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Spline Solutions for Nonlinear Fourth-Order Two-Point Boundary Value Problems

By

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Abstract

Methods of order 2 and 4 are developed for the continuous approximation of the solution of a nonlinear fourth-order two-point boundary value problem. Numerical results are briefly summarized to demonstrate the practical usefulness of the methods.

§ 1. Introduction and Description of Methods

We shall consider smooth approximation of the solution of the following two point boundary value problem:

$$x^{(4)}(t)=f(t, x(t)), \qquad 0\leqq t\leqq 1, \tag{1.1}$$

$$M_0\begin{pmatrix} x(0) \\ x'(0) \\ x''(0) \\ x^{(3)}(0) \end{pmatrix}+M_1\begin{pmatrix} x(1) \\ x'(1) \\ x''(1) \\ x^{(3)}(1) \end{pmatrix}=d, \tag{1.2}$$

where d is a 4-vector, and M_0, M_1 are constant 4×4 matrices. The function $f(t, x)$ is defined and four times continuously differentiable in a region D of (t, x)-space intercepted by two lines $t=0$ and $t=1$.

The problem of this type arises in the plate deflection theory. Finite difference methods and collocation methods are developed and analysed for the restrictive linear case by Papamichael [1] and Usmani [4, 5].

In this paper we shall assume that the problem (1.1)-(1.2) has an isolated solution $\hat{x}(t)$ satisfying the internality condition

$$U=\{(t, x): |x-\hat{x}(t)|\leqq\delta, t\in[0, 1]\}\subset D \qquad \text{for some } \delta>0. \tag{1.3}$$

The object of this paper is to show the existence and convergence of spline approximations to the solution of (1.1) - (1.2) on this assumption. The solution $\hat{x}(t)$ is isolated if and only if

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(1.4) $$G=M_0\Phi(0)+M_1\Phi(1)=M_0+M_1\Phi(1)$$

is nonsingular, where $\Phi(t)$ is the solution of the initial value problem:

(1.5) $$\Phi'=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \sigma & 0 & 0 & 0 \end{pmatrix}\Phi\,, \qquad \Phi(0)=E\,,$$

($\sigma(t)=f_x(t,\ \hat{x}(t))$ and E is the unit matrix).

Now making use of B-spline $Q_{m+1}(t)=(1/m!)\sum_{i=0}^{m}(-1)^i\binom{m+1}{i}(t-i)_+^m$, we consider a quintic spline function of the form:

(1.6) $$x_h(t)=\sum_{i=-5}^{n-1}\alpha_i Q_6(t/h-i) \qquad (nh=1)$$

with undetermined coefficients $(\alpha_{-5},\ \alpha_{-4},\ \cdots,\ \alpha_{n-1})$. The above $x_h(t)$ will be an approximate solution to the problem (1.1)-(1.2) if it satisfies

(1.7) $$x_h^{(4)}=P_1 f(t,\ x_h(t))\,, \qquad 0\leqq t\leqq 1\,,$$

subject to the same boundary conditions (1.2). Here P_1 is an operator defined by $(P_1 g)(t)=\sum_{i=0}^{n} g_i L_i(t)$, where $L_i(t)$ is a piecewise linear function with the property $L_i(t_j)=\delta_{ij}$, $t_j=jh$. Any two piecewise linear functions coincide with each other if and only if they coincide at the nodes, therefore we see that equation (1.7) is equivalent to the following $n+1$ equations:

(1.8) $$\begin{aligned} F_i(\alpha)=&(1/h^4)(\alpha_{i-1}-4\alpha_{i-2}+6\alpha_{i-3}-4\alpha_{i-4}+\alpha_{i-5}) \\ &-f(t_i,\ (1/120)(\alpha_{i-1}+26\alpha_{i-2}+66\alpha_{i-3}+26\alpha_{i-4}+\alpha_{i-5}))=0\,, \\ &i=0(1)n\,. \end{aligned}$$

The boundary conditions give four equations towards the determination of the unknowns:

