On a Compact Complex Manifold in C
without Holomorphic 2-Forms

By

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Introduction

Recall that a compact complex manifold X is said to be in C if it is a meromorphic image of a compact Kähler manifold (cf. [2]). Then by Chow's lemma [10] X is in fact a holomorphic image of a compact Kähler manifold. In this note we are concerned with the following:

Problem. Let X be a compact complex manifold in C. Suppose that there exists no nonzero holomorphic 2-form on X. Then is X Moishezon?

We shall obtain the following partial result (Proposition 2): For X as above let $f: X^* \to Y$ be (a holomorphic model of) an algebraic reduction of X. Then $a(f) = k(f) = 0$ where $a(f)$ (resp. $k(f)$) is the algebraic (resp. Kummer) dimension of f (cf. [6]). In particular the irregularity $q(X_y^*)$ of any smooth fiber X_y^* vanishes. Moreover X_y^* is not bimeromorphic to a K3 surface. The result is used in [6].

The arrangement of this note is as follows. We gather some preliminary material in Section 1. Also the relation of our problem with some fundameutal problems on the theory of compact Kähler manifolds and manifolds in C will be explained. In Section 2 we prove that a smooth fiber space of complex tori always admits a Kähler polarization, provided that it can be compactified to a morphism of compact complex manifolds in C. Finally in Section 3 we shall show Proposition 2 mentioned above using the results obtained in Section 2 and in [6] [7].

In this note complex manifolds are assumed to be paracompact and connected. For a surjective morphism $h: X \to Y$ of complex manifolds we shall write dim $h = \dim X - \dim Y$. A fiber space is a proper surjective morphism with connected fibers.

§ 1. Preliminaries

a) Let X be a compact complex manifold in C. Then the Hodge to de

Rham spectral sequence $E_1^{p,q} := H^q(X, \Omega_X^p) \Rightarrow H^k(X, C)$, $p+q=k$, degenerates and we have the Hodge decomposition

(*) $H^k(X, C) = \bigoplus_{p+q=k} H^{p,q}(X)$, $\bar{H}^{p,q}(X) = H^{q,p}(X)$

where $H^{p,q}(X) = F^p(H^k(X, C)) \cap \bar{F}^{k-p}(H^k(X, C))$ with F the Hodge filtration ($^-$denotes the complex conjugate) (cf. [3]). Then we set $h^{p,q}(X) := \dim H^{p,q}(X)$ $= \dim H^q(X, \Omega_X^p)$. The assumption of our problem is thus equivalent to: $h^{0,2}(X)$ $= h^{2,0}(X) = 0$. We further set $H^{s,s}(X, R) = H^{s,s}(X) \cap H^{2s}(X, R)$, $s \geq 0$. Let $H^{1,1}(X, Z) = j(H^2(X, Z)) \cap H^{1,1}(X, R)$ where $j: H^2(X, Z) \to H^2(X, R)$ is the natural homomorphism. Then as in the Kähler case $H^{1,1}(X, Z)$ is precisely the image of the *real* chern class map $c_1: H^1(X, \mathcal{O}_X^*) \to H^2(X, R)$ (cf. [4] § 1).

   b) We call a class $\omega \in H^{1,1}(X, R)$ (or $H^2(X, R)$) a *Kähler class* if it is represented by a Kähler form, i.e., the real closed (1,1)-form associated to a Kähler metric. We call a class $\Omega \in H^{s,s}(X, R)$ a *positive class* if it is represented by a positive form $\hat{\Omega}$ (in the sense of Stoll) (cf. [2] Def. 2.1). Here $\hat{\Omega}$ is said to be a *positive form* if for any $x \in X$ and any linearly independent vectors $e_1, \cdots, e_s \in T_x$, $\hat{\Omega}_x((\sqrt{-1})^{-s} e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_s \wedge \bar{e}_s) > 0$ where T_x is the tangent space of X at x. From the definition it follows readily that the exterior product $\omega^s = \omega \wedge \cdots \wedge \omega$ (s times) of a Kähler class ω is a positive class, that the set of positive classes form an open cone in $H^{s,s}(X, R)$, and also that a positive form in the sense of Stoll is a fortiori a positive form in the sense of Lelong [14]; $\hat{\Omega}$ is Lelong positive if $\hat{\Omega}_x((\sqrt{-1})^{-s} e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_s \wedge \bar{e}_s) \geq 0$ for any x and e_i as above. (In this note we call a positive form in the sense of Stoll simply a positive form.)

   c) We refer to Lelong ([14] p. 65 ff.) for the definition of a *positive current* on a complex manifold. We only recall that 1) a direct image of a positive current by a proper morphism is again a positive current, 2) a positive form in the sense of Lelong, and a fortiori a positive form in the sense of Stoll, is a positive current, and 3) a real closed current α of type (1, 1) is positive if and only if locally at each point it is written in the form $\alpha = dd^c \varphi$ where φ is a plurisubharmonic function and $d^c = \sqrt{-1}(\bar{\partial} - \partial)$ with $d = \partial + \bar{\partial}$ the type decomposition of d (a positive current is by definition real of type (1, 1)).

