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On a Holomorphic Fiber Bundle with Meromorphic Structure

By

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Introduction

Let $f: X \to Y$ be a proper surjective morphism of compact complex manifolds. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. Let $X_U = f^{-1}(U)$ and let $f_U: X_U \to U$ be the induced morphism. Assume that f_U is a holomorphic fiber bundle with typical fiber F and the structure group H. Let $G_U \to U$ be the holomorphic fiber bundle associated with f_U with typical fiber H with the adjoint action of H on itself so that G_U acts naturally on X_U over U. Let $I_U \to U$ be the principal H-bundle associated to f_U. Then G_U acts naturally on I_U over U also. We say that f_U is a holomorphic fiber bundle with meromorphic structure if there exists a compact complex space G^* (resp. I^*) over Y containing G_U (resp. I_U) as a Zariski open subset such that the action of G_U on X_U (resp. I_U) extends 'meromorphically' to that of G^* on X (resp. I^*). Then in this paper we shall prove the following: Suppose that f_U is a holomorphic fiber bundle with meromorphic structure for some G^* and I^* as above. Then

1) there exists a 'generic quotients' X/G^* of X by G^* over Y, and

2) X/G^* is bimeromorphic to the product space $(F/H) \times Y$ where F/H is a generic quotient of F by H.[1]

Actually in this paper, these results are obtained in a more general setting of comparing two proper morphisms $f_i: X_i \to Y$, $i=1, 2$, over Y having isomorphic general fibers (cf. Theorems 1 and 2); the above special case corresponds to the case where one of the f_i is isomorphic to the projection $p: F \times Y \to Y$. (This generalization is in a sense parallel with Grothendieck's generalization [7] of the theory of fiber bundles to the theory of general fiber spaces with structure sheaf.)

Section 1 is preliminary, and in Section 2 we prove Theorems 1 and 2 mentioned above. Then in Section 3 we shall give some general examples which appear naturally in the study of the structure of compact complex manifolds in $\mathcal{C}$ [5]; indeed, the application to these examples is the principal motivation for this paper. Finally in Section 4, as a reference for [5], we gather some results

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1) The assumption on I_U is unnecessary for the assertion 1).

related to the subject of this paper.

In this paper a complex variety means a reduced and irreducible complex space. Let $f: X \to Y$ be a proper surjective morphism of complex varieties. Then we write $f \in \mathcal{C}/Y$ if there exist a proper Kähler morphism $g: Z \to Y$ (cf. [4]) and a surjective meromorphic Y-map $\psi: Z \to X$.

§1. Preliminaries and Basic Definitions

1.1. a) Let Y be a complex space. Then a *relative complex Lie group over* Y is a complex space G over Y with a holomorphic section $e: Y \to G$ (the identity section) and Y-morphisms $\mu = \mu_{G/Y}: G \times_Y G \to G$, and $\iota = \iota_{X/Y}: G \to G$ (relative group multiplication and inversion) satisfying the usual axioms of group law (cf. [11], Def. 0.1). Then a *relative complex Lie subgroup* of G is a complex subspace H of G which itself is a relative complex Lie group over Y with respect to the 'restrictions' of e, μ and ι to H. Let $f: X \to Y$ be a morphism of complex spaces and G a relative complex Lie group over Y. Then a *relative* (biholomorphic) *action of* G *on* X over Y is a Y-morphism $\sigma: G \times_Y X \to X$ satisfying the usual axioms of operation (cf. [11], Def. 0.3).

b) Let $f: X \to Y$ and $f': X' \to Y$ be proper surjective flat morphisms of complex spaces. Let (An/Y) be the category of complex spaces over Y. Then we define the contravariant functor $\mathbf{Isom}_Y(X, X'): (\mathrm{An}/Y) \to (\mathrm{Sets})$ by the following formula; $\mathbf{Isom}_Y(X, X')(\tilde{Y}):=$the set of $\tilde{Y}$-isomorphisms $\varphi: X \times_Y \tilde{Y} \to X' \times_Y \tilde{Y}$ where $\tilde{Y} \in (\mathrm{An}/Y)$. Let $D_{X \times_Y X'/Y} \to Y$ be the relative Douady space associated to the morphism $f \times_Y f': X \times_Y X' \to Y$. Then $\mathbf{Isom}_Y(X, X')$ is represented by a Zariski open subset $\mathrm{Isom}_Y(X, X')$ of $D_{X \times_Y X'/Y}$ (cf. Schuster [13]). We set $\mathrm{Aut}_Y X := \mathrm{Isom}_Y(X, X)$. Then $\mathrm{Aut}_Y X$ has the natural structure of a complex Lie group over Y, acting naturally on X over Y.

When Y is a point, we write $\mathrm{Isom}(X, X')$ and $\mathrm{Aut}\, X$ instead of $\mathrm{Isom}_Y(X, X')$ and $\mathrm{Aut}_Y X$ respectively. $\mathrm{Aut}\, X$ is thus the automorphism group of X as a complex Lie group in the usual sense.

For any $\tilde{Y} \in (\mathrm{An}/Y)$ we have the natural isomorphisms $\mathrm{Isom}_Y(X, X') \times_Y \tilde{Y} \cong \mathrm{Isom}_{\tilde{Y}}(\tilde{X}, \tilde{X}')$ and $(\mathrm{Aut}_Y X) \times_Y \tilde{Y} \cong \mathrm{Aut}_{\tilde{Y}} \tilde{X}$ where $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{X}' = X' \times_Y \tilde{Y}$ (cf. [8], [13]). In particular we have for each $y \in Y$, $\mathrm{Isom}_Y(X, X')_y \cong \mathrm{Isom}(X_y, X'_y)$ and $(\mathrm{Aut}_Y X)_y \cong \mathrm{Aut}\, X_y$.

c) $\mathrm{Aut}_Y X$ and $\mathrm{Aut}_Y X'$ act naturally on $\mathrm{Isom}_Y(X, X')$ over Y (from the right in the case of $\mathrm{Aut}_Y X$). In relation to these actions we shall define the notion of principai subspace of $\mathrm{Isom}_Y(X, X')$ in a rather primitive way.

i) When Y is a point, then with respect to this action $\mathrm{Isom}(X, X')$ becomes a principal homogeneous space under either of $\mathrm{Aut}\, X$ and $\mathrm{Aut}\, X'$, i.e., for any $h \in \mathrm{Isom}(X, X')$ the induced maps $\sigma_h: \mathrm{Aut}\, X \to \mathrm{Isom}(X, X')$, $\sigma_h(g) = hg$, and $\sigma'_h: \mathrm{Aut}\, X' \to \mathrm{Isom}(X, X')$, $\sigma'_h(g') = g'h$, are isomorphic where $g \in \mathrm{Aut}\, X$ and $g' \in \mathrm{Aut}\, X'$. We shall call any isomorphism $\mathrm{Aut}\, X \to \mathrm{Isom}(X, X')$ obtained in this way *admissible*. The composition $h_* := \sigma_h'^{-1}\sigma_h$: Aut $X \to$ Aut X' is an isomorphism of complex Lie groups, and is given by $h_*(g) = hgh^{-1}$, $g \in$ Aut X. Hence $h(gx) = h_*(g)h(x)$ for any $g \in$ Aut X and $x \in X$. Now let $I \subseteqq$ Isom X be a subspace. Then I is called *principal* if there exist complex Lie subgroups $G \subseteqq$ Aut X and $G' \subseteqq$ Aut X' such that $G \cong I$ and $G' \cong I$ under some admissible isomorphisms. In this case G and G' are said to be *associated to* I.

ii) In the general case let $I \subseteqq \mathrm{Isom}_Y(X, X')$ be any analytic subset. Assume that X and Y are varieties. Then I is called *principal* if there exist relative complex Lie subgroups $G \subseteqq \mathrm{Aut}_Y X$ and $G' \subseteqq \mathrm{Aut}_Y X'$ such that for each $y \in Y$, I_y is principal with the associated subgroups $G_y \subseteqq$ Aut X_y and $G'_y \subseteqq$ Aut X_y. In this case we call G and G' *associated to* I.

1.2. a) We use the following terminology. Let $h: Z \to Y$ be a proper morphism of complex varieties and $V \subseteqq Y$ a Zariski open subset. Let $A \subseteqq h^{-1}(V)$ be an analytic subset whose closure $\bar{A}$ in Z is analytic. Then the *essential closure* A^* of A in Z (over Y) is the union of those irreducible components of $\bar{A}$ which are mapped surjectively onto Y. Clearly, if $V' \subseteqq V$ is another Zariski open subset, then the essential closure of $A \cap h^{-1}(V')$ in Z coincides with A^*. Moreover, if $\bar{A}$ is proper over Y, there exists a Zariski open subset $U \subseteqq Y$ such that for any $y \in U$, $A_y \subseteqq A_y^*$ and A_y^* is the closure of A_y. In fact, since A^* is the closure of $A \cap A^*$, it suffices to show the assertion with A^* replaced by $\bar{A}$. In this case the proof is standard.

b) Let $f: X \to Y$ and $f': X' \to Y$ be proper surjective morphisms of complex varieties (not necessarily flat). Let $U \subseteqq Y$ be a Zariski open subset over which both f and f' are flat [1]. Then $\mathrm{Isom}_U(X_U, X'_U)$ is Zariski open in $D_{X \times_Y X'/Y} \supseteqq D_{X_U \times_U X'_U/Y}$.

Definition 1. $\mathrm{Isom}_Y^*(X, X')$ is the essential closure of $\mathrm{Isom}_U(X_U, X'_U)$ in $D_{X \times_Y X'/Y}$ over Y. We set $\mathrm{Aut}_Y^* X := \mathrm{Isom}_Y^*(X, X)$. When Y is a point, we simply write $\mathrm{Isom}^*(X, X')$ and $\mathrm{Aut}^* X$.

