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A Mathematical One-Dimensional One-Phase Model of Superccoling Solidification

By

Ryo KOBAYASHI*

Abstract

This paper gives a mathematical study for the supercooling solidification of materials consisting of pure elements or compounds. Solidification is a complex phenomenon, and its process is not yet entirely understood. By simplifying physical characteristics of the supercooling solidification phenomenon, a mathematical model is derived for the one-dimensional/one-phase case. This model is given as a new type of moving boundary problem. The existence and uniqueness theorem is then proved.

§ 1. Introduction

The Stefan problem is well known as a mathematical model describing solidification or melting of materials. In the Stefan problem, the phase is determined by the temperature distribution, and the temperature on the solid/liquid interface is equal to the equilibrium temperature. In view of these properties, the Stefan problem cannot describe supercooling solidification, then in order to deal with such phenomena we need another model in which the physical characteristics of supercooling solidification are taken into consideration.

In various solidification phenomena, the speed of the growth of the solid phase is determined by several factors, for example, the supercooling temperature on the solid/liquid interface, the shape of the solid/liquid interface and the crystalline anisotropy. The most important one is the supercooling temperature on the solid/liquid interface which gives the driving force of the solidification (see [5], [6]).

In this paper, we attempt to describe a supercooling solidification under the hypothesis that the speed of the solid/liquid interface is determined only by the supercooling temperature on the interface. Under this hypothesis T. Nogi has proved the existence and uniqueness theorem of the one-dimensional/two-phase problem (see [3]).

Here we consider the case in which the liquid phase is uniformly supercooled throughout the process. The temperature distribution on the solid phase and the

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* Department of Applied Mathematics and Physics Fuculty of Engineering, Kyoto University, Kyoto 606.

position of the solid/liquid interface are unknown. Hence such process yields one-phase problem. In the sequel we cofine ourselves to the one-dimensional/one-phase problem.

We shall solve the equations describing the conservation law of heat energy and the motion of the interface. The solution is constructed by a difference scheme in which the time width is variable so that the free boundary consists of mesh points at each step. Then we will obtain a solution untill the time when the supercooling on the solid/liquid interface vanishes. To prove the uniqueness theorem, we convert the above equations to an integral equation, and then apply the fixed point theorem to it.

§ 2. Mathematical Model

As noted in the introduction, we cofine ourselves to the one dimensional process of supercooling solidification.

Let the initial position of the solid-liquid interface be at $x=l$ and $\phi(x)$ $(0\leqq x\leqq l)$ the temperature distribution on the solid phase at the initial time $t=0$. Let the position of the interface and the temperature distribution on the solid phase at time t be denoted by $y(t)$ and $u(x, t)$, respectively. Since we assume that the liquid phase is uniformly supercooled, the temperature distribution on it is constant, say zero; accordingly, the equilibrium temperature u_e is positive.

Suppose that the relation of the speed of the interface and the supercooling temperature on it is given by a function F, which is monotone increasing, Lipschitz continuous with Lipschitz constant K_1 and $F(0)=0$. Such restrictions on F are naturally derived from a physical consideration on solidification (see [5]).

Let the latent heat of the material be L, and let the boundary condition be given by the Dirichlet data $f(t)$ at $x=0$. Then we obtain the following system of equations.

(2.1)

$$u_t(x, t)=u_{xx}(x, t) \qquad 0<x<y(t), \quad t>0 \tag{2.1.1}$$

$$\dot{y}(t)(L-u(y(t), t))=u_x(y(t), t) \qquad t>0 \tag{2.1.2}$$

$$\dot{y}(t)=F(u_e-u(y(t), t)) \qquad t>0 \tag{2.1.3}$$

$$u(0, t)=f(t) \qquad t\geqq 0 \tag{2.1.4}$$

$$u(x, 0)=\phi(x) \qquad 0\leqq x\leqq l \tag{2.1.5}$$

$$y(0)=l. \tag{2.1.6}$$

Remark 2.2. The temperature and its partial derivative at the boundary $u(y(t), t)$ and $u_x(y(t), t)$ are understood in the sense of left limits $\lim_{x\to y(t)-0} u(x, t)$ and $\lim_{x\to y(t)-0} u_x(x, t)$, respectively.

Remark 2.3. For simplicity, we assume that the density, the heat capacity and the heat conduction coefficient are all equal to one. The general case can be reduced to (2.1) by suitably changing the variables.

Assumptions 2.4. The following conditions are imposed:

(2.4.1) $u_e < L$; (this means that the material is not overly supercooled)

(2.4.2) there exists $M>0$ such that $-M \leqq f(t) \leqq u_e$, $-M \leqq \phi(x) \leqq u_e$ for all t, x;

(2.4.3) f and ϕ are Lipschitz continuous with Lipschitz constant K_2;

(2.4.4) $f(0)=\phi(0)$.

Let us state what we mean by the solution of (2.1).

Definition 2.5. A pair (u, y) is a solution of (2.1) on $[0, T]$ if it satisfies the following conditions:

(2.5.1) $y \in C^1[0, T]$, $y(t)>0$ for $t \in [0, T]$ and y satisfies (2.1.6);

(2.5.2) u is a continuous function on $\{(x, t); 0 \leqq x \leqq y(t), 0 \leqq t \leqq T\}$ which satisfies (2.1.3), (2.1.4), (2.1.5) and $u(x, t) < u_e$;

(2.5.3) u_t, u_{xx} exist and are continuous on $\{(x, t); 0<x<y(t), 0<t<T\}$ and satisfies (2.1.1);

(2.54.) for almost all $t \in [0, T]$, (2.1.2) holds.

§ 3. Existence of the Solution

In this section, we construct a solution of (2.1) by taking the limit of the sequence of the solutions of the difference scheme which approximates (2.1).

3.1. Difference Scheme

Let h be the uniform space width. In the sequel we take only those h that makes $l/h=J$ an integer. Let k_n be the variable time width; k_n is determined at each step so that the approximated free boundary consists of the mesh points. Denote the discrete coordinates by (x_j, t_n), where $x_j = jh$ and $t_n = \sum_{m=1}^{n} k_m$; u_j^n and y_n correspond to $u(x_j, t_n)$ and $y(t_n)$, respectively. We employ the following notations for the usual divided differences:

$$u_{jx}^n = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad u_{j\bar{x}}^n = \frac{1}{h}(u_j^n - u_{j-1}^n),$$

$$u_{jx\bar{x}}^n = \frac{1}{h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n), \quad u_{j\bar{t}}^n = \frac{1}{k_n}(u_j^n - u_j^{n-1}), \text{ etc.}$$

In our scheme, the temperature distribution is obtained from an implicit scheme of the heat conservation law; the free boundary is explicitly obtained once k_n is determined. Our basic scheme is the following:

$$(3.1)\quad \begin{cases} u^n_{j\bar{t}}=u^n_{x\bar{x}} & j=1,2,\cdots,J+n-1,\ n=1,2,\cdots & (3.1.1)\\ \dfrac{h}{k_n}(L-u^n_{J+n})=u^n_{J+n\bar{x}} & n=1,2,\cdots & (3.1.2)\\ \dfrac{h}{k_n}=F(u_e-u^{n-1}_{J+n-1}) & n=1,2,\cdots & (3.1.3)\\ u^n_0=f^n & n=0,1,\cdots & (3.1.4)\\ u^0_j=\phi_j & j=0,1,\cdots,J & (3.1.5)\end{cases}$$

where $f^n=f(t_n)$ and $\phi_j=\phi(x_j)$.

