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On the Existence of Solutions to Time-Dependent Hartree-Fock Equations

By

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§ 1. Introduction and Summary

The approximate methods in the quantum mechanical many body problems lead us to interesting non-linear equations. Consider an N-body Schrödinger equation ($N\geq 2$)

$$(1.1)\qquad i\frac{\partial}{\partial t}\Psi(t)=H_N\Psi(t)\,,$$

$$H_N=\sum_{j=1}^{N}(-\Delta_j+Q(x_j))+\sum_{i<j}V(x_i-x_j)\,,$$

where $x_j=(x_j^1,\ x_j^2,\ x_j^3)\in\boldsymbol{R}^3$, $\Delta_j=\sum_{i=1}^{3}(\partial/\partial x_j^i)^2$ and $Q(x)$, $V(x)$ are real functions such that $V(x)=V(-x)$. If the system obeys the Fermi statistics, it is natural to treat (1.1) in the anti-symmetric subspace of $L^2(\boldsymbol{R}^{3N})$. Taking note of this anti-symmetry and using the variational principle, Dirac ([3], [4]) has derived the following time-dependent version of Hartree-Fock equation in order to obtain an approximate solution of (1.1):

$$(1.2)\qquad i\frac{\partial}{\partial t}u(t)=Hu(t)+K(u(t))\,,$$

where the unknown $u(t)={}^t(u_1(x,t),\cdots,u_N(x,t))$ is a $\boldsymbol{C}^N$-valued function of $x=(x_1,\ x_2,\ x_3)\in\boldsymbol{R}^3$ and $t\geq 0$,

$$(1.3)\qquad H=-\Delta+Q(x)\,,$$

$$(1.4)\qquad K(u(t))(x)=\int_{\boldsymbol{R}^3}V(x-y)U(x,y,t)\overline{u(y,t)}dy\,,$$

$$(1.5)\qquad U(x,y,t)=(U_{jk}(x,y,t))\quad (\text{the } N\times N \text{ matrix}),$$

$$(1.6)\qquad U_{jk}(x,y,t)=u_j(x,t)u_k(y,t)-u_k(x,t)u_j(y,t)\,.$$

Chadam and Glassey [2] have proved the existence of global solutions to (1.2), when $Q(x)$, $V(x)$ are Coulomb potentials: $Q(x)=-Z/|x|$, $V(x)=1/|x|$, which is practically most important. In this paper, we show that their results can be extended to the more general class of potentials.

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Let $L^p = L^p(\boldsymbol{R}^3)$ and $\mathcal{H}^m = \mathcal{H}^m(\boldsymbol{R}^3)$ denote the usual Lebesgue space and the Sobolev space of order m, respectively. Their norms are written as $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\mathcal{H}^m}$. For Banach spaces X and Y, $\boldsymbol{B}(X; Y)$ denotes the totality of bounded linear operators from X to Y. Now, we shall state the assumptions imposed on $Q(x)$ and $V(x)$:

(A-1) *$Q(x)$ is a real function and is split into two terms $Q_1(x)$ and $Q_2(x)$: $Q(x) = Q_1(x) + Q_2(x)$, where $Q_1 \in L^2$, $Q_2 \in L^\infty$.*

(A-2) *$V(x)$ is a real function such that $V(x) = V(-x)$, and is split into two parts: $V(x) = V_1(x) + V_2(x)$, where $V_1 \in L^2$, $V_2 \in L^\infty$.*

(A-3) *As the multiplication operator, V belongs to $\boldsymbol{B}(\mathcal{H}^1; L^2)$.*

