

A Cut-Free Sequential System for the Propositional Modal Logic of Finite Chains

By

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Abstract

The main purpose of this paper is to give a cut-free Gentzen-type sequential system for **K4.3G** of finite chains. The cut-elimination theorem is proved both model-theoretically and proof-theoretically.

§1. Introduction

There are thousands of modal logics, only a bit of which enjoy Gentzen-type sequential formulations. Modal logics with cut-free sequential systems are even fewer and it is often a challenging problem to find out such pleasant formulations to a given modal logic. See Zeman [7] for the general reference and Sato [5] for an example of recent such attempts. The main purpose of this paper is to give a cut-free sequential system for **K4.3G** of Gabbay [2, §25].

Formulas (of **K4.3G**) are constructed from propositional variables p and \perp (falsity) by using \supset (implication) and \Box (necessity). Other connectives like \wedge (conjunction), \vee (disjunction) and \neg (negation) can be introduced as defined symbols in the usual manner. A *structure* (for **K4.3G**) is a quadruple (S, R, s_0, D_s) , where

- (1) S is a nonempty finite set.
- (2) R is an irreflexive transitive binary relation on S such that either xRy or yRx for any distinct $x, y \in S$.
- (3) $s_0 \in S$.
- (4) For any $s \in S$, D_s assigns a truth-value 0 or 1 to every propositional variable.

Given a structure (S, R, s_0, D_s) , the truth-value $\|A\|_s$ of a formula A at $s \in S$ is defined inductively as follows:

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- (1) $\|p\|_s = D_s(p)$ for any propositional variable p .
- (2) $\|\perp\|_s = 0$.
- (3) $\|A \supset B\|_s = 1$ iff $\|A\|_s = 0$ or $\|B\|_s = 1$.
- (4) $\|\Box A\|_s = 1$ iff for any $t \in S$ such that sRt , $\|A\|_t = 1$.

If $\|A\|_{s_0} = 1$ for any structure (S, R, s_0, D_s) , then A is called *valid*, notation: $\models A$.

K4.3G can be axiomatized by the classical propositional calculus plus the following axioms and inference rules.

- (A1) $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- (A2) $\Box A \supset \Box \Box A$
- (A3) $\Box(\Box A \supset A) \supset \Box A$
- (A4) $\Box(\Box A \supset B) \vee \Box(B \wedge \Box B \supset A)$
- (R1) $\frac{A}{\Box A}$

We write $\vdash_{\mathbf{K4.3G}} A$ if A is provable in the above formal system.

Theorem 1.1. *For any formula A , $\vdash_{\mathbf{K4.3G}} A$ iff $\models A$.*

In the next section we present our sequential system **SK4.3G** and establish its cut-freeness semantically while its purely syntactic proof is given in Section 3. Finally we admit that this paper was inspired by a cut-free sequential system of Leivant [4] for the modal logic **K4G** of finite partial orders but its subtle error in the proof of the cut-elimination theorem is corrected in our more general context.

§ 2. Cut-free System for K4.3G

A *sequent* is an ordered pair (Γ, \mathcal{A}) of (possibly empty) finite sets of formulas, which we usually denote by $\Gamma \rightarrow \mathcal{A}$. We use such self-explanatory notations as $A, \Gamma \rightarrow \mathcal{A}, B$ for $\{A\} \cup \Gamma \rightarrow \mathcal{A} \cup \{B\}$ and $\Box \Gamma$ for $\{\Box A : A \in \Gamma\}$ freely.