Remark 1. 1) $\mathrm{Isom}_Y^*(X, X')$ and $\mathrm{Aut}_Y^* X$ is independent of the choice of U as above and depends only on f and f' (cf. a)).

2) Let $\varphi: X_U \to X'_U$ be a Y-isomorphism represented by a unique holomorphic section $s: U \to \mathrm{Isom}_U(X_U, X'_U)$. Then φ extends to a bimeromorphic Y-map $\varphi^*: X \to X'$ if and only if s extends to a meromorphic section $s^*: Y \to \mathrm{Isom}_Y^*(X, X')$.

3) The relative group multiplication $\mu_U: \mathrm{Aut}_U X_U \times_U \mathrm{Aut}_U X_U \to \mathrm{Aut}_U X_U$ and inversion $\iota_U: \mathrm{Aut}_U X_U \to \mathrm{Aut}_U X_U$ of relative complex Lie groups $\mathrm{Aut}_U X_U$ over U, and the natural relative action $\sigma_U: \mathrm{Aut}_U X_U \times_U X_U \to X_U$ of $\mathrm{Aut}_U X_U$ on X_U over U extend to meromorphic maps $\mu^*: \mathrm{Aut}_Y^* X \times_Y \mathrm{Aut}_Y^* X \to \mathrm{Aut}_Y^* X$, $\iota^*: \mathrm{Aut}_Y^* X \to \mathrm{Aut}_Y^* X$ and $\sigma^*: \mathrm{Aut}_Y^* X \times_Y X \to X$ respectively. Moreover the identity section $e_U: U \to \mathrm{Aut}_U X_U$ extends to a meromorphic section $e^*: Y \to \mathrm{Aut}_Y^* X$.

4) Let $\nu: \tilde{Y} \to Y$ be any proper surjective morphism of complex varieties. Set $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{X}' = X' \times_Y \tilde{Y}$. Then we have the natural isomorphisms

$\mathrm{Isom}_Y^*(X, X')\times_Y\tilde{Y}\cong\mathrm{Isom}_{\tilde{Y}}^*(\tilde{X}, \tilde{X}')$ and $\mathrm{Aut}_Y^*X\times_Y\tilde{Y}\cong\mathrm{Aut}_{\tilde{Y}}^*\tilde{X}$.

5) Suppose that $f, f'\in\mathcal{C}/Y$. Then for any relatively compact open subset $V\subseteqq Y$ any irreducible component of $\mathrm{Isom}_Y^*(X_V, X'_V)$ and $\mathrm{Aut}_Y^*X_V$ is proper over Y. This follows from [4].

1.3. a) Let $f: X\to Y$ be a proper morphism of complex varieties.

Definition 2. Let $G^*\subseteqq\mathrm{Aut}_Y^*X$ be an analytic subset such that any irreducible component of G^* is mapped surjectively onto Y. Then we call G^* (by abuse of language) a *relative quasi-meromorphic* (*Lie*) *subgroup* of Aut_Y^*X if there exists a Zariski open subset $U\subseteqq Y$ such that f is flat over U and that $G_U:=G^*\cap\mathrm{Aut}_U X_U$ is dense in G^* and is a relative Lie subgroup of $\mathrm{Aut}_U X_U$ over U. If, further, G^* is proper over Y, we call G^* a *relative meromorphic* (*Lie*) *subgroup* of Aut_Y^*X.

Remark 2. 1) If Y reduces to a point, G^*, or more properly, $G=G^*\cap\mathrm{Aut}\,X$, is called a (quasi-)meromorphic subgroup of Aut^*X (cf. [3]).

2) If G^* is a relative quasi-meromorphic subgroup and G_U is as above, then the relative group law $G_U\times_U G_U\to G_U$, $G_U\to G_U$ (cf. 1.1) and the relative action $\sigma_U: G_U\times_U X_U\to X_U$ extend to meromorphic Y-maps $G^*\times_Y G^*\to G^*$, $G^*\to G^*$ and $\sigma^*: G^*\times_Y X\to X$ respectively. Moreover the identity section $e_U: U\to G_U$ extends to a meromorphic section $e^*: Y\to G^*$. This follows from Remark 1, 3).

b) Let $f: X\to Y$ and $f': X'\to Y$ be proper surjective morphisms of complex varieties.

Definition 3. Let I^* be any analytic subspace of $\mathrm{Isom}_Y^*(X, X')$. Then we say that I^* is a *quasi-meromorphic principal subspace* if there exist relative quasi-meromorphic subgroups $G^*\subseteqq\mathrm{Aut}_Y^*X$ and $G^{*\prime}\subseteqq\mathrm{Aut}_Y^*X'$ and a Zariski open subset $U\subseteqq Y$ over which both f and f' are flat, such that $I_U:=I^*\cap\mathrm{Isom}_U(X_U, X'_U)$ is dense in I^* and $I_U\subseteqq\mathrm{Isom}_U(X_U, X'_U)$ is principal with the associated relative Lie subgroups $G_U:=G^*\cap\mathrm{Aut}_U X_U$ and $G'_U:=G^{*\prime}\cap\mathrm{Aut}_U X'_U$ (cf. 1.1 c)). In this case we call G^* (resp. $G^{*\prime}$) *associated to* I^*. I^* is called a *meromorphic principal subspace* if further it is proper over Y. In the latter case the associated G^* and $G^{*\prime}$ are also proper over Y and hence are relative meromorphic subgroups of Aut_Y^*X and Aut_Y^*X' respectively.

Remark 3. 1) Let G^* be a relative meromorphic subgroup of Aut_Y^*X. Then the following conditions are equivalent. a) G^* is associated to some meromorphic principal subspace I^*. b) Let $\bar{I}_Y^*(X, X'):=\mathrm{Isom}_Y^*(X, X')/G^*$ be a relative generic quotient of $\mathrm{Isom}_Y^*(X, X')$ by G^* over Y with the natural projection $\varepsilon: \bar{I}_Y^*(X, X')\to Y$ (cf. Definition 5 and Theorem 1 below). Then ε admits a meromorphic section $s: Y\to\bar{I}_Y^*(X, X')$. Moreover in this case I^* is given by $I^*=\pi^{-1}(s(Y))$ and G^* is given by the union of those irreducible components of $p_1(\Gamma\cap(\mathrm{Aut}_Y^*X\times_Y I^*\times_Y I^*))$ which are mapped surjectively onto Y, where $\pi: \mathrm{Isom}_Y^*(X, X')$

$\to \bar{I}_Y^*(X, X')$ is the natural meromorphic projection, $\Gamma \subseteq \mathrm{Aut}_Y^* X \times_Y \mathrm{Isom}_Y^*(X, X') \times_Y \mathrm{Isom}_Y^*(X, X')$ is the closure of the graph of the action of $\mathrm{Aut}_U X_U$ on $\mathrm{Isom}_U(X_U, X'_U)$ and p_1 is the projection to the first factor $\mathrm{Aut}_Y^* X$. In particular I^* and G^* determine each other uniquely. The analogous fact holds of course for a meromorphic subgroup $G^{*\prime} \subseteq \mathrm{Aut}_Y^* X$.

c) We consider the special case of b) where $X' = Y \times F$ for a compact complex variety F and $f' : Y \times F \to Y$ is the natural projection. Then we have the natural isomorphisms

$$\mathrm{Aut}_Y X' \cong Y \times \mathrm{Aut}\, F \quad \text{and} \quad \mathrm{Aut}_Y^* X' \cong Y \times \mathrm{Aut}^* F.$$

Definition 4. Let I^* be a (quasi-)meromorphic principal subspace of $\mathrm{Isom}_Y^*(X, X')$, and $G^* \subseteq \mathrm{Aut}_Y^* X$ and $G^{*\prime} \subseteq \mathrm{Aut}_Y^* X'$ the associated relative (quasi-) meromorphic subgroups. Then we call I^* *admissible* (with the associated meromorphic subgroup H^*) if $G^{*\prime}$ is of the form $G^{*\prime} = Y \times H^*$ for some (quasi-) meromorphic subgroup $H^* \subseteq \mathrm{Aut}^* F$.

Suppose that I^* is admissible as above and set $H = H^* \cap \mathrm{Aut}\, F$. Take a Zariski open subset $U \subseteq Y$ as in Definition 3. Then it is immediate to see that $f_U : X_U \to U$ is a holomorphic fiber bundle with structure group H. In this case we say that f is *a holomorphic fiber bundle over U which is (quasi-)meromorphic with respect to f and with (quasi-)meromorphic structure group H.* We note that in this case the natural map $I_U \to U$ is the principal bundle associated to f_U.

