilon_k)$ and $E_k=E_{\varepsilon_k}^{n_k}$ we have $\lim_{i\to\infty}\nu(\bigcup_{k\geqq i}E_k)>0$. Hence $\limsup_{k\to\infty}E_k\neq\emptyset$. Let us take a point $x\in\limsup_{k\to\infty}E_k$. Then it is easy to see that $\limsup_{k\to\infty}T^{n_k}(E)\supset\{x\}\times\Omega$ a. e. Hence we have $\limsup_{k\to\infty}T^{n_k+i}(E)\supset R_x^i\times\Omega$ a. e., and so there exists m_i such that $T^{m_i}(E)\supset R_x^i\times\Omega$ a. e. Since $\nu(R_x^i)=\lim_{n\to\infty}P^{*n}(\delta_x)(R_x^i)$ uniformly in x and i by Theorem 5.1, we have $\lim_{i\to\infty}\nu(R_x^i)=1$. Therefore we obtain that $\lim_{i\to\infty}m(T^{m_i}(E))=1$ and so $\lim_{n\to\infty}m(T^n(E))=1$ because $m(T^n(E))$ is increasing in n.

We will make the following remark about exactness.

Remark 5.2. a) If $\nu\in\mathcal{J}_{\Pi}$ is Π-exact then it is $\Pi^{(n)}$-ergodic for all $n\geqq 1$. The same fact holds also for $m\in\mathcal{J}_T$.

b) If $m\in\mathcal{J}_T$ is T-exact then the natural extension of (T, m) is a Kolmogorov transformation.

c) If $m\in\mathcal{J}_T$ is T-exact, (T, m) is mixing of k-th order, that is if $\min_i|n_{i+1}-n_i|\rightarrow\infty$ then

$$\lim m(T^{n_1}E_1\cap T^{n_2}E_2\cap\cdots\cap T^{n_k}E_k)=\prod_{i=1}^{k}m(E_i)$$

for any measurable sets $E_1, E_2, \cdots, E_k$, for all $k\geqq 1$.

Any ergodic invariant probability measure can be represented by exact invariant measures as follows.

Theorem 5.3. a) *For any* $\nu\in\mathcal{E}_{\Pi}$, *there exist a positive integer* $n=n(\nu)$ *and a measure* $\bar{\nu}\in\mathcal{J}_{\Pi^{(n)}}$ *which is* $\Pi^{(n)}$-*exact and*

$$\nu=\frac{1}{n}\sum_{i=0}^{n-1}P^{*i}(\bar{\nu}).$$

b) *For any* $m\in\mathcal{E}_T^b$, *there exist a positive integer* $n=n(m)$ *and a measure* $\bar{m}\in\mathcal{J}_{T^n}^b$ *which is* T^n-*exact and*

$$m=\frac{1}{n}\sum_{i=0}^{n-1}T^{*i}(\bar{m})$$

where T^* *is the dual operator of* T *defined by* $T^*(\bar{m})(B)=\bar{m}(T^{-1}(B))$.

Proof. a) If ν is $\Pi^{(n)}$-ergodic for all $n\geqq 1$, then ν itself is Π-exact (Lemma 5.1). If ν is not $\Pi^{(n)}$-ergodic for some n, let n_1 be the minimum of such n. Then there exists $\nu_1\in\mathcal{E}_{\Pi^{(n_1)}}$ such that $S_{\nu_1}\subset S_\nu$. Let $\bar{\nu}=n_1^{-1}\sum_{i=0}^{n_1-1}P^{*i}(\nu_1)$, then $\bar{\nu}$ is Π-ergodic with $S_{\bar{\nu}}\subset S_\nu$, which shows $\bar{\nu}=\nu$. And also we obtain that $\{\overline{S_{P^{*i}(\nu_1)}}\,; i=0, 1, \cdots, n_1-1\}$ are disjoint. If ν_1 is $\Pi^{(n_1)}$-exact, then we obtain the assertion a). If ν_1 is not $\Pi^{(n_1n)}$-ergodic for some n, we obtain $n_2>n_1$ and $\nu_2\in\mathcal{E}_{\Pi^{(n_2)}}$ such that $\nu=n_2^{-1}\sum_{i=0}^{n_2-1}P^{*i}(\nu_2)$ and $\{\overline{S_{P^{*i}(\nu_2)}}\,; i=0, 1, \cdots, n_2-1\}$ are disjoint. And inductively we obtain $n_k>n_{k-1}$ and $\nu_k\in\mathcal{E}_{\Pi^{(n_k)}}$ as above until ν_k becomes $\Pi^{(n_k)}$-exact. If ν_k is $\Pi^{(n_k)}$-exact, then we obtain the assertion a). Now take an connected open subset O of S_ν. From the relation $\nu=n_k^{-1}\sum_{i=0}^{n_k-1}P^{*i}(\nu_k)$, we have that $S_\nu=\bigcup_{i=0}^{n_k-1}S_{P^{*i}(\nu_k)}$ and $\nu(S_{P^{*i}(\nu_k)})=n_k^{-1}$. Because $\{\overline{S_{P^{*i}(\nu_k)}}\,; i=0, 1, \cdots, n_k-1\}$ are disjoint and O is connected, we have that $O\subset S_{P^{*i}(\nu_k)}$ for some i, so we obtain that $n_k\leqq 1/\nu(O)$. So ν_k must be $\Pi^{(n_k)}$-exact for some k, which shows the assertion a). The assertion b) follows from a) and Lemma 5.3.