Our sequential formal system **SK4.3G** (“S” for “Sequential”) consists of the following axioms and inference rules:

Axioms: $A \rightarrow A$
 $\perp \rightarrow$

Rules: $\frac{\Gamma \rightarrow \mathcal{A}}{\Pi, \Gamma \rightarrow \mathcal{A}, \mathcal{A}}$ (thin)
 $\frac{\Gamma \rightarrow \mathcal{A}, A \quad B, \Pi \rightarrow \mathcal{A}}{A \supset B, \Gamma, \Pi \rightarrow \mathcal{A}, \mathcal{A}}$ ($\supset L$)
 $\frac{A, \Gamma \rightarrow \mathcal{A}, B}{\Gamma \rightarrow \mathcal{A}, A \supset B}$ ($\supset R$)

$$\frac{\{\Gamma, \Box\Gamma, \Box\Pi \rightarrow \Pi, \Box A: \Pi \cup A = \Delta, \Pi \cap A = \emptyset \text{ and } \Pi \neq \emptyset\}}{\Box\Gamma \rightarrow \Box\Delta} \quad (\text{GL4.3})$$

, where $\Delta \neq \emptyset$ in (GL4.3).

It is easy to show that the following rules are admissible in **SK4.3G**.

$$\begin{array}{c} \frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta} \quad (\neg\text{L}) \\ \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A} \quad (\neg\text{R}) \\ \left. \begin{array}{c} \frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \\ \frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \end{array} \right\} \quad (\wedge\text{L}) \\ \frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B} \quad (\wedge\text{R}) \\ \frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta} \quad (\vee\text{L}) \\ \left. \begin{array}{c} \frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B} \\ \frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \vee B} \end{array} \right\} \quad (\vee\text{R}) \end{array}$$

If $\Gamma \rightarrow \Delta$ is provable in **SK4.3G**, we write $\vdash_{\text{SK4.3G}} \Gamma \rightarrow \Delta$. We notice that the rule (GL4.3) has the variable number of upper sequents, depending on the number $|\Delta|$. If $|\Delta|=1$, our rule (GL4.3) degenerates into the rule (GL) of Leivant [4].

$$\frac{\Gamma, \Box\Gamma, \Box A \rightarrow A}{\Box\Gamma \rightarrow \Box A} \quad (\text{GL})$$

If $|\Delta|=2$, then the rule (GL4.3) goes as follows:

$$\frac{\Gamma, \Box\Gamma, \Box A, \Box B \rightarrow A, B \quad \Gamma, \Box\Gamma, \Box A \rightarrow A, \Box B \quad \Gamma, \Box\Gamma, \Box B \rightarrow \Box A, B}{\Box\Gamma \rightarrow \Box A, \Box B}$$

To deepen the reader's understanding of the rule (GL4.3), we shall show that $\vdash_{\text{SK4.3G}} \Box(\Box A \supset B) \vee \Box(B \wedge \Box B \supset A)$.

We have the following proof π_1 of the sequent $\Box(\Box A \supset B), \Box(B \wedge \Box B \supset A) \rightarrow \Box A \supset B, B \wedge \Box B \supset A$.

$$\frac{\frac{\frac{B \rightarrow B}{\Box(\Box A \supset B), \Box(B \wedge \Box B \supset A), \Box A, B \rightarrow A, B} \quad (\text{thin})}{\Box(\Box A \supset B), \Box(B \wedge \Box B \supset A), \Box A, B \wedge \Box B \rightarrow A, B} \quad (\wedge\text{L})}{\Box(\Box A \supset B), \Box(B \wedge \Box B \supset A), \Box A \rightarrow B, B \wedge \Box B \supset A} \quad (\supset\text{R})}{\Box(\Box A \supset B), \Box(B \wedge \Box B \supset A) \rightarrow \Box A \supset B, B \wedge \Box B \supset A} \quad (\supset\text{R})$$

We have the following proof π_2 of the sequent $\Box(\Box A \supset B) \rightarrow \Box A \supset B, \Box(B \wedge \Box B \supset A)$.