§ 2. Relative Generic Quotients and Related Results

2.1. We generalize the generic quotient theorem by a meromorphic group in [3] to a relative case.

Theorem 1. *Let $f : X \to Y$ be a proper surjective morphism of complex varieties. Let $G^* \subseteq \mathrm{Aut}_Y^* X$ be any relative meromorphic subgroup over Y. Then there exists a unique subspace $\bar{X} \subseteq D_{X/Y}$[2] having the following properties: Let $\rho : Z \to \bar{X}$ be the universal family $\rho_{X/Y} : Z_{X/Y} \to D_{X/Y}$ restricted to $\bar{X}$, i.e., $Z = Z_{X/Y} \times_{D_{X/Y}} \bar{X}$. Then: 1) the natural Y-morphism $\pi : Z \to X$ is bimeromorphic, and 2) there exists a Zariski open subset $V \subseteq \bar{X}$ such that for any $v \in V$, the corresponding subspace $Z_v \subseteq X_y$ is a closure of an orbit Z_v^0 of the group G_y acting on X_y, where $y = \bar{f}(v)$ ($\bar{f} : \bar{X} \to Y$ being the natural map) and $G_y = G_y^* \cap \mathrm{Aut} X_y$.*

Proof. Define a meromorphic Y-map $\Phi : G^* \times_Y X \to X \times_Y X$ by $\Phi(g, x) = (\sigma^*(g, x), x)$. Let $R := \Phi(G^* \times_Y X) \subseteq X \times_Y X$. Let $p : R \to X$ be induced by the second projection $p_2 : X \times_Y X \to X$;

2) the relative Douady space associated to f.

$$\begin{array}{c} R \subseteq X \times_Y X \\ \searrow^{p} \ \downarrow p_2 \\ X \end{array}$$

Let $\tau: X \to D_{X/Y}$ be the universal meromorphic Y-map associated to this diagram where τ is holomorphic exactly on the Zariski open subset over which p is flat (cf. [2], Lemma 5.1). Let $\bar{X}$ be the image of τ. Let $\bar{\tau}: X \to \bar{X}$ be the resulting meromorphic Y-map. We claim that this $\bar{X}$ has the desired property. Take a Zariski open subset $U \subseteq Y$ such that both f and $G^* \to Y$ are flat over U, and that for each $y \in U$, G_y is dense in G_y^* (cf. 1.2 a)). Now we consider Φ as a meromorphic map over X where $X \times_Y X$ is over X by p_2. Take a Zariski open subset $W \subseteq X$ such that for each $x \in W$, if we set $y = f(x)$, then $y \in U$, and with respect to the natural identification $(G^* \times_Y X)_x = G_y^*$ and $(X \times_Y X)_x = X_y$, Φ induces a meromorphic map $\Phi_x: G_y^* \to X_y$. Then we have for $x \in W$, $R_x = \Phi_x(G_y^*) = \overline{\Phi_x(G_y)} = \overline{G_y x}$ as a subspace of X_y, where $G_y x$ is the orbit of x under G_y and $\overline{G_y x}$ its closure. In particular for any $\bar{w} \in \tau(W)$, $Z_{\bar{w}}$ is a closure of an orbit of G_y. Since $\tau(W)$ contains a nonempty Zariski open subset, say V, 2) follows. It remains to show that π is bimeromorphic. Restricting W if necessary we may assume that $p_W: R_W \to W$ is flat [1] so that for any $x \in W$, we have $\dim(G_{f(x)} x) = m$ for a fixed integer $m \geqq 0$. (W is a union of 'regular orbits'. cf. [11]) Then just as in the proof of the absolute case (cf. Theorem 4.1 of [3]) we can show that π is isomorphic on $\rho^{-1}(V) \cap \pi^{-1}(W)$. Thus π is bimeromorphic.

It remains to show the uniqueness of $\bar{X}$. In fact, from 1) and 2) alone we deduce easily the following: 1) There exists a Zariski open subset $W_1 \subseteq X$ such that a) for every $x \in W_1$ with $f(x) = y$, the point $d(x) \in D_{X/Y, y}$ corresponding to the subspace $\overline{G_y x}$ belongs to $\bar{X}$, and b) the set $\{d(x); x \in W_1\}$ forms a Zariski open subset of $\bar{X}$. Uniqueness clearly follows from this. q. e. d.

Definition 5. We call the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ q\ } & \bar{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \bar{f}} & \\ & Y & \end{array} \qquad q = \rho\pi^{-1}$$

or simply, the meromorphic Y-map $q: X \to \bar{X}$, or $\bar{X}$ itself, the *relative generic quotient* of X by G^* over Y. We often denote $\bar{X}$ symbolically by X/G^*.

Proposition 1. *Let* $f: X \to Y$ *and* $G^* \subseteq \mathrm{Aut}_Y^* X$ *be as in Theorem* 1.

1) *Let* $\tilde{Y} \to Y$ *be a surjective morphism of complex varieties. Let* $\tilde{X} = X \times_Y \tilde{Y}$ *and* $\tilde{G}^* = G^* \times_Y \tilde{Y}$. *Then* $\tilde{G}^*$ *is a relative meromorphic subgroup of* $\mathrm{Aut}_{\tilde{Y}}^* \tilde{X} \cong \mathrm{Aut}_Y^* X \times_Y \tilde{Y}$ *and the relative generic quotient* $\tilde{X}/\tilde{G}^*$ *of* $\tilde{X}$ *by* $\tilde{G}^*$ *over* $\tilde{Y}$ *is isomorphic over* $\tilde{Y}$ *to the pull-back* $(X/G^*) \times_Y \tilde{Y}$.

2) *Suppose that* Y *is a complex variety over another complex variety* T *with a surjective morphism* $h: Y \to T$. *Then there exists a Zariski open subset* $U \subseteq T$

such that for any $t \in U$ i) G_t^* *is a relative meromorphic subgroup of* $\mathrm{Aut}^*_{Y_t} X_t$ *over* Y_t *and* ii) $(X/G^*)_t = X_t/G_t^*$ *as a subspace of* $(D_{X/Y})_t = D_{X_t/Y_t}$, *where we consider any complex space over* Y *naturally as a complex space over* T *via* h. *In particular if* $Y=T$ *then for each* $y \in U \subseteq Y$, $(X/G^*)_y$ *is the generic quotient of* X_y *by* G_y^*. *As a special case of this, if* $X/G^* \to Y$ *is bimeromorphic, then there exists a Zariski open subset* $W \subseteq X$ *with* $f(W) \subseteq U$ *such that if* $y \in U$ *then* G_y *acts almost homogeneously on* X_y *and its unique Zariski open orbit coincides with* W_y.

3) *If* $G_y^* = G_y$ *is a complex torus for* $y \in U$, *and* G_y *acts freely on* X_y *then* $q_y : X_y \to X_y/G_y$ *is a holomorphic fiber bundle and hence* $q_U : X_U \to (X/G^*)_U$ *is holomorphic and smooth.*

Proof. In view of the uniqueness assertion of Theorem 1 the verification of 1) is straightforward. For the first assertion of 2) it suffices to take U in such a way that for any $t \in U$, $\pi_t : Z_t \to X_t$ is bimeromorphic and $V_t^- = \bar{X}_t$ where V_t^- is the closure of V_t in $\bar{X}_t$ in the notation of Theorem 1. Here, restricting U if necessary, we may assume further that G_U is smooth over U. Then, when $X/G^* \to Y$ is bimeromorphic, if we set $A = \{x \in X_U ; \dim G_U(x) > t-r\}$ ($t = \dim G_U/U$, $r = \dim f$) where $G_U(x)$ is the stabilizer of $G_{U, f(x)}$ at $x \in X_{f(x)}$, then $W := X_U - A$ is easily seen to satisfy the above condition. In 3) that q is a fiber bundle is due to Holmann [10], § 5. Since $\dim X_{\bar{x}}$ is constant on $\bar{X}_U$, from this follows the last assertion.

2.2. Let $f : X \to Y$ and $f' : X' \to Y$ be proper surjective morphism of complex varieties. Let $U \subseteq Y$ be a Zariski open subset over which both f and f' are flat. Then by the universality of the relative Douady space we have the natural transformation of functors $\psi : \mathbf{Isom}_U(X_U, X'_U) \to \mathbf{Isom}_U(D_{X/U}, D_{X'/U})$. Let $I^* \subseteq \mathrm{Isom}^*(X, X')$ be an analytic subset such that $I_U := I^* \cap \mathrm{Isom}_U(X_U, X'_U)$ is dense in I^*. Let $B \subseteq D_{X/Y}$ and $B' \subseteq D_{X'/Y}$ be analytic subspaces which are proper over Y and are flat over U. Now we assume the following condition; (*) the image of $I_U \subseteq \mathrm{Isom}_U(X_U, X'_U)$ by ψ is contained in the subfuctor $\mathbf{Isom}_U((D_{X_U/U}, B_U), (D_{X_U/U}, B'_U))$ (cf. 3.1 a) below for the notation) where we identify $\mathrm{Isom}_U(X_U, X'_U)$ with the functor it represents. Then composed with the natural projection $\mathbf{Isom}_U((D_{X_U/U}, B_U), (D_{X'_U/U}, B'_U)) \to \mathrm{Isom}_U(B_U, B'_U)$ we get a U-map $\psi : I_U \to \mathrm{Isom}_U(B_U, B'_U)$. It is immediate to see that ψ is indeed a morphism of complex spaces and that ψ extends to a meromorphic Y-map $\psi^* : I^* \to \mathrm{Isom}^*_Y(B, B')$. The condition (*) is fulfilled if for each $y \in U$ and for each $h \in I_{U, y} \subseteq \mathrm{Isom}(X_y, X'_y)$ we have $D_y h(B_y) = B'_y$ where $D_y h \in \mathrm{Isom}(D_{X_y}, D_{X'_y})$ is the element canonically induced by h.

