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Relative Algebraic Reduction and Relative Albanese Map for a Fiber Space in $\mathcal{C}$

By

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Introduction

Let $f: X \to Y$ be a fiber space of compact complex manifolds, i.e., f is surjective with connected fibers. Let $U \subsetneqq Y$ be a Zariski open subset over which f is smooth. Then for each $y \in U$ we have the Albanese map $\phi_y: X_y \to \mathrm{Alb}\, X_y$ of $X_y := f^{-1}(y)$. Under a suitable condition, e.g., if X_y is a manifold in $\mathcal{C}$ (i.e., X_y is a meromorphic image of a compact Kähler manifold), then the collection $\{\mathrm{Alb}\, X_y\}$ can be put together to form a complex manifold $\mathrm{Alb}\, X_U/U$ over U and $\{\phi_y\}$ to form a holomorphic map $\phi_U: X_U \to \mathrm{Alb}\, X_U/U$ over U where $X_U = f^{-1}(U)$. Then the main problem to be treated in this paper is the following: When can we compactify $\mathrm{Alb}\, X_U/U$ to a compact complex manifold $\mathrm{Alb}^* X/Y$ over Y so that ϕ_U extends to a meromorphic map $\phi: X \to \mathrm{Alb}^* X/Y$ over Y? (Here we do not require any good property for $\mathrm{Alb}^* X/Y$; any compactification is enough for our purpose.) We shall show in this paper that this is the case if i) the total space X is in $\mathcal{C}$, and ii) any smooth fiber X_y is Moishezon, (after a possible restriction of U). Moreover it turns out that in this case $\mathrm{Alb}^* X/Y$ is again in $\mathcal{C}$ and the pair $(\phi, \mathrm{Alb}^* X/Y)$ is unique up to bimeromorphic equivalences. We call ϕ briefly the relative Albanese map for f. One notable property of ϕ we prove is that it is Moishezon in the sense that it is bimeromorphic to a projective morphism. Thus, in a sense, the relative Albanese variety $\mathrm{Alb}^* X/Y$ may be considered as the obstruction for a fiber space with general fiber Moishezon to be a Moishezon morphism.

We follow the method of Grothendieck [13] in algebraic geometry, constructing $\mathrm{Alb}\, X_U/U$ as a component of the relative Picard variety $\mathrm{Pic}\,((\mathrm{Pic}_\gamma X_U/U)/U)$ of some component $\mathrm{Pic}_\gamma X_U/U$ of the relative Picard variety $\mathrm{Pic}\, X_U/U$ of X_U over U. Here $\mathrm{Pic}\, X_U/U$ (or at least a good part of it) in turn is constructed as a flat quotient of the space $\mathrm{Div}\, X_U/U$ of relative divisors on X_U over U.

Our first step is thus to construct a natural completion $\mathrm{Div}^* X/Y$ of $\mathrm{Div}\, X_U/U$ over Y, where the assumption that $X \in \mathcal{C}$ is essential to guarantee that each irreducible component of $\mathrm{Div}^* X/Y$ is compact (Section 2). The second step is then to complete $\mathrm{Pic}\, X_U/U$ to a complex variety $\mathrm{Pic}^* X/Y$ over Y such that the

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natural morphism $\mu_{X_U/U}$: Div $X_U/U \to$ Pic X_U/U extends to a meromorphic map $\mu^*_{X/Y}$: Div* $X/Y \to$ Pic* X/Y (Section 3). This will be done through a simple but useful lemma (Lemma 9). Here, however, for the method of [13] to be applicable it is necessary to show that after passing to another bimeromorphic model of X (which is admissible because of the bimeromorphic invariance of Albanese map) any general fiber of f becomes projective. This is also done in Section 2. The final step is the construction of Alb* X/Y from Pic* X/Y and will be given in Section 4.

Though it is expected that the second condition of X_y being Moishezon is irrelevant for the existence theorem, our method gives no idea for the general case.

In Section 2, in relation with our study of the space Div* X/Y we also develop the theory of relative algebraic reduction, i. e., we show that for any fiber space $f: X \to Y$ in $\mathcal{C}$ we can always construct a compact complex manifold Z over Y and surjective meromorphic map $g: X \to Z$ such that for 'general' $y \in Y$ g induces a meromorphic map $g_y: X_y \to Z_y$ which is an algebraic reduction of X_y. We note that this theory of relative algebraic reduction has also been developped by Campana [3] independently. Both relative Albanese maps and relative algebraic reductions provide us with fundamental tools for our investigation of the structure of compact complex manifolds in $\mathcal{C}$ in [11], which was actually the motivation for this paper. The results in this paper were announced in [8] and [8a].

Notations and Convention. A *complex variety* means a reduced and irreducible complex space. As above a *fiber space* is a proper surjective holomorphic map with general fiber irreducible. A compact complex space X is said to be in the class $\mathcal{C}$ if X_{red}, the underlying reduced subspace of X, is a meromorphic image of a compact Kähler manifold (cf. [5]). (Notation $X \in \mathcal{C}$) A Zariski open subset of a complex variety is always assumed to be nonempty. Let $f: X \to Y$ be a morphism of complex space. Then for any morphism $\tilde{Y} \to Y$ we often write $X_{\tilde{Y}} = X \times_Y \tilde{Y}$ and $f_{\tilde{Y}}: = f \times_Y id_{\tilde{Y}}: X_{\tilde{Y}} \to \tilde{Y}$. Let $f': X' \to Y$ be another complex space and $g: X \to X'$ a meromorphic map over Y. Then for any open subset $U \subseteq Y$ we often denote by g_U the restriction of g to X_U; $g_U = g|_{X_U}: X_U \to X'_U$.

Let $f: X \to Y$ and $f': X' \to Y$ be morphisms of compact reduced complex spaces. Suppose that Y is a variety and any irreducible component of X and X' is mapped surjectively onto Y. Let $U \subseteq Y$ be a Zariski open subset over which f is flat. Let $\tilde{U} = f'^{-1}(U)$. Then the closure of $X \times_U \tilde{U}$ in $X \times_Y X'$ is analytic and is independent of the choice of U as above. Then we call this closure the *strict pull-back* of X by f' and denote it by $X \dot{\times}_Y X'$. Then it is readily verified that $X \dot{\times}_Y X' \cong X' \dot{\times}_Y X$ with respect to the natural isomorphism $X \times_Y X' \cong X' \times_Y X$ so that the formation of $X \dot{\times}_Y X'$ is symmetric in X and X'. We denote the induced morphisms $X \dot{\times}_Y X' \to X'$ and $X \dot{\times}_Y X' \to Y$ by $f \dot{\times}_Y X'$ and $f \dot{\times}_Y f'$ respectively. We note that the above definition extends naturally to those X and X' which are unions of compact complex varieties satisfying the above conditions.

§ 1. Preliminaries

In this section, mainly to fix notations, we shall review the generalities on relative Picard varieties Pic X/Y, the space of relative divisors Div X/Y, and the relative Albanese map Alb X/Y, for a proper smooth morphism $f: X \to Y$ (1.1-1.4); we also introduce the notion of s. ampleness of a line bundle and give a certain description of Albanese map in the absolute case. We denote by (An/Y') the category of complex spaces over Y.

1.1. Pic X/Y. Let $f: X \to Y$ be a proper smooth morphism of complex varieties.

a) Define a contravariant functor **Pic** X/Y : (An/Y)→(Sets) by **Pic** $X/Y(Y')$: $=\Gamma(Y', R^1 f_{Y'*}\mathcal{O}^*_{X\times_Y Y'})$ where $f_{Y'}=f\times_Y id_{Y'}: X\times_Y Y'\to Y'$. Then **Pic** X/Y is represented by a commutative complex Lie group Pic X/Y over Y. (See Bingener [2], and when f is locally projective, Grothendieck [14].)