$$\frac{\frac{\frac{A \rightarrow A \quad (\text{thin})}{\Box(\Box A \supset B), \Box A \supset B, \Box A, A, \Box(B \wedge \Box B \supset A), B \wedge \Box B \rightarrow A} \quad (\supset R)}{\Box(\Box A \supset B), \Box A \supset B, \Box A, A, \Box(B \wedge \Box B \supset A) \rightarrow B \wedge \Box B \supset A} \quad (\supset R)}{\frac{\frac{\Box(\Box A \supset B), \Box A \rightarrow \Box(B \wedge \Box B \supset A)}{\Box(\Box A \supset B), \Box A \rightarrow B, \Box(B \wedge \Box B \supset A)} \quad (\text{thin})}{\Box(\Box A \supset B) \rightarrow \Box A \supset B, \Box(B \wedge \Box B \supset A)} \quad (\supset R)} \quad (\text{GL4.3})$$

We have the following proof π_3 of the sequent $\Box(B \wedge \Box B \supset A) \rightarrow \Box(\Box A \supset B), B \wedge \Box B \supset A$.

$$\frac{\frac{\frac{B \rightarrow B \quad (\text{thin})}{\Box(B \wedge \Box B \supset A), B \wedge \Box B \supset A, \Box B, B, \Box(\Box A \supset B), \Box A \rightarrow B} \quad (\supset R)}{\Box(B \wedge \Box B \supset A), B \wedge \Box B \supset A, \Box B, B, \Box(\Box A \supset B) \rightarrow \Box A \supset B} \quad (\supset R)}{\frac{\frac{\Box(B \wedge \Box B \supset A), \Box B \rightarrow \Box(\Box A \supset B)}{\Box(B \wedge \Box B \supset A), \Box B \rightarrow \Box(\Box A \supset B), A} \quad (\text{thin})}{\Box(B \wedge \Box B \supset A), B \wedge \Box B \rightarrow \Box(\Box A \supset B), A} \quad (\wedge L)} \quad (\text{GL4.3})}{\Box(B \wedge \Box B \supset A) \rightarrow \Box(\Box A \supset B), B \wedge \Box B \supset A} \quad (\supset R)$$

Therefore

$$\frac{\left. \left. \left. \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} \pi_1 \quad \left. \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} \pi_2 \quad \left. \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} \pi_3 \right.}{\frac{\rightarrow \Box(\Box A \supset B), \Box(B \wedge \Box B \supset A)}{\rightarrow \Box(\Box A \supset B) \vee \Box(B \wedge \Box B \supset A)} \quad (\vee R)} \quad (\text{GL4.3})$$

A sequent $\Gamma \rightarrow \Delta$ is called *realizable* if for some structure (S, R, s_0, D_s) , $\|A\|_{s_0} = 1$ for any $A \in \Gamma$ and $\|B\|_{s_0} = 0$ for any $B \in \Delta$. A sequent $\Gamma \rightarrow \Delta$ which is not realizable is called *valid*, notation: $\models \Gamma \rightarrow \Delta$.

Theorem 2.1. (Soundness Theorem). *For any sequent $\Gamma \rightarrow \Delta$, if $\vdash_{\text{SK4.3G}} \Gamma \rightarrow \Delta$, then $\models \Gamma \rightarrow \Delta$.*

Proof. By induction on a proof of $\Gamma \rightarrow \Delta$.

Corollary 2.2. (Consistency). *The empty sequent \rightarrow is not provable in SK4.3G.*

Theorem 2.3. (Completeness Theorem). *For any sequent $\Gamma \rightarrow \Delta$, if $\models \Gamma \rightarrow \Delta$, then $\vdash_{\text{SK4.3G}} \Gamma \rightarrow \Delta$.*

Proof. Let $\Gamma \rightarrow \Delta$ be the given sequent. We denote by Ω the set of all subformulas occurring in a formula of $\Gamma \cup \Delta$. A sequent $\Pi \rightarrow \Lambda$ is called Ω -saturated if it satisfies the following conditions:

- (1) $\not\vdash_{\text{SK4.3G}} \Pi \rightarrow \Lambda$.
- (2) $\Pi \cup \Lambda \subseteq \Omega$.

