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Derivations in C^*-Algebras Commuting with Compact Actions

By

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Abstract

Let (A, G, α) be a C^*-dynamical system such that G is compact, and let δ be a closed *-derivation in A commuting with α. If $A^\alpha \cap \mathcal{D}(\delta) \subset \ker \delta$ and A^α is contained in the center of A, then δ is the generator of a strongly continuous one-parameter group of *-automorphisms of A commuting with α.

§1. Introduction.

We shall consider the problem of whether a closed *-derivation in a C^*-algebra commuting with an action of a locally compact group is the generator of a strongly continuous one-parameter group of *-automorphisms of the C^*-algebra.

S. Sakai [14] first proposed this question and proved that a non-zero closed derivation in $C(\boldsymbol{T})$ commuting with translations by elements of $\boldsymbol{T}$ is a constant multiple of differentiation. The same result for $C_0(\boldsymbol{R})$ has been obtained by C. J. K. Batty [4] and F. Goodman [8]. F. Goodman [9] and H. Nakazato [13] generalized it to every locally compact group and compact group, respectively. In [9] the case that a locally compact group acts on its homogeneous space has also been considered.

In the present paper we answer this problem in the affirmative for every ergodic action of a compact group. We shall also give some related results on non-ergodic actions of compact groups.

§2. Notation and Preliminaries

Let α be a strongly continuous representation of a locally compact group G on a Banach space X which is bounded. Then α induces a continuous representation of $L^1(G)$ on X defined as follows: for $f \in L^1(G)$, $a \in X$ and $\phi \in X^*$,

$$\langle \alpha(f)a, \phi \rangle = \int f(g) \langle \alpha_g(a), \phi \rangle dg,$$

where dg denotes a left Haar measure of G; see [3].

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Suppose that G is compact. Let γ be an element of $\hat{G}$, the set of all equivalence classes of irreducible continuous unitary representations of G, and let $U^{\gamma}\in\gamma$. Since the representation space of U^{γ} is of finite dimension, U_g^{γ} may correspond to a unitary matrix $(u_{ij}^{\gamma}(g))$, so that u_{ij}^{γ} are continuous on G. $\mathrm{Tr}\,(U^{\gamma})$, denoted by χ_{γ}, is called the character of γ and coincides with $\sum_i u_{ii}^{\gamma}$. We denote $\alpha(\bar{\chi}_{\gamma})X$ by $X^{\alpha}(\gamma)$ and call this a minimal spectral subspace. $C(G)^{\lambda}(\gamma)$ is spanned by u_{ij}^{γ} and $C(G)$ is topologically spanned by $C(G)^{\lambda}(\gamma)$ with all γ in $\hat{G}$, where λ denotes the left regular representation of G on $C(G)$. It follows from these facts that the $X^{\alpha}(\gamma)$ span X. If, moreover, X is a *-algebra and the α_g are *-automorphisms, then $\sum_{\gamma\in\hat{G}} X^{\alpha}(\gamma)$ is a dense *-subalgebra of X, because $\sum_{\gamma\in\hat{G}} C(G)^{\lambda}(\gamma)$ is a *-subalgebra of $C(G)$; see [11]. If T is a closed linear mapping in X commuting with α, then T commutes with $\alpha(\bar{\chi}_{\gamma})$ and hence $\alpha(\bar{\chi}_{\gamma})\mathcal{D}(T)\subset X^{\alpha}(\gamma)\cap\mathcal{D}(T)$ and $T(\alpha(\bar{\chi}_{\gamma})\mathcal{D}(T))\subset X^{\alpha}(\gamma)$. Moreover the $\sum_{\gamma\in\hat{G}} X^{\alpha}(\gamma)\cap\mathcal{D}(T)$ is dense in the Banach space $\mathcal{D}(T)$, that is, a core for T.

By a C^*-dynamical system we mean a triplet (A, G, α) of a C^*-algebra A, a locally compact group G and a strongly continuous representation α of G by *-automorphisms of A. If G is compact, then $\int \alpha_g dg$ is a faithful projection of norm one onto the fixed-point algebra A^{α} under α, denoted by E.

