Publ. RIMS, Kyoto Uuiv.
19 (1983), 203-205

Remark to the Previous Paper "Ergodic Decomposition of Quasi-Invariant Measures"

Dedicated to Professor Hisaaki Yoshizawa on his 60th birthday

By

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In [1] the author derived a canonical decomposition of measures on $\boldsymbol{R}^\infty$ which was listed as Theorem 4.2. Under the same notations as in [1], it states that for any probability measure μ on $\mathfrak{B}(\boldsymbol{R}^\infty)$, there exist a family of tail-trivial probability measures $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}$ on $\mathfrak{B}(\boldsymbol{R}^\infty)$ and a measurable map p from $(\boldsymbol{R}^\infty, \mathfrak{B}_\infty)$ to $(\boldsymbol{R}^1, \mathfrak{B}(\boldsymbol{R}^1))$ such that $\mu(B\cap p^{-1}(E))=\int_E \mu^\tau(B)dp\mu(\tau)$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. Moreover if μ is $\boldsymbol{R}_0^\infty$-quasi-invariant, then $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}$ also can be chosen as $\boldsymbol{R}_0^\infty$-quasi-invariant measures. Starting from this fundamental fact, we proceeded to the following general problem. *Let $\boldsymbol{R}_0^\infty\subset\Phi\subset\boldsymbol{R}^\infty$, and Φ be a complete separable metric linear topological space whose topology is stronger than the usual topology of $\boldsymbol{R}^\infty$. If μ is Φ-quasi-invariant, then does the same hold for almost all μ^τ?* In the case that $\boldsymbol{R}_0^\infty$ is not dense in Φ, it was easily shown that this problem is negative in general. However in the case that $\boldsymbol{R}_0^\infty$ is dense in Φ, it was left as an open problem. In this paper we shall show that it is also negative, even if $\Phi=l^2$, by constructing a suitable μ.

First of all, we shall introduce some necessary notations for our discussions. For a general probability measure p on $\mathfrak{B}(\boldsymbol{R}^\infty)$, we put $p_t(B)=p(B-t)$ for all $t\in\boldsymbol{R}^\infty$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. And we call a set $T_p\equiv\{t\in\boldsymbol{R}^\infty | p_t$ is equivalent with $p\}$ the admissible set for p. Let g be the canonical Gaussian measure on $\mathfrak{B}(\boldsymbol{R}^\infty)$. That is, g is the product-measure of 1-dimensional Gaussian measures with mean 0 and variance 1. It is well known that $T_g=l^2$. (For example, see [2].) And let λ be the Lebesgue measure on the interval (0, 1]. We shall sometimes write $d\tau$ in place of $d\lambda$. Using indicator functions $\chi_{n,k}(\tau)$ of the intervals $\left(\frac{k-1}{n}, \frac{k}{n}\right]$ ($n=1, 2, \cdots, k=1, \cdots, n$), we define a map $\phi(\tau)=(\phi_h(\tau))_h$ from (0, 1] to $\boldsymbol{R}^\infty$, $\phi_h(\tau)=\{\sqrt{n}\,\chi_{n,k}(\tau)+1\}\tau$, if $h=2^{-1}n(n-1)+k$ $(1\leqq k\leqq n)$. It is easy to see that $\int_0^1 \phi_h^2(\tau)d\tau\leqq 4$. Hence,

Communicated by H. Araki, March 2, 1982.
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(1) for any fixed $a=(a_h)_h\in l^2$, $\sum_{h=1}^{\infty}a_h^2\phi_h^2(\tau)<\infty$ for λ-$a.e.\tau$.

Next using a map V_τ; $x=(x_h)_h\in\boldsymbol{R}^\infty\mapsto(\phi_h(\tau)^{-1}x_h)_h\in\boldsymbol{R}^\infty$, we consider a image measure $V_\tau g$ for each $\tau\in(0, 1]$. The admissible set of $V_\tau g$ becomes, $T_{V_\tau g}=V_\tau T_g=V_\tau l^2=\{x\in\boldsymbol{R}^\infty\mid\sum_{h=1}^{\infty}\phi_h^2(\tau)x_h^2<\infty\}\subset l^2$. Therefore from (1),

(2) for any fixed $a=(a_h)_h\in l^2$, $a\in T_{V_\tau g}$ holds for λ-$a.e.\tau$.

However $\{\phi_h(\tau)\}_h$ is not a bounded sequence, so

(3) for any $\tau\in(0, 1]$, $T_{V_\tau g}\subsetneqq l^2$.