(3) For any $A \in \Omega - (II \cup A)$, $\vdash_{\text{SK4.3G}} \Pi \rightarrow A$, A and $\vdash_{\text{SK4.3G}} A, II \rightarrow A$.

Assuming that $\nVdash_{\text{SK4.3G}} \Gamma \rightarrow \mathcal{A}$, we shall show that $\nVdash \Gamma \rightarrow \mathcal{A}$. We denote by $W(\Omega)$ the set of all Ω -saturated sequents. Since $\nVdash_{\text{SK4.3G}} \Gamma \rightarrow \mathcal{A}$, the sequent $\Gamma \rightarrow \mathcal{A}$ can be extended to some $\Gamma_0 \rightarrow \mathcal{A}_0 \in W(\Omega)$. For any set Σ of formulas, $(\Sigma)_{\square}$ denotes the set of all formulas A such that $\square A \in \Sigma$. If $(\mathcal{A}_0)_{\square} = \emptyset$, then let $S = \{\Gamma_0 \rightarrow \mathcal{A}_0\}$. If $(\mathcal{A}_0)_{\square} \neq \emptyset$, $\nVdash_{\text{SK4.3G}} \square(\Gamma_0)_{\square} \rightarrow \square(\mathcal{A}_0)_{\square}$. Therefore, taking the rule (GL4.3) into consideration, there exist two sets Σ_1, Σ_2 of formulas such that:

- (1) $\Sigma_1 \neq \emptyset$.
- (2) $\Sigma_1 \cup \Sigma_2 = (\mathcal{A}_0)_{\square}$.
- (3) $\Sigma_1 \cap \Sigma_2 = \emptyset$.
- (4) $\nVdash_{\text{SK4.3G}} (\Gamma_0)_{\square}, \square(\Gamma_0)_{\square}, \square \Sigma_1 \rightarrow \Sigma_1, \square \Sigma_2$.

The sequent $(\Gamma_0)_{\square}, \square(\Gamma_0)_{\square}, \square \Sigma_1 \rightarrow \Sigma_1, \square \Sigma_2$ can be extended to some $\Gamma_1 \rightarrow \mathcal{A}_1 \in W(\Omega)$. We notice that:

- (1) $(\Gamma_0)_{\square} \subset (\Gamma_1)_{\square}$ (By \subset we denote the proper inclusion).
- (2) $(\Gamma_0)_{\square} \subseteq \Gamma_1$.
- (3) $(\mathcal{A}_0)_{\square} \subseteq \mathcal{A}_1 \cup (\mathcal{A}_1)_{\square}$.

If $(\mathcal{A}_1)_{\square} = \emptyset$, then we let $S = \{\Gamma_0 \rightarrow \mathcal{A}_0, \Gamma_1 \rightarrow \mathcal{A}_1\}$. If $(\mathcal{A}_1)_{\square} \neq \emptyset$, we repeat the above process. In any case we finally obtain a sequence $\{\Gamma_i \rightarrow \mathcal{A}_i\}_{i=0}^k$ of $W(\Omega)$ such that:

- (1) $(\Gamma_0)_{\square} \subset (\Gamma_1)_{\square} \subset \dots \subset (\Gamma_k)_{\square}$.
- (2) $(\Gamma_i)_{\square} \subseteq \Gamma_{i+1}$ for any $0 \leq i < k$.
- (3) $(\mathcal{A}_i)_{\square} \subseteq \mathcal{A}_{i+1} \cup (\mathcal{A}_{i+1})_{\square}$ for any $0 \leq i \leq k-1$.
- (4) $(\mathcal{A}_i)_{\square} \neq \emptyset$ for any $i < k$ and $(\mathcal{A}_k)_{\square} = \emptyset$.