   Let α be a closed positive current of type (1, 1) on a complex manifold X. Let $S_\infty(\alpha) = \{x \in X; \varphi(x) = -\infty\}$ where $\alpha = dd^c \varphi$ in a neighborhood of x (the condition is independent of the choice of such a φ). Let A be a submanifold of X with the inclusion $\iota: A \to Z$. Then the restriction $\iota^* \alpha$ of α to A as a current is defined as follows; for any $x \in X$ we write $\alpha = dd^c \varphi$ as above in a neighborhood of x. Then $\iota^* \alpha = dd^c(\iota^* \varphi)$ in a neighborhood of x. Note that $\iota^* \alpha$ is again a positive current on A. Let $\bar{\alpha} \in H^{1,1}(X, R)$ be the class of α. Then the following is proved in [5] Lemma 2.2: If $A \nsubseteq S_\infty(\alpha)$, $\iota^* \bar{\alpha} \in H^{1,1}(A, R)$ is represented by $\iota^* \alpha$ in the sense defined above.

   d) Let $f: X \to Y$ be a surjective morphism of compact complex manifolds

in C. Then the induced morphism $f^*: H^k(Y, \boldsymbol{C}) \rightarrow H^k(X, \boldsymbol{C})$ is injective and is compatible with the Hodge decomposition (*). In fact, if X is Kähler with a Kähler class ω and if $r = \dim f$, then $f_* L^r$ gives (up to a constant factor) the left inverse to f^* where $f_*: H^{k+2r}(X, \boldsymbol{C}) \rightarrow H^k(Y, \boldsymbol{C})$ is the Gysin homomorphism and $L^r = \omega^r \wedge$ is the cup product operator with ω^r (cf. [3]). It follows in partic- ular that $f_*: H^{r+1, r+1}(X, \boldsymbol{R}) \rightarrow H^{1,1}(Y, \boldsymbol{R})$ is surjective.

On the other hand, let $U \subsetneqq Y$ be a Zariski open subset over which f is smooth. Then it is easy to see that for any closed positive form $\hat{\Omega}$ of type $(r+1, r+1)$ on X, its direct image $f_* \hat{\Omega}$ as a current is given by a Kähler form on U when restricted to U.

Remark. In the above notation let $\Omega \in H^{r+1, r+1}(X, \boldsymbol{R})$ be the class of $\hat{\Omega}$. Then $h_* \Omega \neq 0$ in $H^{1,1}(Y, \boldsymbol{R})$. In fact, since $h_* \Omega$ is represented by $h_* \hat{\Omega}$ which is a positive current, from the Hodge decomposition (*) for $k=1$ it follows (cf. [4] § 1) that if $h_* \Omega = 0$, then $h_* \hat{\Omega} = dd^c \varphi$ for some plurisubharmonic function φ on X. Then φ is constant since X is compact. Hence $h_* \hat{\Omega} = dd^c \varphi = 0$, a con- tradiction. In particular for any compact complex manifold Y in C, $h^{1,1}(Y) \neq 0$. (Take $f: X \rightarrow Y$ as above with X Kähler with a Kähler form $\hat{\omega}$ and set $\hat{\Omega} = \hat{\omega}^{r+1}$.)

e) Suppose that X is bimeromorphic to a compact Kähler manifold, say, X'. Then our problem is true for X. In fact, since $h^{0,2}(X') = h^{0,2}(X) = 0$, by a theorem of Kodaira [13] X' is projective. Hence X is Moishezon.

f) Let X be a reduced complex space. Then X is called a *Kähler space* if there exist an open covering $\{U_i\}$ of X and a system of strictly plurisubharmonic functions $\{\varphi_i\}$ with each φ_i defined on U_i such that $\varphi_i - \varphi_j$ is pluriharmoic on $U_i \cap U_j$, so that $\{dd^c \varphi_i\}$ defines a real closed $(1, 1)$-form on X. We call any such form a *Kähler form* on X. (For the more detail see [2].)