A linear mapping δ in a C^*-algebra A is said to be a derivation if its domain $\mathcal{D}(\delta)$ is a dense subalgebra of A and if $\delta(ab)=a\delta(b)+\delta(a)b$ for any $a, b\in\mathcal{D}(\delta)$. A derivation δ is said to be a *-derivation if $\mathcal{D}(\delta)$ is self-adjoint and $\delta(a^*)=\delta(a)^*$ for any $a\in\mathcal{D}(\delta)$. If a *-derivation is a generator in A, then it is clearly the generator of a strongly continuous one-parameter group of *-automorphisms of A.

Let ϕ be a G-invariant state of A and $\{\pi_{\phi}, \mathcal{H}_{\phi}, \xi_{\phi}\}$ the GNS-representation associated with ϕ. By α^{ϕ} and U^{ϕ} we denote a strongly continuous representation of G on $\pi_{\phi}(A)$ and a continuous unitary representation of G on $\mathcal{H}_{\phi}$ such that $\alpha_g^{\phi}\circ\pi_{\phi}=\pi_{\phi}\circ\alpha_g$, $U_g^{\phi}a\xi_{\phi}=\alpha_g^{\phi}(a)\xi_{\phi}$ and $\alpha_g^{\phi}(a)=U_g^{\phi}aU_g^{\phi*}$ for any $a\in\pi_{\phi}(A)$. We note that the fixed-point algebra of $\pi_{\phi}(A)$ under α^{ϕ} coincides with $\pi_{\phi}(A^{\alpha})$, provided that G is compact.

If δ is a derivation in A and ϕ is a state of A with $\phi\circ\delta=0$, then δ induces a derivation δ_{ϕ} in $\pi_{\phi}(A)$ with the domain $\pi_{\phi}(\mathcal{D}(\delta))$ such that $\delta_{\phi}(\pi_{\phi}(a))=\pi_{\phi}(\delta(a))$ for any $a\in\mathcal{D}(\delta)$. In fact, for any $a\in\ker\pi_{\phi}\cap\mathcal{D}(\delta)$, $b\in\mathcal{D}(\delta)$ and $c\in\mathcal{D}(\delta)^*$,

$$
\begin{aligned}
&(\pi_{\phi}(\delta(a))\pi_{\phi}(b)\xi_{\phi}\,|\,\pi_{\phi}(c)\xi_{\phi})\\
&\qquad=\phi(c^*\delta(a)b)=\phi(\delta(c^*ab)-c^*a\delta(b)-\delta(c^*)ab)\\
&\qquad=-\phi(c^*a\delta(b))-\phi(\delta(c^*)ab)\\
&\qquad=-(\pi_{\phi}(a)\pi_{\phi}(\delta(b))\xi_{\phi}\,|\,\pi_{\phi}(c)\xi_{\phi})-(\pi_{\phi}(a)\pi_{\phi}(b)\xi_{\phi}\,|\,\pi_{\phi}(\delta(c^*)^*)\xi_{\phi})\\
&\qquad=0\,,
\end{aligned}
$$

and so $\delta(\ker\pi_{\phi}\cap\mathcal{D}(\delta))\subset\ker\pi_{\phi}$.

Throughout this paper, let (A, G, α) denote a C^*-dynamical system with a compact group G.

§ 3. Derivations Commuting with Ergodic Actions

Lemma 3.1. *Let δ be a derivation in A commuting with α and E such that $A^{\alpha}\cap\mathscr{D}(\delta)\subset\ker\delta$.*

Then every G-invariant state ϕ of A is invariant under δ, i.e., $\phi\circ\delta=0$, and hence δ and δ_{ϕ} are closable and these closures are derivations in A and $\pi_{\phi}(A)$ commuting with α and α^{ϕ}, respectively.

Proof. The assumption implies that for any $a\in\mathscr{D}(\delta)$,

$$\phi(\delta(a))=\phi\circ E(\delta(a))=\phi(\delta(E(a)))=0\,.$$

Since there is a separating family of G-invariant states of A, it follows from [13] Corollary 2.3 that δ is closable. It is clear that the closure of δ is a derivation in A commuting with α. These facts are therefore valid for δ_{ϕ}.

Q. E. D.

Theorem 3.2. *Let δ be a closed *-derivation in A commuting with α such that $A^{\alpha}\cap\mathscr{D}(\delta)\subset\ker\delta$. Suppose that for any $\gamma\in\hat{G}$, $A^{\alpha}(\gamma)$ is of finite dimension.*

Then δ is a generator.