Set $S = \{\Gamma_0 \rightarrow \mathcal{A}_0, \dots, \Gamma_k \rightarrow \mathcal{A}_k\}$. We let $(\Gamma_i \rightarrow \mathcal{A}_i)R(\Gamma_j \rightarrow \mathcal{A}_j)$ iff $i < j$. Let $s_0 = \Gamma_0 \rightarrow \mathcal{A}_0$. We define $D_{\Gamma_i \rightarrow \mathcal{A}_i}$ as follows:

$D_{\Gamma_i \rightarrow \mathcal{A}_i}(p) = 1$ iff $p \in \Gamma_i$ for any propositional variable p .

It is not difficult to show by induction on $A \in \Omega$ that for any $0 \leq i \leq k$,

- (1) $\|A\|_{\Gamma_i \rightarrow \mathcal{A}_i} = 1$ if $A \in \Gamma_i$.
- (2) $\|A\|_{\Gamma_i \rightarrow \mathcal{A}_i} = 0$ if $A \in \mathcal{A}_i$.

In particular, we can conclude that $\Gamma_0 \rightarrow \mathcal{A}_0$ is realizable and so is $\Gamma \rightarrow \mathcal{A}$. This completes the proof.

Corollary 2.4. For any formula A , $\vdash_{\text{K4.3G}} A$ iff $\vdash_{\text{SK4.3G}} \rightarrow A$.

Corollary 2.5. The following inference rule is admissible in **SK4.3G**.

$$\frac{\Gamma \rightarrow \mathcal{A}, A \quad A, II \rightarrow \mathcal{A}}{\Gamma, II \rightarrow \mathcal{A}, A} \quad (\text{cut})$$

$$\frac{\frac{\frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\frac{\Box \Gamma, \Gamma, B \rightarrow B}{\Box \Gamma \rightarrow \Box B}} \text{ (GL4.3)}$$

$$\frac{\Box \Gamma, \Gamma, B \rightarrow B}{\Box \Gamma \rightarrow \Box B} \text{ (thin L)}$$

$$\frac{\Box \Gamma, \Gamma, B \rightarrow B}{A, \Box \Gamma \rightarrow \Box B}$$

If A is not of the form $\Box C$, then (thin L) degenerates into (thin) $_{\Box}$ which is of course admissible. If A is of the form $\Box C$, the above proof figure is transformed into:

$$\frac{\frac{\frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\frac{\Box \Gamma, \Gamma, B \rightarrow B}{\Box C, \Box \Gamma, \Gamma, \Box B \rightarrow B}} \text{ (thin L)}$$

$$\frac{\Box \Gamma, \Gamma, B \rightarrow B}{\Box C, \Box \Gamma, \Gamma, \Box B \rightarrow B} \text{ (thin L)}$$

$$\frac{\Box C, C, \Box \Gamma, \Gamma, \Box B \rightarrow B}{\Box C, \Box \Gamma \rightarrow \Box B} \text{ (GL4.3)}$$

Therefore the induction process works well.

Lemma 3.3. *The following rule (thin R) is admissible in SK4.3G'.*

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A} \text{ (thin R)}$$

Proof. It is sufficient to deal with proof figures which contain only one (thin R) as the last inference.

$$\frac{\frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}} \text{ (thin R)}$$

The proof is carried out by double induction mainly on the formula A and secondly on the length of longest threads of the proof of $\Gamma \rightarrow \Delta$. Here we deal only with a special case of (GL4.3) being the last inference of the proof of $\Gamma \rightarrow \Delta$.

$$\frac{\frac{\frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\cdot} \quad \frac{\cdot}{\cdot} \quad \frac{\cdot}{\cdot}}{\frac{\Box \Gamma, \Gamma, \Box B \rightarrow B}{\Box \Gamma \rightarrow \Box B}} \text{ (GL4.3)}$$

$$\frac{\Box \Gamma, \Gamma, \Box B \rightarrow B}{\Box \Gamma \rightarrow \Box B} \text{ (thin R)}$$

$$\frac{\Box \Gamma, \Gamma, \Box B \rightarrow B}{\Box \Gamma \rightarrow \Box B, A}$$