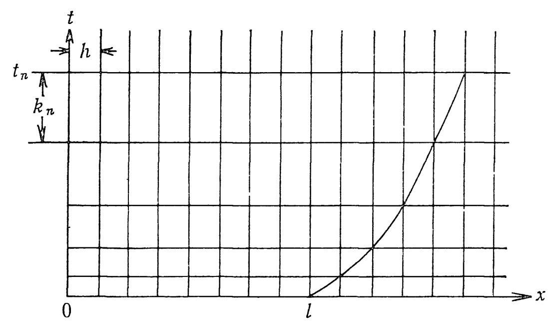

Fig. 3.2.

If $u^{n-1}_{J+n-1}<u_e$, then k_n is determined by (3.13.). And then $u^n_0, u^n_1, \cdots, u^n_{J+n}$ by determined by (3.1.1), (3.1.2) and (3.1.4). So we can find k_n, u^n_j ($j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, N$) which satisfies (3-1) as long as $u^n_{J+n}<u_e$ for $n=0, 1, \cdots, N-1$.

Remark 3.3. It may occur that $u^N_{J+N}\geqq u_e$. In this case we cannot solve (3.1) any more. It is considered that the supercooling on the free boundary vanished.

For the solution of (3.1), the following proposition holds.

Proposition 3.4. *If* u^n_j ($j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, N$) *satisfies* (3.1) (*note that* $u^n_{J+n}<u_e$ *for* $n=0, 1, \cdots, N-1$), u^n_j *cannot attain its maximum or minimum on the inner mesh points*; *i.e.* $\{(x_j, t_n);\ j=1, 2, \cdots, J+n-1,\ n=1, 2, \cdots, N\}$.

Proof. If u^n_j attain its maximum or minimum at the inner mesh point (x_{j_0}, t_{n_0}), $u^n_j=\text{constant}$ for $j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, n_0$ by the maximum principle. From (3.1.2) the constant must be L, which contradicts $\phi_j<u_e$. □

Corollary 3.5. $u^N_{J+N}<L$.

Proof. If $u^N_{J+N}\geqq L$, then u^N_{J+N} must be the maximum value since $u^n_j<u_e<L$ for any other mesh points on the parabolic boundary. By proposition (3.4), $u^N_{J+N-1}<u^N_{J+N}$ must hold, hence $u^N_{J+N\bar{x}}>0$. On the other hand, $u^N_{J+N\bar{x}}\leqq 0$ provided $u^N_{J+N}\geqq L$ from (3.1.2), which is a contradiction. □

Corollary 3.6. $u_j^n \geqq -M$ for $j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, N$.

Proof. Since $u_{J+n}^n < L$, $u_{J+n\bar{x}}^n > 0$ for $n=0, 1, \cdots, N$ from (3.1.2). Therefore u_j^n cannot attain its minimum on $\{(x_{J+n}, t_n);\ n=1, 2, \cdots, N\}$. Hence u_j^n attains its minimum on $\{(0, t_n);\ n=0, 1, \cdots, N\}$ or $\{(x_j, 0);\ j=0, 1, \cdots, J\}$, on which $u_j^n \geqq -M$. □

Corollary 3.7. $h/k_n \leqq \kappa$ for $n=1, 2, \cdots, N$ where $\kappa = K_1(u_e+M)$.

Proof. Obvious from (3.6) and Lipschitz continuity of F. □

Next we need an estimate for u_{jx}^n.

Proposition 3.8. *There exists a positive constant* $C_1 = C_1(t_N)$ *such that*

$$|u_{jx}^n| \leqq C_1 \quad \text{for} \quad j=0, 1, \cdots, J+n-1,\ n=0, 1, \cdots, N.$$

Proof. Set $C_1 = \mathrm{Max}\{K_2(1+l/2),\ (L+M)/l + K_2(l+\kappa t_N)/2,\ \kappa(L+M)\}$. Fix n_0 $(=0, 1, \cdots, N)$ and define $\zeta_j^{n;n_0}$ $(j=0, 1, \cdots, J+n,\ n=0, 1, \cdots, n_0)$ by

$$\zeta_j^{n;n_0} = f^{n_0} + K_2(t_{n_0} - t_n) + C_1 x_j - \frac{K_2}{2} x_j^2 .$$

Clearly, $\zeta_{j\bar{t}}^{n;n_0} = \zeta_{jx\bar{x}}^{n;n_0}$ holds. Also, our evaluation on C_1 gives $\zeta_j^{n;n_0} \geqq u_j^n$ on the parabolic boundary. Therefore it can be shown that $\zeta_j^{n;n_0} \geqq u_j^n$ for $j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, n_0$, by applying the maximum principle to $\zeta_j^{n;n_0} - u_j^n$.

Since $\zeta_0^{n_0;n_0} = u_0^{n_0}$ and $\zeta_1^{n_0;n_0} \geqq u_1^{n_0}$, $u_{0x}^{n_0} \leqq \zeta_{0x}^{n_0;n_0} \leqq C_1$. Similarly $u_{0x}^{n_0} \geqq -C_1$, and hence we obtain $|u_{0x}^n| \leqq C_1$ for $n=0, 1, \cdots, N$. Furthermore, $|u_{jx}^0| \leqq C_1$ for $j=0, 1, \cdots, J-1$ and $|u_{J+n-1\,x}^n| \leqq C_1$ for $n=1, 2, \cdots, N$. Applying the maximum principle to u_{jx}^n, we get the desired conclusion $|u_{jx}^n| \leqq C_1$. □

To construct a local solution we consider the variation of the solution of (3.1) along the free boundary.

Lemma 3.9. *Let* v_i^n *be a function defined on the mesh points* $\{(x_i, t_n);\ i=0, 1, \cdots, I,\ n=0, 1, \cdots, N\}$, *where* $x_i = ih$, $t_n = \sum_{m=1}^{n} k_m$ *and* $r = Ih$. *If* $v_{i\bar{t}}^n = v_{ix\bar{x}}^n$ *for* $i=1, 2, \cdots, I-1$; $n=1, 2, \cdots, N$ *and* $|v_i^n| \leqq B$ *for* $i=0, 1, \cdots, I$; $n=0, 1, \cdots, N$, *then the following estimate holds,*

$$|v_{ix}^n| \leqq C_2 B\Big[\frac{1}{\sqrt{t_n}}\log\Big(1+\frac{r}{\sqrt{t_n}}\Big) + \frac{1}{x_i}\log\Big(1+\frac{\sqrt{t_n}}{x_i}\Big) + \frac{1}{r-x_i}\log\Big(1+\frac{\sqrt{t_n}}{r-x_i}\Big)\Big] \tag{3.10}$$

for $i=1, 2, \cdots, I-1$, $n=1, 2, \cdots, N$.