Theorem 2. *Let* $I^* \subseteq \mathrm{Isom}^*_Y(X, X')$ *be a meromorphic principal subspace with the associated relative meromorphic subgroups* $G^* \subseteq \mathrm{Aut}^*_Y X$ *and* $G^{*\prime} \subseteq \mathrm{Aut}^*_Y X'$. *Let* $\bar{X} = X/G^*$ *and* $\bar{X}' = X'/G^{*\prime}$ *be the respective relative generic quotients over* Y. *Then there exists a natural bimeromorphic* Y-*map* $\bar{X} \to \bar{X}'$ *which is isomorphic* *over some Zariski open subset of Y.*

Proof. Take Zariski open subsets $V\subseteqq\bar{X}$ and $V'\subseteqq\bar{X}'$ as in 2) of Theorem 1. Restricting V and V' we may assume that the following conditions are satisfied: 1) $\bar{f}(V)=\bar{f}'(V')$, and if we denote this set by U, then U is nonsingular and Zariski open in Y, where $\bar{f}: \bar{X}\to Y$ and $\bar{f}': \bar{X}'\to Y$ are the natural morphisms, 2) both f and f' are flat over U, and 3) for each $y\in U$, a) G_y^* is a meromorphic subgroup of Aut^*X_y and $\bar{X}_y$ is the generic quotient of X_y by G_y^*, and the similar condition for $G_y^{*\prime}$ and $\bar{X}_y'$ is true (cf. Proposition 1), b) V_y is dense in $\bar{X}_y$ and c) the induced map $\pi_y: Z_y\to X_y$ is bimeromorphic where Z is as in Theorem 1. Let $I_U=\mathrm{Isom}_U(X_U, X_U')\cap I^*$. Take any $y\in U$ and any $h=h_y\in I_y:=I_{U,y}\subseteqq\mathrm{Isom}(X_y, X_y')$. We shall first show that $D_yh(\bar{X}_y)=\bar{X}_y'$ where D_yh is defined just before the theorem. Let $\bar{X}_y''=D_yh(\bar{X}_y)\subseteqq D_{X'_y}$ and $V_y''=D_yh(V_y)\subseteqq\bar{X}_y''$. Then h induces an isomorphism of the following diagrams

$$\begin{array}{ccc} X_y \xleftarrow{\pi_y} & Z_y & \qquad X_y' \xleftarrow{\pi_y''} & Z_y'':=Z_{X/Y}\times_{D_{X/Y}}\bar{X}_y'' \\ & \downarrow & & \downarrow \\ & \bar{X}_y & & \bar{X}_y'' \end{array}$$

By the uniqueness of the generic quotient in Theorem 1 it suffices to show that $\bar{X}_y''$ satisfies the conditions of that theorem for f_y' and $G_y^{*\prime}$. Since π_y'' is bimeromorphic as well as π_y, 1) is satisfied. We set $G_y=G_y^*\cap\mathrm{Aut}\,X_y$ and $G_y'=G_y^{*\prime}\cap\mathrm{Aut}\,X_y'$. For 2) it suffices to show that for any point $v''\in V_y''$, $Z_{v''}''$ is a closure of an orbit of G_y' when $Z_{v''}''$ is considered as a subspace of X_y' via π_y''. In fact, take $v\in V$ with $D_yh(v)=v''$. Then $Z_{v''}''=h(Z_v)$, which is the closure of $h(Z_v^0)$ where Z_v^0 is a G_y-orbit on X_y. Then, since h_y is (G_y, G_y')-equivariant with respect to the homomorphism $h_{y*}: G_y\to G_y'$ (cf. 1.1 c)), $h(Z_v^0)$ is an orbit of G_y' as was desired.

Thus D_yh induces an element of $\mathrm{Isom}(\bar{X}_y, \bar{X}_y')$ which we shall denote by the same letter D_yh. Hence by the remark just before the theorem we have obtained a U-morphism $\phi: I_U\to\mathrm{Isom}_U(\bar{X}_U, \bar{X}_U')$ which extends to a meromorphic Y-map $\phi^*: I^*\to\mathrm{Isom}_Y^*(\bar{X}, \bar{X}')$. Next we show that $D_y(h)=D_y(h')$ for any $h, h'\in I_y$, $y\in U$. It suffices to show that $D_yh(v)=D_yh'(v)$ for any $v\in V_y$ since V_y is dense in $\bar{X}_y$. In fact, since $g(Z_v)=Z_v$ for any $g\in G_y$ and $h'^{-1}h\in G_y$, $h(Z_v)=h'h'^{-1}h(Z_v)=h'(Z_v)$, or equivalently, $D_yh(v)=D_yh'(v)$ as was desired. Since $I_U\to U$ is surjective it follows that $\phi(I_U)\subseteqq\mathrm{Isom}_U(\bar{X}_U, \bar{X}_U')$ gives a holomorphic section to $\mathrm{Isom}_U(\bar{X}_U, \bar{X}_U')\to U$, U being nonsingular, and hence, $\phi^*(I^*)\subseteqq\mathrm{Isom}_Y^*(\bar{X}, \bar{X}')$ a meromorphic section to $\mathrm{Isom}_Y^*(\bar{X}, \bar{X}')\to Y$. Hence by Remark 1, 2) $\bar{X}$ and $\bar{X}'$ are bimeromorphic over Y by a bimeromorphic map which is isomorphic over U.

q. e. d.

In Theorem 2 assume that there exists a Y-isomorphism $\phi: X'\to Y\times F$ for some compact complex variety F. Let $H^*\subseteqq\mathrm{Aut}^*F$ be a meromorphic subgroup. Then we say that ϕ is *admissible* with respect to (I^*, H^*) if ϕ induces an isomorphism $G^{*\prime}\cong Y\times H^*$. Then, if ϕ and ϕ' are Y-isomorphisms $X'\cong Y\times F$ which are admissible with respect to (I^*, H^*), then $\phi'\phi^{-1}$ induces a Y-automorphism of $p_1: Y\times F\to Y$, i.e., gives a holomorphic map $Y\to \mathrm{Aut}\,F$, whose image is contained in H where $H=H^*\cap \mathrm{Aut}\,F$. This implies that the set of admissible Y-isomorphisms is naturally a principal homogeneous space under the group $\mathrm{Hol}(Y, H)$, the space of holomorphic maps of Y to H. From this observation we get the following:

Lemma 1. *Suppose that there exists a Y-isomorphism $\phi: X'\to Y\times F$ which is admissible with respect to (I^*, H^*), so that we have the natural isomorphism $X'/G^{*\prime}\cong Y\times(F/H^*)$. Then the composite meromorphic map $X\to X/G^*\cong X'/G^{*\prime}\cong Y\times(F/H^*)\to F/H^*$ is independent of the choice of the admissible isomorphism ϕ.*

Definition 6. We call the meromorphic map $X\to\bar{F}:=F/H^*$ defined in the lemma, or any meromorphic map which is bimeromorphic to it, a *canonical meromorphic map* associated to f and to H^*.

Clearly we have $\dim\bar{F}=\dim p$ where $p: X/G^*\to Y$ is the natural map.

§ 3. Examples of Relative Quasi-Meromorphic Subgroups

3.1. $\mathrm{Isom}_Y^*((X, A), (X', A'))$ *and* $\mathrm{Aut}_Y^*(X, A)$. Let $f: X\to Y$ and $f': X'\to Y$ be proper morphisms of complex varieties. Let $A=(A_1, \cdots, A_m)$ and $A'=(A'_1, \cdots, A'_m)$ be sequences of analytic subspaces of X and X' respectively.

a) Suppose first that f and f' are flat and that A'_α are all flat over Y with respect to f'. Then we define a subfunctor $\mathbf{Isom}_Y((X, A), (X', A')): (\mathrm{An}/Y)\to(\mathrm{Sets})$ of $\mathbf{Isom}_Y(X, X')$ as follows; $\mathbf{Isom}_Y((X, A), (X', A'))(\tilde{Y})=\{\varphi\in\mathbf{Isom}_Y(X, X')(\tilde{Y})$; φ induces isomorphisms of $A_\alpha\times_Y\tilde{Y}$ and $A'_\alpha\times_Y\tilde{Y}$ for all $\alpha\}$.

Lemma 2. $\mathbf{Isom}_Y((X, A), (X', A'))$ *is represented by a unique analytic subspace* $\mathrm{Isom}_Y((X, A), (X', A'))$ *of* $\mathrm{Isom}_Y(X, X')$.

Proof. Let $I=\mathrm{Isom}_Y(X, X')$ and $\xi: X\times_Y I\to X'\times_Y I$ the universal I-isomorphism. Let $\bar{A}_{\alpha,I}:=\xi(A_\alpha\times_Y I)$. Then by [12] Prop. 1, there exists a unique analytic subspace $T\subseteq I$ such that for any morphism $u: T'\to I$ of complex spaces $\bar{A}_{\alpha,I}\times_I T'=A'_{\alpha,I}\times_I T'$, where $A'_{\alpha,I}:=A'_\alpha\times_Y I$, as a subspace of $X'_I\times_I T'$ if and only if u factors through T. (In fact, apply [12] Prop. 1 to the morphism $X'\times_Y I\to I$ and to the coherent analytic sheaves $\mathcal{E}:=\mathcal{O}_{A'_{\alpha,I}}$ and $\mathcal{F}:=\mathcal{O}_{A'_{\alpha,I}\cap\bar{A}_{\alpha,I}}$.) Then it is easy to see that T represents the functor $\mathbf{Isom}_Y((X, A), (X', A'))$.

We then set $\mathrm{Aut}_Y(X, A)=\mathrm{Isom}_Y((X, A), (X, A))$. $\mathrm{Aut}_Y(X, A)$ is a relative complex Lie subgroup of $\mathrm{Aut}_Y X$ over Y.

b) In the general case, let $U\subseteq Y$ be a Zariski open subset such that X, X', and A'_α are all flat over U [1].

Lemma 3. *The closure* I^- *of* $\mathrm{Isom}_U((X_U, A_U), (X'_U, A'_U))$ *in* $D_{X\times_Y X/Y}$ *is analytic, where* $A_U=(A_{1,U}, \cdots, A_{m,U})$ *and* $A'_U=(A'_{1,U}, \cdots, A'_{m,U})$.