Here we should note that any $f \in L^2$ can be split into two parts $f = f_1 + f_2$ where $f_1 \in L^2 \cap L^p$ for any p such that $1 \leq p \leq 2$, $f_2 \in L^\infty$. Indeed, we have only to take $f_1(x) = f(x)$ ($|f(x)| \geq 1$), $f_1(x) = 0$ ($|f(x)| < 1$) and $f_2(x) = f(x) - f_1(x)$. This fact can be written formally as

$$L^2 + L^\infty = L^2 \cap L^p + L^\infty \qquad (1 \leq p \leq 2).$$

Let p ($1 \leq p \leq 2$) be arbitrarily fixed, and $f \in L^2 + L^\infty$. Then, as above, one can easily see that for any $\varepsilon > 0$, f can be split into two parts f_1 and f_2, where

$$f = f_1 + f_2,$$

$$\|f_1\|_{L^2} + \|f_1\|_{L^p} < \varepsilon, \qquad f_2 \in L^\infty.$$

We shall frequently use this relation in the later arguments.

Under the assumption (A-1), the differential operator H restricted to $C_0^\infty(\boldsymbol{R}^3)$ (the smooth functions of compact support in $\boldsymbol{R}^3$) is essentially self-adjoint and the domain of its self-adjoint realization, which we also denote by H, is equal to $\mathcal{H}^2$. Then the equation (1.2) can be transformed into the integral equation

$$u(t) = e^{-itH}u(0) - i\int_0^t e^{-i(t-s)H}K(u(s))ds\,. \tag{1.7}$$

By the solution of (1.2), we mean an $\mathcal{H}^2$-valued continuous function of $t \geq 0$ verifying the integral equation (1.7).

The result of this paper is summarized in the following

Theorem. (1) *Under the assumptions* (A-1) *and* (A-2), *for any Cauchy data $u(0) \in \mathcal{H}^2$, there exists a unique local solution of* (1.2).

(2) *Under the assumptions* (A-1), (A-2) *and* (A-3), *for any Cauchy data $u(0) \in \mathcal{H}^2$, there exists a unique global solution to* (1.2).

The proof of the above theorem is carried out along the line of Chadam and Glassey [2]. For the local existence, it suffices to show that the non-linear term $K(u)$ is locally Lipshitz continuous in $\mathcal{H}^2$. As for the global existence, we have only to obtain some a-priori estimate of the solution $u(t)$, which can be proved by using the energy conservation law.

We shall end this section by giving an example of V satisfying (A-3).

Example. *Let $V(x)$ be split into three terms: $V(x)=V_1(x)+V_2(x)+V_3(x)$, where $|V_1(x)|\leq C/|x|$ for a constant $C>0$, $V_2\in L^3$ and $V_3\in L^\infty$. Then $V\in \boldsymbol{B}(\mathcal{H}^1; L^2)$ as the multiplication operator.*

Indeed, by the well-known inequality, we have

$$\|V_1 f\|_{L^2}\leq C\|f(x)/|x|\|_{L^2}\leq \text{Const.}\|\nabla f\|_{L^2}\leq \text{Const.}\|f\|_{\mathcal{H}^1}.$$

One can also see that

$$\|V_2 f\|_{L^2}\leq \|V_2\|_{L^3}\|f\|_{L^6}\leq \text{Const.}\|V_2\|_{L^3}\|f\|_{\mathcal{H}^1},$$

where we have used the well-known Sobolev inequality

$$\|f\|_{L^6}\leq \text{Const.}\|f\|_{\mathcal{H}^1},$$

(see e. g. [6] p. 12). These observations show that V verifies the assumption (A-3).

§ 2. Existence of Local Solutions

Let $A(W; f, g, h)$ be the operator defined by

$$A(W; f, g, h)(x)=f(x)\int_{R^3} W(x-y)g(y)h(y)dy. \tag{2.1}$$

Lemma 2.1. *We have the following estimates:*

(1) $\|A(W; f, g, h)\|_{L^2}\leq \|W\|_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$,

(2) $\|A(W; f, g, h)\|_{L^2}\leq \text{Const.}\|W\|_{L^2}\|f\|_{L^2}\|g\|_{\mathcal{H}^2}\|h\|_{L^2}$,

(3) $\|A(W; f, g, h)\|_{L^2}\leq \text{Const.}\|W\|_{L^{3/2}}\|f\|_{L^2}\|g\|_{\mathcal{H}^1}\|h\|_{\mathcal{H}^1}$,

(4) $\|A(W; f, g, h)\|_{L^2}\leq \|W\|_{B(\mathcal{H}^1; L^2)}\|f\|_{L^2}\|g\|_{\mathcal{H}^1}\|h\|_{L^2}$,

where $\|\cdot\|_{B(\mathcal{H}^1; L^2)}$ denotes the operator norm of W as the multiplication operator from $\mathcal{H}^1$ to L^2.