Proof. See [3]. □

Proposition 3.11. *Let* $h \leqq l/9$. *If* u_j^n $(j=0, 1, \cdots, J+n,\ n=0, 1, \cdots, N)$ *is a solution of* (3.1) *and* $t_n \leqq \mathrm{Min}\{1, l^2/9\}$, *then there exists a positive constant* C_3 *such* *that*

$$|u_{J+N}^N-u_J^0|\leqq C_3\sqrt{t_N}\,. \tag{3.12}$$

Proof. We can easily show $2(\sqrt{t_N}+h)\leqq l-h$. Put $I'=\left[\frac{\sqrt{t_N}}{h}\right]+1$ and $r'=I'h$, then $J-2I'=\frac{l}{h}-2\left(\left[\frac{\sqrt{t_N}}{h}\right]+1\right)\geqq\frac{l-2(\sqrt{t_N}+h)}{h}\geqq 1$ and $\sqrt{t_N}<r'\leqq\sqrt{t_N}+h$' so we can define $v_i^n=u_{J-2I'+i\bar{x}}^n$ for $i=0, 1, \cdots, 2I'$, $n=0, 1, \cdots, N$. From (3.8) $|v_i^n|\leqq C_1=C_1(1)$, and from (3.1.1) $v_{i\bar{t}}^n=v_{ix\bar{x}}^n$. Applying Lemma 3.9 to v_i^n and setting $i=I'$, we have

$$\begin{aligned}&|u_{J-I'\bar{t}}^n|=|u_{J-I'x\bar{x}}^n|\\&\leqq C_1C_2\left[\frac{1}{\sqrt{t_n}}\log\left(1+\frac{2r'}{\sqrt{t_n}}\right)+\Big)+\frac{2}{r'}\log\left(1+\frac{\sqrt{t_n}}{r'}\right)\right]\\&\leqq C_1C_2\left[\frac{1}{\sqrt{t_n}}\log\left(1+\frac{2r'}{\sqrt{t_n}}\right)+\frac{2}{r'}\log 2\right].\end{aligned}$$

Hence

$$\begin{aligned}&|u_{J-I'}^N-u_{J-I'}^0|\leqq\sum_{n=1}^N k_n|u_{J-I'\bar{t}}^n|\\&\leqq C_1C_2\left[\int_0^{t_N}\frac{1}{\sqrt{t}}\log\left(1+\frac{2r'}{\sqrt{t}}\right)dt+(2\log 2)\frac{t_N}{r'}\right].\end{aligned}$$

Since

$$\int_0^{t_N}\frac{1}{\sqrt{t}}\log\left(1+\frac{2r'}{\sqrt{t}}\right)dt\leqq 4r'\int_2^\infty\frac{\log(1+\tau)}{\tau^2}\,d\tau\,,$$

and

$$\sqrt{t_N}\leqq r'\leqq\sqrt{t_N}+h\leqq\sqrt{t_N}+\kappa k_n\leqq(1+\kappa)\sqrt{t_N}\,,$$

there exists a positive number C' such that $|u_{J-I'}^N-u_{J-I'}^0|\leqq C'\sqrt{t_N}$. Hence

$$\begin{aligned}|u_{J+N}^N-u_J^0|&\leqq|u_{J+N}^N-u_{J-I'}^N|+|u_{J-I'}^N-u_{J-I'}^0|+|u_{J-I'}^0-u_J^0|\\&\leqq C_1(r'+\kappa t_N)+C'\sqrt{t_N}+C_1r'\,.\end{aligned}$$

The conclusion immediately follows. □

We next prove the existence of the local solution of the difference scheme (3.1).

Proposition. 3.13. *For all* $\varepsilon>0$ *such that* $\phi(l)<u_e-\varepsilon$, *there exist positive numbers* T, h_0 *satisfying the following property*: *For all* $h<h_0$, (3.1) *has a solution* u_j^n ($j=0, 1, \cdots, J+n$, $n=0, 1, \cdots, N$), *where* $t_N\geqq T$ *and* $u_{J+n}^n<u_e-\varepsilon$ *for* $n=0, 1, \cdots, N$.

Proof. Set $T=\mathrm{Min}\{1/2,\ l^2/18,\ ((u_e-\varepsilon-\phi(l))/C_3)^2/2\}$, $h_0=\mathrm{Min}\{l/9,\ F(\varepsilon)T\}$, and fix $h<h_0$. Suppose $u_{J+n}^n<u_e-\varepsilon$ for $n=0, 1, \cdots, N-1$ and $t_{N-1}<T$. Since $F(\varepsilon)\leqq h/k_n$ for $n=1, 2, \cdots, N$, $t_N=t_{N-1}+k_N\leqq 2T$, and therefore $t_N\leqq\mathrm{Min}\{1,\ l^2/9\}$ and $\sqrt{t_N}\leqq(u_e-\varepsilon-\phi(l))/C_3$. Applying (3.12), $|u_{J+N}^N-\phi(l)|<u_e-\varepsilon-\phi(l)$, hence $u_{J+N}^N<u_e-\varepsilon$. Therefore, by solving (3.1) step by step we finally get a solution u_j^n $(j=0, 1, \cdots, J+n,\ n=0, 1, \cdots, N)$ such that $t_N\geqq T$ and $u_{J+n}^n<u_e-\varepsilon$ for $n=0, 1, \cdots, N$. □

Proposition 3.14. *Let* u_j^n $(j=0, 1, \cdots, J+n,\ n=0, 1, \cdots, N)$ *be a solution obtained in Proposition* (3.13), *then there exists a positive constant* C_4 *such that*

$$\sum_{n=1}^{N}k_n\sum_{j=1}^{J+n-1}h(u_{j\bar{t}}^n)^2\leqq C_4\,. \tag{3.15}$$