Proof. Take a proper modification $\sigma: \tilde{Y}\to Y$ such that σ gives an isomorphism of $\sigma^{-1}(U)$ and U and that the strict transforms $\tilde{X}$ and $\tilde{A}_\alpha$ (resp. $\tilde{X}'$ and $\tilde{A}'_\alpha$) of X and A_α in $X\times_Y\tilde{Y}$ (resp. of X' and A'_α in $X'\times_Y\tilde{Y}$) respectively are all flat over $\tilde{Y}$ [9]. Then by Lemma 2 $\tilde{I}=\mathrm{Isom}_{\tilde{Y}}((\tilde{X}, \tilde{A}), (\tilde{X}', \tilde{A}'))$, $\tilde{A}=(\tilde{A}_1, \cdots, \tilde{A}_m)$, $\tilde{A}'=(\tilde{A}'_1, \cdots, \tilde{A}'_m)$, is realized as an analytic subspace of $\mathrm{Isom}_{\tilde{Y}}(\tilde{X}, \tilde{X}')$. Let $\hat{I}$ be the union of those irreducible components of $\tilde{I}$ whose images in $\tilde{Y}$ intersect with $\sigma^{-1}(U)$. Then the image of $\hat{I}$ in $D_{X\times_Y X'/Y}$ by the natural proper morphism $\hat{I}\subseteq D_{\tilde{X}\times_{\tilde{Y}}\tilde{X}'/\tilde{Y}}\cong D_{X\times_Y X'/Y}\times_Y\tilde{Y}\to D_{X\times_Y X'/Y}$ is nothing but I^-.

Definition 7. $\mathrm{Isom}^*_Y((X, A), (X', A'))$ is the essential closure of $\mathrm{Isom}_U((X_U, A_U), (X'_U, A'_U))$ in $D_{X\times_Y X'/Y}$. We set $\mathrm{Aut}^*_Y(X, A)=\mathrm{Isom}^*_Y((X, A), (X, A))$. When Y is a point, we write $\mathrm{Aut}^*(X, A)$ for $\mathrm{Aut}^*_Y(X, A)$.

Remark 4. $\mathrm{Aut}^*_Y(X, A)$ is a relative quasi-meromorphic subgroup of $\mathrm{Aut}^*_Y X$, and $I^*=\mathrm{Isom}^*_Y((X, A), (X', A'))$ is a quasi-meromorphic principal subspace with the associated quasi-meromorphic subgroups $\mathrm{Aut}^*_Y(X, A)$ and $\mathrm{Aut}^*_Y(X', A')$. This follows immediately from the definitions.

c) In b) assume further that X' is of form $X'=Y\times F$ for some compact complex variety F and $f': X'\to Y$ is the natural projection as in 1.3 c). Suppose that there exists a sequence $B=(B_1, \cdots, B_m)$ of subspaces of F such that $A'_\alpha=Y\times B_\alpha\subseteq X'$. Then $\mathrm{Aut}^*_Y(X', A')\cong Y\times\mathrm{Aut}^*(F, B)$. Thus $\mathrm{Isom}^*_Y((X, A), (X', A'))$ is admissible, if it is not empty (Definition 4). In general, let $I^*\subseteq\mathrm{Isom}^*_Y(X, X')$ be a meromorphic principal subspace. Suppose that I^* is admissible with the associated meromorphic subgroup $H^*\subseteq\mathrm{Aut}^*F$ and that $I^*\subseteq\mathrm{Isom}^*_Y((X, A), (X', A'))$. Then $f_{X,A}: (X, A)\to Y$ is a holomorphic fiber bundle over U in the sense that for each $y\in U$ there exist a neighborhood $y\in V$ and a trivialization $X_V\cong V\times F$ which sends A_α onto $V\times B_\alpha$ isomorphically. In this case we say that $f_{X,A}$ is a holomorphic fiber bundle over U which is meromorphic with respect to f (and with meromorphic structure group H).

3.2. $\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}$ *and* $\mathrm{Aut}^*_Y X_\omega$.

a) Let $f: X\to Y$ and $f': X'\to Y$ be proper smooth morphisms of complex varieties. Let $\omega\in\Gamma(Y, R^2f_*\boldsymbol{R})$ and $\omega'\in\Gamma(Y, R^2f'_*\boldsymbol{R})$ be fixed elements. Then we define a subfunctor $\mathbf{Isom}_Y(X, X')_{\omega,\omega'}$ of $\mathbf{Isom}_Y(X, X')$ as follows, $\mathbf{Isom}_Y(X, X')_{\omega,\omega'}(\tilde{Y})=\{\varphi\in\mathbf{Isom}_Y(X, X')(\tilde{Y}); \varphi^*\omega'_{\tilde{Y}}=\omega_{\tilde{Y}}\}$ where $\omega_{\tilde{Y}}$ (resp. $\omega'_{\tilde{Y}}$) is the pull-back of ω (resp. ω') to $X\times_Y\tilde{Y}$ (resp. $X'\times_Y\tilde{Y}$).

Lemma 4. $\mathbf{Isom}_Y(X, X')_{\omega,\omega'}$ *is represented by a unique analytic subspace* $\mathrm{Isom}_Y(X, X')_{\omega,\omega'}$ *of* $I=\mathrm{Isom}_Y(X, X')$ *which is a union of connected components.*

Proof. Let $\xi: X\times_Y I\to X'\times_Y I$ be the universal I-isomorphism. Let $y\in Y$ be any point and $I_{y,\gamma}$ be any connected component of I_y. For $t\in I_y$ let $\xi_t: X_y\to X'_y$

be the isomorphism induced by ξ. Then if $\xi_{t_0}^*\omega'_{I,t_0}=\omega_{I,t_0}$ for some $t_0\in I_{y,\gamma}$, then $\xi_t^*\omega'_{I,t}=\omega_{I,t}$ for all $t\in I_{y,\gamma}$. From this the assertion follows readily.

We set $\mathrm{Aut}_Y X_\omega=\mathrm{Isom}_Y(X, X)_{\omega,\omega}$.

b) In general let $g: Z\to Y$ be any proper smooth morphism of complex varieties. Then any real closed C^∞ 2-form α on Z determines a unique section $\bar{\alpha}\in\Gamma(Y, R^2g_*\boldsymbol{R})$ such that the class of α_y equals $\bar{\alpha}_y$ in $H^2(Z_y, \boldsymbol{R})$.

Proposition 2. *Let f, f', ω, ω' be as in* a). *Suppose that there exists a real closed C^∞ 2-form β (resp. β') on X (resp. X') with $\bar{\beta}=\omega$ (resp. $\bar{\beta}'=\omega'$), which restricts to a Kähler form on each fiber of f (resp. f'). Then the closure $\bar{I}$ of* $\mathrm{Isom}_Y(X, X')_{\omega,\omega'}$ *in $D_{X\times_Y X'/Y}$ is proper over Y.*

For the proof we need a general result. Let $f: X\to Y$ be a smooth morphism of complex varieties and β a C^∞ 2-form on X which restricts to a positive $(1,1)$-form on each fiber of f. Let $D_{X/Y}$ be the relative Douady space of X over Y and $A\subseteq D_{X/Y}$ an analytic subset. Let $\delta: A\to Y$ be the natural morphism. Then we say that A is bounded with respect to β if there exist a dense Zariski open subset $V\subseteq A$, a positive constant R and an integer $q\geqq 0$ such that for any $d\in V$ the corresponding subspace $Z_d\subseteq X_{\delta(d)}$ is reduced and is of pure dimension q and that if $\mathrm{vol}(Z_d):=\int_{Z_d}\beta_{\delta(d)}^q$ is the volume of Z_d with respect to $\beta_{\delta(d)}$ (the restriction of β to $X_{\delta(d)}$), then $\mathrm{vol}(Z_d)\leqq R$.

Proposition 3. *Let $A\subseteq D_{X/Y}$ be as above. Suppose that for any relatively compact open subset $U\subseteq Y$, the restriction $A_U=A\cap D_{X_U/U}$ of A over U is bounded with respect to β_U. Then A is proper over Y.*

Proof. Follows immediately from Propositions 4.1 and 3.4 of [2]. (The proof there clearly applies also to β as above.)

Proof of Proposition 2. In view of a) it is clear that $\bar{I}$ is a union of irreducible components of $D_{X\times_Y X'/Y}$. To show the properness we shall apply Proposition 3 to $A=\bar{I}$, by considering $f\times_Y f': X\times_Y X'\to Y$ and C^∞ 2-form $\tilde{\beta}_0:=\tilde{\beta}+\tilde{\beta}'$ on $X\times_Y X'$ instead of f and β in the proposition respectively. Here $\tilde{\beta}$ and $\tilde{\beta}'$ are the natural pull-backs to $X\times_Y X'$ of β and β' respectively. Then we have to show that on any relatively compact open subset of Y, $\bar{I}$ is bounded with respect to $\tilde{\beta}_0$. Let $V:=\mathrm{Isom}(X, X')_{\omega,\omega'}\subseteq\bar{I}$. Then for any $d\in V$ the associated subspace $Z_d\subseteq X_y\times X'_y$, $y=\delta(d)$, equals the graph Γ_h of the isomorphism $h=h_d: X_y\to X'_y$ corresponding to d, where $\delta: I\to Y$ is the natural morphism. Hence $Z_d\cong X_y$. Moreover, since $h_d^*\omega'_y=\omega_y$ we calculate easily that

$$\mathrm{vol}(Z_d)=\int_{Z_d}\tilde{\beta}_{0,y}^q=(q+1)\int_{X_y}\beta_y^q$$

where $q=\dim X_y$ (cf. the proof of Theorem 4.8 in [3]). Thus $\mathrm{vol}(Z_d)$ depends only on $y=\delta(d)$ and is a continuous function of y. Hence it is bounded on any relatively compact open subset of Y as was desired. q. e. d.

c) In general let $g: Z\to Y$ be a proper morphism of complex varieties. Then we call $\alpha\in\Gamma(Y, R^2f_*\boldsymbol{R})$ a relative Kähler class if the restriction $\alpha_y\in H^2(X_y, \boldsymbol{R})$ of α_y to each X_y is a Kähler class, i. e., represented by a Kähler form. Using Proposition 2 we have shown in [6] the following:

Proposition 4. *Let f, f', ω, ω' be as in* a). *Suppose that ω and ω' are relative Kähler classes. Then $\bar{I}$ is proper over Y.*

Proof. See [6], Proposition 4.

d) Let $f: X\to Y$ and $f': X'\to Y$ be generically smooth proper morphisms of complex varieties. Let $U\subseteq Y$ be a Zariski open subset over which both f and f' are smooth. Let $\omega\in\Gamma(Y, R^2f_*\boldsymbol{R})$ and $\omega'\in\Gamma(Y, R^2_*f'\boldsymbol{R})$ be fixed elements.