We denote by $b=b_{X/Y}$: Pic $X/Y\to Y$ the structural morphism, and write $m=m_{X/Y}$: Pic $X/Y\times_Y$Pic $X/Y\to$Pic X/Y for the relative group multiplication and $\iota_{X/Y}$: Pic $X/Y\to$Pic X/Y for the relative group inversion as a complex Lie group over Y. (For relative complex Lie group over Y, see [10] or [20].) Then we set $a=a_{X/Y}$: Pic $X/Y\times_Y$Pic $X/Y\to$Pic X/Y, $a=m(id_{\text{Pic } X/Y}\times_Y \iota_{X/Y})$ (the relative subtraction). When Y is a point, we write Pic X for Pic X/Y. We have then the natural isomorphism Pic $X\cong H^1(X, \mathcal{O}^*_X)$.

b) *Functorial properties of* Pic X/Y. i) For any complex space $\tilde{Y}$ over Y we have the natural isomorphism P: Pic $(X\times_Y\tilde{Y}/\tilde{Y})\cong$Pic $X/Y\times_Y\tilde{Y}$. In particular for any $y\in Y$, (Pic $X/Y)_y$ is naturally identified with Pic X_y so that each point $p\in$Pic X/Y represents a unique line bundle L_p on $X_{b(p)}$. ii) Let $f': X'\to Y$ be another proper smooth morphism and $g: X'\to X$ a Y-morphism. Then g induces a natural Y-homomorphism g^*: Pic $X/Y\to$Pic X'/Y.

c) By the definition of Pic X/Y there exists a universal section $l\in\Gamma(\text{Pic } X/Y, R^1 f_{\text{Pic } X/Y*}\mathcal{O}^*_{X\times_Y \text{Pic } X/Y})$ where $f_{\text{Pic } X/Y}: =f\times_Y id_{\text{Pic } X/Y}: X\times_Y$Pic X/Y $\to$Pic X/Y. In particular for any complex space $\tilde{Y}$ over Y and an invertible sheaf $\mathcal{L}$ on $X\times_Y\tilde{Y}$ there exists a unique Y-morphism $\tau: \tilde{Y}\to$Pic X/Y such that the pull-back of l by τ coincides with the image of $\mathcal{L}$ in $\Gamma(\tilde{Y}, R^1 f_{\tilde{Y}*}\mathcal{O}^*_{X\times_Y\tilde{Y}})$. We call τ the *universal Y-morphism defined by $\mathcal{L}$*.

d) When f admits a holomorphic section $s: Y\to X$ we have **Pic** $X/Y(Y')$ =the set of invertible sheaves $\mathcal{L}$ on $X\times_Y Y'$ together with a fixed isomorphism $s'^*\mathcal{L}\cong\mathcal{O}_{Y'}$ where $s'=s\times_Y id_{Y'}$ (cf. [13]). In this case the corresponding universal invertible sheaf $\mathcal{L}$ on $X\times_Y$Pic X/Y is called the *relative Poincaré sheaf associated to s*. When Y is a point, giving an s is equivalent to giving a fixed point $o\in X$.

In this case we call $\mathcal{L}$ the (*normalized*) *Poincaré sheaf associated to* $o\in X$. In the general case let p_i: Pic $X/Y\times_Y$Pic $X/Y\to$Pic X/Y be the projections to the i-th factors. Let $\mathcal{L}_i=(id_X\times p_i)^*\mathcal{L}$ which are invertible sheaves on $\hat{X}:=X\times_Y$Pic $X/Y\times_Y$Pic X/Y. Then $a_{X/Y}$ is nothing but the universal Y-morphism defined by $\mathcal{L}_1\otimes_{\mathcal{O}}\mathcal{L}_2^{-1}$, $\mathcal{O}=\mathcal{O}_{\hat{X}}$.

1.2. Div X/Y *and* Pic X/Y. a) Let $f: X\to Y$ be as in 1.1. Define a contravariant functor **Div** X/Y: (An/Y)$\to$(Sets) by **Div** $X/Y(Y')$=the set of all effective relative divisors $Z\subseteq X\times_Y Y'$ over Y' where a relative divisor is a Cartier divisor which is fiat over Y'. Then **Div** X/Y is represented by a Zariski open subset Div X/Y of $D_{X/Y}$ which is a union of connected components where $D_{X/Y}$ is the relative Douady space of X over Y (cf. [5]). We write $\delta=\delta_{X/Y}$: Div $X/Y\to Y$ for the structural morphism.

b) Let $Z_{X/Y}\subseteq X\times_Y$Div X/Y be the universal relative divisor over Div X/Y. We note that Div X/Y has the natural structure of a relative complex semigroup over Y induced by the universality. We denote by $\tilde{m}_{X/Y}$: Div $X/Y\times_Y$Div X/Y $\to$Div X/Y the corresponding multiplication. When Y is a point, we write Div X for Div X/Y.

c) Let $[Z_{X/Y}]$ be the line bundle on $X\times_Y$Div X/Y defined by $Z_{X/Y}$. Then we denote by $\mu_{X/Y}$: Div $X/Y\to$Pic X/Y the universal Y-morphism defined by $[Z_{X/Y}]$. Suppose that f admits a holomorphic section so that the relative Poincaré sheaf $\mathcal{L}$ exists. Then by Grothendieck ([13], exposé 232, Th. 4.3) there exists a coherent analytic sheaf Q on Pic X/Y such that Div X/Y is isomophic to the projective variety associated to Q (cf. Fischer [4] p. 55). Q is in fact given by $(V_{\mathrm{Pic}\,X/Y})^{-1}(\mu_{X/Y})_+L^\vee$ in the notation of Schuster [22]*) where $L^\vee$ denotes the line bundle dual to the line bundle corresponding to $\mathcal{L}$. In general, since f admits a local section at any point of Y, this implies that $y\in Y$ has a neighborhood V over which $\mu_{X/Y}$ is projective, and that the fiber over any $p\in$Pic X/Y is isomorphic to the projective space $\boldsymbol{P}(\Gamma(X_{b(p)}, L_p)):=(\Gamma(X_{b(p)}, L_p)-\{0\})/\boldsymbol{C}^*$. In particular if $\dim\Gamma(X_{b(p)}, L_p)=k+1$ is independent of $p\in N$ for some open subset $N\subseteq$Pic X/Y, then $\mu_{X/Y}$ is a holomorphic $\boldsymbol{P}^k$-bundle when restricted over N.

1.3. Pic X/Y *in a special case.* We now consider Pic X/Y in the case where f is a fiber space and $X_y\in\mathcal{C}$ for all $y\in Y$. In this case a direct construction of Pic X/Y is known (cf. [14]), and is roughly described as follows.

a) *The construction.* We set $E_1=R^1f_*\boldsymbol{C}/H^{1,0}$ and $E_2=R^2f_*\boldsymbol{C}/(H^{2,0}\oplus H^{1,1})$ where $H^{p,q}$ is the Hodge subbundles of type (p, q). Then E_i are holomorphic vector bundles over Y such that $\mathcal{O}_Y(E_i)\cong R^if_*\mathcal{O}_X$ naturally. Set $L_i=R^if_*\boldsymbol{Z}$, $i=1, 2$. Then the inclusion $\boldsymbol{Z}\subseteq\boldsymbol{C}$ induces the natural homomorphisms j_i: $L_i\to E_i$

*) For any complex space X, V_X: Coh$_X\to$Lin$_X$ denotes the natural anti-equivalence where Coh$_X$ (resp. Lin$_X$) is the category of coherent analytic sheaves (resp. of linear fiber spaces) on X (cf. [4], 1.6, [22], §3).

where we consider L_i, E_i as relative complex Lie groups over Y. Then it turns out that j_1 is injective. We set $\mathrm{Pic}_0 X/Y := E_1/L_1$. $\mathrm{Pic}_0 X/Y$ is thus smooth over Y with $(\mathrm{Pic}_0 X/Y)_y = \mathrm{Pic}_0 X_y$ for $y \in Y$. Moreover we obtain an exact sequence

$$0 \longrightarrow \mathrm{Pic}_0 X/Y \longrightarrow \mathrm{Pic}\, X/Y \xrightarrow{c_1} L_2 \xrightarrow{j_2} E_2$$

of relative complex Lie groups over Y such that taking the sheaves of germs of holomorphic sections of these groups we obtain an exact sequence of $\mathcal{O}_Y$-modules

$$0 \longrightarrow R^1 f_* \mathcal{O}_X / R^1 f_* \boldsymbol{Z} \longrightarrow R^1 f_* \mathcal{O}_X^* \longrightarrow R^2 f_* \boldsymbol{Z} \longrightarrow R^2 f_* \mathcal{O}_X$$

coming from the usual exponential sequence $0 \to \boldsymbol{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ where for $p \in \mathrm{Pic}\, X/Y$, $c_1(p) = c_1(L_p) \in H^2(X_{b(p)}, \boldsymbol{Z}) = (R^2 f_* \boldsymbol{Z})_{b(p)}$.

b) *Essential component*. Let L_γ, $\gamma \in \Gamma'(f)$, be the set of connected componets of L_2. Here a special index $0 \in \Gamma'(f)$ is specified by the condition that L_0 is the zero section of $\varepsilon: L_2 \to Y$. Let $L_\gamma(\underline{0}) := j_2^{-1}(\underline{0}) \cap L_\gamma$, where $\underline{0}$ denotes the zero section of E_2. Let $\mathrm{Pic}_\gamma X/Y := c_1^{-1}(L_\gamma(\underline{0}))$ (in compatible with the above definition of $\mathrm{Pic}_0 X/Y$). Then c_1 induces a proper smooth morphism $c_1(\gamma): \mathrm{Pic}_\gamma X/Y \to L_\gamma(\underline{0})$, the fiber over $q \in L_\gamma(\underline{0})$ being isomorphic to a connected component $\mathrm{Pic}_{\gamma(q)} X_y$ of $\mathrm{Pic}\, X_y$ consisting of those line bundles whose chern class is $q \in L_{\gamma, y} \cong H^2(X_y, \boldsymbol{Z})$ where $y = \varepsilon(q)$. In particular if $j_2(L_\gamma) = \underline{0}$, i.e., $L_\gamma = L_\gamma(\underline{0})$, then $\mathrm{Pic}_\gamma X/Y$ is connected and the natural map $b_\gamma: \mathrm{Pic}_\gamma X/Y \to Y$ is smooth since $L_\gamma \to Y$ is unramified. We call such a component $\mathrm{Pic}_\gamma X/Y$ an *essential component* of $\mathrm{Pic}\, X/Y$. An essential component is precisely a component which is mapped surjectively onto Y.