g) A possible approach to our problem would be the following. Take a compact Kähler manifold Z and a surjective morphism $h: Z \rightarrow X$. Let $r = \dim h$. Let ω be a Kähler class on Z and set $\Omega = \omega^{r+1}$. We have $h_* \Omega \in H^{1,1}(X, \boldsymbol{R})$. Under our condition that $h^{2,0}(X) = 0$, we have $H^2(X, \boldsymbol{R}) = H^{1,1}(X, \boldsymbol{R})$ so that $H^2(X, \boldsymbol{Q}) = H^{1,1}(X, \boldsymbol{Q})$ is dense in $H^{1,1}(X, \boldsymbol{R})$. On the other hand, since $h_*: H^{r+1, r+1}(Z, \boldsymbol{R}) \rightarrow H^{1,1}(X, \boldsymbol{R})$ is surjective (cf. d)), we can find a sequence $\{\Omega_n\}_{n=1,2,\ldots}$, $\Omega_n \in H^{r+1, r+1}(Z, \boldsymbol{R})$, converging to Ω (with respect to the standard topology of $H^{r+1, r+1}(X, \boldsymbol{R})$) such that $h_* \Omega_n \in H^{1,1}(X, \boldsymbol{Q})$. Since Ω is a positive class, Ω_n also is positive if n is sufficiently large (cf b)). Take and fix such an n. Let q be a positive integer such that $q h_* \Omega_n \in H^{1,1}(X, \boldsymbol{Z})$. Then we can find a holomorphic line bundle L on X such that $c_1(L) = q h_* \Omega_n$ (cf. a)). Then the problem would be to show that there are sufficiently many holomorphic sections to $L^{\otimes m}$ for sufficiently large $m > 0$. It would also be interesting to show that $c_1(L)^s > 0$ where $s = \dim X$.

h) Finally we shall explain how our problem is related to fundamental problems in the theory of compact Kähler manifolds and manifolds in C.

Problem. 1) (Hironaka [11]). Let $f: X \to Y$ be a surjective flat morphism of compact complex spaces. Suppose that X is Kähler. Then is Y again Kähler?

2) Let X be a compact complex manifold in C. Then is X bimeromorphic to a compact Kähler manifold?

As we have remarked in [2] (Remark 4.4) the following holds.

Lemma 1. *The affirmative answer to* 1) *implies that of* 2).

Proof. Let X be as in 2). Then we can find a compact Kähler manifold Z and a surjective morphism $h: Z \to X$. Take a flattening $\tilde{h}: \tilde{Z} \to \tilde{X}$ of h which is obtained by blowing up X (Hironaka [10]). Taking resolution we may assume that $\tilde{X}$ is non-singular. Then since the natural map $\tilde{Z} \to Z$ is projective, $\tilde{Z}$ again is a Kähler space (cf. [2]). Thus if 1) is true, then $\tilde{X}$ is a Kähler manifold, and so X is bimeromorphic to a compact Kähler manifold.   q. e. d.

Together with e) the lemma gives the implications: 1) $\to$ 2) $\to$ Our problem.

§2. Kählerian Polarization for a Smooth Fiber Space of Complex Tori

We shall begin with some lemmas.

Lemma 2. *Let* $f: X \to Y$, $g: Z \to Y$ *and* $h: Z \to X$ *be morphisms of compact complex manifolds in* C *with* $g = fh$. *Let* $r = \dim h$. *Let* Ω *be a positive class in* $H^{r+1, r+1}(Z, \mathbf{R})$. *Let* $\omega = h_* \Omega \in H^{1,1}(X, \mathbf{R})$. *Then there exists a dense subset* $V \subseteq Y$ *such that* $Y - V$ *is of Lebesgue measure zero and that* $h_{y*} \Omega_y = \omega_y$ *in* $H^{1,1}(X_y, \mathbf{R})$ *for any* $y \in V$.

Proof. Let $U \subseteq Y$ be a Zariski open subset over which both f and g are smooth. Let $\hat{\Omega}$ be a closed positive form representing Ω. Then the closed positive current $\hat{\omega} = h_* \hat{\Omega}$ represents the classe ω. Thus by Section 1, c) for any $y \in U$ the class ω_y is represented by the restriction $\hat{\omega}_y$ of $\hat{\omega}$ to X_y. Hence we have only to prove the existence of V as in the lemma such that $h_{y*} \hat{\Omega}_y = \hat{\omega}_y$ as a current on X_y for any $y \in V$. For this we use Federer's theory of slicing currents [1] as is presented in [9] and [12].