Definition 8. $\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}$ is the essential closure of $\mathrm{Isom}_U(X_U, X'_U)_{\omega_U,\omega'_U}$ in $\mathrm{Isom}^*_Y(X, X')$. We set $\mathrm{Aut}^*X_\omega=\mathrm{Isom}^*_Y(X, X)_{\omega,\omega}$.

Remark 5. $\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}$ and $\mathrm{Aut}^*_YX_\omega$ are unions of irreducible components of $\mathrm{Isom}^*_Y(X, X')$ and Aut^*_YX respectively (cf. Lemma 4).

Proposition 5. *Suppose that ω_U, ω'_U are relative Kähler classes, and that $f, f'\in\mathcal{C}/Y$. Then* $\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}$ *is proper over Y. Thus* $\mathrm{Aut}^*_YX_\omega$ *and* $\mathrm{Aut}^*_YX'_{\omega'}$ *are meromorphic subgroups of* Aut^*_YX *and* Aut^*_YX' *respectively and* $\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}$ *is a meromorphic principal subspace with the associated meromorphic subgroups* $\mathrm{Aut}^*_YX_\omega$ *and* $\mathrm{Aut}^*_YX'_{\omega'}$.

Proof. By Proposition 4 $\mathrm{Isom}_U(X_U, X'_U)_{\omega_U,\omega'_U}$ has only finitely many irreducible components, say $I_{1,U}, \cdots, I_{k,U}$, which are mapped surjectively onto U. Then $\mathrm{Isom}^*_Y(X, X)_{\omega,\omega'}$ is the union of the closures I_j of $I_{j,U}$. Since $f, f'\in\mathcal{C}/Y$, $f\times_Yf'\in\mathcal{C}/Y$, and hence each I_j are proper over Y by [4]. Thus the first assertion follows. The second assertion then follows readily from the definition of these spaces.

3.3. a) Let $f: X\to Y$, $f': X'\to Y$, $U\subseteq Y$, ω and ω' be as in Proposition 5. Let $A=(A_1, \cdots, A_m)$, $A'=(A'_1, \cdots, A'_m)$ be as in 3.1.

Definition 9. We set

$$\mathrm{Isom}^*_Y((X, A), (X', A'))_{\omega,\omega'}:=\mathrm{Isom}^*_Y(X, X')_{\omega,\omega'}\cap\mathrm{Isom}^*_Y((X, A), (X', A'))$$

and

$$\mathrm{Aut}^*_Y(X, A)_\omega:=\mathrm{Aut}^*_YX_\omega\cap\mathrm{Aut}^*_Y(X, A).$$

Remark 6. 1) $\mathrm{Isom}^*_Y((X, A), (X', A'))_{\omega,\omega'}$ is a meromorphic principal subspace with the associated meromorphic subgroups $\mathrm{Aut}^*_Y(X, A)_\omega$ and $\mathrm{Aut}^*_Y(X', A')_{\omega'}$.

2) There exists a Zariski open subset $U\subseteq Y$ such that

$$(\mathrm{Isom}^*_Y((X, A), (X', A'))_{\omega,\omega'})_y=\mathrm{Isom}^*((X_y, A_y), (X'_y, A_y))_{\omega_y,\omega'_y}$$

for any $y \in U$.

3) Let $\nu: \tilde{Y} \to Y$ be a surjective morphism of complex varieties. Let $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{A} = (A_1 \times_Y \tilde{Y}, \cdots, A_m \times_Y \tilde{Y})$. Let $\tilde{\omega}$ be the pull-back of ω to $\tilde{X}$. Then $\mathrm{Aut}_Y^*(X, A)_\omega \times_Y \tilde{Y} \cong \mathrm{Aut}_{\tilde{Y}}^*(\tilde{X}, \tilde{A})_{\tilde{\omega}}$ with respect to the natural isomorphism $\mathrm{Aut}^*{}_Y X \times_Y \tilde{Y} \cong \mathrm{Aut}_{\tilde{Y}}^* \tilde{X}$.

In fact, since $\mathrm{Isom}_Y^*(X, X')_{\omega, \omega'}$ is a union of irreducible components (Remark 5) it follows that $\mathrm{Isom}_Y^*((X, A), (X', A'))_{\omega, \omega'}$ is the essential closure of $\mathrm{Isom}_U(X_U, X'_U)_{\omega_U, \omega'_U} \cap \mathrm{Isom}_U((X_U, A_U), (X'_U, A'_U))$. From this together with Remark 4 and Proposition 5, 1) follows. 2) is standard (cf. 1.2 a)). For 3) it suffices to see that $\mathrm{Aut}_Y^*(X, A) \times_Y \tilde{Y} \cong \mathrm{Aut}_{\tilde{Y}}^*(\tilde{X}, \tilde{A})$ and $(\mathrm{Aut}_Y^* X_\omega) \times_Y \tilde{Y} \cong \mathrm{Aut}_{\tilde{Y}}^* \tilde{X}_{\tilde{\omega}}$. Since ν is surjective, this follows from the isomorphisms $\mathrm{Aut}_U(X_U, A_U) \times_U \tilde{U} \cong \mathrm{Aut}_{\tilde{U}}(\tilde{X}_{\tilde{U}}, \tilde{A}_{\tilde{U}})$ and $(\mathrm{Aut}_U X_U)_{\omega_U} \times_U \tilde{U} \cong (\mathrm{Aut}_{\tilde{U}} X_{\tilde{U}})_{\omega_{\tilde{U}}}$ where $\tilde{U} = \nu^{-1}(U)$.

b) Consider the special case where $X' = Y \times F$ for some compact complex variety F and $f': X' \to Y$ is the natural projection. Let $B = (B_1, \cdots, B_m)$ be a sequence of subspaces of F as in 3.1 c). Suppose that ω' is of the form $\omega' = p^* \omega_0$ for some Kähler class ω_0 on F where $p: X' \to F$ is the natural projection. Then:

Proposition 6. *If* $\mathrm{Isom}_Y^*((X, A), (X', A'))_{\omega, \omega'} \neq \emptyset$, *then* $f_{X,A}$ *is a holomorphic fiber bundle over* U *which is meromorphic with respect to* f *and with meromorphic structure group* $\mathrm{Aut}(F, B)_{\omega_0}$ *in the sence of* 3.1 c).

Proof. We have $\mathrm{Aut}_Y^*(X', A')_{\omega'} \cong Y \times \mathrm{Aut}^*(F, B)_{\omega_0}$ and hence $\mathrm{Isom}_Y^*((X, A), (X', A'))_{\omega, \omega'}$ is admissible. Thus the proposition follows from 3.1 c).

3.4. Let $f: X \to Y$ be a proper flat morphism of complex varieties. Let $\mathrm{Aut}_{Y,0} X$ be the unique irreducible component of $\mathrm{Aut}_Y X$ which contains the identity section $e(Y)$. Then it is easy to see that $\mathrm{Aut}_{Y,0} X$ is a relative complex Lie subgroup of $\mathrm{Aut}_Y X$.

Lemma 5. *Suppose that* $f \in \mathcal{C}/Y$. *Then there exists a Zariski open subset* $U \subseteqq Y$ *such that* $(\mathrm{Aut}_{Y,0} X)_y = \mathrm{Aut}_0 X_y$ *for each* $y \in U$ *where* $\mathrm{Aut}_0 X_y$ *is the identity component of* $\mathrm{Aut}\, X_y$.

Proof. Let $\mu: \mathrm{Aut}_{Y,0} X \to Y$ be the natural morphism. Let $r = \dim \mu$, and $V = \{y \in Y; \dim_{e(y)} \mu^{-1}(y) = r$, and Y is smooth at $y\}$. Then V is Zariski open in Y. Moreover μ is smooth at every point of $e(V)$ and hence $\mathrm{Aut}_{Y,0} X$ is smooth along $e(V)$. Let $A = \mathrm{Aut}_{Y,0} X$ and $n: \tilde{A} \to A$ the normalization. Since n is isomorphic along $e(V)$, e lifts to a meromorphic section $\tilde{e}$ to $\tilde{\mu}: \tilde{A} \to Y$. On the other hand, since $f \in \mathcal{C}/Y$, $\tilde{\mu}$ is proper [4]. Let $b: \tilde{A} \to \tilde{Y}$, $c: \tilde{Y} \to Y$ be the Stein factorization of $\tilde{\mu}$. Then $b\tilde{e}$ gives a meromorphic section to c. Hence the fiber of $\tilde{\mu}$ is connected. Since $\tilde{A}$ is normal, this implies that the general fiber of $\tilde{\mu}$, and hence of μ, is irreducible. Thus for general $y \in Y$, A_y is the closure of $\mathrm{Aut}_0 X_y$. Hence the assertion follows.