Now consider a measure μ defined by $\mu(B)=\int_0^1 V_\tau g(B)d\tau$. We shall derive a canonical decomposition of μ. Take an arbitrary $\tau\in(0, 1]$ and fix it. Then for each n, there exists a unique k_n which satisfies $\chi_{n,k_n}(\tau)=1$. Put $h_n=2^{-1}n(n-1)+k_n$. Applying the law of large numbers for g, we have

$$\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n+1)}^2}{2^{-1}n(n+1)}=1 \quad\text{and}\quad \lim_{n\to\infty}\frac{x_{h_1}^2+\cdots+x_{h_n}^2}{2^{-1}n(n+1)}=0 \quad\text{for } g\text{-}a.e.x.$$

Thus, $\displaystyle\lim_{n\to\infty}\frac{(\sqrt{1}+1)^2x_{h_1}^2+\cdots+(\sqrt{n}+1)^2x_{h_n}^2}{2^{-1}n(n+1)}=0$ for $V_\tau g$-$a.e.x$.

It follows that

$$(4)\quad \lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n+1)}^2}{2^{-1}n(n+1)}=\tau^{-2} \quad\text{for } V_\tau g\text{-}a.e.x.$$

Define $p(x)=\displaystyle\lim_{n\to\infty}\left\{\frac{x_1^2+\cdots+x_{2^{-1}n(n+1)}^2}{2^{-1}n(n+1)}\right\}^{-1/2}$, if the limit exists, and $p(x)=0$, otherwise. Then we have $p(x)=\tau$ for $V_\tau g$-$a.e.x$, equivalently $V_\tau g(p^{-1}(E))=\chi_E(\tau)$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$. As $V_\tau g$ is a measure of product-type, so it is tail-trivial. Hence we have $\mu(B\cap p^{-1}(E))=\int_E V_\tau g(B)d\tau$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, and we have reached a canonical decomposition $[V_\tau g, p]$ of μ. Now, from (2) it is obvious that μ is l^2-quasi-invariant (in fact, $T_\mu=l^2$) ,while (3) shows that $V_\tau g\equiv\mu^\tau$ is not l^2-quasi-invariant for any $\tau\in(0, 1]$. Therefore by Theorem 4.3 in [1] (corresponding to the uniqueness of canonical decompositions), there does not exist any canonical decompositions of μ whose factor measures are almost all l^2-quasi-invariant. Moreover by Proposition 5.5 in [1], μ can never be written as a superposition of l^2-quasi-invariant and l^2-ergodic measures.

Finally, we shall add some arguments concerning with a continuity of characteristic function $\hat{\nu}(a)=\int\exp(i\langle x, a\rangle)d\nu(x)$ of a probability measure ν on $\mathfrak{B}(\boldsymbol{R}^\infty)$ and the canonical decomposition $[\nu^\tau, q]$. It is interesting to observe whether a continuity of $\hat{\nu}$ will be transmitted to corresponding $\widehat{\nu^\tau}$s. However it is also false in general. Such a counte-rexample is constructed in a similar manner.

This time, we put $\phi_h(\tau)=\{n^{1/4}\chi_{n,k}(\tau)+1\}\tau$, S_τ; $x=(x_h)_h\in \boldsymbol{R}^\infty \mapsto (x_h\phi_h(\tau))_h\in \boldsymbol{R}^\infty$, and $\nu(B)=\int_0^1 S_\tau g(B)d\tau$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. Then we can derive that $\nu=[S_\tau g, q]$ by a similar argument to the previous one, where $q(x)=\left\{\lim_{n\to\infty}\frac{x_1^2+\cdots+x^2_{2^{-1}n(n+1)}}{2^{-1}n(n+1)}\right\}^{1/2}$, if the limit exists and $q(x)=0$, otherwise. We have $\widehat{S_\tau g}(a)=\exp(-1/2\sum_{h=1}^\infty a_h^2\phi_h^2(\tau))$ for all $a\in\boldsymbol{R}_0^\infty$. As $\sup_h \phi_h(\tau)=\infty$, so $\widehat{S_\tau g}$ is not l^2-continuous. While, $|1-\hat{\nu}(a)| =\int_0^1 \{1-\exp(-1/2\sum_{h=1}^\infty a_h^2\phi_h^2(\tau))\}\,d\tau\leqq 1/2\int_0^1 \sum_{h=1}^\infty a_h^2\phi_h^2(\tau)d\tau\leqq 2\sum_{h=1}^\infty a_h^2$. Hence $\hat{\nu}$ is l^2-continuous.

References

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[2] Umemura, Y., Measures on infinite dimensional vector spaces, *ibid.*, **1** (1965), 1-47.