We denote by $\{\mathrm{Pic}_\gamma X/Y\}$, $\gamma \in \Gamma(f)$, the set of essential components of $\mathrm{Pic}\, X/Y$. When Y is a point we write $\mathrm{Pic}_\gamma X$ instead of $\mathrm{Pic}_\gamma X/Y$. In this case we have the natural identification of $\Gamma(f)$ with the Neron-Severi group $NS(X)$ of X, the group of the first chern classes of line bundles on X.

c) *Some remarks*. i) Let $Y' \subseteqq Y$ be any Zariski open subset. Then the restriction $\mathrm{Pic}_\gamma X/Y \to \mathrm{Pic}_\gamma X_{Y'}/Y'$ sets up a natural bijective correspondence between the sets of essential components of $\mathrm{Pic}\, X/Y$ and $\mathrm{Pic}\, X_{Y'}/Y'$ where $X_{Y'} = X \times_Y Y'$. In particular we can naturally identify $\Gamma(f_{Y'})$ with $\Gamma(f)$.

ii) If for some $\gamma \in \Gamma(f)$, $L_\gamma \to Y$ is finitely unramified of degree m we can associate canonically to L_γ another connected component $L_{a(\gamma)}$ for which $L_{a(\gamma)} \to Y$ is isomorphic; if $L_{\gamma, y} = \{t_1(y), \cdots, t_m(y)\}$, then $y \to \sum_{i=1}^{m} t_i(y)$ defines a holomorphic section $s: Y \to L_2$ and we simply set $L_{a(\gamma)} = s(Y)$. In this case the corresponding $\mathrm{Pic}_{a(\gamma)} X/Y \to Y$ is a smooth fiber space.

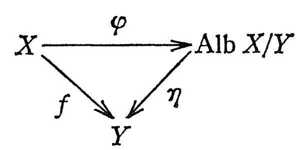
iii) Let $\nu: \tilde{Y} \to Y$ be a surjective morphism of complex varieties. Then we have a unique map $\Gamma(\nu): \Gamma(\tilde{f}) \to \Gamma(f)$ determined by the condition that $P_\gamma(\nu)$: $\mathrm{Pic}_\gamma X/Y \times_Y \tilde{Y} \cong \coprod_{\tilde{\gamma}} \mathrm{Pic}_{\tilde{\gamma}} \tilde{X}/\tilde{Y}$, $\tilde{\gamma} \in \Gamma(\nu)^{-1}(\gamma)$, with respect to the isomorphism P in 1.1 b) where $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{f}: \tilde{X} \to \tilde{Y}$ is the natural morphism.

iv) Let $f': X'\to Y$ and $g: X'\to X$ be as in 1.1 b) ii). Then we have the natural morphism $\Gamma(g): \Gamma(f)\to\Gamma(f')$ by the conidition that $g^*(\mathrm{Pic}_\gamma X/Y) \subseteq \mathrm{Pic}_{\Gamma(g)\gamma} X'/Y$. Of course we have $\Gamma(g)(0)=0$.

v) For $\gamma\in\Gamma(f)$ we set $\mathrm{Div}_\gamma X/Y=(\mu^{-1}_{X/Y}(\mathrm{Pic}_\gamma X/Y))_{red}$ and $\mu_\gamma := \mu_{X/Y}|_{\mathrm{Div}_\gamma X/Y}$: $\mathrm{Div}_\gamma X/Y \to \mathrm{Pic}_\gamma X/Y$. We denote by $Z_\gamma=(Z_{X/Y})_\gamma$ the universal divisor restricted to $\mathrm{Div}_\gamma X/Y$. $m_{X/Y}$, $a_{X/Y}$ defined in 1.1 a) defines Y-morphsims $m_{\gamma,\gamma'}$: $\mathrm{Pic}_\gamma X/Y \times_Y \mathrm{Pic}_{\gamma'} X/Y \to \mathrm{Pic}_{\gamma+\gamma'} X/Y$ (resp. $a_{\gamma,\gamma'}: \mathrm{Pic}_\gamma X/Y \times_Y \mathrm{Pic}_{\gamma'} X/Y \to \mathrm{Pic}_{\gamma-\gamma'} X/Y$) where $\gamma, \gamma'\in\Gamma(f)$; in this way $\Gamma(f)$ itself has the natural structure of an additive group with the identity $0\in\Gamma(f)$.

1.4. *Relative Albanese map* (*smooth case*).

Definition 1. Let $f: X\to Y$ be a smooth fiber space of complex varieties. Then a *relative Albanese map for* f is a commutative diagram of complex varieties



where η is a smooth fiber space with any fiber a complex torus and φ is a Y-morphism, with the following universal property: Let Y' be any complex variety over Y, $T'\to Y'$ any smooth morphism with any fiber a complex torus and $\varphi': X\times_Y Y'\to T'$ any Y'-morphism. Then there exists a unique Y'-morphism $h': \mathrm{Alb}\, X/Y\times_Y Y'\to T'$ such that $\varphi'=h'\varphi_{Y'}$. We often call φ itself the *relative Albanese map for* f and $\mathrm{Alb}\, X/Y$ the *relative Albanese variety for* f.

From the definition the following is true: (P) *For any* $y\in Y$, $\varphi_y: X_y \to (\mathrm{Alb}\, X/Y)_y$ *is isomorphic to the Albanese map* $\varphi(y): X_y\to \mathrm{Alb}\, X_y$ *of* X_y. On the existence of the relative Albanese variety we have the following:

Proposition 1. *Let* $f: X\to Y$ *be a smooth fiber space of complex varieties such that* $X_y\in\mathcal{C}$ *for some* $y\in Y$. *Then a relative Albanese map for* f *exists. Moreover it is up to isomorphisms uniquely characterized by the property* (*P*) *above.*

See [12] for the proof. In fact, we need the proposition only in the case where $X_y\in\mathcal{C}$ for all $y\in Y$ and in this case the construction is easy, though we need not the construction itself here (cf. [13] 236 and the proof of Theorem 1 below).

1.5. *s. ampleness.* Let X be a compact complex manifold. We call $\gamma\in NS(X)$, or any line bundle $L\in\mathrm{Pic}_\gamma X$, *s. ample* (sufficiently ample) if L' is very ample and $H^i(X, L')=0$, $i>0$, for any $L'\in\mathrm{Pic}_\gamma X$. For any ample line bundle L_1 its high multiple is always s. ample (cf. Kodaira [17]). Note that the definition of

s. ampleness naturally extends to any compact complex space (not necessarily reduced).

Let $f: X \to Y$ be a smooth fiber space of compact complex varieties in $\mathcal{C}$. Then an essential component $\mathrm{Pic}_\gamma X/Y$, or the index $\gamma \in \Gamma(f)$ itself, is called s. ample if there exists a Zariski open subset $U_\gamma \subseteq Y$ such that for any $p \in b_\gamma^{-1}(U_\gamma)$ the corresponding line bundle L_p on $X_{b(p)}$ is s. ample. In general, if L_p is s. ample for all $p \in N$ where N is an open subset of $\mathrm{Pic}\, X/Y$, then $\dim \Gamma(X_{b(p)}, L_p) = k+1$ is independent of $p \in N$. Hence by 1.2 c) $\mu_\gamma: \mathrm{Div}_\gamma X/Y \to \mathrm{Pic}_\gamma X/Y$ is a holomorphic $\boldsymbol{P}^k$-bundle over N. In particular this is the case with $N = b_\gamma^{-1}(U)$ if γ is s. ample.