Proof. Let

$$B(W; g, h)(x)=\int_{R^3} W(x-y)g(y)h(y)dy.$$

Then we have only to estimate $\|B(W; g, h)\|_{L^\infty}$.

(1) easily follows from the Schwarz inequality.

To show (2), we note the Sobolev inequality:

$$\|g\|_{L^\infty}\leq \text{Const.}\|g\|_{\mathcal{H}^2}.$$

Then we have

$$\begin{aligned}|B(W; g, h)(x)| &\leq \|W\|_{L^2}\|g\|_{L^\infty}\|h\|_{L^2}\\ &\leq \text{Const.}\|W\|_{L^2}\|g\|_{\mathcal{H}^2}\|h\|_{L^2}.\end{aligned}$$

(3) follows from the Hölder and Sobolev inequalities:

$$|B(W;g,h)(x)|\leq\|W\|_{L^{3/2}}\|g\|_{L^6}\|h\|_{L^6}$$
$$\leq\text{Cont.}\|W\|_{L^{3/2}}\|g\|_{\mathcal{H}^1}\|h\|_{\mathcal{H}^1}.$$

(4) can be proved as follows:

$$|B(W;g,h)(x)|\leq\|W(x-\cdot)g(\cdot)\|_{L^2}\|h\|_{L^2}$$
$$\leq\|W\|_{B(\mathcal{H}^1;L^2)}\|g\|_{\mathcal{H}^1}\|h\|_{L^2},$$

where we have used the fact that $\|W(x-\cdot)g(\cdot)\|_{L^2}=\|W(\cdot)g(x-\cdot)\|_{L^2}\leq\|W\|_{B(\mathcal{H}^1;L^2)}$ $\cdot\|g(x-\cdot)\|_{\mathcal{H}^1}=\|W\|_{B(\mathcal{H}^1;L^2)}\|g\|_{\mathcal{H}^1}$. □

We introduce the following notations

$$p_1(f,g,h)=\|f\|_{\mathcal{H}^2}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^2}, \tag{2.2}$$

$$p_2(f,g,h)=\|f\|_{\mathcal{H}^2}\|g\|_{\mathcal{H}^1}\|h\|_{\mathcal{H}^1}+\|f\|_{\mathcal{H}^1}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^1}+\|f\|_{\mathcal{H}^1}\|g\|_{\mathcal{H}^1}\|h\|_{\mathcal{H}^2}. \tag{2.3}$$

Lemma 2.2. *We have*:

(1) $\|A(V;f,g,h)\|_{\mathcal{H}^2}\leq\text{Const.}(\|V_1\|_{L^2}+\|V_1\|_{L^{3/2}}+\|V_2\|_{L^\infty})p_1(f,g,h)$.

(2) $\|A(V;f,g,h)\|_{\mathcal{H}^2}\leq\text{Const.}(\|V_1\|_{L^{3/2}}+\|V_2\|_{L^\infty}+\|V\|_{B(\mathcal{H}^1;L^2)})p_2(f,g,h)$.