Proof. We shall prove that $\|(1+\lambda\delta)(a)\|\geqq\|a\|$ for any $a\in\mathscr{D}(\delta)$ and $\lambda\in\boldsymbol{R}$. Let ϕ be a G-invariant state of A, so that $\phi\circ\delta=0$ by Lemma 3.1. Defining H by $Ha\xi_{\phi}=-i\delta_{\phi}(a)\xi_{\phi}$, it is easy that H is a symmetric operator commuting with U^{ϕ} and $\delta_{\phi}(a)=\overline{[i\overline{H}, a]}$ for any $a\in\mathscr{D}(\delta_{\phi})$. Since $\pi_{\phi}(\alpha(\bar{\chi}_{\gamma})a)\xi_{\phi}=U^{\phi}(\bar{\chi}_{\gamma})(\pi_{\phi}(a)\xi_{\phi})$ for any $a\in A$, $U^{\phi}(\bar{\chi}_{\gamma})\pi_{\phi}(A)\xi_{\phi}$ is of finite dimension, and so coincides with $\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)$. Since $U^{\phi}(\bar{\chi}_{\gamma})\mathscr{D}(\overline{H})$ is contained in $\mathscr{D}(\overline{H})$ and dense in $\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)$, we have $\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)\subset\mathscr{D}(\overline{H})$. Hence, by the injectivity of $1+i\overline{H}$, $(1+i\overline{H})\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)=\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)$ or $(1+i\overline{H})\sum_{\gamma\in\hat{G}}\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)=\sum_{\gamma\in\hat{G}}\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)$. Since $\sum_{\gamma\in\hat{G}}\mathscr{H}_{\phi}^{U^{\phi}}(\gamma)$ is dense in $\mathscr{H}_{\phi}$, it follows that $\overline{H}$ is self-adjoint. $(\mathrm{Ad}\, e^{it\overline{H}})$ is therefore a σ-weakly continuous one-parameter group of *-automorphisms of $\mathscr{B}(\mathscr{H}_{\phi})$. Denoting by δ' its generator, we have $\|(1+\lambda\delta')(a)\|\geqq\|a\|$ for any $a\in\mathscr{D}(\delta')$ and $\lambda\in\boldsymbol{R}$. Since $\delta_{\phi}(a)=\overline{[i\overline{H}, a]}=\delta'(a)$ for any $a\in\mathscr{D}(\delta_{\phi})$, we have $\|\pi_{\phi}((1+\lambda\delta)(a))\|\geqq\|\pi_{\phi}(a)\|$ for any $a\in\mathscr{D}(\delta)$ and $\lambda\in\boldsymbol{R}$.

Now there exists a separating family (ϕ_{ι}) of G-invariant states of A, so that $\bigoplus_{\iota}\pi_{\phi_{\iota}}$ is faithful. It follows from the above facts that for any $a\in\mathscr{D}(\delta)$ and $\lambda\in\boldsymbol{R}$,

$$\begin{aligned}\|(1+\lambda\delta)(a)\|&=\|\bigoplus_{\iota}\pi_{\phi_{\iota}}((1+\lambda\delta)(a))\|\\&=\sup_{\iota}\|\pi_{\phi_{\iota}}((1+\lambda\delta)(a))\|\\&\geqq\sup_{\iota}\|\pi_{\phi_{\iota}}(a)\|\\&=\|a\|\,.\end{aligned}$$

Since $\alpha(\bar{\chi}_{\gamma})\mathscr{D}(\delta)$ is dense in $A^{\alpha}(\gamma)$ and $A^{\alpha}(\gamma)$ is of finite dimension, $A^{\alpha}(\gamma)=$

$\alpha(\bar{\chi}_\gamma)\mathscr{D}(\delta)\subset\mathscr{D}(\delta)$. For any $\lambda\in\boldsymbol{R}$, since $1+\lambda\delta$ is injective, we have $(1+\lambda\delta)A^\alpha(\gamma)=A^\alpha(\gamma)$ and so $(1+\lambda\delta)\mathscr{D}(\delta)$ is dense in A. Thus it follows from the Hille-Yoshida theorem that δ is a generator. Q.E.D.