1.6. *Description of Albanese map.* a) Let X be a projective manifold. Fix a base point $o \in X$. Let $\mathcal{L} \to X \times \mathrm{Pic}\, X$ be the Poincaré sheaf associated to X and $o \in X$. Then $\mathcal{L}_0 \to X \times \mathrm{Pic}_0 X$, considered as a family of invertible sheaves on $\mathrm{Pic}_0 X$ parametrized by X, defines the universal morphism $\psi: X \to \mathrm{Pic}_0 \mathrm{Pic}_0 X$. Then ψ is naturally identified with an Albanese map $\psi_X: X \to \mathrm{Alb}\, X$ of X (cf. [13] exposé 236). Let $f: X \to X'$ be a morphism of projective manifolds. Since Pic_0 is contravariant, we get a homomorphism $F: \mathrm{Pic}_0 \mathrm{Pic}_0 X \to \mathrm{Pic}_0 \mathrm{Pic}_0 X'$. Then we have $F\psi_X = \psi_{X'} f$.

b) Fix an s. ample $\gamma \in NS(X)$ so that $\mathrm{Div}_\gamma X$ is a holomorphic $\boldsymbol{P}^k$-bundle over $\mathrm{Pic}_\gamma X$ for some $k > 0$. Let $Z_\gamma \subseteq X \times \mathrm{Div}_\gamma X$ be the universal divisor. To obtain another description of Albanese map first we prove the following:

Lemma 1. *Consider* $Z_{\gamma,o} \subseteq \{o\} \times \mathrm{Div}_\gamma X \subseteq \mathrm{Div}_\gamma X$ *as a divisor on* $\mathrm{Div}_\gamma X$ *and set* $\tilde{Z}_{\gamma,o} = X \times Z_{\gamma,o} \subseteq X \times \mathrm{Div}_\gamma X$. *Let* $\mathcal{F}_\gamma = \mathcal{O}([Z_\gamma])$ *and* $\mathcal{F}'_\gamma = \mathcal{O}([\tilde{Z}_{\gamma,o}])$ *where* $\mathcal{O} = \mathcal{O}_{X \times \mathrm{Div}_\gamma X}$. *Then* $\mathcal{F}_\gamma \otimes_{\mathcal{O}} \mathcal{F}'^{\vee}_\gamma \cong (id_X \times \mu_\gamma)^* \mathcal{L}_\gamma$, *so that in particular* $(id_X \times \mu_\gamma)_* (\mathcal{F}_\gamma \otimes_{\mathcal{O}} \mathcal{F}'^{\vee}_\gamma) \cong \mathcal{L}_\gamma$ *where* $\mathcal{F}'^{\vee}_\gamma = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}'_\gamma, \mathcal{O})$.

Proof. Let $\mathcal{E}_\gamma = \mathcal{F}_\gamma \otimes_{\mathcal{O}} \mathcal{F}'^{\vee}_\gamma$. Since $\mathcal{E}_\gamma$ is trivial when restricted to each fiber of $id_X \times \mu_\gamma$, and $id_X \times \mu_\gamma$ is a $\boldsymbol{P}^k$-bundle, there exists a unique invertible sheaf $\mathcal{M}_\gamma$ on $X \times \mathrm{Pic}_\gamma X$ such that $\mathcal{E}_\gamma = (id_X \times \mu_\gamma)^* \mathcal{M}_\gamma$. It suffices to show that $\mathcal{L}_\gamma \cong \mathcal{M}_\gamma$. By the definitions of these sheaves and of Z_γ we infer readily that 1) for any $p \in \mathrm{Pic}_\gamma X$, $\mathcal{M}_{\gamma,p} \cong \mathcal{O}_X([Z_{\gamma,d}]) \cong \mathcal{L}_{\gamma,p}$ on $X = X \times \{p\} = X \times \{d\}$ where $d \in (\mathrm{Div}_\gamma X)_p$ is an arbitrary point, and 2) $\mathcal{L}_{\gamma,o} \cong \mathcal{O}_{\mathrm{Pic}_\gamma X} \cong \mathcal{M}_{\gamma,o}$ on $\mathrm{Pic}_\gamma X = \{o\} \times \mathrm{Pic}_\gamma X$. From this it follows immediately that $\mathcal{M}_\gamma \cong \mathcal{L}_\gamma$. q. e. d.

c) In the notation of b) Z_γ is a relative divisor also over X since γ is s. ample. Let $\varphi: X \to \mathrm{Div}\,(\mathrm{Div}_\gamma X)$ be the associated universal morphism which factors through $\mathrm{Div}_\delta(\mathrm{Div}_\gamma X) \subseteq \mathrm{Div}\,(\mathrm{Div}_\gamma X)$ for a unique $\delta \in NS(\mathrm{Div}_\gamma X)$. The resulting morphism $X \to \mathrm{Div}_\delta \mathrm{Div}_\gamma X$ will still be denoted by φ.

Lemma 2. *Let* $\psi' := \mu_\delta \varphi: X \to \mathrm{Pic}_\delta(\mathrm{Div}_\gamma X)$ *where* $\mu_\delta: \mathrm{Div}_\delta(\mathrm{Div}_\gamma X) \to \mathrm{Pic}_\delta(\mathrm{Div}_\gamma X)$ *is the natural morphism. Then* ψ' *is an Albanese map of* X.

Proof. Let $\psi_\gamma: X \to \mathrm{Pic}_0(\mathrm{Pic}_\gamma X)$ be the morphism defined by the universality of the Poincaré sheaf $\mathcal{L}_\gamma \to X\times \mathrm{Pic}_\gamma X$, i. e., $\psi_\gamma(x)$ is the point corresponding to the invertible sheaf $\mathcal{L}_{\gamma,x}$ on $\mathrm{Pic}_\gamma X$ (which has the zero chern class). Let $j: \mathrm{Pic}_\delta(\mathrm{Div}_\gamma X) \to \mathrm{Pic}_0(\mathrm{Div}_\gamma X)$ be the isomorphism defined by the subtraction by $\psi'(o)$. Let $\eta_\gamma: \mathrm{Pic}_0(\mathrm{Pic}_\gamma X) \to \mathrm{Pic}_0((\mathrm{Div}_\gamma X)$ be the isomorphism induced by μ_γ. Then we show that $\psi'':=\eta_\gamma^{-1} j\psi': X\to \mathrm{Pic}_0\mathrm{Pic}_\gamma X$ coincides with ψ_γ above. First, let F_x be the line bundle on $\mathrm{Pic}_\gamma X$ corresponding to $\psi'(x) \in \mathrm{Pic}_\delta(\mathrm{Pic}_\gamma X)$ and Z_x the divisor on $\mathrm{Div}_\gamma X$ corresponding to $\varphi(x) \in \mathrm{Div}_\delta(\mathrm{Div}_\gamma X)$ so that $F_x \cong [Z_x]$. Then from the definition of ψ'' it follows that $\psi''(x)$ is the unique line bundle M_x on $\mathrm{Pic}_\gamma X$ satisfying $\mathcal{F}_x \otimes \mathcal{F}_o^{-1} \cong \mu_\gamma^* \mathcal{M}_x$ where $\mathcal{F}_x = \mathcal{O}_{\mathrm{Div}_\gamma X}(F_x)$ and $\mathcal{M}_x = \mathcal{O}_{\mathrm{Pic}_\gamma X}(M_x)$. Then it suffices to show that $\mathcal{M}_x \cong \mathcal{L}_{\gamma,x}$, which is in fact the case by virtue of Lemma 1. Since ψ_γ is naturally isomorphic to $\psi_X: X\to \mathrm{Pic}_0\mathrm{Pic}_0 X$, ψ_γ is an Albanese map of X by a). q. e. d.

§ 2. The Structure of Div*X/Y and Relative Algebraic Reduction

2.1. Div*X/Y. Let $f: X\to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. We write $X_U := f^{-1}(U)$ and $f_U := f|_{f^{-1}(U)}: X_U \to U$.

a) We shall fix some notations which will be used also in Section 3.

i) Let $\mathrm{Pic}_\gamma X_U/U$, $\gamma \in \Gamma(f_U)$, be the essential components of Pic X_U/U (cf. 1.3). In view of 1.3 c) i) if U' is another Zariski open subset over which f is smooth, we can naturally identify the index sets $\Gamma(f_U)$ and $\Gamma(f_{U'})$. So in what follows we may, and we shall, denote $\Gamma(f_U)$ for any U as above by $\Gamma(f)$.

ii) Let $\nu: \tilde{Y} \to Y$ be a surjective morphism with $\tilde{Y}$ a compact complex variety in $\mathcal{C}$. We set $\tilde{X} = X\times_Y \tilde{Y}$, $\tilde{U} = \nu^{-1}(U)$ and $\tilde{f} = f\times_Y \tilde{Y}: \tilde{X} \to \tilde{Y}$. Then we obtain a natural map $\Gamma(\nu): \Gamma(\tilde{f}) \to \Gamma(f)$ by 1.3 c) iii), in view of the above definition of $\Gamma(f)$ and $\Gamma(\tilde{f})$.

iii) Let $f': X' \to Y$ be another fiber space of compact complex varieties in $\mathcal{C}$ which is smooth over U. Let $g: X' \to X$ be a meromorphic Y-map which is holomorphic over U. Then in view of 1.3 c) iv) and the definition of $\Gamma(f)$ and $\Gamma(f')$, g induces a unique map $\Gamma(g): \Gamma(f) \to \Gamma(f')$.