Bowen [2] conjectured that if one-dimensional smooth map has an absolutely continuous invariant probability measure then the measure will have a loosely Bernoulli property. The above theorem shows that any ergodic invariant probability measure of any smooth random dynamical system is "loosely exact".

§ 6. Asymptotic Behavior of ω-Orbit and Unique Ergodicity

In this section, we will investigate the uniqueness of Π-invariant measure through the consideration of the asymptotic behavior of ω-orbit $O(f; (x, \omega))=\{f^{n(\omega)}(x); n\geqq 0\}$. The notations and the assumptions are the same as in the preceding sections. Especially Π_x^ω and Π_x are the ones defined in Definition 5.1.

Lemma 6.1. *Assume that there exists a point* $(x, \omega)\in M\times\Omega$ *such that* $O(f; (x, \omega))$ *is dense in* M. *Then* $|\mathcal{I}_\Pi|=$the cardinality of $\mathcal{I}_\Pi=1$ *and the unique invariant probability measure* $\nu\in\mathcal{I}_\Pi$ *is* Π*-exact. Hence* $|\mathcal{I}_T^b|=1$ *and* $\nu\times\boldsymbol{b}$ *is* T*-exact.*

Proof. Take any $\nu\in\mathcal{I}_\Pi$. We may assume that $x\in S_\nu$. By Lemma 5.2, it is enough to prove that $\bar{S}_\nu=M$. For any $y\in M$ and any $\varepsilon>0$, there exists an n such that $d_M(f^{n(\omega)}(x), y)<\varepsilon/2$. By Assumption 4.2, we can choose ω' such that $x\xrightarrow{n}f^{n(\omega')}(x)$ and $d_M(f^{n(\omega')}(x), f^{n(\omega)}(x))<\varepsilon/2$. Thus $d_M(f^{n(\omega')}(x), y)<\varepsilon$, and $f^{n(\omega')}(x)\in S_\nu$ by Lemma 4.2. Hence $y\in\bar{S}_\nu$ and so $\bar{S}_\nu=M$.

Lemma 6.2. *For any* $\nu\in\mathcal{E}_\Pi$ *and any* $x\in S_\nu$, *we have* $\Pi_x^\omega=\{\nu\}$ *for almost all* ω.

Proof. By Theorem 3.1, $m=\nu\times\boldsymbol{b}$ is T-ergodic. Using the ergodic theorem for (T, m), we get that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i(x,\omega)}=m\,, \qquad \text{for a. e. } (x, \omega).$$

Since $T^i(x, \omega)=(f^{i(\omega)}(x), \sigma^i(x))$, we obtain that $\Pi_x^\omega=\{\nu\}$ for a. e. (x, ω). Let $\Omega_x=\{\omega; \Pi_x^\omega=\{\nu\}\}$, then $\boldsymbol{b}(\Omega_x)=1$ for a. e. x. On the other hand, for any $x\in S_\nu$, we have $b(\Omega_{f_\lambda(x)})=1$ for a. e. $\lambda\in\Lambda$, because $f_*^x(\gamma)$ is absolutely continuous with respect to μ. Hence $\boldsymbol{b}(\Omega_x)=\int_\Lambda\boldsymbol{b}(\Omega_{f_\lambda(x)})d\gamma(\lambda)=1$ for any $x\in S_\nu$.

Theorem 6.1. *For any* $x\in M$ *and for almost all* $\omega\in\Omega$, *there exists a* $\nu(x, \omega)\in\mathcal{E}_\Pi$ *such that* $\Pi_x^\omega=\{\nu(x, \omega)\}$.