Proof. Let $I_j(V)$ $(j=1,2,\cdots,6)$ be defined as follows:

$$I_1(V)=A(V;\Delta f,g,h),$$
$$I_2(V)=2\sum_{i=1}^{3}A(V;\partial f/\partial x_i,\partial g/\partial x_i,h),$$
$$I_3(V)=2\sum_{i=1}^{3}A(V;\partial f/\partial x_i,g,\partial h/\partial x_i),$$
$$I_4(V)=A(V;f,\Delta g,h),$$
$$I_5(V)=2\sum_{i=1}^{3}A(V;f,\partial g/\partial x_i,\partial h/\partial x_i),$$
$$I_6(V)=A(V;f,g,\Delta h).$$

Then we have

$$\Delta A(V;f,g,h)=\sum_{j=1}^{6}I_j(V).$$

Lemma 2.1 (2) implies that

$$\|I_1(V_1)\|_{L^2}\leq C\|V_1\|_{L^2}\|f\|_{\mathcal{H}^2}\|g\|_{\mathcal{H}^2}\|h\|_{L^2}, \tag{2.4}$$

$$\|I_4(V_1)\|_{L^2}\leq C\|V_1\|_{L^2}\|f\|_{L^2}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^2}, \tag{2.5}$$

$$\|I_6(V_1)\|_{L^2}\leq C\|V_1\|_{L^2}\|f\|_{L^2}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^2}. \tag{2.6}$$

Also Lemma 2.1 (3) shows that

$$\|I_2(V_1)\|_{L^2}\leq C\|V_1\|_{L^{3/2}}\|f\|_{\mathcal{H}^1}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^1}, \tag{2.7}$$

$$\|I_3(V_1)\|_{L^2}\leq C\|V_1\|_{L^{3/2}}\|f\|_{\mathcal{H}^1}\|g\|_{\mathcal{H}^1}\|h\|_{\mathcal{H}^2}, \tag{2.8}$$

$$\|I_5(V_1)\|_{L^2}\leq C\|V_1\|_{L^{3/2}}\|f\|_{L^2}\|g\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^2}. \tag{2.9}$$

We have, therefore,

$$\|A(V_1; f, g, h)\|_{\mathcal{H}^2} \leq C\|(1-\Delta)A(V_1; f, g, h)\|_{L^2} \leq C(\|V_1\|_{L^2}+\|V_1\|_{L^{3/2}})p_1(f, g, h). \tag{2.10}$$

Using Lemma 2.1 (1), we can show as above that

$$\|A(V_2; f, g, h)\|_{\mathcal{H}^2} \leq C\|V_2\|_{L^\infty} p_2(f, g, h). \tag{2.11}$$

(2.10) together with (2.11) proves the assertion (1).

In view of Lemma 2.1 (4), we have

$$\|I_1(V)\|_{L^2}+\sum_{j=4,5,6}\|I_j(V)\|_{L^2} \leq C\|V\|_{B(\mathcal{H}^1; L^2)} p_2(f, g, h),$$

which together with (2.7) and (2.8) shows the assertion (2). □

Lemma 2.3. *Under the assumption* (A-2), *the non-linear term K is locally Lipshitz continuous in $\mathcal{H}^2$. That is, for any bounded set B in $\mathcal{H}^2$, there exists a constant $C=C(B)>0$ such that*

$$\|K(u)-K(v)\|_{\mathcal{H}^2} \leq C\|u-v\|_{\mathcal{H}^2}$$

if $u, v \in B$.

Proof. Let $K_j(u)$ be the j-th component of $K(u)$. Then $K_j(u)$ can be written as

$$K_j(u)=\sum_{k=1}^{N}\{A(V; u_j, u_k, \bar{u}_k)-A(V; u_k, u_j, \bar{u}_k)\}$$

(see (1.4)). Thus to prove the Lipshitz continuity of $K_j(u)$, we have only to show that of $A(V; u_j, u_k, \bar{u}_k)$ and $A(V; u_k, u_j, \bar{u}_k)$, which can be proved by using the multi-linearity of $A(V; \cdot, \cdot, \cdot)$. For example,

$$\begin{aligned} &A(V; u_j, u_k, \bar{u}_k)-A(V, v_j, v_k, \bar{v}_k) \\ &=A(V; u_j-v_j, u_k, \bar{u}_k)+A(V; v_j, u_k-v_k, \bar{u}_k)+A(V; v_j, v_k, \overline{u_k-v_k}). \end{aligned}$$