b) The Zariski open subset $\mathrm{Div}_\gamma(X_U/U) \subseteq D_{X_U/U,red}$ is also Zariski open in $D_{X/Y,red}$ (cf. 1.2 a)). Let $\mathrm{Div}_\gamma^- X/Y$ be the closure of $\mathrm{Div}_\gamma X_U/U$ in $D_{X/Y,red}$ and $\mathrm{Div}_\gamma^* X/Y$ the union of those irreducible components of $\mathrm{Div}_\gamma^- X/Y$ which are mapped surjectively onto Y. Then it is readily seen that $\mathrm{Div}_\gamma^* X/Y$ is independent of the choice of U as above (as a subspace of $D_{X/Y}$). Let Z_γ^* be the closure of $Z_\gamma \subseteq X_U \times_U \mathrm{Div}_\gamma X_U/U$ in $X\times_Y \mathrm{Div}_\gamma^* X/Y$ which is again analytic and proper over Y. We call Z_γ^* the meromorphic universal relative divisor for each γ. Z_γ^* neither depends on the choice of U as above. We further set $\mathrm{Div}^* X/Y = \bigcup_{\gamma\in\Gamma(f)} \mathrm{Div}_\gamma^* X/Y$. We also recall that $\mathrm{Div}_\gamma^* X/Y$ is a compact complex space in $\mathcal{C}$ (cf. [6]).

c) As follows easily from the definition the formation of Div*X/Y has the following properties.

i) If $\nu: \tilde{Y}\to Y$ is as in a) ii), then we have the natural isomorphism $D_{\gamma}^{*}(\nu)$: $\mathrm{Div}_{\gamma}^{*}X/Y\dot{\times}_{Y}\tilde{Y}\cong\cup_{\tilde{\gamma}}\mathrm{Div}_{\tilde{\gamma}}^{*}\tilde{X}/\tilde{Y}$, $\tilde{\gamma}\in\Gamma(\nu)^{-1}(\gamma)$, and hence $\mathrm{Div}^{*}X/Y\dot{\times}_{Y}\tilde{Y}\cong\mathrm{Div}^{*}\tilde{X}/\tilde{Y}$ with respect to the natural isomorphism $D_{X/Y}\times_{Y}\tilde{Y}\cong D_{\tilde{X}/\tilde{Y}}$ where $\dot{\times}$ denotes the strict pull-back (cf. Convention).

ii) Let $f': X'\to Y$ and $g: X'\to X$ be as in a) iii) above. Let $\delta_{\gamma}^{*}: \mathrm{Div}_{\gamma}^{*}X/Y\to Y$ be the structure morphism. If for general $d\in\mathrm{Div}_{\gamma}^{*}X/Y$, $g(X')_{y}\not\subseteq Z_{\gamma,d}^{*}$ $(y=\delta_{\gamma}^{*}(d))$, then g induces a natural meromorphic Y-map $\tilde{g}_{\gamma}^{*}: \mathrm{Div}_{\gamma}^{*}X/Y\to\mathrm{Div}_{\gamma'}^{*}X'/Y$ with $\gamma'=\Gamma(g)\gamma$ and hence a meromorphic Y-map $\tilde{g}^{*}: \mathrm{Div}^{*}X/Y\to\mathrm{Div}^{*}X'/Y$.

iii) There exists a meromorphic Y-map $\tilde{m}_{\gamma,\gamma'}^{*}: \mathrm{Div}_{\gamma}^{*}X/Y\times_{Y}\mathrm{Div}_{\gamma'}^{*}X/Y\to\mathrm{Div}_{\gamma+\gamma'}^{*}X/Y$, $\gamma, \gamma'\in\Gamma(f)$, which is bimeromorphic over U to the U-morphism $\tilde{m}_{\gamma,\gamma'}: \mathrm{Div}_{\gamma}X_{U}/U\times_{U}\mathrm{Div}_{\gamma'}X_{U}/U\to\mathrm{Div}_{\gamma+\gamma'}X_{U}/U$ induced by $\tilde{m}_{X/Y}$ (cf. 1.2 b)).

2.2. *Some lemmas on s. ample components.*

Lemma 3. *Let $f: X\to T$ be a proper morphism of compact complex varieties. Let $\mathcal{F}$ be a coherent analytic sheaf on X. Suppose that there exists a Zariski open subset $U\subseteq T$ such that $\mathcal{F}$ is invertible on X_U and f is flat on X_U. Suppose further that there exists $o\in U$ such that $\mathcal{F}_o$ is s. amples on X_o. Then there exists a Zariski open subset $V\subseteq T$ such that $V\subseteq U$ and $V=\{t\in U; \mathcal{F}_t$ is s. ample on $X_t\}$.*

Proof. If f is flat and $\mathcal{F}$ is invertible on the whole X, then by the Zariski openness of very ampleness and the upper semicontinuity of cohomology dimension on the fibers, it is immediate to see that the set $T':=\{t\in T; \mathcal{F}_t$ is s. ample$\}$ itself is Zariski open. Then we have only to set $V=T'\cap U$. In the general case take a proper modification $\sigma_1: X_1\to X$ such that the strict transform $\mathcal{F}_1$ of $\mathcal{F}$ on X_1 is invertible and that σ_1 gives an isomorphism of $\sigma_1^{-1}(X_U)$ and X_U [21]. Let $\eta: T_2\to T$ be a proper modification such that $\eta|_{\eta^{-1}(U)}: \eta^{-1}(U)\to U$ is isomorphic and that the strict transform X_2 of X_1 in $X_1\times_T T_2$ is flat over T_2 ([15]). Let $\mathcal{F}_2$ be the pull-back of $\mathcal{F}_1$ to X_2. Then $T_2':=\{t\in T_2; \mathcal{F}_{2,t}$ is s. ample$\}$ is Zariski open in T_2 as above. Then we have only to set $V=\eta(T_2'\cap\eta^{-1}(U))$. q. e. d.

Lemma 4. *Let $f: X\to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let $U\subseteq Y$ be a Zariski open subset over which f is smooth. Let $\mathrm{Pic}_{\gamma}X_U/U$ be an essential component of $\mathrm{Pic}\,X_U/U$. If there exists a point $p\in\mathrm{Pic}_{\gamma}X_U/U$ such that the corresponding line bundle L_p on $X_{b(p)}$ is s. ample, then $\mathrm{Pic}_{\gamma}X_U/U\to U$ is proper, smooth and γ is s. ample.*

Proof. Let $y=b(p)$. Write for simplicity $P_{\gamma}=\mathrm{Pic}_{\gamma}X_U/U$. Let $P_{\gamma,y,1}$ be a connected component of $P_{\gamma,y}$ with $p\in P_{\gamma,y,1}$ so that every point of $P_{\gamma,y,1}$ corresponds to an s. ample line bundle on $X_{b(p)}$. Then there exists a Zariski open neighborhood N of $P_{\gamma,y,1}$ in P_{γ} such that for any $q\in N$, the corresponding line bundle L_q on $X_{b(q)}$ is s. ample (cf. the proof of the previous lemma). Then since $\mu_{\gamma}(\mathrm{Div}_{\gamma}X_U/U)\supseteq N$ and $\delta_{X_U/U}=b_{X_U/U}\mu_{X_U/U}$, $\delta_{\gamma}^{*}(\mathrm{Div}_{\gamma}^{*}X/Y)$ contains an open subset of Y. Hence, being an analytic subset of Y, it coincides with Y. Write $D_\gamma^*=\mathrm{Div}_\gamma^* X/Y$. By the previous lemma applied to $X\dot{\times}_Y D_\gamma^*\to D_\gamma^*$ there exists a Zariski open subset $V\subseteq D_\gamma^*$ such that $V=\{d\in \mathrm{Div}_\gamma X_U/U\,;\,[Z_{X/Y}]_d$ is s. ample on $X_{\delta(d)}\}$. On the other hand, by the definition of s. ampleness for any $u\in U$ and any connected component $D_{\gamma,u,k}^*$ of $D_{\gamma,u}^*$ we have either $D_{\gamma,y,k}^*\cap V=\emptyset$ or $D_{\gamma,y,k}^*\subseteq V$. Then it is easy to find a Zariski open subset $U_\gamma\subseteq Y$ contained in U such that L_p is s. ample for any $p\in b_\gamma^{-1}(U_\gamma)$. Thus γ is s. ample. Finally since $D_{\gamma,U}^*\to P_\gamma$ is surjective and $D_{\gamma,U}^*$ is proper over U, P_γ is proper and smooth over U (cf. 1.3 b)). q. e. d.