(1.9) $$\begin{aligned} \begin{pmatrix} F_{-2}(\alpha) \\ F_{-1}(\alpha) \\ F_{n+1}(\alpha) \\ F_{n+2}(\alpha) \end{pmatrix} =& M_0\begin{pmatrix} (1/120)(\alpha_{-1}+26\alpha_{-2}+66\alpha_{-3}+26\alpha_{-4}+\alpha_{-5}) \\ (1/24h)(\alpha_{-1}+10\alpha_{-2}-10\alpha_{-4}-\alpha_{-5}) \\ (1/6h^2)(\alpha_{-1}+2\alpha_{-2}-6\alpha_{-3}+2\alpha_{-4}+\alpha_{-5}) \\ (1/2h^3)(\alpha_{-1}-2\alpha_{-2}+2\alpha_{-4}-\alpha_{-5}) \end{pmatrix} \\ &+M_1\begin{pmatrix} (1/120)(\alpha_{n-1}+26\alpha_{n-2}+66\alpha_{n-3}+26\alpha_{n-4}+\alpha_{n-5}) \\ (1/24h)(\alpha_{n-1}+10\alpha_{n-2}-10\alpha_{n-4}-\alpha_{n-5}) \\ (1/6h^2)(\alpha_{n-1}+2\alpha_{n-2}-6\alpha_{n-3}+2\alpha_{n-4}+\alpha_{n-5}) \\ (1/2h^3)(\alpha_{n-1}-2\alpha_{n-2}+2\alpha_{n-4}-\alpha_{n-5}) \end{pmatrix}-d=0\,. \end{aligned}$$

The number of undetermined coefficients is $n+5$ and the conditions (1.8) - (1.9) give the requisite number of equations. Corresponding to $\hat{x}(t)$, one can determine uniquely a quintic spline function of the form

$$\hat{x}_h(t)=\sum_{i=-5}^{n-1}\hat{\alpha}_i Q_6(t/h-i) \tag{1.10}$$

so that

$$\begin{aligned} \hat{x}_h(t_i)&=\hat{x}(t_i), \qquad i=0(1)n, \\ \hat{x}_h^{(4)}(t_i)&=\hat{x}^{(4)}(t_i), \qquad i=0,\ 1,\ n-1,\ n. \end{aligned} \tag{1.11}$$

On using consistency relation:

$$\begin{aligned} &(1/120)\{\hat{x}_h^{(4)}(t_{i+2})+26\hat{x}_h^{(4)}(t_{i+1})+66\hat{x}_h^{(4)}(t_i)+26\hat{x}_h^{(4)}(t_{i-1})+\hat{x}_h^{(4)}(t_{i-2})\} \\ &\quad=(1/h^4)\{\hat{x}_h(t_{i+2})-4\hat{x}_h(t_{i+1})+6\hat{x}_h(t_i)-4\hat{x}_h(t_{i-1})+\hat{x}_h(t_{i-2})\} \end{aligned} \tag{1.12}$$

we have

$$\hat{x}_h^{(4)}(t_i)=\hat{x}^{(4)}(t_i)+O(h^2), \qquad i=0(1)n,$$

from which follows

$$\|\hat{x}_h^{(m)}-\hat{x}^{(m)}\|=\max|\hat{x}_h^{(m)}(t)-\hat{x}^{(m)}(t)|=O(h^{6-m}), \qquad m=0(1)4 \tag{1.13}$$

by the repeated use of Rolle's theorem. Hence we have the estimate of $\|F(\hat{\alpha})\|$ of the form

$$\|F(\hat{\alpha})\|=O(h^2), \tag{1.14}$$

where, for any finite dimensional vector, we shall denote its maximum norm by $\|\cdot\|$.

Next we consider a sextic spline function $z_h(t)$ of the form

$$z_h(t)=\sum_{i=-6}^{n-1}\beta_i Q_7(t/h-i) \tag{1.15}$$

with undetermined coefficients $(\beta_{-6}, \beta_{-5}, \cdots, \beta_{n-1})$. The above $z_h(t)$ will be an approximate solution if it satisfies

$$z_h^{(4)}(t)=P_2 f(t, z_h(t)), \qquad 0\leqq t\leqq 1, \tag{1.16}$$

subject to the same boundary conditions (1.2). Here P_2 is an operator defined by $(P_2 g)(t)=\sum_{i=-2}^{n-1}\gamma_i Q_3(t/h-i)$ so that

$$\begin{cases} (P_2 g)(t_{i+1/2})=(1/8)(\gamma_i+6\gamma_{i-1}+\gamma_{i-2})=g_{i+1/2}, \qquad i=0(1)n-1, \\ (P_2 g)(t_0)=(1/2)(\gamma_{-1}+\gamma_{-2})=g_0, \\ (P_2 g)(t_n)=(1/2)(\gamma_{n-1}+\gamma_{n-2})=g_n. \end{cases} \tag{1.17}$$