Proof. Take a $\nu\in\Pi_x$. The measure ν is represented as $\nu=\sum_i\alpha_i\nu_i$, $\alpha_i>0$, $\nu_i\in\mathcal{E}_\Pi$. Since $S_\nu=\bigcup_i S_{\nu_i}$ a. e. (μ) and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P^{*k}(\delta_x)(S_\nu)=1\,,$$

for a. e. ω there exist n and i such that $f^{n(\omega)}(x)\in S_{\nu_i}$. Then, from Lemma 6.2, we have that $\Pi_x^\omega=\{\nu_i\}$.

Definition 6.1. Let X be a closed subset of M. The set X is called a *U-attractor* of the random system (f, M, Λ, γ), if 1) for any open set $G\neq\varnothing$ there exists an $\omega\in\Omega$ such that

$$\lim_{n\to\infty} d_M(f^{n(\omega)}(G), X)=0,$$

where the metric $d_M(A, B)$ of A and $B\subset M$ is defined by

$$d_M(A, B)=\sup_{y\in B}\inf_{x\in A} d_M(x, y)+\sup_{x\in A}\inf_{y\in B} d_M(x, y),$$

and 2) there exists an $(x, \omega')\in X\times\Omega$ such that $\overline{O(f; (x, \omega'))}=X$.

Theorem 6.2. *If the random system (f, M, Λ, γ) has a U-attractor X, then $|\mathcal{I}_\Pi|=1$ and for the unique Π-invariant probability measure we have $\bar{S}_\nu\supset X$.*

Proof. Let $\nu\in\mathcal{E}_\Pi$. Since S_ν is open, there exists an $\omega\in\Omega$ such that $d_M(f^{n(\omega)}(S_\nu), X)\to 0$. Therefore for any $\varepsilon>0$ there exist an $n>0$ and an $x\in S_\nu$ such that $d_M(f^{n(\omega)}(x), X)<\varepsilon$. We can choose $\omega'\in\Omega$ such that $x\xrightarrow{n} f^{n(\omega')}(x)$ and $d_M(f^{n(\omega')}(x), X)<\varepsilon$. Hence by Lemma 4.2, $\bar{S}_\nu\cap X\neq\emptyset$. Take a $y\in\bar{S}_\nu\cap X$ and let $U=R_{R_y}$ then $U\cap X\neq\emptyset$ and $R_U=R_y\subset S_\nu$. By the definition of U-attractor there exist an $\omega\in\Omega$ and a $z\in U$ which satisfy $\overline{O(f; (z, \omega))}=X$. Hence for any $x\in X$ and any $\varepsilon>0$, there exists an ω' such that $d_M(f^{n(\omega')}(z), x)<\varepsilon$ and $z\xrightarrow{n} f^{n(\omega')}(z)$. This means that $\bar{S}_\nu\supset X$. If ν and $\nu'\in\mathcal{E}_\Pi$ are distinct then $\bar{S}_\nu\cap\bar{S}_{\nu'}=\emptyset$. Hence $|\mathcal{E}_\Pi|=1$ and so $|\mathcal{I}_\Pi|=1$.

Example 6.1. Let $f(x, \lambda)=\lambda x(1-x)$. If λ_1, λ_2 satisfy $1<\lambda_1<\lambda_2\leqq 4$ and $(\lambda_1-1)/\lambda_1\geqq f^2_{\lambda_1}(f_{\lambda_2}(1/2))$, then we can take $M=[f_{\lambda_1}(f_{\lambda_2}(1/2)), f_{\lambda_2}(1/2)]$ and $\Lambda=[\lambda_1, \lambda_2]$. Let γ be any probability measure on Λ with a continuous positive density. Then it is easy to see that the random system (f, M, Λ, γ) satisfies Assumptions 4.1-4.3. We claim that this random system is uniquely ergodic.

Indeed, Guckenheimer [5] proved that for a fixed λ, f_λ has an attractive periodic orbit or the partition $\{[0, 1/2), [1/2, 1]\}$ is a generator of f_λ. He also proved that if there is an attractive periodic orbit in $[0, 1]$ then it absorbs almost all points in $[0, 1]$. Hence if $f_\lambda(x)$ has an attractive periodic orbit for some $\lambda\in[\lambda_1, \lambda_2]$, then the periodic orbit becomes a U-attractor of the random system. Thus, in this case $|\mathcal{I}_\Pi|=1$ by Theorem 6.2. If f_λ has a generator for some λ, then for any open interval I there exists an n such that $f^n_\lambda(I)$ contains 1/2. Hence for any $\nu\in\mathcal{I}_\Pi$, $1/2\in S_\nu$. This means that $|\mathcal{I}_\Pi|=1$.