Proof. Let us rewrite the left-hand side of (3.14) as follows:

$$\begin{aligned}\sum_{n=1}^{N}k_n\sum_{j=1}^{J+n-1}h(u_{j\bar{t}}^n)^2&=\sum_{n=1}^{N}k_n\sum_{j=1}^{J+n-1}hu_{j\bar{t}}^n u_{jx\bar{x}}^n\\&=\sum_{n=1}^{N}k_n\Big\{u_{J+n-1\,\bar{t}}^n u_{J+n-1\,x}^n-f_{\bar{t}}^n u_{0x}^n-\sum_{j=0}^{J+n-2}hu_{jx\bar{t}}^n u_{jx}^n\Big\}\\&=\sum_{n=1}^{N}k_n u_{J+n-1\,\bar{t}}^n u_{J+n-1\,x}^n-\sum_{n=1}^{N}k_n f_{\bar{t}}^n u_{0x}^n\\&\quad-\frac{1}{2}\sum_{n=1}^{N}k_n\sum_{j=0}^{J+n-2}h\Big\{((u_{jx}^n)^2)_{\bar{t}}+\frac{1}{k_n}(u_{jx}^n-u_{jx}^{n-1})^2\Big\}\\&=\sum_{n=1}^{N}k_n u_{J+n-1\,\bar{t}}^n u_{J+n-1\,x}^n-\sum_{n=1}^{N}k_n f_{\bar{t}}^n u_{0x}^n-\frac{1}{2}\sum_{j=0}^{J+N-2}h(u_{jx}^n)^2\\&\quad+\frac{1}{2}\sum_{j=0}^{J-1}h(\phi_{jx})^2+\frac{1}{2}\sum_{j=J}^{J+N-2}h(u_{jx}^{j-J+1})^2-\frac{1}{2}\sum_{n=1}^{N}k_n^2\sum_{j=0}^{J+n-2}h(u_{jx\bar{t}}^n)^2\,.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n=1}^{N}k_n\sum_{j=1}^{J+n-1}h(u_{j\bar{t}}^n)^2&\leqq\sum_{n=1}^{N}k_n u_{J+n-1\,\bar{t}}^n u_{J+n-1\,x}^n+\sum_{n=1}^{N}h(u_{J+n-1\,x}^n)^2\\&\quad-\sum_{n=1}^{N}k_n f_{\bar{t}}^n u_{0x}^n+\frac{1}{2}\sum_{j=0}^{J-1}h(\phi_{jx})^2\\&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}\,.\end{aligned}$$

Then

$$|\mathrm{III}|\leqq K_2C_1\,,$$

$$|\mathrm{IV}|\leqq\frac{1}{2}K_2^2 l\,.$$

$$\mathrm{I}+\mathrm{II}=\sum_{n=1}^{N}(L-u_{J+n}^n)F(u_e-u_{J+n-1}^{n-1})(u_{J+n}^n-u_{J+n-1}^{n-1})\,.$$

Define

$$\tilde{F}(u)=\int_0^u(L-v)F(u_e-v)dv\,,$$

then

$$\tilde{F}(u^n_{J+n})-\tilde{F}(u^{n-1}_{J+n-1})=(L-u^n_{J+n})F(u_e-u^{n-1}_{J+n-1})(u^n_{J+n}-u^{n-1}_{J+n-1})+R_n$$

where $R_n=O(k_n)$.

Hence

$$|\mathrm{I}+\mathrm{II}|\leqq|\tilde{F}(u^N_{J+N})-\tilde{F}(u^0_J)|+C''.$$

Since $\tilde{F}(u)$ is bounded on $[-M, u_e]$, (3.15) holds. □

3.2. Construction of the Solution

To construct a solution of (2.1) we need some preliminaries. Let us denote the domain of a function f by Dom f. We write

$$f_n \longrightarrow\!\!\!\!\to f \quad \text{on} \quad A$$

if the sequence of the functions $\{f_n\}$ converges to f uniformly on A. Let D be an open set in $\boldsymbol{R}^m$, and let $\{D_n\}$ be a sequence of subsets of the same space. If for every compact subset K of D there exists a positive integer N_K such that $D_n \supset K$ for all $n \geqq N_K$, the sequence $\{D_n\}$ is said to *cover* D. A sequence of functions $\{f_n\}$ *covers* D, if {Dom f_n} *covers* D. Suppose $\{f_n\}$ *covers* D and f is a function with domain D. We write

$$f_n \overset{c}{\longrightarrow\!\!\!\!\to} f \quad \text{on} \quad D$$

if for all compact subset K of D, $\{f_n\}_{n \geqq N_K}$ converges to f uniformly on K.

Lemma 3.16. *Suppose $\{f_n\}$ covers D, and for every compact subset K of D, $\{f_n\}_{n \geqq N_K}$ is uniformly bounded and equicontinuous on K. Then there exists a continuous function f defined on D and a subsequence $\{g_n\}$ of $\{f_n\}$ such that*

$$g_n \overset{c}{\longrightarrow\!\!\!\!\to} f \quad \text{on} \quad D.$$

Proof. This lemma is an easy consequence of the Ascoli-Arzera theorem. □

To extend the solution u^n_j of (2.1) to a continuous function, we linearly interpolate it according to the triangle partition as Fig. 3.17.

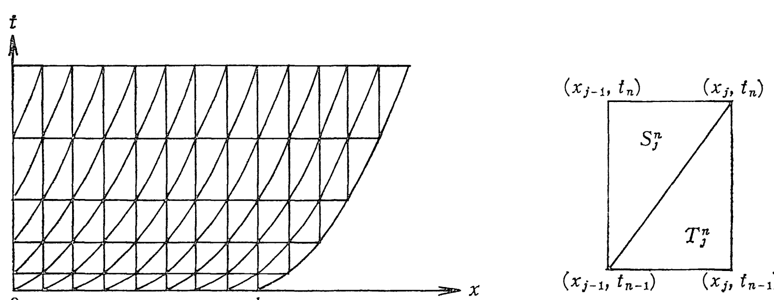

Fig. 3.17.

Let us call the triangle whose vertices are $(x_{j-1}, t_{n-1})(x_j, t_{n-1})$ and (x_j, t_n),

$(x_{j-1}, t_{n-1})(x_{j-1}, t_n)$ and (x_j, t_n), T_j^n and S_j^n, respectively. Then the interpolated function $u(x, t)$ is given by the following formula:

$$u(x, t)=u_{j-1}^{n-1}+u_{j\bar{x}}^{n-1}(x-x_{j-1})+u_{j\bar{t}}^{n}(t-t_{n-1}), \qquad (x, t)\in T_j^n,$$
$$=u_{j-1}^{n-1}+u_{j\bar{x}}^{n}(x-x_{j-1})+u_{j-1\,\bar{t}}^{n}(t-t_{n-1}), \qquad (x, t)\in S_j^n.$$

The domain of u is $D=\bigcup_{\substack{1\leq j\leq J+n-1\\ 1\leq n\leq N}} T_j^n \cup \bigcup_{\substack{1\leq j\leq J+n\\ 1\leq n\leq N}} S_j^n$. The partial derivatives u_x and u_t are represented by the divided differences $u_{j\bar{x}}^n$ and $u_{j\bar{t}}^n$, respectively:

$$u_x(x, t)=u_{j\bar{x}}^{n-1}, \qquad (x, t)\in \mathring{T}_j^n$$
$$=u_{j\bar{x}}^{n}, \qquad (x, t)\in \mathring{S}_j^n$$
$$u_t(x, t)=u_{j\bar{t}}^{n}, \qquad (x, t)\in \mathring{T}_j^n$$
$$=u_{j-1\bar{t}}^{n}, \qquad (x, t)\in \mathring{S}_j^n.$$

Remark 3.18. Note that u_x indicates the partial derivatives of the interpolated function u, and u_{jx}^n, $u_{j-1\,x}^n$ etc. the divided differences of the solution u_j^n of the difference scheme.