If A is not of the form $\Box C$, then (thin R) degenerates into (thin) $_{\Box}$, which is of course admissible. If A is of the form $\Box C$, the above proof figure is trans-

form $\Box B$ that occur as subformulas in $\Gamma_2 \cup \mathcal{A}_2^{3)}$. We denote by $\delta_r^2(\pi)$ the number of formulas of the form $\Box B$ that occur in $\Gamma_2^{4)}$. We denote by $\delta_r(\pi)$ the number $\delta_l^1(\pi) - \delta_r^2(\pi)$. We decree that $\delta(\pi) = \delta_l(\pi) + \delta_r(\pi)$. By $\rho_l(\pi)$ we denote the number of the longest threads that end with the left upper sequent $\Gamma_1 \rightarrow \mathcal{A}_1, A$ and contain the cut formula A consecutively. Similarly we denote by $\rho_r(\pi)$ the number of the longest threads that end with the right upper sequent $A, \Gamma_2 \rightarrow \mathcal{A}_2$ and contain the cut formula A consecutively. By $\rho(\pi)$ we mean the number $\rho_l(\pi) + \rho_r(\pi)$. Now our proof proceeds by triple induction mainly on $\gamma(\pi)$, secondly on $\delta(\pi)$ and thirdly on $\rho(\pi)$. Since our proof is not by the usual double induction on $\gamma(\pi)$ and $\rho(\pi)$, we should be careful enough even in dealing with classical cases.

(1) $\rho(\pi) = 2$: Since in rule (GL4.3) the antecedent of the lower sequent is contained in that of every upper sequent, A can be of the form $\Box B$ only when the right upper sequent $\Box B, \Gamma_2 \rightarrow \mathcal{A}_2$ is an axiom sequent. In this case $\Gamma \rightarrow \mathcal{A}$ is also an axiom sequent. Therefore the only nontrivial case we must consider goes as follows:

$$\frac{\frac{\frac{\text{---}}{\Gamma_1 \rightarrow \mathcal{A}_1, B} \quad \frac{\text{---}}{\Gamma_1 \rightarrow \mathcal{A}_1, A \supset B}}{\Gamma_1 \rightarrow \mathcal{A}_1, B} \quad (\supset R) \quad \frac{\frac{\text{---}}{\Gamma_2 \rightarrow \mathcal{A}_2, A} \quad \frac{\text{---}}{B, \Gamma_2 \rightarrow \mathcal{A}_2}}{A \supset B, \Gamma_2 \rightarrow \mathcal{A}_2} \quad (\text{cut}) \quad (\supset L)'}{\Gamma \rightarrow \mathcal{A}} \quad (\text{cut})$$

$(\Gamma = \Gamma_1 \cup \Gamma_2 \text{ and } \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2)$

This proof figure π is transformed into:

$$\frac{\frac{\frac{\text{---}}{\Gamma_2 \rightarrow \mathcal{A}_2, A} \quad \frac{\text{---}}{A, \Gamma_1 \rightarrow \mathcal{A}_1, B}}{\Gamma \rightarrow \mathcal{A}, B} \quad (\text{cut}) \quad \frac{\text{---}}{B, \Gamma_2 \rightarrow \mathcal{A}_2}}{\Gamma \rightarrow \mathcal{A}} \quad (\text{cut})$$

Since the grades of A and B are smaller than that of $A \supset B$, the induction process works well.

(2) $\rho(\pi) > 2$: There are several nontrivial cases, which we shall consider case by case in the following:

(2a) π is of the following form:

$$\frac{\frac{\frac{\text{---}}{\Gamma_1 \rightarrow \mathcal{A}_1, B, A} \quad \frac{\text{---}}{C, \Gamma_1 \rightarrow \mathcal{A}_1, A}}{B \supset C, \Gamma_1 \rightarrow \mathcal{A}_1, A} \quad (\supset L) \quad \frac{\text{---}}{A, \Gamma_2 \rightarrow \mathcal{A}_2}}{B \supset C, \Gamma \rightarrow \mathcal{A}} \quad (\text{cut})$$

$(\Gamma = \Gamma_1 \cup \Gamma_2 \text{ and } \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2)$

1)-4) Repetition is not counted.