The Lipshitz continuity then follows from Lemma 2.2 (1). □

The assumption (A-1) shows that Q is infinitesimally small with respect to $H_0=-\Delta$. That is, for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$\|Qf\|_{L^2} \leq \varepsilon\|H_0 f\|_{L^2}+C_\varepsilon\|f\|_{L^2} \qquad (f \in \mathcal{H}^2)$$

(see e.g. [7]). Therefore, for sufficiently large $\lambda>0$, we can find a constant $C>0$ such that

$$C\|f\|_{\mathcal{H}^2} \leq \|(H+\lambda)f\|_{L^2} \leq C^{-1}\|f\|_{\mathcal{H}^2}. \tag{2.12}$$

In view of (2.12), one can easily see that e^{-itH} is uniformly bounded and strongly continuous in $\mathcal{H}^2$ for $t \in \boldsymbol{R}$. Thus using the Lipshitz continuity of $K(u)$, one can solve the integral equation (1.7) locally.

Theorem 2.4 (Local Existence). *Assume* (A-1) *and* (A-2). *Then for any bounded set B in $\mathcal{H}^2$, there exists a constant $T=T(B)>0$ such that the solution* *of* (1.2) *exists uniquely for* $t\in[0, T]$ *if* $u(0)\in B$.

§ 3. Existence of Global Solutions

In this section, we shall assume that the solution $u(t)={}^t(u_1(t), \cdots, u_N(t))$ of (1.2) exists for $t\in[0, T)$ and derive its properties. In the followings, (,) denotes the inner product of L^2 and also $(L^2)^N$, and C's denote various constants independent of T.

The first important property has already been obtained by Dirac [4].

Theorem 3.1. $\frac{d}{dt}(u_j(t), u_k(t))=0$ *for any* j, k.

Proof. Using the equation (1.2), we have

$$i\frac{d}{dt}(u_j(t), u_k(t))=\{(Hu_j(t), u_k(t))-(u_j(t), Hu_k(t))\} + \{(K_j(u(t)), u_k(t))-(u_j(t), K_k(u(t)))\}.$$

The first term of the right-hand side vanishes because of the self-adjointness of H. In view of (1.4), we have

$$(K_j(u(t)), u_k(t))=\iint V(x-y)u_j(x, t)\overline{u_k(x, t)}|u(y, t)|^2 dxdy - \sum_{n=1}^{N}\iint V(x-y)u_n(x, t)\overline{u_k(x, t)}u_j(y, t)\overline{u_n(y, t)}dxdy .$$

Therefore we have

$$\begin{aligned}(u_j(t), K_k(u(t)))&=\overline{(K_k(u(t)), u_j(t))}\\ &=\iint V(x-y)\overline{u_k(x, t)}u_j(x, t)|u(y, t)|^2 dxdy\\ &\quad - \sum_{n=1}^{N}\iint V(x-y)\overline{u_n(x, t)}u_j(x, t)\overline{u_k(y, t)}u_n(y, t)dxdy .\end{aligned}$$

If we interchange the variables x and y, and take into account of the property: $V(x)=V(-x)$, we can see that $(u_j(t), K_k(u(t)))=(K_j(u(t)), u_k(t))$, which shows that $\frac{d}{dt}(u_j(t), u_k(t))=0$. □

Corollary 3.2. $\|u_j(t)\|_{L^2}=\|u_j(0)\|_{L^2}$ $(j=1, \cdots, N)$.

We also prepare the following lemma.