Let us state the existence theorem of the local solution of (2.1).

Theorem 3.19. *For all $\varepsilon>0$ such that $\phi(l)<u_e-\varepsilon$, there exists a positive T and a solution (u, y) of* (2.1) *on $[0, T]$ satisfying $u(y(t), t)\leqq u_e-\varepsilon$ for all $t\in[0, T]$.*

Before going into the proof, we have to prepare some notations and a lemma.

To represent the dependence of a solution of (3.1) on h, denote the solution of (3.1) as u_{hj}^n; similarly we write k_n^h, t_n^h and y_n^h etc. Let $y_h(t)$ be a function obtained by linearly interpolating y_n^h. Take $T>0$ given by (3.13) and set

$$D_h=\{(x, t);\ 0\leqq x\leqq y_h(t),\ 0\leqq t\leqq T\},$$
$$\bar{D}_h=\{(x, t);\ 0\leqq x\leqq y_h(t)-h,\ 0\leqq t\leqq T\},$$
$$\bar{\bar{D}}_h=\{(x, t);\ 0\leqq x\leqq y_h(t)-h,\ k_1^h\leqq t\leqq T\}.$$

By interpolating u_{hj}^n, u_{hjx}^n and $u_{hj\bar{t}}^n$ in the way previously stated, we get u_h, u_h and $\bar{\bar{u}}_h$. The functions u_h, $\bar{u}_h$ and $\bar{\bar{u}}_h$ are defined on D_h, $\bar{D}_h$ and $\bar{\bar{D}}_h$, respectively.

Lemma 3.20. *Let u_{hj}^n be the solution of the equation*

$$u_{hj\bar{t}}^n=u_{hjx\bar{x}}^n,$$

and let u_h be the interpolating function of u_{hj}^n. If u_h is uniformly bounded and covers an open set D, then the following holds. For each compact subset K of D, an arbitrarily high order divided dcfference of u_{hj}^n is uniformly bounded on K.

Proof. This lemma is proved by the similar method to the one in [4] using the estimate (3.10). □

Proof of Theorem 3.19. In the first step we constract the solution (u, y) as the limit of the subsequences of $\{u_h\}$ and $\{y_h\}$, and in the following steps we show that (u, y) satisfies the conditions of the solution of (2.1).

First step: Since

(3.21) $$l \leqq y_h(t) \leqq l+\kappa T \quad \text{for all} \quad t \in [0, T]$$

and

(3.22) $$|y_h(t'')-y_h(t')| \leqq \kappa |t''-t'| \quad \text{for all} \quad t', t'' \in [0, T],$$

y_h is uniformly bounded and equicontinuous on $[0, T]$. Therefore, by the Ascoli-Arzela theorem there exists a subsequence $\{y_{h'}\}$ of $\{y_h\}$ and a continuous function y on $[0, T]$ such that

(3.23) $$y_{h'} \longrightarrow\!\!> y \quad \text{on} \quad [0, T].$$

From (3.21), (3.22) and (3.23)

(3.24) $$l \leqq y(t) \leqq l+\kappa T \quad \text{for all} \quad t \in [0, T]$$

and

(3.25) $$|y(t'')-y(t')| \leqq \kappa |t''-t'| \quad \text{for all} \quad t', t'' \in [0, T].$$

Obviously

(3.26) $$y(0)=l\,.$$

Set

(3.27) $$D=\{(x, t)\,;\; 0<x<y(t),\; 0<t<T\},$$

which is an open set in $x-t$ plane. Clearly $\{D_{h'}\}$, $\{\bar{D}_{h'}\}$ and $\{\bar{\bar{D}}_{h'}\}$ *cover* D.

Since $\{u_{h'}\}$ is uniformly bounded and *covers* D, for every compact subset K of D, $\{\bar{u}_{h'}\}$ and $\{\bar{\bar{u}}_{h'}\}$ is uniformly bounded on K from Lemma (3.20). Therefore $\{u_{h'}\}$ is equicontinuous on K. By applying Lemma (3.16) to $\{u_{h'}\}$ and D, there exists a subsequence $\{u_{h''}\}$ of $\{u_{h'}\}$, and a continuous function u on D such that

$$u_{h''} \xrightarrow{c}\!\!> u \quad \text{on} \quad D.$$

Similarly there exists a subsequence $\{\bar{u}_{h'''}\}$ of $\{\bar{u}_{h''}\}$, and a continuous function $\bar{u}$ on D such that

$$\bar{u}_{h'''} \xrightarrow{c}\!\!> \bar{u} \quad \text{on} \quad D\,;$$

and there exists a subsequence $\{\bar{\bar{u}}_\nu\}$ of $\{\bar{\bar{u}}_{h'''}\}$, and a continuous function $\bar{\bar{u}}$ on D such that

$$\bar{\bar{u}}_\nu \xrightarrow{c}\!\!> \bar{\bar{u}} \quad \text{on} \quad D.$$

We thus obtained subsequences $\{y_\nu\}$, $\{u_\nu\}$, $\{\bar{u}_\nu\}$ and $\{\bar{\bar{u}}_\nu\}$ of $\{y_h\}$, $\{u_h\}$, $\{\bar{u}_h\}$ and $\{\bar{\bar{u}}_h\}$ such that

(3.28) $$y_\nu \longrightarrow\!\!> y \quad \text{on} \quad [0, T],$$

(3.29) $$u_\nu \xrightarrow{c}\!\!> u \quad \text{on} \quad D,$$

(3.30) $$\bar{u}_\nu \xrightarrow{c}\!\!\!\!\!\to \bar{u} \quad \text{on} \quad D,$$

(3.31) $$\bar{\bar{u}}_\nu \xrightarrow{c}\!\!\!\!\!\to \bar{\bar{u}} \quad \text{on} \quad D.$$

Second step: In this step we prove

(3.32) $$u_t=\bar{\bar{u}}, \quad u_x=\bar{u} \quad \text{and} \quad \bar{u}_x=\bar{\bar{u}} \quad \text{on} \quad D;$$

which guarantees

(3.33) $$u_t=u_{xx} \quad \text{on} \quad D.$$

Here we show only $u_t=\bar{\bar{u}}$, then other equations are shown by a similar method.