§7. Stability of Invariant Measure under Random Perturbation

Let (f, M, Λ) be the one as in Section 4. Fix a $\lambda_0\in\Lambda$. We consider an infinite family $\{\Lambda_\delta; \delta>0\}$ of submanifolds of Λ such that $\lambda_0\in\Lambda_\delta$ for all $\delta>0$ and $d(\Lambda_\delta)=\sup_{\lambda,\lambda'\in\Lambda_\delta} d_\Lambda(\lambda, \lambda')\leqq\delta$. The restriction $f|_{M\times\lambda_\delta}$ of f to $M\times\Lambda_\delta$ is also denoted by f. We assume that a probability measure γ_δ on Λ_δ is given and $(f, M, \Lambda_\delta, \gamma_\delta)$ satisfies the assumptions in Section 4 for each $\delta>0$. We will investigate the asymptotic behavior of random systems $(f, M, \Lambda_\delta, \gamma_\delta)$ as $\delta\to 0$. Especially we will prove that the invariant measure is stable if f_{λ_0} has an attractor, which is defined as follows.

Definition 7.1. A closed subset X of M is called an *attractor* of f_{λ_0}, if there exists an open set $U\supset X$ such that $\overline{f_{\lambda_0}^{n+1}(U)}\subset$(the interior of $f_{\lambda_0}^{n}(U)$) for all $n>0$ and $X=\bigcap_{n>0}f_{\lambda_0}^{n}(U)$.

Let us denote by Π_δ the Markov chain corresponding to the random system $(f, M, \Lambda_\delta, \gamma_\delta)$.

Lemma 7.1 (*Stability of attractor*). *If f_{λ_0} has an attractor X, then for any $\varepsilon>0$ there exists a $\delta_0>0$ such that for all $\delta<\delta_0$ we have a $\nu_\delta\in\mathcal{E}_{\Pi_\delta}$ satisfying*

$$d_M(S_{\nu_\delta}, X)<\varepsilon .$$

Proof. Let U be an open set stated in the definition of the attractor X. Take an n such that $d_M(f_{\lambda_0}^{n}(U), X)<\varepsilon$ and an $\varepsilon'>0$ such that $\varepsilon'<d_M(f_{\lambda_0}^{n+1}(U)$, (the interior of $f_{\lambda_0}^{n}(U)^c$)). By the continuity of f and the compactness of M, there exists a $\delta_0>0$ such that $d(f^x(\Lambda_\delta))<\varepsilon'$ for all $x\in M$ and $\delta<\delta_0$. Then for any $x\in f_{\lambda_0}^{n}(U)$ and $y\in R_x$ (about Λ_δ), we have that $d_M(y, f_{\lambda_0}(x))<\varepsilon'$ and $f_{\lambda_0}(x)\in f_{\lambda_0}^{n+1}(U)$. Hence $y\in f_{\lambda_0}^{n}(U)$. Thus for some $\nu_\delta\in\Pi_\delta$, $S_{\nu_\delta}\subset f_{\lambda_0}^{n}(U)$ and hence $d_M(S_{\nu_\delta}, X)<\varepsilon$.

Lemma 7.2. *If $\nu_\delta\in\mathcal{I}_{\Pi_\delta}$ converges weakly to ν as $\delta\to 0$, then ν is f_{λ_0}-invariant.*

Proof. For any continuous function g on M, we have

$$\lim_{\delta\to 0}\int g(f_{\lambda_0}(x))d\nu_\delta(x)=\int g(f_{\lambda_0}(x))d\nu(x) .$$

By the continuity of f, we have

$$\lim_{\delta\to 0}\left|\int g(f_{\lambda_0}(x))d\nu_\delta(x)-\iint g(f_\lambda(x))d\gamma_\delta(\lambda)d\nu_\delta(x)\right|=0 .$$

Hence we obtain

$$\lim_{\delta\to 0}\left|\int g(f_{\lambda_0}(x))d\nu(x)-\iint g(f_\lambda(x))d\gamma_\delta(\lambda)d\nu_\delta(x)\right|=0$$

and so

$$\int g(f_{\lambda_0}(x))d\nu(x)=\int g(x)d\nu(x) .$$

Theorem 7.1. *Let $\{(f, M, \Lambda_\delta, \gamma_\delta), \delta>0\}$ be a family of random systems such that each $(f, M, \Lambda_\delta, \gamma_\delta)$ satisfies Assumptions* 4.1–4.3, *$\lambda_0\in\Lambda_\delta$ for a fixed λ_0 and for all $\delta>0$, and $d(\Lambda_\delta)\leqq\delta$. Assume that each $(f, M, \Lambda_\delta, \gamma_\delta)$ has a U-attractor and f_{λ_0} has an attractor X. Then for any $\delta>0$ $|\mathcal{I}_{\Pi_\delta}|=1$ and writing $\mathcal{I}_{\Pi_\delta}=\{\nu_\delta\}$ we have that any limiting measure ν of $\{\nu_\delta\}$ is f_{λ_0}-invariant and the support of ν is included in X.*

Proof. The theorem follows from Theorem 6.2, Lemmas 7.1 and 7.2.