For $y \in Y$ let $\langle \hat{\Omega}, g, y \rangle$ (resp. $\langle \hat{\omega}, f, y \rangle$) be the slice of the current $\hat{\Omega}$ (resp. $\hat{\omega}$) at y by g (resp. f) (if it exists), which is a current of degree $2r+2$ (resp. 2) on Z (resp. X) with support contained in Z_y (resp. X_y). (See [12] Def. 2.3.2 for the precise definition.) Since $\hat{\Omega}$ is a C^∞ form and g is smooth over U, $\langle \hat{\Omega}, g, y \rangle$ in fact exists for any $y \in U$ and $\langle \hat{\Omega}, g, y \rangle = j_{y*} \hat{\Omega}_y$ (cf. [12] p. 197) where $j_y: Z_y \to Z$ is the inclusion, and $j_{y*} \hat{\Omega}_y$ is the direct image as a current. Then by [9] Lemma 1.19 (together with the partition of unity and [12] 2.3.3 (3)), $\langle \hat{\omega}, f, y \rangle$ also exists and we have $\langle \hat{\omega}, f, y \rangle = h_* \langle \hat{\Omega}, g, y \rangle$ for any $y \in U$. Now take a locally finite countable open covering $\{U_i\}$ of X_U such that 1) if $W_i = f(U_i)$ then $f|_{U_i}: U_i \to W_i$ is isomorphic to the projection $W_i \times G_i \to W_i$ where G_i is a

domain of C^s, $s=\dim f$, and 2) $\hat{\omega}=dd^c\varphi_i$ for some plurisubharmonic function φ_i on U_i. Consider then φ_i as a locally integrable function on X by defining $\varphi_i\equiv0$ outside U_i. (Passing to a suitable refinement of $\{U_i\}$ if necessary this is always achieved.) Then we can find a subset $V_i\subseteq U$ with $Y-V_i$ Lebesgue measure zero such that 1) for any $y\in V_i$ the slice $\langle\varphi_i, f, y\rangle$ of φ_i by f exists, 2) φ_{iy} is locally integrable on X_y and 3) $\langle\varphi_i, f, y\rangle=j'_{y*}\varphi_{iy}$ where $j'_y: X_y\to X$ is the natural inclusion and φ_{iy} is the restriction of φ_i to X_y as a function (cf. [9] 1.3.6). Further by [9] Lemma 1.18 (and its proof), for any $y\in V_i$ the slice $\langle dd^c\varphi_i, f, y\rangle$ exists and $\langle dd^c\varphi_i, f, y\rangle=dd^c\langle\varphi_i, f, y\rangle=j'_{y*}(dd^c\varphi_{iy})$. Thus we have $\langle\hat{\omega}, f, y\rangle=j'_{y*}\hat{\omega}_y$ on U_1. (Slice is local with respect to X [12] 2.3.3 (4).) Thus if we set $V=\cap_i V_i$, its complement is again of Lebesgue measure zero, and for any $y\in V$, $\langle\hat{\omega}, f, y\rangle=j'_{y*}\hat{\omega}_y$ on X. Thus for any $y\in V$, $j'_{y*}\hat{\omega}_y=h_*j_{y*}\hat{\Omega}_y=j'_{y*}h_{y*}\hat{\Omega}_y$ and hence $\hat{\omega}_y=h_{y*}\hat{\Omega}_y$ as was desired.   q. e. d.

Lemma 3. *Let $f: X\to Y$ be a smooth fiber space of complex manifolds with every fiber X_y a complex torus. Suppose that there exist a dense subset $W\subseteq Y$ and a class $\omega\in\Gamma(Y, R^2f_*\mathbf{R})$ such that $\omega_y\in H^2(X_y, \mathbf{R})$ is a Kähler class for any $y\in W$. Then ω_y is a Kähler class also for any $y\in Y$.*

Proof. For any $o\in Y$ take a contractible neighbourhood $o\in V$, a holomorphic section $h: V\to X$ to f over V, and a C^∞ family $\{g_y\}_{y\in V}$ of translation invariant Kähler metrics g_y on the fiber X_y of f over V. Then by Lemma in [15, p.196] we can find a real C^∞ closed 2-form β on X_V such that for any $y\in Y$ the restriction β_y of β to X_y represents the class $\omega_y\in H^2(X_y, \mathbf{R})$ and β_y is harmonic with respect to g_y. Since X_y is a complex torus, β_y is naturally identified with the induced Hermitian form B_y on T_y and ω_y is a Kähler class if and only if B_y is positive definite, where T_y is the tangent space of X_y at $h(y)$. Then since B_y depends differentiably on y and B_y is positive definite for $y\in W$, B_y is positive semidefinite everywhere. On the other hand, since $\omega_y^n\ne0$ for $y\in W$, $\omega_y^n\ne0$ for any $y\in Y$ where $n=\dim f$. The latter implies the nondegeneracy of B_y, $y\in V$. Hence B_y is positive definite for any $y\in V$. Since $o\in Y$ was arbitrary, ω_y is a Kähler class for any $y\in Y$.   q. e. d.