Lemma 3.3. $\mathrm{Re}(K(u(t)), \frac{\partial}{\partial t}u(t))=\frac{1}{4}\frac{d}{dt}(K(u(t)), u(t))$, *where* Re *means the real part.*

Proof. $\left(K(u(t)), \frac{\partial}{\partial t}u(t)\right)$ is split into two parts I_1 and I_2, where

$$I_1=\sum_{j=1}^{N}\iint V(x-y)u_j(x,t)\overline{\left(\frac{\partial}{\partial t}u_j(x,t)\right)}|u(y,t)|^2dxdy\,,$$

$$I_2=-\sum_{j,k}\iint V(x-y)u_k(x,t)u_j(y,t)\overline{u_k(y,t)}\overline{\frac{\partial}{\partial t}u_j(x,t)}dxdy\,.$$

Therefore, we can see that

$$\mathrm{Re}\,I_1=\frac{1}{2}\iint V(x-y)\left(\frac{\partial}{\partial t}|u(x,t)|^2\right)|u(y,t)|^2dxdy\,.$$

Exchanging the variables x and y suitably, we can rewrite this as

$$\mathrm{Re}\,I_1=\frac{1}{4}\frac{d}{dt}\iint V(x-y)|u(x,t)|^2|u(y,t)|^2dxdy\,.$$

Similarly,

$$\mathrm{Re}\,I_2=-\frac{1}{2}\sum_{j,k}\iint V(x-y)u_k(x,t)u_j(y,t)\overline{u_k(y,t)}\overline{\frac{\partial}{\partial t}u_j(x,t)}dxdy$$

$$-\frac{1}{2}\sum_{j,k}\iint V(x-y)\overline{u_k(x,t)}\overline{u_j(y,t)}u_k(y,t)\frac{\partial}{\partial t}u_j(x,t)dxdy$$

$$=-\frac{1}{4}\frac{d}{dt}\sum_{j,k}\iint V(x-y)u_k(x,t)u_j(y,t)\overline{u_j(x,t)}\overline{u_k(y,t)}dxdy\,.$$

Thus we have

$$\mathrm{Re}\left(K(u(t)),\frac{\partial}{\partial t}u(t)\right)=\frac{1}{4}\frac{d}{dt}\iint V(x-y)^t\overline{u(x,t)}U(x,y,t)\overline{u(y,t)}dxdy$$

$$=\frac{1}{4}\frac{d}{dt}(K(u(t)),u(t))\,. \qquad \square$$

We can now prove an important theorem concerning the conservation of energy.

Theorem 3.4 (The Energy Conservation Law). *Let $E(t)$ be defined by $E(t)=(Hu(t),u(t))+\frac{1}{2}(K(u(t)),u(t))$. Then, $E(t)=E(0)$.*

Proof. By the equation (1.2), we have

$$i\left(\frac{\partial}{\partial t}u(t),\frac{\partial}{\partial t}u(t)\right)=\left(Hu(t),\frac{\partial}{\partial t}u(t)\right)+\left(K(u(t)),\frac{\partial}{\partial t}u(t)\right).$$

Taking the real part, we have

$$\mathrm{Re}\left(Hu(t),\frac{\partial}{\partial t}u(t)\right)+\mathrm{Re}\left(K(u(t)),\frac{\partial}{\partial t}u(t)\right)=0\,.$$

Since $\mathrm{Re}\left(Hu(t),\frac{\partial}{\partial t}u(t)\right)=\frac{1}{2}\frac{d}{dt}(Hu(t),u(t))$, in view of Lemma 3.3, we have

$$\frac{d}{dt}\left\{(Hu(t),u(t))+\frac{1}{2}(K(u(t)),u(t))\right\}=0\,. \qquad \square$$

Lemma 3.5. *Under the assumption* (A-1), *for sufficiently large $\lambda>0$, there* *exists a constant* $C>0$ *such that*

$$C\|f\|_{\mathcal{H}^1}^2 \leq ((H+\lambda)f,\ f) \leq C^{-1}\|f\|_{\mathcal{H}^1}^2 .$$

Proof. We have only to show that for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|(Qf,\ f)| \leq \varepsilon(H_0 f,\ f) + C_\varepsilon\|f\|_{L^2}^2 \tag{3.1}$$

$(H_0=-\Delta)$. By the Hölder and Sobolev inequalities

$$|(Q_1 f,\ f)| \leq \|Q_1\|_{L^{3/2}}\|f\|_{L^6}^2 \leq C\|Q_1\|_{L^{3/2}}\|f\|_{\mathcal{H}^1}^2 .$$

We also have

$$|(Q_2 f,\ f)| \leq \|Q_2\|_{L^\infty}\|f\|_{L^2}^2 .$$

Since $\|Q_1\|_{L^{3/2}}$ can be made arbitrarily small, we see (3.1). □

Lemma 3.6 (A-priori $\mathcal{H}^1$ Bound). *If we assume* (A-1) *and* (A-2), *we have* $\|u(t)\|_{\mathcal{H}^1} \leq C$ *for a suitable constant* $C>0$.