To prove $u_t=\bar{\bar{u}}$, it is sufficient to show that

(3.34) $$u(x, t')-u(x, t'')=\int_{t'}^{t''}\bar{\bar{u}}(x, t)dt$$

for all (x, t'), $(x, t'')\in D$ where $t'<t''$. Let us estimate the difference of $u_{\nu t}$ and $\bar{\bar{u}}_\nu$. Since

(3.35) $$u_{\nu t}=u^n_{\nu j\bar{t}} \quad \text{on} \quad T^n_{\nu j}\cup S^n_{\nu j+1}$$

and $\bar{\bar{u}}_\nu$ is a linearly interpolated function of $u^n_{\nu j\bar{t}}$,

(3.36) $$\sup_{T^n_{\nu j}\cup S^n_{\nu j}}|u_{\nu t}-\bar{\bar{u}}_\nu|=\text{Max}\{|u^n_{\nu j\bar{t}}-u^{n-1}_{\nu j-1\bar{t}}|,\ |u^n_{\nu j\bar{t}}-u^{n-1}_{\nu j\bar{t}}|,\ |u^n_{\nu j\bar{t}}-u^n_{\nu j+1\bar{t}}|\}.$$

From Lemma (3.20) there exists a positive number C_K such that

(3.37) $$|u^n_{\nu j\bar{t}\bar{t}}|\leqq C_K \quad \text{and} \quad |u^n_{\nu j\bar{t}x}|\leqq C_K \quad \text{on} \quad K,$$

where K is a compact subset of D whose interior includes a segment connecting (x, t') and (x, t''). Therefore

(3.38) $$|u_{\nu t}(x, t)-\bar{\bar{u}}_\nu(x, t)|\leqq(1+1/F(\varepsilon))C_K h_\nu$$

on the segment. Then

(3.39) $$|u_\nu(x, t'')-u_\nu(x, t')-\int_{t'}^{t''}\bar{\bar{u}}_\nu(x, t)dt|\leqq(1+1/F(\varepsilon))C_K h_\nu|t''-t'|,$$

and by taking the limit as $\nu\to\infty$, we obtain (3.34).

Third step: In this step we prove

(3.40) $$\lim_{x\to+0} u(x, t)=f(t) \qquad \text{for} \quad t>0,$$

and

(3.41) $$\lim_{t\to+0} u(x, t)=\phi(x) \qquad \text{for} \quad 0<x<l.$$

From (3.8), $|\bar{u}_\nu|\leqq C_1$ and consequently

(3.42) $$|u_x|\leqq C_1.$$

Then

(3.43) $$|u(x, t)-f(t)|\leqq|u(x, t)-u_\nu(x, t)|+C_1x+|u_\nu(0, t)-f(t)|,$$

and by taking the limit as $\nu\to\infty$, it follows that

$$(3.44)\qquad |u(x,t)-f(t)|\leqq C_1 x \quad \text{for} \quad t>0,$$

which implies (3.40).

In order to prove (3.41) we apply Lemma 3.9 to $u^n_{\nu j\bar{x}}$. Then we get the following: for $x\in(0, l)$ there exists a positive number $C(x)$ such that

$$(3.45)\qquad |u_\nu(x,t)-u_\nu(x,0)|\leqq -C(x)\sqrt{t}\log t$$

for sufficiently small t. Then

$$(3.46)\qquad |u(x,t)-\phi(x)|\leqq |u(x,t)-u_\nu(x,t)|-C(x)\sqrt{t}\log t+|u_\nu(x,0)-\phi(x)|,$$

and by taking the limit as $\nu\to\infty$, it follows that

$$(3.47)\qquad |u(x,t)-\phi(x)|\leqq -C(x)\sqrt{t}\log t\,.$$

Therefore (3.41) holds.

In view of the Lipschitz continuity of f and ϕ, and $f(0)=\phi(0)$; u can be extended to a continuous function on $\{(x,t);\ 0\leqq x<y(t),\ 0\leqq t<T\}$.

Fourth step: Here we show that $\lim_{t\to y(t)-0} u(x,t)$ exists and is a continuous function of t (denote it $u(y(t),t)$), and

$$(3.48)\qquad \dot{y}(t)=F(u_e-u(y(t),t))$$

and

$$(3.49)\qquad u(y(t),t)\leqq u_e-\varepsilon\,.$$

The existence of $\lim_{t\to y(t)-0} u(x,t)$ is immediately follows from (3.42). Put $v(t)=u(y(t),t)$ and $v_\nu(t)=u_\nu(y_\nu(t),t)$, and let us show

$$(3.50)\qquad v_\nu \xrightarrow{c}\!\!\!\to v \quad \text{on} \quad (0,T).$$

Fix $\varepsilon>0$ and $[t', t'']\subset(0,T)$, then for $t\in[t', t'']$

$$(3.51)\qquad |v(t)-v_\nu(t)|\leqq 2C_1\varepsilon+|u(y(t)-\varepsilon,t)-u_\nu(y(t)-\varepsilon,t)|+C_1|y(t)-y_\nu(t)|.$$

Since $\{(y(t)-\varepsilon,t);\ t\in[t', t'']\}$ is a compact subset of D,

$$(3.52)\qquad |u(y(t)-\varepsilon,t)-u_\nu(y(t)-\varepsilon,t)|\leqq C_1\varepsilon$$

for sufficiently large ν. And

$$(3.53)\qquad |y(t)-y_\nu(t)|\leqq\varepsilon$$

for sufficiently large ν. Therefore, for such ν it holds that

$$(3.54)\qquad |v(t)-v_\nu(t)|<4C_1\varepsilon \quad \text{for all} \quad t\in[t', t''].$$

It means (3.50), which implies the continuity of $v(t)=u(y(t),t)$ on $(0,T)$, and (3.49). Also from (3.12)

$$(3.55)\qquad |u(y(t),t)-\phi(l)|\leqq C_3\sqrt{t} \quad \text{for} \quad t\in[0,T].$$

These facts guarantee that u can be extended continuously to $\{(x,t);\ 0\leqq x\leqq y(t),\ 0\leqq t\leqq T\}$.

Next we show

$$y(t'')-y(t')=\int_{t'}^{t''}F(u_e-u(y(t),t))dt \tag{3.56}$$

for $t', t''\in(0, T)$ where $t'<t''$. From (3.1.3)

$$\frac{h_\nu}{k_n^\nu}=F(u_e-v_\nu(t_{n-1}^\nu)) \tag{3.57}$$

then

$$y_\nu(t'')-y_\nu(t')=\sum h_\nu=\sum F(u_e-v_\nu(t_{n-1}^\nu))k_n^\nu \tag{3.58}$$

Taking the limit as $\nu\to\infty$, we obtain (3.56) and then (3.48).