Lemma 4. *Let $\nu: \tilde{X}\to X$ be a generically finite and surjective morphism of compact complex manifolds in C. Let $f: \tilde{X}\to T$ be a surjective morphism of $\tilde{X}$ onto a complex torus T. Let $r=\dim f$. Then for any positive class $\Omega\in H^{r+1,r+1}(X, \mathbf{R})$, $\omega=f_*\nu^*\Omega$ is a Kähler class on T.*

Proof. Let α be the unique translation invariant closed (1, 1)-form belonging to the class ω. Then with respect to a suitable global coordinates on T, α can be written in the form $\alpha=\sqrt{-1}\sum_{i=1}^{n}e_i dz_i\wedge d\bar{z}_i$, where e_i are real numbers with $e_1\le\cdots\le e_n$ and $n=\dim T$. It suffices to show that $e_1>0$. Set β

$=(\sqrt{-1})^{n-1}\prod_{i=2}^{n}dz_i\wedge d\bar{z}_i$ and let ϕ be the corresponding cohomology class. Then

$(\omega\wedge\phi)[T]=\int_{T}\alpha\wedge\beta=e_1 C$ where $[T]$ denotes the evaluation on the fundamental

class of T and $C=(\sqrt{-1})^n\int_{T}\prod_{i=1}^{n}dz_i\wedge d\bar{z}_i$ is a positive constant. On the other

hand, by the definition of ω, $(\omega\wedge\phi)[T]=\int_{\tilde{X}}\nu^*\hat{\Omega}\wedge f^*\beta$ where $\hat{\Omega}$ is a positive form

representing the class Ω. Since $\hat{\Omega}$ is a positive form and at general points of $\tilde{X}$, ν is locally biholomorphic and f is locally a product, it is easy to see that the right-hand side is strictly positive. Hence $e_1 > 0$ as was desired.

<div align="right">q. e. d.</div>

Proposition 1. *Let* $f : X\rightarrow Y$ *be a fiber space of compact complex manifolds in* C. *Let* $U\subseteqq Y$ *be a Zariski open subset over which* f *is smooth. Suppose that* X_y *is a complex torus for any* $y\in U$. *Let* Z *be a compact Kähler manifold,* $h : Z\rightarrow X$ *a surjective morphism,* $\omega\in H^{1,1}(Z, \boldsymbol{R})$ *a Kähler class and* $\Omega=\omega^{r+1}$ *where* $r=\dim h$. *Let* $\bar{\omega}=h_*\Omega\in H^{1,1}(X, \boldsymbol{R})$. *Then the restriction* $\bar{\omega}_y\in H^{1,1}(X_y, \boldsymbol{R})$ *of* $\bar{\omega}$ *to* X_y *is a Kähler class for any* $y\in U$.

Proof. Let $g=fh : Z\rightarrow Y$. Let W be a Zariski open subset of Y over which both f and g are smooth. By Lemma 2 there exists a dense subset $V\subseteqq Y$ such that $\bar{\omega}_y=h_{y*}\Omega_y$ for all $y\in V$. Then by Lemma 4 for $y\in W\cap V$, $\bar{\omega}_y$ is a Kähler class. Hence by Lemma 3 $\bar{\omega}_y$ are Kähler classes for all $y\in U$.   q. e. d.

§3. Algebraic Reduction and Vanishing of $h^{2,0}$

In what follows for a compact complex manifold X we shall denote by $a(X)$ the algebraic dimension of X. Further a compact complex manifold X is said to be *Kummer* if it is bimeromorphic to the quotient variety T/G where T is a complex torus and $G\subseteqq\operatorname{Aut} T$ is a finite group. Here if, further, $\operatorname{codim} B\geqq 2$ where B is the analytic subset of those points $t\in T$ whose stabilizer G_t is nontrivial, then X is said to be *bimeromorphically quasi-hyperelliptic* (cf. [6]).