Proof. Theorem 3.4 and Corollary 3.2 show that

$$((H+\lambda)u(t),\ u(t)) + \frac{1}{2}(K(u(t)),\ u(t)) = E(0) + \lambda\|u(0)\|_{L^2}^2 .$$

Choosing λ large enough and taking note of Lemma 3.5, we have

$$\|u(t)\|_{\mathcal{H}^1}^2 \leq C(1+\|K(u(t))\|_{L^2}) ,$$

where we have again used Corollary 3.2. Now, $K(u(t))$ can be divided into two parts $K^{(1)}(u(t))$ and $K^{(2)}(u(t))$, where

$$K^{(j)}(u(t)) = \int V_j(x-y)U(x,\ y,\ t)\overline{u(y,\ t)}dy .$$

Lemma 2.1 (1) shows that

$$\|K^{(2)}(u(t))\|_{L^2} \leq C\|V_2\|_{L^\infty}\|u(t)\|_{L^2}^3 \leq C\|V_2\|_{L^\infty} .$$

Lemma 2.1 (3) implies that

$$\begin{aligned}\|K^{(1)}(u(t))\|_{L^2} &\leq C\|V_1\|_{L^{3/2}}\|u(t)\|_{L^2}\|u(t)\|_{\mathcal{H}^1}^2 \\ &\leq C\|V_1\|_{L^{3/2}}\|u(t)\|_{\mathcal{H}^1}^2 .\end{aligned}$$

Since $\|V_1\|_{L^{3/2}}$ can be made arbitrarily small, we have for small $\varepsilon>0$

$$\|u(t)\|_{\mathcal{H}^1}^2 \leq \varepsilon\|u(t)\|_{\mathcal{H}^1}^2 + C_\varepsilon ,$$

proving the present lemma. □

We can now obtain an a-priori bound of $\|u(t)\|_{\mathcal{H}^2}$.

Lemma 3.7 (A-priori $\mathcal{H}^2$ Estimate). *The assumptions* (A-1), (A-2) *and* (A-3) *imply that*

$$\|u(t)\|_{\mathcal{H}^2} \leq M\exp(Mt) ,$$

for a suitable constant $M>0$.

Proof. Since e^{-itH} is uniformly bounded in $\mathscr{H}^2$, we have by the integral equation (1.7),

$$\|u(t)\|_{\mathscr{H}^2}\leq C\Big(1+\int_0^t\|K(u(s))\|_{\mathscr{H}^2}ds\Big).$$

In view of Lemma 2.2 (2) and Lemma 3.5, we have

$$\|K(u(t))\|_{\mathscr{H}^2}\leq C\|u(t)\|_{\mathscr{H}^1}^2\|u(t)\|_{\mathscr{H}^2}\leq C\|u(t)\|_{\mathscr{H}^2}.$$

We have thus obtained the integral inequality

$$\|u(t)\|_{\mathscr{H}^2}\leq M\Big(1+\int_0^t\|u(s)\|_{\mathscr{H}^2}ds\Big).$$

The assertion of the lemma now follows from the well-known Gronwall's inequality. □

Since we have obtained the apriori estimate of $u(t)$, we can easily prove the global existence of solutions to (1.2) by the standard arguments.

Theorem 3.8 (Global Existence). *Assume* (A-1), (A-2) *and* (A-3). *Then for any Cauchy data* $u(0)\in\mathscr{H}^2$, *there exists a unique global solution to* (1.2).

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