Let $f: X\to Y$ be a fiber space of complex varieties. We say that f is *locally projective* if for any $y\in Y$ there exists a neighborhood $y\in V$ such that $f_V: X_V\to V$ is projective. We say that f is *generically locally projective* if there exists a Zariski open subset $U\subseteq Y$ such that $f_U: X_U\to U$ is locally projective.

Lemma 5. *Let $f: X\to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let U be a Zariski open subset over which f is smooth. Suppose that f is projective and smooth over an open subset $W\subseteq U$. Then there exists an s. ample component* $\mathrm{Pic}_\gamma X_U/U$ *of* $\mathrm{Pic}\, X_U/U$ *such that* $\mathrm{Pic}_\gamma X_U/U\to U$ *is a smooth fiber space. In particular f is generically locally projective.*

Proof. Fix $y\in W$. Let L be a line bundle of X_W which is very ample with respect to $f_W: X_W\to W$ (restricting W if necessary). Replacing L by its high multiple we may assume that $L|_{X_y}$ is s. ample. Let $s: W\to \mathrm{Pic}\, X_W/W$ be the holomorphic section defined by L. Let $\mathrm{Pic}_\gamma X_U/U$, $\gamma\in\Gamma(f_U)$, be the unique connected component of $\mathrm{Pic}\, X_U/U\supseteq \mathrm{Pic}\, X_W/W$ containing $s(W)$. Then by Lemma 4 γ is s. ample. If $\mathrm{Pic}_\gamma X_U/U\to U$ is not a fiber space, we have only to replace $\mathrm{Pic}_\gamma X_U/U$ by $\mathrm{Pic}_{a(\gamma)} X_U/U$ (cf. 1.3 c)). In fact, we easily check that each point $p\in b_{a(\gamma)}^{-1}(U_\gamma)$ corresponds to an s. ample line bundle on $X_{b(p)}$ with U_γ as in 1.5. It follows that $f_{U_\gamma}: X_{U_\gamma}\to U_\gamma$ is locally projective. q. e. d.

Lemma 6. *Let $f: X\to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let $U\subseteq Y$ be a Zariski open subset over which f is smooth. Suppose that f is generically locally projective. Then for any $\gamma\in\Gamma(f)$ we can find s. ample elements α, $\beta\in\Gamma(f)$ such that $\gamma=\alpha-\beta$ and that $a_{\alpha\beta}$:* $\mathrm{Pic}_\alpha X_U/U \times_U \mathrm{Pic}_\beta X_U/U\to \mathrm{Pic}_\gamma X_U/U$ *is a fiber space.*

Proof. Take and fix an s. ample α' according to Lemma 5 so that in particular $\mathrm{Pic}_{\alpha'} X_U/U$ is a fiber space over U. Fix any $p\in \mathrm{Pic}_{\alpha'} X_U/U$ such that the corresponding line bundle L_p is s. ample on $X_{b(p)}$. Let $y=b(p)$. Take any $q\in(\mathrm{Pic}_\gamma X_U/U)_y$. Then $L_p^{\otimes n}\otimes L_q$ is s. ample on X_y for a sufficiently large n. Then we set $\alpha=n\alpha'$ and $\beta=\alpha+\gamma$. Then α and β are s. ample. This is clear for α and is true for β by Lemma 4 since $L_p^{\otimes n}\otimes L_q=L_r$ for some $r\in(\mathrm{Pic}_\beta X_U/U)_y$. Finally, since $\mathrm{Pic}_\alpha X_U/U$ is a fiber space over U as well as $\mathrm{Pic}_{\alpha'} X_U/U$, it follows readily that $a_{\alpha\beta}$ also is a fiber space. q. e. d.

2.3. *Local projectivity of* $\mathrm{Div}_\gamma^* X/Y$.

Lemma 7. *Let $f: X \to Y$ be a generically smooth fiber space of complex varieties. Let $V \subseteq Y$ be an open subset such that X_y are smooth and projective for all $y \in V$. Then f is projective over some open subset of V.*

Proof. By assumption for any $y \in V$ there exist an irreducible component $D(y)$ of $\mathrm{Div}^- X/Y$ (where $\mathrm{Div}^- X/Y$ is the closure of $\mathrm{Div}_U X_U/U$ in $D_{X/Y}$ with U as in 2.1) and a point $d = d(y) \in D(y)_y$ such that the corresponding divisor Z_d on X_y is ample. Then we have $V \subseteq \bigcup_{y \in V} \delta(D(y))$. Since $\mathrm{Div}^- X/Y$ is countable (cf. 1.3), by Baire argument $V \subseteq \delta(D(y_0))$ for some $y_0 \in V$. By the Zariski openness of the ampleness there exists a neighborhood W of $d(y_0)$ in $D(y_0)$ such that the divisor Z_d is ample on $X_{\delta(d)}$ for all $d \in W$. Take any open subset $V_0 \subseteq V$ on which we can find a holomorphic section $V_0 \to W$. Then it is immediate to see that f is projective over V_0. q. e. d.

Proposition 2. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Then any irreducible component of* $\mathrm{Div}^* X/Y$ *is generically locally projective over Y.*

Proof. Let D_k^* be any irreducible component of $\mathrm{Div}^* X/Y$. D_k^* is a compact complex variety in $\mathcal{C}$ and the natural morphism $D_k^* \to Y$ is surjective. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. Let $\mu_{k,U}: D_{k,U}^* \to \mathrm{Pic}_\gamma X_U/U$ be induced by $\mu_{X_U/U}$ where $D_k^* \subseteq \mathrm{Div}_\gamma^* X/Y$. Let $B_k := \mu_{k,U}(D_{k,U}^*) \subseteq \mathrm{Pic}_\gamma X_U/U$. Since $\mu_{X_U/U}$ is projective over any open subset $V \subseteq U$ over which f admits a holomorphic section (cf. 1.2 c)), it suffices by Lemma 5 to show that the analytic set B_k is projective over some open subset of U. Let $r_U: \tilde{B}_k \to B_k$ be a resolution. Take an open subset U' of U such that $\tilde{B}_k$ is smooth over U'. On the other hand, $(D_{k,U}^*)_y$ is projective as a compact subspace of $(\mathrm{Div}\, X/Y)_y = \mathrm{Div}\, X_y$ [9]. Hence each fiber of $\tilde{B}_{k,U'} \to U'$ is Moishezon. So there exists a relative Albanese map $\varphi: \tilde{B}_{k,U'} \to A = \mathrm{Alb}(\tilde{B}_{k,U'}/U')$ for $\tilde{B}_{k,U'} \to U'$ with a smooth structure morphism $\eta: A \to U'$, such that each fiber of η is an abelian variety. Then by the universality of the relative Albanese map we have a unique U'-morphism $h: A \to \mathrm{Pic}\, X_{U'}/U'$ such that $h\varphi = ir_{U'}$ where $i: B_{k,U'} \to \mathrm{Pic}\, X_{U'}/U'$ is the inclusion. Let $\bar{A} = h(A) \subseteq \mathrm{Pic}\, X_{U'}/U'$. Then $\bar{A}$ is smooth over U' and each fiber is an abelian variety. Hence by Lemma 7 $\bar{A} \to U'$ is projective over some open subset W of U'. As a subspace of $\bar{A}_{U'}$, B_k is a fortiori projective over W as was desired.

q. e. d.

2.4. *Relative algebraic dimension.* a) Let X be a compact complex space and L a line bundle on X. Let $\kappa(X, L)$ be the L-dimension of X in the sense of Iitaka (cf. [23]). The following is shown in Lieberman-Sernesi [19]: *Let f: $X \to Y$ be a flat fiber space of complex spaces. Let L be a line bundle on X and $k \geq 0$ an integer. Then the set $Y_k = \{y \in Y; \kappa(X_y, L_y) \geq k\}$ is a union of at most* *countably many analytic subvarieties of* Y.