Let $f : X\rightarrow\bar{X}$ be a proper bimeromorphic morphism of complex manifolds. Then by [4, §1] we have a natural homomorphism $f_* : H^1(X, \mathcal{O}_X^*)\rightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}^*)$ such that $f_* f^*=$ identity and that the following diagram is commutative

$$
\begin{array}{ccc}
H^1(X, \mathcal{O}_X^*) & \xrightarrow{\ c_1\ } & H^2(X, \boldsymbol{R}) \\
f_*\downarrow & & f_*\downarrow \\
H^1(\bar{X}, \mathcal{O}_{\bar{X}}^*) & \xrightarrow{\ c_1\ } & H^2(\bar{X}, \boldsymbol{R})
\end{array}
$$

Lemma 5. *Let* $h : Z\rightarrow X$ *be a surjective morphism of compact complex manifolds. Suppose that there exist a positive class* $\Omega\in H^{r+1,r+1}(Z, \boldsymbol{R})$ *and a holomorphic line bundle* L *on* X *such that* $h_*\Omega=c_1(L)$ *in* $H^{1,1}(X, \boldsymbol{R})$. *1) If* X *is bimero-*

morphically quasi-hyperelliptic, then X is Moishezon. In particular X cannot be Kummer with $a(X)=0$ unless dim $X=0$. *2) If X is bimeromorphic to a K3 surface, then $a(X)>0$.*

Proof. 1) Suppose that X is bimeromorphic to T/G with T and G as above. Let $\varepsilon: T \to X$ be the corresponding meromorphic map. Let $u: \tilde{T} \to T$ be a proper modification with $\tilde{T}$ nonsingular such that $\bar{\varepsilon}=\varepsilon u: \tilde{T} \to X$ is holomorphic. Let $\tilde{Z}$ be a resolution of $Z \times_X \tilde{T}$ and $\varphi: \tilde{Z} \to Z$, $\tilde{h}: \tilde{Z} \to \tilde{T}$ the natural morphisms. We have $h\varphi=\bar{\varepsilon}\tilde{h}$. Let m be the degree of φ. Then we have $mh_*\Omega=h_*\varphi_*\varphi^*\Omega$ $=\bar{\varepsilon}_*\tilde{h}_*\varphi^*\Omega$. Let $\tilde{\omega}=\tilde{h}_*\varphi^*\Omega$ and $\omega=u_*\tilde{\omega}=(u\tilde{h})_*\varphi^*\Omega$. Then by Lemma 4 ω is a Kähler class on T. On the other hand, we have $c_1(L^{\otimes m})=\bar{\varepsilon}_*\tilde{\omega}$, and $\varepsilon^*L \cong u_*u^*\varepsilon^*L$ $\cong u_*\bar{\varepsilon}^*L$. Then $c_1(\varepsilon^*L^{\otimes m})$ is given by $u_*\bar{\varepsilon}^*\bar{\varepsilon}_*\tilde{\omega}$. We now show that $u_*\bar{\varepsilon}^*\bar{\varepsilon}_*\tilde{\omega}$ $=\sum_{g \in G} g^*\omega$. Take Zariski open subsets $U \subseteqq T$ and $V \subseteqq X$ such that codim $(T-U)$ $\geqq 2$, ε is holomorphic on U, $\varepsilon(U)=V$ and $\varepsilon|_U: U \to V$ is an unramified covering which is Galois with Galois group G. Then we have only to show the above equality on U. Let $\omega_U=\omega|_U$. Then we have $(u_*\bar{\varepsilon}^*\bar{\varepsilon}_*\tilde{\omega})|_U=(u_*u^*\varepsilon^*\varepsilon_*u_*\tilde{\omega})|_U$ $=\varepsilon^*\varepsilon_*\omega_U=\sum_{g \in G} g^*\omega_U=(\sum_{g \in G} g^*\omega)|_U$ as was desired. Since $\sum_{g \in G} g^*\omega$ is a Kähler class as well as ω this shows that ε^*L is ample on T. Hence T is projective and X is Moishezon. The final assertion follows from the fact that Kummer manifold of algebraic dimension zero is necessarily bimeromorphically quasi-hyperelliptic (cf. [6] Remark 6.1).