Fifth step: In this step we prove

$$\lim_{x\to y(t)-0} u_x(x, t)=\dot{y}(t)(L-u(y(t), t)) \tag{3.59}$$

for almost all $t\in[0, T]$. From (3.15),

$$\sum_n k_n \sum_j h(u_{\nu j\bar{t}}^n)^2\leqq C_4, \tag{3.60}$$

then

$$\int_0^T dt\int_0^{y(t)} u_t(x, t)^2 dx=\int_0^T dt\int_0^{y(t)} u_{xx}(x, t)^2 dx\leqq C_4. \tag{3.61}$$

By Fubini's theorem,

$$\int_0^{y(t)} u_{xx}(x, t)^2 dx<\infty \qquad \text{for almost all} \quad t\in[0, T], \tag{3.62}$$

and for such t, $u_x(x, t)$ is Hölder continuous of order 1/2 in the space variable x. Therefore $\lim_{x\to y(t)-0} u_x(x, t)$ exists for almost every t. Put

$$\phi_\delta(t)=u_x(y(t)-\delta, t) \qquad \text{for small} \quad \delta>0, \tag{3.63}$$

and

$$\chi(t)=\dot{y}(t)(L-u(y(t), t)). \tag{3.64}$$

Let us prove the following: for arbitrary $[t', t'']\subset(0, T)$,

$$\phi_\delta \longrightarrow \chi \quad \text{in} \quad L^2[t', t''] \quad \text{as} \quad \delta\longrightarrow +0, \tag{3.65}$$

which assures the desired result. Set

$$\bar{\phi}_\nu(t)=\bar{u}_\nu(y_\nu(t)-h_\nu, t). \tag{3.66}$$

First of all, let us show

$$\bar{\phi}_\nu \longrightarrow\!\!\!\!\to \chi \quad \text{on} \quad [t', t'']. \tag{3.67}$$

From (3.1.2) and (3.1.3),

$$F(u_e-v_\nu(t_{n-1}^\nu))(L-v_\nu(t_n^\nu))=\bar{\phi}_\nu(t_n^\nu). \tag{3.68}$$

Since $v_\nu \longrightarrow\!\!\!\!\to v$ on $[t', t'']$, for arbitrary $\varepsilon>0$ we can take sufficiently large ν so that

$$|\bar{\phi}_\nu(t_n^\nu)-F(u_e-v(t_{n-1}^\nu))(L-v(t_n^\nu))|<\varepsilon . \tag{3.69}$$

Also from the uniform continuity of v on $[t', t'']$,

$$|\bar{\phi}_\nu(t_n^\nu)-F(u_e-v(t))(L-v(t))|<2\varepsilon \qquad \text{for} \quad t\in[t_{n-1}^\nu, t_n^\nu]. \tag{3.70}$$

Then, for sufficiently large ν, it holds that

$$|\bar{\phi}_\nu(t_n^\nu)-\chi(t)|<2\varepsilon \qquad \text{for} \quad t\in[t_{n-1}^\nu, t_n^\nu]\subset[t', t'']. \tag{3.71}$$

Since $\bar{\phi}_\nu(t)$ is a continuous and piecewise linear function, it follows that

$$|\bar{\phi}_\nu(t)-\chi(t)|<6\varepsilon \qquad \text{for} \quad t\in[t', t''], \tag{3.72}$$

which indicates (3.67).

We next estimate the norm of $\phi_\delta-\bar{\phi}_\nu$ in $L^2[t', t'']$. From (3.60)

$$\int_{t'}^{t''}dt\int_0^{y_\nu(t)-h_\nu}\bar{u}_{\nu x}(x, t)^2dx\leqq C_4 , \tag{3.73}$$

and applying Schwarz inequality, we get

$$|\bar{u}_\nu(y(t)-\delta, t)-\bar{\phi}_\nu(t)|^2\leqq|y_\nu(t)-y(t)+\delta-h_\nu|\int_{y(t)-\delta}^{y_\nu(t)-h_\nu}|\bar{u}_{\nu x}(x, t)|^2dx . \tag{3.74}$$

Integrating both sides from t' to t'', and using (3.73)

$$\int_{t'}^{t''}|\bar{u}_\nu(y(t)-\delta, t)-\bar{\phi}_\nu(t)|^2dt\leqq C_4\sup_{t\in[t', t'']}|y_\nu(t)-y(t)+\delta-h_\nu|. \tag{3.75}$$

Taking the limit as $\nu\to\infty$

$$\int_{t'}^{t''}|\phi_\delta(t)-\chi(t)|^2dt\leqq C_4\delta , \tag{3.76}$$

therefore (3.65) holds. □

We have obtained the local solution of (2.1) by Theorem (3.19). Considering the proof of Theorem (2.1) we can extend the local solution as long as the temperature at the free boundary is less than u_e. Then the global solution (2.1) is obtained.

Theorem 3.77. *There exist a solution* (u, y) *of* (2.1) *on* $[0, T)$ *satisfying* (1) *or* (2):

(1) $T=+\infty$,

(2) $T<+\infty$ *and* $\lim_{t\to T-0}u(y(t), t)=u_e$.

§ 4. Uniqueness

In this section we prove the following theorem.

Theorem 4.1. *The solution of* (2.1) *is unique.*

Before going into the proof, we need several preliminary results. In (2.1), replace u, ϕ and f by u_e-u, $u_e-\phi$ and u_e-f, respectively. Setting $k=L-u_e$, we get

(4.2)
$$u_t(x, t)=u_{xx}(x, t) \qquad 0<x<y(t),\ t>0 \qquad (4.2.1)$$
$$\dot{y}(t)(k+u(y(t), t))=-u_x(y(t), t) \qquad t>0 \qquad (4.2.2)$$
$$\dot{y}(t)=F(u(y(t), t)) \qquad t>0 \qquad (4.2.3)$$
$$u(0, t)=f(t) \qquad t\geqq 0 \qquad (4.2.4)$$
$$u(x, 0)=\phi(x) \qquad 0\leqq x\leqq l \qquad (4.2.5)$$
$$y(0)=l\,. \qquad (4.2.6)$$

In order to express the solution to (4.2) in terms of Green's formula, we put

$$w(x, t)=\exp\left(-\frac{x^2}{4t}\right) \qquad t>0$$
$$=0 \qquad t\leqq 0$$

and

$$g(x, t\,; \xi, \tau)=w(x-\xi, t-\tau)-w(x+\xi, t-\tau)\,.$$

Then we obtain

$$u(x, t)=\int_0^l g(x, t\,; \xi, 0)\phi(\xi)d\xi+\int_0^t g_\xi(x, t\,; 0, \tau)f(\tau)d\tau$$
$$+\int_l^{y(t)} g(x, t\,; \xi, s(\xi))u(\xi, s(\xi))d\xi$$
$$+\int_0^t g(x, t\,; y(\tau), \tau)u_\xi(y(\tau), \tau)d\tau$$
$$-\int_0^t g_\xi(x, t\,; y(\tau), \tau)u(y(\tau), \tau)d\tau\,,$$

where $s(x)$ is the inverse function of $y(t)$. Also, in view of the boundary conditions (4.2.2) and (4.2.3), the sum of the third and the fourth term is equal to

$$-k\int_0^t g(x, t\,; y(\tau), \tau)F(u(y(\tau), \tau))d\tau\,.$$

Then, by setting $u(y(\tau), \tau)=v(\tau)$, we obtain

(4.3)
$$u(x, t)=\int_0^l g(x, t\,; \xi, 0)\phi(\xi)d\xi+\int_0^t g_\xi(x, t\,; 0, \tau)f(\tau)d\tau$$
$$-k\int_0^t g(x, t\,; y(\tau), \tau)F(v(\tau))d\tau-\int_0^t g_\xi(x, t\,; y(\tau), \tau)v(\tau)d\tau\,.$$