Let $f: X \to Y$ be a proper surjective morphism of complex varieties. Let

$U \subseteq Y$ be a Zariski open subset over which f is smooth. Then we denote by $\mathrm{Aut}^{\sharp}_{Y,0}X$ the closure of $\mathrm{Aut}_{U,0}X_U$ in $\mathrm{Aut}^{\sharp}_{Y}X$. This is independent of the choice of U as above. $\mathrm{Aut}^{\sharp}_{Y,0}X$ is a relative meromorphic subgroup if $f \in \mathcal{C}/Y$.

Proposition 7. *Let $f: X \to Y$ be a proper morphism of complex spaces. Let $U \subseteq Y$ be a Zariski open subset.* 1) *Suppose that f is smooth over U with each fiber a complex torus and that f admits a holomorphic section $e_U: U \to X_U$ on U. Then $f_U: X_U \to U$ has the unique structure of a complex Lie group over U with e_U the identity section.* 2) *Suppose further that X, Y are varieties, $f \in \mathcal{C}/Y$ and that e_U extends to a meromorphic section $e^*: Y \to X$. Then the group law of X_U over U extends meromorphically over Y.*

Proof. 1) Restricting the natural relative action $\sigma_U: (\mathrm{Aut}_{U,0}X_U) \times_U X_U \to X_U$ to $(\mathrm{Aut}_{U,0}X_U) \times_U e_U(U) \cong \mathrm{Aut}_{U,0}X_U$ we get an isomorphism $\eta_U: \mathrm{Aut}_{U,0}X_U \cong X_U$ (cf. Appendix). Hence 1) follows. (For the uniqueness see [11], Cor. 6.6.) 2) Similarly, restricting $\sigma^*: \mathrm{Aut}^{\sharp}_{Y,0}X \times_Y X \to X$ to $\mathrm{Aut}^{\sharp}_{Y,0}X \times_Y e(Y)$, which is bimeromorphic to $\mathrm{Aut}^{\sharp}_{Y,0}X$ we get a natural bimeromorphic map $\mathrm{Aut}^{\sharp}_{Y,0}X \to X$ extending η_U. Then 2) follows from Remark 1, 3). q. e. d.

3.5. In concluding this section, as an application of Theorem 2 combined with the consideration of this section, we shall prove a proposition which is used in [5].

Let $g: X \to Y$, $h: Y \to T$ be fiber spaces[3] of complex varieties. Let $A = (A_1, \cdots, A_m)$ be a sequence of analytic subspaces of X. Suppose that 1) there exist Zariski open subsets $U \subseteq T$, $V \subseteq Y$ with $h(V) \subseteq U$ such that for any $u \in U$, $g_u = g_{u, X_u, A_u}: (X_u, A_u) \to Y_u$ is a holomorphic fiber bundle over $V_u \subseteq Y_u$ which is meromorphic with respect to g_u (cf. 3.1, c)) and 2) there exists a holomorphic section $s: T \to Y$ with $s(T) \cap V \neq \emptyset$. Suppose further that g is Kähler (cf. [4]) so that in particular we can find a relative Kähler class $\omega \in \Gamma(Y, R^2 g_* \boldsymbol{R})$ over Y. Then by Proposition 6 if $s(u) \in V$ we can take $G^*(u) := \mathrm{Aut}^*(X_{s(u)}, A_{s(u)})_{\omega_{s(u)}}$ as a meromorphic structure group of g_u (considering $(X_{s(u)}, A_{s(u)})$ as a typical fiber of the bundle). Then we shall prove the following:

Proposition 8. *Under the above situation there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ a\ } & Z \\ {\scriptstyle g}\downarrow & h & \downarrow{\scriptstyle b} \\ Y & \xrightarrow{\quad} & T \end{array}$$

where a is a surjective meromorphic map and b is a fiber space of complex varieties, such that if we restrict U smaller, then for each $u \in U$, Z_u is a generic quotient $X_u/G^*(u)$ of X_u by $G^*(u)$ and a induces a canonical meromorphic map $a_u: X_u \to Z_u$ associated to g_u and $G^*(u)$ (cf. Def. 6).

3) A fiber space is a proper surjective morphism with general fiber irreducible.

Proof. Let $\hat{X} := X \times_Y T$ where T is over Y via s. Let $X' := \hat{X} \times_T Y$ and $g' : X' \to Y$ the natural map. Let $\hat{\omega}$ (resp. ω') be the pull-back of ω (resp. $\hat{\omega}$) to $\hat{X}$ (resp. X'). Then g' is a Kähler morphism with a relative Kähler class $\omega' \in \Gamma(Y, R^2 g'_* \boldsymbol{R})$. Let $\hat{A}_i := A_i \times_Y T \subseteq \hat{X}$ and $A'_i := \hat{A}_i \times_T Y \subseteq X'$. Let $I^* := \mathrm{Isom}_Y^\sharp((X, A), (X', A'))_{\omega, \omega'}$ where $A := (A_1, \cdots, A_m)$ and $A' := (A'_1, \cdots, A'_m)$. Then by Remark 6, 1) I^* is a principal meromorphic subspace to which $G^* := \mathrm{Aut}_Y^\sharp(X, A)_\omega$ and $G^{*\prime} := \mathrm{Aut}_Y^\sharp(X', A')_{\omega'}$ are associated. Let $\bar{X} = X/G^*$ (resp. $\bar{X}' = X'/G^{*\prime}$) be the relative generic quotient of X by G^* (resp. X' by $G^{*\prime}$) over Y. Then by Theorem 2 there exists a canonical bimeromorphic map $\eta : \bar{X} \to \bar{X}'$ over Y. On the other hand, by Remark 6, 3) $G^{*\prime} = \hat{G}^* \times_T Y$ where $\hat{G}^* := \mathrm{Aut}_T^\sharp(\hat{X}, \hat{A})_{\hat{\omega}}$, $\hat{A} = (\hat{A}_1, \cdots, \hat{A}_m)$. Further we have the natural meromorphic map $\pi : \bar{X}' \to \hat{X}/\hat{G}^*$ over T (cf. Proposition 1, 1)). Let $Z := \hat{X}/\hat{G}^*$ and define $a : X \to Z$ by the composite meromorphic map $\pi\eta q : X \to Z$ where $q : X \to \bar{X}$ is the quotient meromorphic map. Let $b : Z \to T$ be the natural surjective morphism. Then we have $hf = ba$. We claim that the resulting diagram meets the requirement of the proposition. In fact, restricting U smaller, we have that for each $u \in U$, G_u^* is a relative meromorphic subgroup of $\mathrm{Aut}_{Y_u}^\sharp X_u$ over Y_u and $\bar{X}_u = X_u/G_u^*$ (cf. Proposition 1, 2)), where X_u/G_u^* is a relative generic quotient of X_u by G_u^* over Y_u. Further we have $G_{s(u)}^* = G^*(u)$ and $Z_u := (\hat{X}/\hat{G}^*)_u \cong X_{s(u)}/G^*(u)$. Combining these facts we see readily from our construction that for sufficiently small U, the induced meromorphic map $a_u : X_u \to Z_u$ is a canonical meromorphic map associated to $G^*(u)$. q. e. d.

§ 4. $\mathbf{BHol}_Y^\sharp(X, X')$

a) Let $f : X \to Y$ and $f' : X' \to Y$ be proper flat morphisms of complex varieties. Let $\mathbf{Hol}_Y(X, X')$ be the contravariant functor $(\mathrm{An}/Y) \to (\mathrm{Sets})$ defined by $\mathbf{Hol}_Y(X, X')(\tilde{Y}) :=$ the set of $\tilde{Y}$-morphisms $\psi : X \times_Y \tilde{Y} \to X' \times_Y \tilde{Y}$. Then $\mathbf{Hol}_Y(X, X')$ is represented by a unique Zariski open subset $\mathrm{Hol}_Y(X, X')$ of the relative Douady space $D_{X \times_Y X'/Y}$ with $\mathrm{Isom}_Y(X, X') \subseteq \mathrm{Hol}_Y(X, X')$ (cf. [13]).

Suppose for simplicity that both f and f' are smooth with connected fibers. Let $\mathrm{BHol}_Y(X, X') := \bigcup_{y \in Y} \mathrm{BHol}_Y(X, X')_y$ where $\mathrm{BHol}_Y(X, X')_y := \{h \in \mathrm{Hol}_Y(X, X')_y ;$ $h(y)$ is bimeromorphic$\}$, where $h(y) : X_y \to X'_y$ is a morphism corresponding to h. Then $\mathrm{BHol}_Y(X, X')$ is Zariski open in $D_{X \times_Y X'/Y}$ (cf. [2], Lemma 5.5). We see that for any open subset $W \subseteq Y$ there is a natural bijective correspondence between the set of holomorphic sections of $\mathrm{BHol}_Y(X, X') \to Y$ on W and the set of bimeromorphic morphisms $X \to X'$ over W.