2) We assume that X is bimeromorphic to a K3 surface. Let $\bar{X}$ be the minimal model of X and $v: X \to \bar{X}$ the natural morphism. Then the line bundle v_*L has the chern class $v_*h_*\Omega$. Hence replacing X, L, $h_*\Omega$ by $\bar{X}$, v_*L and $(vh)_*\Omega$ if necessary we may assume that X is minimal. We shall derive a contradiction assuming that $a(X)=0$. Let $U \subseteqq X$ be a Zariski open subset of X over which h is smooth. Let $C=X-U$ and C_i, $1 \leqq i \leqq m$, the irreducible components of C. Then by Remark 1 of [7] we can find a Kähler class $\omega \in H^{1,1}(X, \boldsymbol{R})$ such that $\omega|_U=h_*\Omega|_U$. On the other hand, in the local cohomology exact sequence

$$\longrightarrow H^2_C(X, \boldsymbol{R}) \xrightarrow{\varphi} H^2(X, \boldsymbol{R}) \longrightarrow H^2(U, \boldsymbol{R}) \longrightarrow$$

the image of φ is generated by the chern classes of the line bundles $[C_i]$ defined by those C_i with dim $C_i=1$. Hence we can write $\omega=h_*\Omega+\sum_{i=1}^{m} r_i c_1([C_i])$ for some $r_i \in \boldsymbol{R}$. Then take $q_i \in \boldsymbol{Q}$ sufficiently near to r_i so that $\omega':=h_*\Omega+\sum_{i=1}^{m} q_i c_1([C_i])$ is still a Kähler class. Then if we take an integer $q>0$ such that qq_i are integers, then the line bundle $L^{\otimes q} \otimes \prod_{i=1}^{m} ([C_i])^{\otimes qq_i}$ is ample on X. This is a contradiction.   q. e. d.

Let $f: X \to Y$ be a fiber space of compact complex manifolds. Let $a(f)$ be the (relative) algebraic dimension of f; for an integer $k \geqq 0$, $a(f)=k$ if and only if $a(X_y)=k$ for 'general' $y \in Y$. Suppose that $a(f)=0$. Then the Kummer

dimension $k(f)$ of f is defined; for an integer $k \geqq 0$, $k(f) = k$ if and only if $a(X_y) = 0$ and $k(X_y) = k$ for 'general' $y \in Y$ where $k(X_y)$ is the Kummer dimension of X_y (cf. [6]). As in [6], here and in what follows 'for 'general' $y \in Y$' means that 'there exists a subset $V \subseteqq Y$ which is a complement of a union of at most countable proper analytic subvarieties of Y such that for $y \in V$'.

Poposition 2. *Let X be a compact complex manifold in C with $h^{0,2}(X) = 0$. Let $f : X^* \to Y$ be a holomorphic model of an algebraic reduction of X. Then $a(f) = k(f) = 0$. Further the general fiber X_y^* of f cannot be bimeromorphic to a K3 surface.*

Proof. Replacing X by X^* we may assume that $X = X^*$.

a) First we show that $a(f) = 0$. By [6, 9.4(7)] after passing to a suitable bimeromorphic model of f we have a decomposition $f = \gamma \alpha g$ of f into three fiber spaces $g : X \to \bar{X}$, $\alpha : \bar{X} \to A$, $\gamma : A \to Y$ such that $a(g) = 0$, that $\dim \gamma \alpha = 0$ if $\dim \gamma = 0$ and that the general fiber of γ is a complex torus. Thus it suffices to show that $\dim \gamma = 0$. So assuming that $\dim \gamma > 0$ we shall derive a contradiction. Since γ is an algebraic reduction of A and $0 = h^{0,2}(X) \geqq h^{0,2}(A) = 0$, replacing f by γ if necessary we may assume that the general fiber of f is a complex torus to get a contradiction. Let Z be a compact Kähler manifold, $h : Z \to X$ a surjective morphism, ω a Kähler class on Z and $\Omega = \omega^{r+1}$ where $r = \dim h$. Let L be a line bundle on X with $c_1(L) = q h_* \Omega_n$ with Ω_n, q as in Section 1, g). Then by Proposition 1 there exists a Zariski open subset $U \subseteqq Y$ such that X_y is a complex torus and $c_1(L)_y$ is a Kähler class on X_y for any $y \in U$, and hence $L_y := L|_{X_y}$ is ample for $y \in U$. This implies that f is Moishezon. Hence X is Moishezon as well as Y, contradicting our assumption that $\dim f > 0$ and the fact that f is an algebraic reduction.