Corollary 3.3. *Under the same assumption as in Theorem 3.2, if δ is a closed derivation in A commuting with α such that $A^\alpha\cap\mathscr{D}(\delta)\subset\ker\delta$, then the *-derivations $1/2(\delta+\delta^*)$ and $1/2i(\delta-\delta^*)$ are pregenerators and δ coincides with the closure of $1/2\overline{(\delta+\delta^*)}+1/2\overline{(\delta-\delta^*)}$, where $\delta^*(a)=\delta(a^*)^*$ for $a\in\mathscr{D}(\delta)^*$.*

Proof. As in Theorem 3.2, we obtain $\sum_{\gamma\in\hat{G}}A^\alpha(\gamma)\subset\mathscr{D}(\delta)$. Since $\sum_{\gamma\in\hat{G}}A^\alpha(\gamma)$ is a dense *-subalgebra of A, $1/2(\delta+\delta^*)$ and $1/2i(\delta-\delta^*)$ are densely defined *-derivations in A commuting with α and E. By Lemma 3.1, they are closable and so pregenerators. Since $\sum_{\gamma\in\hat{G}}A^\alpha(\gamma)$ is a common core of all closed linear mappings commuting with α, δ is the closure of $1/2\overline{(\delta+\delta^*)}+1/2\overline{(\delta-\delta^*)}$. Q.E.D.

If either $A=C(G/H)$ and α is the left regular representation, where H is a closed subgroup of G, or G is abelian and α is ergodic, then it is well known that each minimal spectral subspace is of finite dimension. In fact, in the former, $\dim A^\alpha(\gamma)\leqq(\dim\gamma)^2$ because $C(G/H)$ can be embedded in $C(G)$. In the latter, each minimal spectral subspace is zero or one dimensional. But, for any ergodic action α, [12] Proposition 2.1 assures $\dim A^\alpha(\gamma)\leqq(\dim\gamma)^2$. Therefore we obtain the following theorem:

Theorem 3.4. *Let δ be a closed *-derivation in A commuting with α. If α is ergodic, then δ is a generator.*

In particular, for abelian groups we have the following:

Proposition 3.5. *Suppose that A has a unit, α is ergodic and G is abelian. If δ is a closed derivation in A commuting with α, then there exists a unique homomorphism d of $(\ker\alpha)^\perp$ into $\boldsymbol{C}$ such that δ is the multiplication by $id(\gamma)$ on $A^\alpha(\gamma)$ with $\gamma\in(\ker\alpha)^\perp$; moreover, when δ is a *-derivation, δ is the generator of (α_{g_t}) for some continuous one-parameter subgroup (g_t) of G.*

Conversely, if d is a homomorphism of $(\ker\alpha)^\perp$ into $\boldsymbol{C}$, then there exists a unique closed derivation δ in A such that δ is the multiplication by $id(\gamma)$ on $A^\alpha(\gamma)$ with $\gamma\in(\ker\alpha)^\perp$.

Proof. Since α is ergodic, if $\gamma\in\mathrm{sp}\,\alpha$, then $A^\alpha(\gamma)=\boldsymbol{C}u_\gamma$ for some unitary u_γ. Since $A^\alpha(\gamma)A^\alpha(\gamma')\subset A^\alpha(\gamma+\gamma')$, $\mathrm{sp}\,\alpha$ is a subgroup of $\hat{G}$. Therefore, by [6] Lemma 2.3.8, we have $\mathrm{sp}\,\alpha=(\ker\alpha)^\perp$. With $d(\gamma)=-i\delta(u_\gamma)u_\gamma^*$, d is a homomorphism of $\mathrm{sp}\,\alpha$ into $\boldsymbol{C}$. For, $u_\gamma u_{\gamma'}\in A^\alpha(\gamma+\gamma')$ and $\delta(u_\gamma u_{\gamma'})=u_\gamma\delta(u_{\gamma'})+\delta(u_\gamma)u_{\gamma'}=i(d(\gamma)+d(\gamma'))u_\gamma u_{\gamma'}$.

If γ is a *-derivation, then d is real-valued, and hence, $e^{itd}\in(\mathrm{sp}\,\alpha)\hat{}\simeq G/\ker\alpha$ for any $t\in\boldsymbol{R}$. Therefore, by [2] Lemma B, there exists a continuous one-parameter subgroup (g_t) of G such that $\langle g_t, \gamma\rangle = e^{itd(\gamma)}$. δ is then the generator of (α_{g_t}). In fact,

$$e^{t\delta}u_\gamma = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n(u_\gamma) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(id(\gamma))^n u_\gamma$$
$$= e^{itd(\gamma)}u_\gamma = \langle g_t, \gamma\rangle u_\gamma = \alpha_{g_t}(u_\gamma).$$

Conversely, suppose that d is a homomorphism of $\mathrm{sp}\,\alpha$ into C. Defining δ on $\sum_{\gamma\in\hat{G}} A^\alpha(\gamma)$ by $\delta(\sum_\gamma \lambda_\gamma u_\gamma) = i\sum_\gamma d(\gamma)\lambda_\gamma u_\gamma$, we see easily that δ is a derivation commuting with α and the projection E. By Lemma 3.1, δ is closable and can be uniquely extended to a closed derivation in A commuting with α. Q. E. D.