Lemma 8. *Let* $f: X \to Y$ *be a generically smooth fiber space of complex spaces. Let* Z *be a subspace of* X *of pure codimension* 1. *Let* $U \subseteq Y$ *be a smooth Zariski open subset over which* f *is smooth and* $f|_Z$ *is flat. Let* $k \geqq 0$ *be an integer. Then the set* $A_k(Z) := \{y \in U; \kappa(X_y, [Z_y]) \geqq k\}$ *is a union of at most countably many analytic subsets of* U *whose closures in* Y *are analytic.*

Proof. Let $\sigma: \tilde{X} \to X$ be the blowing up of X with center Z and $\tilde{Z}$ the inverse image of Z in $\tilde{X}$. $\tilde{Z}$ is then a Cartier divisor on $\tilde{X}$ and σ is isomorphic over U. Let $\tilde{L} = [\tilde{Z}]$ be the line bundle defined by $\tilde{Z}$. Let $\tilde{f} = f\sigma: \tilde{X} \to Y$. Then take a proper modification $\varphi: Y' \to Y$ such that φ is isomorphic on $\varphi^{-1}(U)$ and the strict transform X' of $\tilde{X}$ in $\tilde{X} \times_Y Y'$ is flat over Y' (cf. [15]). Let $\psi: X' \to \tilde{X}$ be the natural morphism. Let $L' = \psi^* \tilde{L}$. By our construction we may regard $A_k(Z) \subseteq \varphi^{-1}(U) \subseteq Y'$. Let $A_k(L') := \{y' \in Y'; \kappa(X'_{y'}, L'_{y'}) \geqq k\}$. Then by the result of Liebermann-Sernesi cited above $A_k(L')$ is a union of at most countably many analytic subvarieties $A_k(L')_\nu$ of Y' and $A_k(L') \cap \varphi^{-1}(U) = A_k(Z)$ with respect to the above identification. It follows that the closure of $A_k(Z)$ in Y is a union of those $\varphi(A_k(L)_\nu)$ with $A_k(L)_\nu \cap \varphi^{-1}(U) = \emptyset$. q. e. d.

b) For any compact complex variety we shall denote by $a(X)$ its algebraic dimension (cf. [23]). When X is nonsingular, then $a(X) \geqq k$ if and only if there exists a line bundle L on X with $\kappa(X, L) \geqq k$.

Proposition 3. *Let* $f: X \to Y$ *be a generically smooth fiber space of compact complex varieties in* $\mathcal{C}$. *Let* $U \subseteq Y$ *be a Zariski open subset over which* f *is smooth. For any integer* $k \geqq 0$ *we set* $A_k := \{y \in U; a(X_y) \geqq k\}$. *Then* A_k *is at most a countable union of analytic subsets of* U *whose closures in* Y *are analytic.*

Proof. Let $\mathrm{Div}^- X/Y$ be the closure of $\mathrm{Div}\, X_U/U$ in $D_{X/Y}$ and $Z^-_{X/Y}$ the closure of the universal relative divisor $Z_{X_U/U}$ in $(\mathrm{Div}^- X/Y) \times_Y X$. Since Z^- is of pure codimension 1 in $(\mathrm{Div}^- X/Y) \times_Y X$, by Lemma 8 the set $B_k(U) = \{d \in \mathrm{Div}\, X_U/U; \kappa(X_{\delta(d)}, [Z_{U,d}]) \geqq k\}$ (where $Z = Z^*_{X/Y}$) is a union of at most countably many analytic subsets $B^0_{k,\nu}$, $\nu \in N$, of $\mathrm{Div}\, X_U/U$ whose closures $B_{k,\nu}$ of $B^0_{k,\nu}$ are analytic in $\mathrm{Div}^- X/Y$. Since $X \in \mathcal{C}$, $B_{k,\nu}$ are all compact. Let $\bar{B}_{k,\nu} = \delta(B_{k,\nu})$ and $\bar{B}_k = \bigcup_\nu \bar{B}_{k,\nu}$. Then by the above remark, for $y \in U$, $a(X_y) \geqq k$ if and only if $y \in \bar{B}_k$, i.e., $A_k = \bar{B}_k$. The proposition follows. q. e. d.

c) Let $f: X \to Y$ be as in Proposition 3. Since $A_k \supseteq A_{k+1}$ and $A_0 = U$, there exists a unique maximal k such that $A_k = U$. By the above proposition this number k is independent of the choice of U as above.

Definition 2. We shall call k the *algebraic dimension of* f, or the *relative algebraic dimension of* X *over* Y and denote it by $a(f)$; $a(f) = k$. It follows from the above proposition that $a(f) = k$ if and only if $a(X_y) = k$ for 'general'

$y \in Y$, i. e., if y is in a complement of at most countably many proper analytic subvarieties of Y.

2.5. *Relative algebraic reduction.*

Definition 3. Let $f: X \to Y$ be a fiber space of compact complex varieties in $\mathcal{C}$. Then a *relative algebraic reduction for f* is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & \bar{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

of compact complex varieties in $\mathcal{C}$ where h is a fiber space and g is a meromorphic fiber space such that 1) $a(h)=a(f)$ and 2) $a(h)=\dim h$. We also call the map $g: X \to \bar{X}$ a relative algebraic reduction of f.

Here and in what follows we call a meromorphic map $g: X \to \bar{X}$ of complex varieties a *meromorphic fiber space* if g is generically surjective and its general fiber is irreducible.

Proposition 4. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Then there exists a relative algebraic reduction $g: X \to \bar{X}$ for f such that $\bar{X}$ is generically locally projective over Y. In particular if $a(f)=$* $\dim f$, *we can always find a bimeromorphic model $f': X' \to Y$ of f which is generically locally projective.*

Proof. Let $W \subseteqq U$ be an open subset on which there exists a holomorphic section s_W to $B_k(U) \to U$ where $B_k(U)$ is as in the proof of Proposition 3 with $k=a(f)$. The holomorphic line bundle $L_W:=(id_{X_W \times_W} s_W)^*[Z_W]$ on $X_W = X_W \times_W W$ satisfies $\kappa(X_y, L_y) \geqq a(f)$ for all $y \in W$ and that the equality holds for 'general' y where $L_y = L_W|_{X_y}$. Then after restricting W if necessary, for some sufficiently large $m>0$, the meromorphic W-map $X_W \to \boldsymbol{P}(f_* \mathcal{L}_W^{\otimes m})$ associated to the coherent sheaf $f_* \mathcal{L}_W^{\otimes m}$ has the property that if $\bar{Z}_W \subseteqq \boldsymbol{P}(f_* \mathcal{L}_W^{\otimes m})$ is the image of the map, then the induced map $\bar{\varphi}_W; X_W \to \bar{Z}_W$ is a meromorphic fiber space and $\dim \bar{h}_W = a(f)$ where $\bar{h}_W: \bar{Z}_W \to W$ is the natural morphism. Let D_α be any irreducible component of $\mathrm{Div}^* X/Y$ containing $s_W(W)$. (Clearly $s_W(W) \subseteqq \mathrm{Div}^* X/Y$). We consider the universal meromorphic Y-map $X \to \mathrm{Div}^* D_\alpha/Y$ associated to the inclusion $Z_\alpha \subseteqq D_\alpha \times_Y X$ where Z_α is considered to be a relative divisor over a Zariski open subset of X (cf. [5], Lemma 5.1). Let $\bar{X}$ be its image, $g: X \to \bar{X}$ the resulting meromorphic Y-map, and $h: \bar{X} \to Y$ the natural morphism. Then from the definition of D_α together with the construction of g it follows that over W we have a unique meromorphic W-map $\eta_W: \bar{X}_W \to \bar{Z}_W$ such that $\bar{\varphi}_W = \eta_W g_W$. On the other hand, by Proposition 2 $\bar{X}$ is generically locally projective over Y.

Hence $\dim h \leqq a(f)$, while we have $a(f) = \dim \bar{h}_W \leqq \dim h$. Thus $\dim h = a(f)$ and η_W must be bimeromorphic. Hence g_W is a meromorphic fiber space as well as $\bar{\varphi}_W$. Thus g also is a meromorphic fiber space. Hence g is a relative algebraic reduction of f. q. e. d.

Remark 1. Using Chow's lemma [15] we may assume in the final assertion that X' is nonsingular and is obtained by a succession of monoidal transformations with nonsingular centers from X.

§ 3. Construction of Pic*X/Y

Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let U be a Zariski open subset of Y over which f is smooth. We assume throughout this section that $a(f) = \dim f$.

The purpose of this section is to associate to each such fiber space a complex space Pic*X/Y over Y with a certain 'meromorphic universal property'. It is roughly an 'extension' of the relative Picard variety Pic $X_U/U \to U$ for the smooth morphism $f_U: X_U \to U$ to the whole Y.