Corollary 7.1. *Let X be an attractor of f_{λ_0} and each random system*

$(f, M, \Lambda_\delta, \gamma_\delta)$, $\delta>0$, have a U-attractor. If $f_{\lambda_0}|X$ is uniquely ergodic and has the invariant measure ν, then for any sequence $\nu_\delta \in \mathcal{I}_{\Pi_\delta}$ the limit measure is ν. Especially, if X is an attractive periodic orbit:

$$X=\{x_p, f_{\lambda_0}(x_p), \cdots, f_{\lambda_0}^{p-1}(x_p)\}, \quad f_{\lambda_0}^p(x_p)=x_p,$$

then the limit measure is

$$\frac{1}{p}\sum_{i=0}^{p-1}\delta(f_{\lambda_0}^i(x_p)), \tag{7.1}$$

where $\delta(x)=\delta_x$.

Remark 7.1. In the situation given in Corollary 7.1, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta(f^{k(\omega_\delta)}(x))$$

for a. e. $\omega_\delta \in \Omega_\delta$ (w. r. t. $\boldsymbol{b}_\delta$) for each $\delta>0$, by Theorems 6.1 and 6.2. Then by Corollary 7.1 ν_δ converges weakly to (7.1).

To summarize these, we introduce the product measure $\boldsymbol{b}=\prod_{\delta>0}\boldsymbol{b}_\delta$ on $\Omega=\prod_{\delta>0}\Omega_\delta$, where $\boldsymbol{b}_\delta=\gamma_\delta^N$. Then for all $x\in M$ and for almost all $\omega=(\omega_\delta)$ (w. r. t. $\boldsymbol{b}$) we have

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta(f^{k(\omega_\delta)}(x))=\frac{1}{p}\sum_{i=0}^{p-1}\delta(f_{\lambda_0}^i(x_p)). \tag{7.2}$$

Thus we may think the right hand side of (7.2) as an asymptotic measure of f_{λ_0}. The notion of asymptotic measures was introduced by D. Ruelle.

Definition 7.2. If an attractive periodic point x_p satisfies (7.2), we call it a *representative* periodic point of f_{λ_0}.

Example 7.1. Let us consider two dimensional map $F_\rho(x, y)=(\rho x(1-x)+\varepsilon xy, \mu y+\varepsilon xy)$, where $0<\mu$, $\varepsilon\ll 1$ and $0<\rho<4$. Notice that the x-axis and the y-axis are invariant under F_ρ and the restriction of F_ρ t_o the x-axis is $g_\rho(x)=\rho x(1-x)$. Let $M=[\delta, 1]\times[-1, 1]$ for $0<\delta\ll 1$. We will show that if $g_\rho(x)$ has an attractive periodic point x_p for a fixed parameter ρ then x_p is a representative periodic point of F_ρ.

Let $(f, M, \Lambda_n, \gamma_n)$, $n\geqq 1$, be a sequence of random systems. Assume that there exists a common value $\lambda_0\in\Lambda_n$ for all $n\geqq 1$ such that $f_{\lambda_0}=F_\rho$ with the parameter ρ mentioned above. It is easy to see that for any open set U in M $\lim_{k\to\infty}d_M(F_\rho^k(U), x\text{-axis})=0$. Therefore for any $\nu_n\in\mathcal{I}_{\nu_n}$ $\bar{S}_{\Pi_n}\cap(x\text{-axis})\neq\emptyset$, and so $V=S_{\nu_n}\cap(x\text{-axis})\neq\emptyset$. By the property of g_ρ there exists k such that $g_\rho^k(V)$ contains 1/2. Hence $|\mathcal{I}_{\Pi_n}|=1$. Moreover S_{ν_n} contains the point $(x_p, 0)$. Thus we get

$$\lim_{n\to\infty}\nu_n=\frac{1}{p}\sum_{i=0}^{p-1}\delta(F_\rho^i(x_p, 0))$$

which implies (7.2).

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