Let $A \subseteq X$ and $A' \subseteq X'$ be any analytic subspaces. Suppose that A' is flat over Y. Then the subfunctor $\mathbf{Hol}_Y((X, A), (X', A'))$ of $\mathbf{Hol}_Y(X, X')$ defined by $\mathbf{Hol}_Y((X, A), (X', A'))(\tilde{Y}) = \{\psi \in \mathbf{Hol}_Y(X, X')(\tilde{Y}) ; \psi(A) = A'\}$ is represented by a unique analytic subspace $\mathrm{Hol}_Y((X, A), (X', A'))$ of $\mathrm{Hol}_Y(X, X')$. This can be shown just in the same way as for Lemma 2. We set $\mathrm{BHol}_Y((X, A), (X', A'))$:

$=\mathrm{Hol}_Y((X, A), (X', A'))\cap \mathrm{BHol}_Y(X, X')$.

b) Let $f: X\to Y$ and $f': X'\to Y$ be generically smooth proper surjective morphisms of complex varieties with connected fibers. Let $U\subseteqq Y$ be a Zariski open subset over which both f and f' are smooth. Then $\mathrm{BHol}_U(X_U, X'_U)$ is Zariski open in $D_{X\times_Y X'/Y}\supseteqq D_{X_U\times_U X'_U/U}$. Let $\mathrm{BHol}^*_Y(X, X')$ be the essential closure (1.2 a)) of $\mathrm{BHol}_U(X_U, X'_U)$ in $D_{X\times_Y X'/Y}$ which is independent of the choice of U. Let $A\subseteqq X$ and $A'\subseteqq X$ be analytic subspaces. Restrict U smaller so that A' is flat over U. Then the closure of $\mathrm{BHol}_U((X_U, A_U), (X'_U, A'_U))$ is analytic in $D_{X\times_Y X'/Y}$ (cf. the proof of Lemma 3). We shall denote the essential closure of $\mathrm{BHol}_U((X_U, A_U), (X'_U, A'_U))$ in $D_{X\times_Y X'/Y}$ by $\mathrm{BHol}^*_Y((X, A), (X', A'))$.

Remark 7. 1) A bimeromorphic morphism $\psi: X_U\to X'_U$ defined on U extends to a bimeromorphic map $\psi^*: X\to X'$ over Y if and only if the corresponding holomorphic section $U\to \mathrm{BHol}_U(X_U, X'_U)$ extends to a meromorphic section $Y\to \mathrm{BHol}^*_Y(X, X')$.

2) If $\tilde{Y}\to Y$ is a surjective morphism of complex varieties, then it is immediate to see that $\mathrm{BHol}^*_Y(X, X')\times_Y\tilde{Y}\cong \mathrm{BHol}^*_{\tilde{Y}}(X\times_Y\tilde{Y}, X\times_Y\tilde{Y})$

3) If $f\in\mathcal{C}/Y$, then after replacing Y by any relatively compact open subset of Y any irreducible component of $\mathrm{BHol}^*_Y(X, X')$ (resp. $\mathrm{BHol}^*_Y((X, A), (X', A'))$ is proper over Y. In particular if X is compact, we need no restriction to a relatively compact subset.

c) We shall include a standard application of Remark 7, 3) as a reference to [5].

Let $f: X\to Y$ and $f': X'\to Y$ be surjective morphisms of compact complex varieties in $\mathcal{C}$. Let $U\subseteqq Y$ be a Zariski open subset over which both f and f' are smooth.

Proposition 9. 1) *Suppose that f and f' admit meromorphic sections $s: Y\to X$ and $s': Y\to X'$ respectively. Suppose further that there exists a U-isomorphism $\eta: X_U\to X'_U$ with $\eta s|_U=s'|_U$. Then if* $\mathrm{Aut}_0(X_u, s(u))=\{e\}$ *for all $u\in U$, then η extends to a bimeromorphic Y-map $\eta^*: X\to X'$.* 2) *Suppose that* $\mathrm{BHol}(X_u, X'_u)$ *(resp.* $\mathrm{Isom}(X_u, X'_u)$*) are nonempty and discrete for all $u\in U$. Then there exists a finite covering $\mu: \tilde{Y}\to Y$ such that $X\times_Y\tilde{Y}$ and $X'\times_Y\tilde{Y}$ is bimeromorphic over $\tilde{Y}$ by a bimeromorphic $\tilde{Y}$-map which is holomorphic (resp. isomorphic) over $\tilde{U}=\mu^{-1}(U)$.*

Proof. 1) Let $I^*=\mathrm{Isom}^*_Y((X, s(Y)), (X', s'(Y)))$ and $I_U=\mathrm{Isom}_U((X_U, s(U)), (X'_U, s'(U)))$. Then η defines a holomorphic section σ to $I_U\to U$. Let I^*_1 be the irreducible component of I^* containing $\sigma(U)$. Since $I_{U,u}=\mathrm{Isom}((X_u, s(u)), (X'_u, s'(u)))$ $\cong \mathrm{Aut}(X_u, s(u))$, from our assumption it follows that I^*_1 is discrete over U. Hence $I^*\to Y$ is generically finite so that it coincides with the closure of $\sigma(U)$. Namely, σ extends to a meromorphic section $Y\to I^*$. Hence the proposition follows from Remark 1, 2). 2) By our assumption we infer readily that there exists an irreducible component $\tilde{Y}$ of $\mathrm{BHol}_Y^*(X, X')$ such that $\tilde{Y} \cap \mathrm{BHol}_U(X_U, X'_U)$ is dense in $\tilde{Y}$ and the natural morphism $\mu: \tilde{Y} \to Y$ is generically finite and surjective. Let $\tilde{X} = X_{\tilde{Y}}$ and $\tilde{X}' = X'_{\tilde{Y}}$. Since $\tilde{Y} \times_Y \tilde{Y} \subseteq \mathrm{BHol}_Y^*(X, X') \times_Y \tilde{Y} \cong \mathrm{BHol}_{\tilde{Y}}^*(\tilde{X}, \tilde{X}')$, $\mathrm{BHol}_{\tilde{Y}}^*(\tilde{X}, \tilde{X}') \to \tilde{Y}$ admits a natural holomorphic section whose image over $\tilde{U}$ is in $\mathrm{BHol}_{\tilde{U}}(\tilde{X}_{\tilde{U}}, \tilde{X}'_{\tilde{U}})$. Hence $f_{\tilde{Y}}$ and $f'_{\tilde{Y}}$ are bimeromorphic. Let $\tilde{Y} \to \tilde{Y}_1 \to Y$ be the Stein factorization of μ. Then replacing $\tilde{Y}$ by $\tilde{Y}_1$ which is bimeromorphic to $\tilde{Y}$ we obtain 2). For Isom the proof is similar. q. e. d.

Remark 8. As is clear from the above proof the conclusion of 2) is true if there exists an analytic subset $\tilde{Y}' \subseteq \mathrm{BHol}_Y^*(X, X')$ (resp. $\mathrm{Isom}_Y^*(X, X')$) such that $\tilde{Y}' \cap \mathrm{BHol}_U(X_U, X'_U)$ (resp. $\tilde{Y}' \cap \mathrm{Isom}_U(X_U, X'_U)$) is dense in $\tilde{Y}'$ and that $\tilde{Y}'_y$, $y \in U$, is discrete. Moreover these results (Proposition 9 and this remark) are true even if the assumption is weakened to: $f, f' \in \mathcal{C}/Y$ (Y may not be compact), except that for 2) we have to replace Y by an arbitrary relatively compact open subset in the conclusion.

Appendix

In this appendix we shall summarize some well-known results on the automorphism group of a complex torus and its relative form.

a) Let T be a complex torus and $o \in T$ a fixed point. Then T has a unique structure of a complex Lie group with identify o. Then we can identify T with $\mathrm{Aut}_0 T$ naturally. Let $\Gamma = H_1(T, \boldsymbol{Z})$ and $H(T) \subseteq \mathrm{Aut}\, T$ the Lie subgroup of isomorphisms of T as a complex Lie group. We note that $H(T) = \mathrm{Aut}\,(T, \{0\})$. Then we have the exact sequence

$$0 \longrightarrow T \longrightarrow \mathrm{Aut}\, T \xrightarrow{\alpha} \mathrm{Aut}\, \Gamma$$

and if H is the image of α, then α induces an isomorphism $H(T) \cong H$. Hence we have the natural semi-direct product decomposition $\mathrm{Aut}\, T = T \cdot H(T)$.

b) Let $f: X \to Y$ be a proper smooth morphism of complex spaces (not necessarily reduced). Suppose that each fiber of f is a complex torus and f admits a holomorphic section $s: Y \to X$. Then X has a unique structure of a relative complex Lie group over Y. In fact we can identify X with $\mathrm{Aut}_{Y,0} X$ in the notation of 3.4 (cf. Proposition 7). Let $H_Y X$ be the relative complex Lie subgroup of $\mathrm{Aut}_Y X$ defined by $H_Y X = \mathrm{Aut}_Y(X, s(Y))$. Then we have $(H_Y X)_y = H(X_y)$ for each $y \in Y$. Let Γ_Y be the local system of abelian groups on Y defined by the presheaf $U \to H_1(X_U, \boldsymbol{Z})$ with U open subsets of Y. Let $r: \mathrm{Aut}_Y \Gamma_Y \to Y$ be the relative automorphism group of $\Gamma_Y \to Y$; r represents the functor $K: (\mathrm{An}/Y) \to (\mathrm{Sets})$ with $K(\tilde{Y})$ = the set of $\tilde{Y}$-automorphisms of $\Gamma \times_Y \tilde{Y}$. $\mathrm{Aut}_Y \Gamma_Y$ is a relative complex Lie group over Y with r locally biholomorphic. Then as in the absolute case we have the exact sequence

$$0 \longrightarrow X \longrightarrow \mathrm{Aut}_Y X \xrightarrow{\alpha_Y} \mathrm{Aut}_Y \Gamma_Y$$

of relative complex Lie groups in the sense that each map is a morphism of complex spaces over Y and induces an exact sequence of complex Lie groups on each fiber. Hence α_Y induces an isomorphism of $H_Y X$ with a relative subgroup of $\mathrm{Aut}_Y \Gamma_Y$, and we have the semi-direct product decomposition

$$\mathrm{Aut}_Y X = X \cdot H_Y X$$

over Y.

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