§4. Non Ergodic Cases

Throughout this section, let δ denote a closed *-derivation in A commuting with α.

We first of all note that $A^\alpha \cap \mathcal{D}(\delta) \subset \ker\delta$ implies $A^\alpha \subset \ker\delta$ and that a G-invariant state of A is the only one if α is ergodic. The former is derived from $A^\alpha = \overline{E(\mathcal{D}(\delta))}$ and the closedness of $\ker\delta$, and the latter from the fact that $\phi(a)1 = E(a)$ for any G-invariant state ϕ and any $a \in A$.

Lemma 4.1. *The following three conditions are equivalent:*

(i) *there exists a separating family of G-ergodic states of A for which α^ϕ is ergodic on $\pi_\phi(A)$;*

(ii) *α^ϕ is ergodic for any G-ergonic state ϕ of A;*

(iii) *A^α is contained in the center of A.*

If one of the above conditions is satisfied, then the set $\mathcal{E}$ of all G-ergodic states of A is closed in the state space of A, and hence locally compact with respect to $\sigma(A^*, A)$-topology, and A^α is isomorphic to $C_0(\mathcal{E})$ in a natural way; moreover the function $\mathcal{E} \ni \phi \mapsto \|\pi_\phi(a)\|$ is upper semi-continuous for any $a \in A$.

Proof. Let (ϕ_ι) be a separating family of G-ergodic states of A such that α^{ϕ_ι} is ergodic, so that $\bigoplus_\iota \pi_{\phi_\iota}$ is faithful. Then for any $a \in A^\alpha$ and $b \in A$, we have

$$\bigoplus_\iota \pi_{\phi_\iota}(ab) = \bigoplus_\iota \phi_\iota(a)\pi_{\phi_\iota}(b) = \bigoplus_\iota \pi_{\phi_\iota}(ba),$$

and so $ab = ba$. Hence (i) implies (iii).

Suppose the condition (iii). If ϕ is a G-ergodic state of A, then $\pi_\phi(A)' \cap \{U_g^\phi \mid g \in G\}' = C1$ and so $\pi_\phi(A^\alpha) \subset C1$. Since $\pi_\phi(A^\alpha)$ coincides with the fixed point algebra of $\pi_\phi(A)$ under α^ϕ, we obtain that α^ϕ is ergodic.

The set of all G-invariant states of A is separating, and so also is the set of all G-ergodic states of A. Thus the conditions (i)-(iii) are equivalent.

Suppose the condition (iii), so that A^α is commutative. The transposed mapping of E is a weakly continuous and affine bijection between the compact set of positive forms on A^{α} of norm $\leqq 1$ and the compact set of G-invariant positive forms on A of norm $\leqq 1$. Therefore $\mathcal{E}$ is homeomorphic to the spectrum of A^{α}, so that $\mathcal{E}$ is locally compact and A^{α} is isomorphic to $C_0(\mathcal{E})$. Moreover, considering the C^*-algebra obtained from A by the adjunction of the unit, it follows that $\mathcal{E}$ is closed in the state space of A.

For each $\phi\in\mathcal{E}$ we have that $\{\psi\circ\pi_{\phi}\,|\,\psi$ is a state of $\pi_{\phi}(A)\}$ coincides with the intersection of ${}^tE^{-1}(\phi)$ and the state space of A, where ${}^tE$ denotes the transposed mapping of E in A^*. If, in fact, a state ψ of A belongs to ${}^tE^{-1}(\phi)$, then $\int\psi(\alpha_g(a))dg=\psi\circ E(a)=\phi(a)=0$ for any positive element $a\in\ker\pi_{\phi}$. Since the function $g\mapsto\psi(\alpha_g(a))$ is continuous and non-negative, it follows that $\psi(a)=0$ for any positive element $a\in\ker\pi_{\phi}$ and also for any $a\in\ker\pi_{\phi}$, and so that $\psi=\psi'\circ\pi_{\phi}$ for some state ψ' of $\pi_{\phi}(A)$.