Since the coefficient matrix of (1.17) is nonsingular, the operator P_2 is well-defined. By a simple calculation, any two quadratic spline functions coincide with each other if and only if they coincide at the mid points $t_{i+1/2}$, $i=0(1)n-1$ and the end points t_i, $i=0, n$. Since $z_h^{(4)}(t)$ and $P_2 f(t, z_h(t))$ are quadratic spline functions, we have the following determining equations $G(\beta)=0$ from (1.16):

$$(1.18)\quad \begin{cases} G_{-2}(\beta)/h=(1/2h^4)(\beta_{-1}-3\beta_{-2}+2\beta_{-3}+2\beta_{-4}-3\beta_{-5}+\beta_{-6}) \\ \qquad -f(t_0,\ (1/720)(\beta_{-1}+57\beta_{-2}+302\beta_{-3}+302\beta_{-4}+57\beta_{-5}+\beta_{-6}))\,, \\ G_i(\beta)=(1/8h^4)(\beta_i+2\beta_{i-1}-17\beta_{i-2}+28\beta_{i-3}-17\beta_{i-4}+2\beta_{i-5}+\beta_{i-6}) \\ \qquad -f(t_{i+1/2},\ (1/46080)(\beta_i+722\beta_{i-1}+10543\beta_{i-2}+23548\beta_{i-3} \\ \qquad +10543\beta_{i-4}+722\beta_{i-5}+\beta_{i-6}))\,, \qquad i=0(1)n-1\,, \\ G_{n-1}(\beta)/h=(1/2h^4)(\beta_{n-1}-3\beta_{n-2}+2\beta_{n-3}+2\beta_{n-4}-3\beta_{n-5}+\beta_{n-6}) \\ \qquad -f(t_n,\ (1/720)(\beta_{n-1}+57\beta_{n-2}+302\beta_{n-3}+302\beta_{n-4}+57\beta_{n-5}+\beta_{n-6})). \end{cases}$$

The boundary conditions give four equations:

$$(1.19)\quad \begin{pmatrix} G_{-4}(\beta) \\ G_{-3}(\beta) \\ G_n(\beta) \\ G_{n+1}(\beta) \end{pmatrix} = M_0 \begin{pmatrix} (1/720)(\beta_{-1}+57\beta_{-2}+302\beta_{-3}+302\beta_{-4}+57\beta_{-5}+\beta_{-6}) \\ (1/120h)(\beta_{-1}+25\beta_{-2}+40\beta_{-3}-40\beta_{-4}-25\beta_{-5}-\beta_{-6}) \\ (1/24h^2)(\beta_{-1}+9\beta_{-2}-10\beta_{-3}-10\beta_{-4}+9\beta_{-5}+\beta_{-6}) \\ (1/6h^3)(\beta_{-1}+\beta_{-2}-8\beta_{-3}+8\beta_{-4}-\beta_{-5}-\beta_{-6}) \end{pmatrix}$$
$$+M_1 \begin{pmatrix} (1/720)(\beta_{n-1}+57\beta_{n-2}+302\beta_{n-3}+302\beta_{n-4}+57\beta_{n-5}+\beta_{n-6}) \\ (1/120h)(\beta_{n-1}+25\beta_{n-2}+40\beta_{n-3}-40\beta_{n-4}-25\beta_{n-5}-\beta_{n-6}) \\ (1/24h^2)(\beta_{n-1}+9\beta_{n-2}-10\beta_{n-3}-10\beta_{n-4}+9\beta_{n-5}+\beta_{n-6}) \\ (1/6h^3)(\beta_{n-1}+\beta_{n-2}-8\beta_{n-3}+8\beta_{n-4}-\beta_{n-5}-\beta_{n-6}) \end{pmatrix} - d.$$

The number of undetermined coefficients is $n+6$ and the conditions (1.18)-(1.19) precisely give the requisite number of equations. Corresponding to $\hat{x}(t)$, one can determine a sextic spline function $\hat{z}_h(t)$ of the form

$$(1.20)\qquad \hat{z}_h(t)=\sum_{i=-6}^{n-1}\hat{\beta}_i Q_7(t/h-i)$$

so that

$$(1.21)\qquad \begin{cases} \hat{z}_h(t_{i+1/2})=\hat{x}(t_{i+1/2})\,, & i=0(1)n-1\,, \\ \hat{z}_h^{(4)}(t_{i+1/2})=\hat{x}^{(4)}(t_{i+1/2})\,, & i=0(1)2,\ n-3(1)n-1\,. \end{cases}$$