The following equation must hold in the limit as $x\to y(t)-0$:

(4.4)
$$v(t)=2\int_0^l g(y(t), t\,; \xi, 0)\phi(\xi)d\xi+2\int_0^t g_\xi(y(t), t\,; 0, \tau)f(\tau)d\tau$$
$$-2k\int_0^t g(y(t), t\,; y(\tau), \tau)F(v(\tau))d\tau-2\int_0^t g_\xi(y(t), t\,; y(\tau), \tau)v(\tau)d\tau\,,$$

where $y(t)=l+\int_0^t F(v(\tau))d\tau\,.$

Note that the last term of the right-hand side of (4.3) becomes

$$\frac{1}{2}v(t)-\int_0^t g_\xi(y(t),\, t\,;\, y(\tau),\, \tau)v(\tau)d\tau\,,$$

due to the discontinuity of double-layer potentials.

We wish to solve this integral equation in a suitable Banach space. Let $C[0,\,\sigma]$ be the Banach space consisting of all real-valued continuous functions on $[0,\,\sigma]$ with supremum norm $\|\cdot\|$, and let $C_A[0,\,\sigma]$ be the closed ball in $C[0,\,\sigma]$ of radius A centered at the origin.

For some A, $\sigma>0$ we can define an operator

$$\Phi:\, C_A[0,\,\sigma]\to C_A[0,\,\sigma]$$

as follows:

$$(\Phi v)(t)=2\int_0^l g(y(t),\, t\,;\, \xi,\, 0)\phi(\xi)d\xi+2\int_0^t g_\xi(y(t),\, t\,;\, 0,\, \tau)f(\tau)d\tau$$
$$-2k\int_0^t g(y(t),\, t\,;\, y(\tau),\, \tau)F(v(\tau))d\tau-2\int_0^t g_\xi(y(t),\, t\,;\, y(\tau),\, \tau)v(\tau)d\tau\,, \tag{4.5}$$

where $y(t)=l+\int_0^t F(v(\tau))d\tau$. Indeed, take A as

$$A=\mathrm{Max}\{4\sup_{0\leqq x\leqq l}|\phi(x)|+1,\ M+u_e\}\,, \tag{4.6}$$

and note that

$$\frac{1}{2}l\leqq y(t)\leqq\frac{3}{2}l\qquad\text{for all}\quad t\in[0,\,\sigma]$$

as long as

$$0<\sigma\leqq\frac{1}{4K_1A}\,. \tag{4.7}$$

Then it readily follows that

$$v\in C[0,\,\sigma]$$

and

$$|(\Phi v)(t)|\leqq 4\sup_{0\leqq x\leqq l}|\phi(x)|+C_5\sup_{0\leqq t\leqq\sigma}|f(t)|\sigma+C_6\sigma^{1/2} \tag{4.8}$$

for all $v\in C_A[0,\,\sigma]$ and $t\in[0,\,\sigma]$. Therefore, by taking σ small enough to satisfy (4.7) and

$$C_5\sup_{0\leqq t\leqq\sigma}|f(t)|\sigma+C_6\sigma^{1/2}\leqq 1\,, \tag{4.9}$$

we see that Φ maps $C_A[0,\,\sigma]$ into itself, thereby insuring the validity of the above definition.

We want to show that Φ becomes a contraction by taking the above σ even smaller. Let

$$y(t)=l+\int_0^t F(v(\tau))d\tau\quad\text{and}\quad y'(t)=l+\int_0^t F(v'(\tau))d\tau\,,$$

for $v,\, v'\in C_A[0,\,\sigma]$. Then

$$(\Phi v)(t)-(\Phi v')(t)=P_1+P_2-P_3-P_4,$$

where

$$P_1=2\int_0^l g(y(t), t; \xi, 0)\phi(\xi)d\xi-2\int_0^l g(y'(t), t; \xi, 0)\phi(\xi)d\xi,$$

$$P_2=2\int_0^t g_\xi(y(t), t; 0, \tau)f(\tau)d\tau-2\int_0^t g_\xi(y'(t), t; 0, \tau)f(\tau)d\tau,$$

$$P_3=2k\int_0^t g(y(t), t; y(\tau), \tau)F(v(\tau))d\tau-2k\int_0^t g(y'(t), t; y'(\tau), \tau)F(v'(\tau))d\tau,$$

$$P_4=2\int_0^t g_\xi(y(t), t; y(\tau), \tau)v(\tau)d\tau-2\int_0^t g_\xi(y'(t), t; y'(\tau), \tau)v'(\tau)d\tau.$$

Estimating P_i $(i=1, 2, 3, 4)$ as in [1], we obtain

$$|P_1|\leqq C_7 \sup_{0\leqq x\leqq l}|\phi(x)|\sigma^{1/2}\|v-v'\|,$$

$$|P_2|\leqq C_8 \sup_{t\geqq 0}|f(t)|\sigma\|v-v'\|,$$

$$|P_3|\leqq C_9\sigma^{1/2}\|v-v'\|,$$

$$|P_4|\leqq C_{10}\sigma^{1/2}\|v-v'\|.$$

Hence if σ satisfies (4.7), (4.9) and

(4.10) $$C_8 \sup|f|\sigma+(C_7 \sup|\phi|+C_9+C_{10})\sigma^{1/2}<1,$$

then

$$\|\Phi v-\Phi v'\|\leqq\theta\|v-v'\| \qquad \text{for all} \quad v, v'\in C_A[0, \sigma],$$

where θ denotes the left-hand side of (4.10). Therefore, if we choose such σ, Φ is a contraction mapping in $C_A[0, \sigma]$. Since $C_A[0, \sigma]$ is closed in $C[0, \sigma]$, the equation $\Phi v=v$ admits a unique solution in $C_A[0, \sigma]$.

Proof of Theorem 4.1. If (u, y) is a solution of (4.1) on $[0, T]$, for A defined in (4-6) and sufficiently small σ,

$$v\in C_A[0, \sigma] \quad \text{and} \quad \Phi v=v \quad \text{where} \quad v(t)=u(y(t), t).$$

By the preceding argument on contraction, such v must be unique, and so is y. This proves the uniqueness of the solution of (4.1) (make use of the maximum principle). The same is true of (2.1). □

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