Denoting by $\mathcal{S}$ the intersection of ${}^tE^{-1}(\mathcal{E})$ and the state space of A, which is locally compact, we deduce from above facts that the self-adjoint portion of A can be embedded isometrically in $C_0(\mathcal{S})$. It follows from the compactness of ${}^tE^{-1}(\phi)\cap\mathcal{S}$ that the function $\mathcal{E}\ni\phi\mapsto\sup\{|f(\psi)|\,|\,\psi\in{}^tE^{-1}(\phi)\cap\mathcal{S}\}$ is upper semi-continuous for any continuous function f on $\mathcal{S}$. Since $\|\pi_{\phi}(a)\|^2=\|\pi_{\phi}(a^*a)\|=\sup\{\psi(\pi_{\phi}(a^*a))\,|\,\psi$ is a state of $\pi_{\phi}(A)\}$, we obtain the upper semi-continuity of the function $\mathcal{E}\ni\phi\mapsto\|\pi_{\phi}(a)\|$ for any $a\in A$. Q.E.D.

Recently, Goodman and Jørgensen answered our problem in the affirmative for commutative C^*-algebras, who also proved Theorem 3.4 independently [10]. The following theorem generalizes their rerult.

Theorem 4.3. *Let δ be a closed *-derivation in A commuting with α. If $A^{\alpha}\cap\mathcal{D}(\delta)\subset\ker\delta$ and A^{α} is contained in the center of A, then δ is a generator.*

Proof. For each G-ergodic state ϕ of A, it follows from Lemma 3.1, Theorem 3.4, and Lemma 4.1 that δ_{ϕ} is a pregenerator, so that $\|(1+\lambda\delta_{\phi})(a)\|\geqq\|a\|$ for any $a\in\mathcal{D}(\delta_{\phi})$ and any $\lambda\in\boldsymbol{R}$. By Lemma 4.1, as in the proof of Theorem 3.2, we have therefore that $\|(1+\lambda\delta)(a)\|\geqq\|a\|$ for any $a\in\mathcal{D}(\delta)$ and any $\lambda\in\boldsymbol{R}$.

Now we show that $(1+\lambda\delta)\mathcal{D}(\delta)$ is dense in A for any $\lambda\in\boldsymbol{R}$. For $\varepsilon>0$, let K be the set of all G-ergodic states ϕ such that $\|\pi_{\phi}(a)\|\geqq\varepsilon$, so that K is compact. For $\phi\in\mathcal{E}$, since δ_{ϕ} is a pregenerator, $\|\pi_{\phi}(a)-(1+\lambda\delta)\pi_{\phi}(b)\|<\varepsilon$ for some $b\in\mathcal{D}(\delta)$. Hence, by Lemma 4.1, there exists a neighbourhood of ϕ on which $\|\pi_{\phi}(a-(1+\lambda\delta)(b))\|<\varepsilon$. Since $A^{\alpha}\simeq C_0(\mathcal{E})$, there exist a continuous partition (c_i) of unity corresponding to a finite open covering (V_i) of K and a family (b_i) in $\mathcal{D}(\delta)$ such that $c_i\in A^{\alpha}$, $\sum\phi(c_i)=1$ on K and $\|\pi_{\phi}(a-(1+\lambda\delta)(b_i))\|<\varepsilon$ on V_i. Hence we have

$$\begin{aligned}&\sup_{\phi\in K}\|\pi_{\phi}(a-(1+\lambda\delta)(\textstyle\sum_i b_ic_i))\|\\&\quad=\sup_{\phi\in K}\|\textstyle\sum_i\phi(c_i)\pi_{\phi}(a-(1+\lambda\delta)(b_i))\|\\&\quad<\varepsilon\,,\end{aligned}$$

so that $\|a-(1+\lambda\delta)(\sum_i b_i c_i)\|<\varepsilon$. Thus $(1+\lambda\delta)\mathcal{D}(\delta)$ is dense in A, which implies that δ is a generator. Q. E. D.

Remark. Suppose that A^α is contained in the center of A. Then, by Lemma 4.1 and [12], every G-ergodic state is a trace. Therefore A has a separating family (ϕ_ι) of tracial states and $(\bigoplus_\iota \pi_{\phi_\iota})(A)''$ is a finite von Neumann algebra. For further results, see [1], [12].