Since $\hat{x}(t)\in C^8[0,\,1]$ due to the assumption that $f(t,\,x)\in C^4_{t,x}(D)$, it is valid that

$$(1.22)\qquad \begin{cases} \hat{z}_h^{(4)}(t_{i+1/2})=\hat{x}^{(4)}(t_{i+1/2})+O(h^4)\,, & i=0(1)n-1\,, \\ \|\hat{z}_h^{(m)}-\hat{x}^{(m)}\|=O(h^{7-m})\,, & m=0(1)4\,. \end{cases}$$

Here we shall consider the above sextic interpolation errors. By the consistency relation of the sextic spline function:

$$(1.23)\qquad \begin{aligned} &(1/46080)\{\hat{z}_h^{(4)}(t_{i+7/2})+722\hat{z}_h^{(4)}(t_{i+5/2})+10543\hat{z}_h^{(4)}(t_{i+3/2}) \\ &+23458\hat{z}_h^{(4)}(t_{i+1/2})+10543\hat{z}_h^{(4)}(t_{i-1/2})+722\hat{z}_h^{(4)}(t_{i-3/2})+\hat{z}_h^{(4)}(t_{i-5/2})\} \\ &=(1/8h^4)\{\hat{z}_h(t_{i+7/2})+2\hat{z}_h(t_{i+5/2})-17\hat{z}_h(t_{i+3/2})+28\hat{z}_h(t_{i+1/2}) \\ &\quad -17\hat{z}_h(t_{i-1/2})+2\hat{z}_h(t_{i-3/2})+\hat{z}_h(t_{i-5/2})\}\,, \end{aligned}$$

we have

$$(1.24)\qquad \hat{z}_h^{(4)}(t_{i+1/2})=\hat{x}^{(4)}(t_{i+1/2})+O(h^4)\,, \qquad i=0(1)n-1\,.$$

By the means of the consistency relation at the end point:

$$(2/3)\hat{z}_h^{(4)}(t_0)=\{169\hat{z}_h(t_{1/2})-647\hat{z}_h(t_{3/2})+898\hat{z}_h(t_{5/2})-502\hat{z}_h(t_{7/2})+53\hat{z}_h(t_{9/2})+29\hat{z}_h(t_{11/2})\}/h^4-(1/5760)\{-15191\hat{z}_h^{(4)}(t_{1/2})+197833\hat{z}_h^{(4)}(t_{3/2})+630898\hat{z}_h^{(4)}(t_{5/2})+302138\hat{z}_h^{(4)}(t_{7/2})+20933\hat{z}_h^{(4)}(t_{9/2})+29\hat{z}_h^{(4)}(t_{11/2})\},$$

(this relation can be proved by substituting the terms: $1, t, \cdots, t^6, (t-t_1)_+^6, (t-t_2)_+^6, \cdots, (t-t_5)_+^6$), we have

(1.25) $$\hat{z}_h^{(4)}(t_0)=\hat{x}^{(4)}(t_0)+O(h^3).$$

Similarly we have

(1.26) $$\hat{z}_h^{(4)}(t_n)=\hat{x}^{(4)}(t_n)+O(h^3).$$

On combining (1.24) - (1.26), we consider the estimation of $\|\hat{z}_h^{(4)}-\hat{x}^{(4)}\|$. Since $\hat{z}_h^{(4)}(t)-\hat{x}^{(4)}(t)=\hat{z}_h^{(4)}(t)-P_2(\hat{x}^{(4)})(t)+(I-P_2)\hat{x}^{(4)}(t)$, let us treat $\hat{z}_h^{(4)}(t)-P_2(\hat{x}^{(4)})(t)$ and $(I-P_2)\hat{x}^{(4)}(t)$, separately. First, we have

$$\hat{z}_h^{(4)}(t)-P_2(\hat{x}^{(4)})(t)=\sum_{i=-2}^{n-1}\gamma_i Q_3(t/h-i)$$

so that

$$\begin{cases}(1/2)(\gamma_{-1}+\gamma_{-2})=\hat{z}_h^{(4)}(t_0)-P_2(\hat{x}^{(4)})(t_0)=O(h^3),\\ (1/8)(\gamma_i+6\gamma_{i-1}+\gamma_{i-2})=\hat{z}_h^{(4)}(t_{i+1/2})-P_2(\hat{x}^{(4)})(t_{i+1/2})=O(h^4), \quad i=0(1)n-1,\\ (1/2)(\gamma_{n-1}+\gamma_{n-2})=O(h^3).\end{cases}$$