Consider the following proof figure π_1 :

$$\frac{\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}}{\frac{\Gamma_1 \rightarrow \mathcal{A}_1, B, A \quad A, \Gamma_2 \rightarrow \mathcal{A}_2}{\Gamma \rightarrow \mathcal{A}, B} \text{ (cut)}}$$

Since $\gamma(\pi_1) = \gamma(\pi)$, $\delta(\pi_1) \leq \delta(\pi)$ and $\rho(\pi_1) < \rho(\pi)$, we have a cut-free proof π'_1 of the sequent $\Gamma \rightarrow \mathcal{A}, B$ by induction hypothesis.

Consider the following proof figure π_2 :

$$\frac{\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}}{\frac{C, \Gamma_1 \rightarrow \mathcal{A}_1, A \quad A, \Gamma_2 \rightarrow \mathcal{A}_2}{C, \Gamma \rightarrow \mathcal{A}} \text{ (cut)}}$$

Since $\gamma(\pi_2) = \gamma(\pi)$, $\delta(\pi_2) \leq \delta(\pi)$ and $\rho(\pi_2) < \rho(\pi)$, we have a cut-free proof π'_2 of the sequent $C, \Gamma \rightarrow \mathcal{A}$ by induction hypothesis. Therefore

$$\frac{\left. \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} \pi'_1 \quad \left. \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} \pi'_2}{\frac{\Gamma \rightarrow \mathcal{A}, B \quad C, \Gamma \rightarrow \mathcal{A}}{B \supset C, \Gamma \rightarrow \mathcal{A}} \text{ (}\supset\text{L)'}}$$

The following three cases are treated similarly to (2a).

(2b) $\rho_l(\pi) \geq 2$ and the last inference of the proof of the left upper sequent of (cut) is (\supset R).

(2c) $\rho_r(\pi) \geq 2$ and the last inference of the proof of the right upper sequent of (cut) is (\supset L)'.

(2d) $\rho_\tau(\pi) \geq 2$ and the last inference of the proof of the right upper sequent of (cut) is (\supset R).

(2e) The last inference of the proofs of both upper sequents of (cut) is (GL4.3):

We deal with the following special case, leaving the general treatment to the reader.

$$\frac{\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}}{\frac{\frac{\frac{\frac{\Box, \Gamma'_1, \Gamma'_1, \Box A, \Box B \rightarrow A, B \quad \Box \Gamma'_1, \Gamma'_1, \Box A \rightarrow A, \Box B \quad \Box \Gamma'_1, \Box B \rightarrow A, B}{\Box \Gamma'_1 \rightarrow \Box A, \Box B} \text{ (GL4.3)} \quad \frac{\frac{\Box \Gamma'_2, \Gamma'_2, \Box A, A, \Box C \rightarrow C}{\Box \Gamma'_2, \Box A \rightarrow \Box C} \text{ (GL4.3)}}{\Box \Gamma \rightarrow \Box B, \Box C} \text{ (cut)}}{\Box \Gamma \rightarrow \Box B, \Box C} \text{ (cut)}}{\Box \Gamma \rightarrow \Box B, \Box C} \text{ (cut)}} \text{ (GL4.3)}$$

$(\Gamma = \Gamma'_1 \cup \Gamma'_2 \text{ and } \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2)$

Consider the following proof figure π_1 :

$$\frac{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array}}{\frac{\frac{\square\Gamma_1, \Gamma_1, \square A, \square B \rightarrow A, B \quad \square\Gamma_2, \Gamma_2, \square A, A, \square C \rightarrow C}{\square\Gamma, \Gamma, \square A, \square B, \square C \rightarrow B, C} \quad (\text{cut})}}{\square\Gamma, \Gamma, \square A, \square B, \square C \rightarrow B, C} \quad (\text{cut})$$

Since the grade of A is smaller than that of $\square A$, there is a cut-free proof π'_1 of $\square\Gamma, \Gamma, \square A, \square B, \square C \rightarrow B, C$ by induction hypothesis.