b) Next we shall show that $k(f) = 0$. By [6, Prop. 2.3] we have a decomposition $f = g_2 g_1$, $g_1 : X \to B$, $g_2 : B \to Y$, of f where g_1 is a meromorphic fiber space and g_2 is a (holomorphic) fiber space such that $k(f) = \dim g_2$ and the general fiber of g_2 is a Kummer manifold (a relative Kummer reduction of f). It suffices to show that $\dim g_2 = 0$. So assuming that $\dim g_2 > 0$ we shall derive a contradiction. For this purpose replacing f by g_2 we may assume that $f = g_2$, so that the general fiber of f is a Kummer manifold. Let $h : Z \to X$, Ω_n, q and L be as in a). By Lemma 2 there exists a dense subset $V \subseteqq Y$ such that $Y - V$ is of Lebesgue measure zero and that for each $y \in V$ Z_y is smooth and $(h_* \Omega_n)_y = h_{y*} \Omega_{n,y}$ where $\Omega_{n,y}$ is the restriction to Ω_n to Z_y. Let $A = \{y \in Y ; f$ is not smooth along X_y or $a(X_y) > 0\}$. Then A is at most countable union of proper analytic subvarieties of Y (cf. [5]). Hence the Lebesgue measure of A is zero. Then there exists a point $y \in V$ such that X_y is a Kummer manifold with $a(X_y) = 0$. Then applying Lemma 5, 1) to $h_y : Z_y \to X_y$, $q \Omega_{n,y}$ and L_y we see that $\dim X_y = 0$. This contradicts our assumption that $\dim f > 0$. Finally the last assertion is also proved in the same way using 2) of Lemma 5 instead of

1).   q. e. d.

Recall that a compact complex manifold X is said to be *simple* if X admits no covering family $\{Z_t\}_{t \in T}$ of proper subvarieties Z_t with dim $Z_t > 0$ (cf. [8]).

Proposition 3. *Let X be a compact complex manifold in C with $h^{0,2}(X)=0$. Then $ca(X) \neq 1, 2$ and if $ca(X)=3$, then the general fiber of a holomorphic model $f : X^* \to Y$ of an algebraic reduction of X is a simple manifold with $k(X)=0$, where $ca(X)=$ dim $X - a(X)$ (the co-algebraic dimension of X).*

Proof. By Proposition 2 we have $a(f)=k(f)=0$. This is impossible if $ca(X)=1$. If $ca(X)=2$, this implies that X_y^* is bimeromorphic to a K3 surface. But this also is impossible by the final assertion of Proposition 2. When $ca(X)=3$, in view of [6] Theorem either X_y^* is simple or is a meromorphic P^1 fiber space over a K3 surface. In the latter case let $f=hg$, $g: X \to Z$, $h: Z \to Y$, be a relative semisimple reduction of f (cf. [8]). Then since the P^1-fibering structure on X_y^* is given by a semisimple reduction of X_y^* if $a(X_y^*)=0$ (cf. [6] [13] 2, b)), the general fiber of h is bimeromorphic to a K3 surface. Now $ca(Z)=2$ and $h^{0,2}(Z)=0$. Hence by what we have proved above we see that this case cannot occur.   q. e. d.

Corollary. *Let X be a compact complex manifold in C with dim $X=3$ and $h^{0,2}(X)=0$. Then X is either Moishezon or simple with $k(X)=0$.*

Finally we shall give a rough discussion on the remaining case of our problem. Let X be a compact complex manifold in C with $h^{2,0}(X)=0$ and dim $X>0$. Then we want to derive a contradiction assuming that X is not Moishezon. Passing to a suitable bimeromorphic model of X we may assume that an algebraic reduction $f : X \to Y$ of X is holomorphic and then that $a(f)=k(f)=0$ by Proposition 2. Next, as in the proof of Lemma 1, by blowing up X we may assume that there exists a compact Kähler space Z and a flat surjective morphism $h : Z \to X$. Then modifying the argument in Section 1, g) a little, we can find a positive form $\hat{\Omega}$ of type $(r+1, r+1)$, $r=$ dim h on Z and a holomorphic line bundle L on X such that $c_1(L)=[h_*\hat{\Omega}]$ where $[h_*\hat{\Omega}]$ is the class of the closed $(1, 1)$-current $h_*\hat{\Omega}$. Then by Lemma 2 for almost all y (with respect to Lebesgue measure) X_y is smooth, Z_y is reduced, $a(X_y)=k(X_y)=0$ and $[h_*\hat{\Omega}_y] = c_1(L_y)$. Then we would get a contradiction if dim $\Gamma(X_y, L_y^{\otimes m}) \geq 2$ for some $m>0$. Summarizing, our problem is true if the following is true: *Let $h : Z \to X$ be a flat surjective morphism of reduced compact complex spaces with X non-singular. Let $r=$ dim h and let $\hat{\Omega}$ be a positive form of type $(r+1, r+1)$ on Z. Let L be a line bundle on X with $c_1(L)=[h_*\hat{\Omega}]$. Then dim $\Gamma(X, L^{\otimes m})>1$ for some $m>0$.*

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Note added in proof. Problem in the introduction is true if, in our reduction of the problem in § 1, g), we can take dim $Z=$ dim X.