Hence we have

(1.27) $$\|\hat{z}_h^{(4)}-P_2(\hat{x}^{(4)})\|=O(h^3).$$

Next we shall consider the error of the quadratic spline interpolation. Let $s(t)=P_2(\hat{x}^{(4)})(t)$, then we have

$$\begin{cases}(1/16)(5s''_{1/2}+s''_{3/2})=(1/2h^2)(2s_0-3s_{1/2}+s_{3/2})\\ \qquad=(1/2h^2)\{2\hat{x}^{(4)}(t_0)-3\hat{x}^{(4)}(t_{1/2})+\hat{x}^{(4)}(t_{3/2})\},\\ (1/8)(s''_{i-1/2}+6s''_{i+1/2}+s''_{i+3/2})=(1/h^2)\{\hat{x}^{(4)}(t_{i-1/2})-2\hat{x}^{(4)}(t_{i+1/2})\\ \qquad+\hat{x}^{(4)}(t_{i+3/2})\}, \quad i=1(1)n-2,\\ (1/16)(5s''_{n-1/2}+s''_{n-3/2})=(1/2h^2)\{2\hat{x}^{(4)}(t_n)-3\hat{x}^{(4)}(t_{n-1/2})+\hat{x}^{(4)}(t_{n-3/2})\},\end{cases}$$

from which follows

$$s''_{i+1/2}=\hat{x}^{(6)}(t_{i+1/2})+O(h), \qquad i=0(1)n-1.$$

Since $s''(t)$ is constant on $[t_i, t_{i+1}]$, by Taylor series expansion we have

$$\|s''-\hat{x}^{(6)}\|_{[t_i, t_{i+1}]}=O(h).$$

By the means of Rolle's theorem, we have

(1.28) $$\|(I-P_2)\hat{x}^{(4)}\|=\|s-\hat{x}^{(4)}\|=O(h^3).$$

Combining (1.27) and (1.28) yields

(1.29) $$\|\hat{z}_h^{(4)}-\hat{x}^{(4)}\|=O(h^3).$$

By the repeated use of Rolle's theorem, we have

(1.30) $$\|\hat{z}_h^{(m)}-\hat{x}^{(m)}\|=O(h^{7-m}), \qquad m=0(1)4.$$

This completes the proof of (1.22). Hence we have the estimate of $\|G(\hat{\beta})\|$ of the form

(1.31) $$\|G(\hat{\beta})\|=O(h^4).$$

Here we remark that the preceding argument can be also applicable to the general fourth order differential equations:

(1.32) $$x^{(4)}(t)=f(t, x(t), x'(t), x''(t), x^{(3)}(t)), \qquad 0\leqq t\leqq 1,$$

subject to the boundary conditions (1.2).

§ 2. Existence and Convergence of Spline Approximations

In this section we shall consider the case using the quintic spline function, since the similar analysis can be applicable to the method using the sextic spline.

Let $J(\alpha)$ be the Jacobian matrix of $F(\alpha)$ with respect to $\alpha=(\alpha_{-5}, \alpha_{-4}, \cdots, \alpha_{n-1})$. In order to investigate the property of $J(\hat{\alpha})$, we consider a linear system:

(2.1) $$J(\hat{\alpha})\xi=\eta$$

where $\xi=(\xi_{-5}, \xi_{-4}, \cdots, \xi_{n-1})$ and $\eta=(\eta_{-2}, \eta_{-1}, \cdots, \eta_{n+2})$. Corresponding to ξ and η, we consider quintic and piecewise linear functions $\phi(t)$ and $\psi(t)$, respectively

(2.2) $$\phi(t)=\sum_{i=-5}^{n-1}\xi_i Q_6(t/h-i) \quad \text{and} \quad \psi(t)=\sum_{i=0}^{n}\eta_i L_i(t).$$