Remark. In [10], it was shown that a closed *-derivation δ in the C^*-algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on a separable Hilbert space $\mathcal{H}$ is a generator if it commutes with an action α of a compact group G on $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})^\alpha \cap \mathcal{D}(\delta) \subset \ker\delta$. This is valid for non-separable Hilbert spaces. The following proof is inspired by Kishimoto.

It suffices to show that the set $\mathcal{U}$ of all unitaries which implement some α_g is weakly compact. Then $\mathcal{U}$ induces an action on $\mathcal{K}(\mathcal{H})$ satisfying the same hypothesis as (G, α). Clearly, the function $U \mapsto UxU^*$ is weakly continuous on the unit ball of $\mathcal{B}(\mathcal{H})$ for any operator x of rank one, and hence for any compact operator. Since $\alpha(G)$ is compact with respect to the topology of weak pointwise convergence, it then follows that for any $U \in \overline{\mathcal{U}}$, U and U^* implement some α_g, so that $\mathcal{U}$ is a unitary. Therefore $\mathcal{U}$ is weakly compact.

Appendix

A. Kishimoto showed, independent of [5], that for a C^*-dynamical system (A, G, α) with a compact abelian group G, a closed *-derivation δ in A commuting with α is a generator if $A^\alpha \cap \mathcal{D}(\delta) \subset \ker\delta$.

We shall describe the outline of its proof. Any element of $A^\alpha(\gamma)\cap\mathcal{D}(\delta)$ can be approximated by such an element a of $A^\alpha(\gamma)\cap\mathcal{D}(\delta)$ that $a=av^*v$ for some $v\in A^\alpha(\gamma)\cap\mathcal{D}(\delta)$. For this, consider the following functions f and g, and notice $A^\alpha(\gamma)A^\alpha(\gamma')\subset A^\alpha(\gamma+\gamma')$: $f(t)=0$ on $[0, \varepsilon]$, $\varepsilon^{-1}(t-\varepsilon)$ on $(\varepsilon, 2\varepsilon)$, and 1 on $[2\varepsilon, \infty)$, $g(t)=\varepsilon^{-1}$ on $[0, \varepsilon]$ and t^{-1} on (ε, ∞). Such an element a is analytic for δ. For, since $av^*\in A^\alpha\subset\ker\delta$, we have $\delta(a)=av^*\delta(v)$ and so $\delta^n(a)=a(v^*\delta(v))^n$. Thus the set of all analytic elements for δ is dense in A. By a slight modification of Theorem 3.2, we then conclude that δ is a generator.

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References

[1] Albeverio, S. and Høegh-Krohn, R., Ergodic actions by compact groups on C^*-algebras, *Math. Zeitschrift*, to appear.

[2] Araki, H., Kastler, D., Takesaki, M. and Haag, R., Extension of KMS states and chemical potential, *Commun. Math. Phys.* **53** (1977), 97-134.

[3] Arveson, W., On groups of automorphisms of operator algebras, *J. Funct. Anal.* **15** (1974), 217-243.

[4] Batty, C. J. K., Derivations in compact spaces, *Proc. London Math. Soc.*, to appear.

[5] Bratteli, O. and Jørgensen, P. E. T., Unbounded derivations tangentail to compact groups of automorphisms, to appear.

[6] Bratteli, O. and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[7] Connes, A., Une classification des facteurs de type III, *Ann. Sci. Ecole Norm. Sup.* **6** (1973), 133-252.

[8] Goodman, F., Closed derivations in commutative C^*-algebras, *J. Funct. Anal.* **39** (1980), 308-346.

[9] ———, Translation invariant closed *-derivations, *Pacific J. Math.*, to appear.

[10] Goodman, F. and Jørgensen, P. E. T., Unbounded derivations commuting with compact group actions, to appear.

[11] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis* II, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[12] Høegh-Krohn, R., Landstad, M. B. and Størmer, E., Compact ergodic groups of automorphisms, preprint, Univ. of Oslo, 1980.

[13] Nakazato, H., Closed *-derivations on compact groups, preprint, 1980.

[14] Sakai, S., The theory of unbounded derivations in operator algebras, Lecture Notes, Univ. of Copenhagen and Univ. of Newcastle upon Tyne, 1977.