Consider the following proof figure π_2 :

$$\frac{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \left. \vphantom{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array}} \right\} \pi'_1}{\frac{\frac{\square\Gamma_1, \Gamma_1, \square B \rightarrow \square A, B \quad \square\Gamma, \Gamma, \square A, \square B, \square C \rightarrow B, C}{\square\Gamma, \Gamma, \square B, \square C \rightarrow B, C} \quad (\text{cut})}}{\square\Gamma, \Gamma, \square B, \square C \rightarrow B, C} \quad (\text{cut})$$

Since $\gamma(\pi_2) = \gamma(\pi)$ and $\delta(\pi_2) < \delta(\pi)$, there is a cut-free proof π'_2 of $\square\Gamma, \Gamma, \square B, \square C \rightarrow B, C$ by induction hypothesis.

Consider the following proof figure π_3 :

$$\frac{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \frac{\frac{\square\Gamma_2, \Gamma_2, \square A, A, \square C \rightarrow C}{\square\Gamma_2, \square A \rightarrow \square C} \quad (\text{GL4.3})}{\frac{\frac{\square\Gamma_1, \Gamma_1, \square B \rightarrow \square A, B \quad \square\Gamma_2, \square A \rightarrow \square C}{\square\Gamma_1, \Gamma_1, \square\Gamma_2, \square B \rightarrow B, \square C} \quad (\text{cut})}}{\square\Gamma_1, \Gamma_1, \square B \rightarrow \square A, B \quad \square\Gamma_2, \square A \rightarrow \square C} \quad (\text{cut})$$

Since $\gamma(\pi_3) = \gamma(\pi)$ and $\delta(\pi_3) < \delta(\pi)$, there is a cut-free proof π'_3 of $\square\Gamma, \Gamma, \square B \rightarrow B, \square C$ by induction hypothesis and Lemma 3.4.

Consider the following proof figure π_4 :

$$\frac{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array}}{\frac{\frac{\square\Gamma_1, \Gamma_1, \square A \rightarrow A, \square B \quad \square\Gamma_2, \Gamma_2, \square A, A, \square C \rightarrow C}{\square\Gamma, \Gamma, \square A, \square C \rightarrow \square B, C} \quad (\text{cut})}}{\square\Gamma_1, \Gamma_1, \square A \rightarrow A, \square B \quad \square\Gamma_2, \Gamma_2, \square A, A, \square C \rightarrow C} \quad (\text{cut})$$

Since $\gamma(\pi_4) < \gamma(\pi)$, there is a cut-free proof π'_4 of $\square\Gamma, \Gamma, \square A, \square C \rightarrow \square B, C$ by induction hypothesis.

Consider the following proof figure π_5 :

$$\frac{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array} \quad \left. \vphantom{\begin{array}{c} \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \\ \text{•} \end{array}} \right\} \pi'_4}{\frac{\frac{\square\Gamma_1 \rightarrow \square A, \square B \quad \square\Gamma, \Gamma, \square A, \square C \rightarrow \square B, C}{\square\Gamma, \Gamma, \square C \rightarrow \square B, C} \quad (\text{cut})}}{\square\Gamma_1 \rightarrow \square A, \square B \quad \square\Gamma, \Gamma, \square A, \square C \rightarrow \square B, C} \quad (\text{cut})$$

Since $\gamma(\pi_5) = \gamma(\pi)$ and $\delta(\pi_5) < \delta(\pi)$, there is a cut-free proof π'_5 of $\square\Gamma, \Gamma, \square C \rightarrow \square B, C$ by induction hypothesis. Therefore