From (2.1), we have

$$\phi^{(4)}(t_i)=\sigma(t_i)\phi(t_i)+\psi(t_i), \qquad i=0(1)n,$$

(2.3) $$M_0\begin{pmatrix}\phi(0)\\ \phi'(0)\\ \phi''(0)\\ \phi^{(3)}(0)\end{pmatrix}+M_1\begin{pmatrix}\phi(1)\\ \phi'(1)\\ \phi''(1)\\ \phi^{(3)}(1)\end{pmatrix}=\begin{pmatrix}\eta_{-2}\\ \eta_{-1}\\ \eta_{n+1}\\ \eta_{n+2}\end{pmatrix}.$$

Since two piecewise linear functions $\phi^{(4)}(t)$ and $P_1(\sigma\phi)(t)+\psi(t)$ coincide at the nodes t_i, $i=0(1)n$, we have

$$\phi^{(4)}(t)=P_1[\sigma\phi]+\psi, \qquad 0\leqq t\leqq 1,$$

that is,

(2.4) $$\phi^{(4)}-\sigma\phi=-(I-P_1)(\sigma\phi)+\psi \qquad (I \text{ the unit operator}).$$

By using the assumption that the problem (1.1)-(1.2) has the isolated solution.

there exists the Green function $H(t, s)$ such that

$$\begin{pmatrix}\phi\\ \phi'\\ \phi''\\ \phi^{(3)}\end{pmatrix}=\Phi G^{-1}\begin{pmatrix}\eta_{-2}\\ \eta_{-1}\\ \eta_{n+1}\\ \eta_{n+2}\end{pmatrix}+\int_0^1 H(\cdot,\ s)\begin{pmatrix}0\\ 0\\ 0\\ R(s)\end{pmatrix}ds, \tag{2.5}$$

$$R(t)=-(I-P_1)(\sigma\phi)+\psi.$$

Here the Green function $H(t, s)$ is given by

$$H(t,\ s)=\begin{cases}\Phi(t)[E-G^{-1}M_1\Phi(1)]\Phi^{-1}(s), & 0\leqq s\leqq t,\\ -\Phi(t)G^{-1}M_1\Phi(1)\Phi^{-1}(s), & s<t\leqq 1,\end{cases} \tag{2.6}$$

(E the unit matrix).

From above, we have

$$\|\phi\|\leqq C[\|\eta\|+\|\psi\|] \quad \text{for } h<h_0 \quad ([3]) \tag{2.7}$$

provided that h_0 is sufficiently small, where C is a generic constant independent of h. Since $\|\phi\|\geqq C\|\xi\|$ and $\|\psi\|\leqq C\|\eta\|$, we have

$$\|\xi\|\leqq C\|\eta\| \quad \text{for } h<h_0. \tag{2.8}$$

By (2.1), this inequality implies the nonsingularity of $J(\hat{\alpha})$ and in addition

$$\|J^{-1}(\hat{\alpha})\|\leqq C \quad \text{for } h<h_0. \tag{2.9}$$

By (1.13), we can choose h_1 such that $\|\hat{x}_h-\hat{x}\|\leqq\delta_0<\delta$ for $h<h_1(\leqq h_0)$. Let $\Omega_{h_1}=\{\alpha: \|\alpha-\hat{\alpha}\|\leqq\delta-\delta_0\}$ and $x_h(t)=\sum_{i=-5}^{n-1}\alpha_i Q_6(t/h-i)$ with $\alpha\in\Omega_{h_1}$, then $(t,\ x_h(t))\in D$. Thus we have, by the means of the mean-value theorem,

$$\|J(\alpha)-J(\beta)\|\leqq C\|\alpha-\beta\| \quad \text{for } \alpha,\ \beta\in\Omega_{h_1}. \tag{2.10}$$

By (1.14), we have already had

$$\|F(\hat{\alpha})\|=O(h^2). \tag{2.11}$$

Thus all the conditions (2.9), (2.10) and (2.11) of Newton-Kantorovitch's theorem are fulfilled. Therefore $F(\alpha)=0$ has one and only one solution $\bar{\alpha}$ in the neighbourhood of $\hat{\alpha}$ (see Rall [1]). Hence we have

Theorem 1. *In a sufficiently small neighbourhood of the isolated solution $\hat{x}(t)$, there exists the quintic spline approximation of the form*

$$\bar{x}_h(t)=\sum_{i=-5}^{n-1}\bar{\alpha}_i Q_6(t/h-i) \tag{2.12}$$

so that

$$\|\bar{x}_h-\hat{x}\|=O(h^2). \tag{2.13}$$

For the derivative of the error, we have

Corollary.

(2.14) $$\|\bar{x}_h^{(m)}-\hat{x}^{(m)}\|=O(h^2), \quad m=1(1)4.$$

Proof. Since $\bar{x}_h^{(4)}(t_i)=f(t_i, \bar{x}_h(t_i))$, we have

$$\|\bar{x}_h^{(4)}-\hat{x}^{(4)}\|=O(h^2).$$

By the means of the well-known inequality:

$$\|g^{(m)}\|\leqq C[\|g\|+\|g^{(4)}\|] \quad \text{for any } g\in C^4[0, 1],$$

we have the desired result.

For the sextic spline approximation, we have

Theorem 2. *In a sufficiently small neighbourhood of the isolated solution $\hat{x}(t)$, there exists the sextic spline approximation:*

(2.15) $$\bar{z}_h(t)=\sum_{i=-6}^{n-1}\bar{\beta}_i Q_7(t/h-i)$$

so that

(i) $\bar{\beta}=(\bar{\beta}_{-6}, \bar{\beta}_{-5}, \cdots, \bar{\beta}_{n-1})$ *is the solution of* $G(\beta)=0$,

(ii) $\|\bar{z}_h^{(m)}-\hat{x}^{(m)}\|=O(h^4), \quad m=0(1)4.$

The proof of Theorem 2 is quite similar to that of Theorem 1. For this reason we omit further details.

§ 3. Numerical Illustration

The following numerical examples are chosen for experimentation.

Example 1.

$$x^{(4)}(t)+tx(t)=-(8+7t+t^3)e^t.$$
$$x(0)=x(1)=0, \quad x'(0)=1, \quad x'(1)=-e.$$

The exact solution $\hat{x}(t)$ is $t(1-t)e^t$.

Example 2.

$$x^{(4)}(t)=3x'^2(t)+4.5x^3(t),$$
$$x(0)=4, \quad x(1)=1, \quad x''(0)=24, \quad x''(1)=1.5.$$

This problem has two isolated solutions such that $\hat{x}(t)=4/(1+t)^2$ and $\hat{x}(0.5)$ $\fallingdotseq -10.53$. We have listed the numerical results for the larger solution $\hat{x}(t)=4/(1+t)^2$.

All the computations were performed in double precision arithmetic in order to keep the rounding errors to a minimum. The observed maximum errors in absolute value for Examples 1 and 2 are displayed in Tables 1 and 2. We remark that by using Richardson's h^2-extrapolation technique the accuracy of our computed solution can be improved at the knots. In Tables 3 and 4, the errors mean the absolute values of the followings:

$\{4\bar{x}_{h/2}(1/2)-\bar{x}_h(1/2)\}/3-\hat{x}(1/2)$ for quintic splines,
$\{16\bar{z}_{h/2}(1/2)-\bar{z}_h(1/2)\}/15-\hat{x}(1/2)$ for sextic splines.

Table 1 (*Example* 1)

	quintic	sextic
$h=1/4$	0.689-3*	0.380-5
1/8	0.172-3	0.222-6
1/16	0.429-4	0.137-7
1/32	0.107-4	0.854-9
1/64	0.268-5	0.536-10

* We write 0.689×10^{-3} by 0.689-3.

Table 2 (*Example* 2)

	quintic	sextic
$h=1/4$	0.209	0.200-1
1/8	0.478-1	0.144-2
1/16	0.117-1	0.933-4
1/32	0.290-2	0.590-5
1/64	0.724-3	0.369-6

Table 3 (*Example* 1)
Errors based on Richardson's extrapolation at $t=1/2$.

	quintic	sextic
$h=1/4$		
	0.859-6	0.168-7
1/8		
	0.452-7	0.168-9
1/16		
	0.269-8	0.177-11
1/32		
	0.158-9	0.853-13
1/64		

Table 4 (*Example* 2)
Errors based on Richardson's extrapolation at $t=1/2$.

	quintic	sextic
$h=1/4$		
	0.569-2	0.201-3
1/8		
	0.417-3	0.337-5
1/16		
	0.270-4	0.535-7
1/32		
	0.170-5	0.858-9
1/64		

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