3.1. First we prove two simple lemmas which provide us with the main technique for construction.

Lemma 9. *Let $f: X \to Y$ be a proper surjective morphism of compact complex varieties in $\mathcal{C}$ which is generically smooth. Suppose that there exist Zariski open subsets $V \subseteqq U \subseteqq Y$, a proper surjective morphism $g: Z \to U$ of complex varieties and a U-morphism $h: X_U \to Z$ which is a fiber space and is flat over Z_V. Then there exists a canonical compactification $Z_V \hookrightarrow Z^*$ of Z_V into a compact complex variety Z^* in $\mathcal{C}$ over Y such that h_V extends to a meromorphic Y-map $h^*: X^* \to Z^*$ which is bimeromorphic over U to h_U.*

Proof. Set $W = Z_V$. Considering $h_W: X_W \to W$, $X_W = (X_U)_W$, as a flat family of subspaces of X over Y with respect to the embedding $h_W \times_Y \iota_W: X_W \to W \times_Y X$ where $\iota_W: X_W \to X$ is the natural inclusion, we get the universal Y-morphism $\tau: W \to D_{X/Y}|_U$, where W is naturally over Y. Then τ is clearly injective and by [16], Lemma 3, it is an open embedding at each point of W. Moreover τ extends to a meromorphic Y-map $\tau^*: Z \to D_{X/Y}|_U$ (cf. [5]). Hence there exists a unique irreducible component D_α of $D_{X/Y,red}$ which contains $\tau^*(Z)$ as a Zariski open subset of $D_{\alpha,U}$. Let

$$\begin{array}{ccc} Z_\alpha & \subseteqq & D_\alpha \times_Y X \\ {\scriptstyle \rho_\alpha}\downarrow & \swarrow & \\ D_\alpha & & \end{array}$$

be the universal family restricted to D_α. By our construction ρ_α restricted to

$\tau(W)$ is naturally isomorphic to h_W. Therefore the natural map $\pi_\alpha: Z_\alpha \to X$ is bimeromorphic, being isomorphic to the inclusion $\iota_W: X_W \to X$ over $W = \tau(W)$. Hence it suffices to take $X^* = Z_\alpha$, $Z^* = D_\alpha$, $h^* = \rho_\alpha$. Finally the canonicity of the compactification means that if $V' \subseteq U \subseteq Y$ is another Zariski open subset such that h_U is flat over $Z_{V'}$, then the resulting complex variety $Z^{*\prime}$ compactifying $Z_{V'}$ via the above procedure is canonically isomorphic to the above Z^*. This is indeed clear from our construction. q. e. d.

Lemma 10. *Let Y be a complex variety. Let $V \subseteq U \subseteq Y$ be Zariski open subsets. Let X_1, X_1', X, X' be reduced complex spaces over Y which are proper over Y. Let $\psi: X_1 \to X_1'$, $h: X_1 \to X$, $h': X_1' \to X'$ be meromorphic Y-maps which are holomorphic over V. We assume that h is surjective and each irreducible component of X is mapped surjectively onto Y. Let X_0 and X_0' be reduced complex spaces over U. Suppose that there exist a meromorphic U-map $\psi_0: X_0 \to X_0'$ and a bimeromorphic U-map $\iota: X_U \to X_0$ (resp. $\iota': X_U' \to X_0'$) which is isomorphic over V such that $\psi_0 \iota h_U = \iota' h_U' \psi_U$. Then there exists a meromorphic Y-map $\bar{\psi}: X \to X'$ such that $\iota' \bar{\psi}_U = \psi_0 \iota$.*

Proof. Let $\Gamma \subseteq X_1 \times_Y X_1'$ be the graph of ψ. Let $\bar{\Gamma} = h \times_Y h'(\Gamma) \subseteq X \times_Y X'$. Then since h and h' are holomorphic over V, h is surjective, and since $\psi_0 \iota h_U = \iota' h_U' \psi_U$, $\bar{\Gamma}_V := \bar{\Gamma} \cap (X_V \times_V X_V)$ coincides with the graph of $\iota'^{-1} \psi_0 \iota |_{X_V}$. Then the closure $\bar{\Gamma}'$ of $\bar{\Gamma}_V$ in $X \times_Y X'$ gives a graph of a meromorphic Y-map $\bar{\psi}: X \to X'$ by virtue of our assumption on X. Moreover again by the above commutativity, over $U \bar{\Gamma}'$ must coincides with the graph of $\iota'^{-1} \psi_0 \iota$. q. e. d.

Corollary. *Let $V \subseteq U \subseteq Y$, X_0, X_0', X, X', ι, ι', and ψ_0 be as above. Let $\nu: \tilde{Y} \to Y$ be a proper surjective morphism of complex varieties. Let $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{X}' = X' \times_Y \tilde{Y}$. Let $\tilde{U} = \nu^{-1}(U)$ and $\tilde{V} = \nu^{-1}(V)$. If there exists a meromorphic $\tilde{Y}$-map $\psi: \tilde{X} \to \tilde{X}'$ which is bimeromorphic to $\psi_0 \times_U \tilde{U}$ over $\tilde{U}$ and is isomorphic to $\psi_0 \times_V \tilde{V}$ over $\tilde{V}$, the conclusion of the above lemma holds true.*

Proof. It suffices to take $X_1 = \tilde{X}$ and $X_1' = \tilde{X}'$ in the above proposition.

Recall that a proper morphism $f: X \to Y$ of complex spaces is called *Moishezon* if it is bimeromorphic to a projective morphism (cf. [6]). We record the following well-known:

Lemma 11. *Let $f: X \to Y$ be a proper morphism of reduced complex spaces. Suppose that there exists a dense Zariski open subset $U \subseteq Y$ such that X_y is a complex projective space for any $y \in U$. Then f is Moishezon.*

Proof. It suffices to show that for any irreducible component Y_i of Y the induced morphism $f_i: f^{-1}(Y_i) \to Y_i$ is Moishezon. So we may assume that Y is irreducible. Restricting U we may assume that U is smooth and that $f_U: X_U \to U$ is flat and hence is smooth. Let $r: \tilde{X} \to X$ be a resolution and $\tilde{f} = fr: \tilde{X} \to Y$. Clearly $\tilde{f}$ satisfies the condition of the lemma. Then the meromorphic Y-map

$\tilde{X}\to \boldsymbol{P}(\tilde{f}_*\mathcal{K}_{\tilde{X}}^{-1})$ is bimeromorphic onto its image, where $\mathcal{K}_{\tilde{X}}$ is the canonical sheaf of $\tilde{X}$. Hence $\tilde{f}$, and hence f also, is Moishezon. q. e. d.

3.2. Let $f: X\to Y$ and $U\subseteqq Y$ be as at the beginning of this section. Then a precise formulation for Pic^*X/Y will now be given in the following:

Definition 4. Let $\{\mathrm{Pic}_\gamma X_U/U\}$, $\gamma\in\Gamma(f)$, be the set of essential components of $\mathrm{Pic}\, X_U/U$ where $\Gamma(f)$ is as in 2.1 a) i). Then we say that Pic^*X/Y *exists* if the following is true. For any $\gamma\in\Gamma(f)$ there exists a compact complex variety $\mathrm{Pic}^*_\gamma X/Y$ in $\mathcal{C}$ over Y with the following properties.

1) $\mathrm{Pic}^*_\gamma X/Y$ and $\mathrm{Pic}_\gamma X_U/U$ are bimeromorphic to each other over U and are isomorphic over some Zariski open subset U_γ of Y with $U_\gamma\subseteqq U$.

2) For any $\nu: \tilde{Y}\to Y$ as in 2.1 a) ii), $\mathrm{Pic}^*_\gamma X/Y\dot{\times}_Y\tilde{Y}$ is naturally bimeromorphic over $\tilde{Y}$ to $\coprod_{\tilde{\gamma}}\mathrm{Pic}^*_{\tilde{\gamma}}\tilde{X}/\tilde{Y}$, $\tilde{X}=X\dot{\times}_Y\tilde{Y}$, where $\tilde{\gamma}\in\Gamma(\nu)^{-1}(\gamma)$.

3) For any $f': X'\to Y$ and $g: X'\to X$ as in 2.1 a) iii) we have a unique meromorphic Y-map $g^*_\gamma: \mathrm{Pic}^*_\gamma X/Y\to\mathrm{Pic}^*_{\gamma'}X'/Y$, $\gamma'=\Gamma(g)(\gamma)$, which is bimeromorphic to the natural U-morphism $g^*_U: \mathrm{Pic}_\gamma X_U/U\to\mathrm{Pic}_{\gamma'}X'_U/U$.

4) There exists a meromorphic Y-map $\mu^*_\gamma: \mathrm{Div}^*_\gamma X/Y\to\mathrm{Pic}^*_\gamma X/Y$ which is bimeromorphic to $\mu_\gamma: \mathrm{Div}_\gamma X_U/U\to\mathrm{Pic}_\gamma X_U/U$ over U (cf. 1.3 c) v)). Moreover μ^*_γ is Moishezon (i. e., any of its holomorphic model is Moishezon).

5) There exists a meromorphic Y-map $m^*_{\gamma,\gamma'}: \mathrm{Pic}^*_\gamma X/Y\dot{\times}_Y\mathrm{Pic}^*_{\gamma'}X/Y\to\mathrm{Pic}^*_{\gamma+\gamma'}X/Y$ (resp. $a^*_{\gamma,\gamma'}: \mathrm{Pic}^*_\gamma X/Y\dot{\times}_Y\mathrm{Pic}^*_{\gamma'}X/Y\to\mathrm{Pic}^*_{\gamma-\gamma'}X/Y$) which is bimeromorphic over U to $m_{\gamma,\gamma'}: \mathrm{Pic}_\gamma X_U/U\times_U\mathrm{Pic}_{\gamma'}X_U/U\to\mathrm{Pic}_{\gamma+\gamma'}X_U/U$ (resp. $a_{\gamma,\gamma'}: \mathrm{Pic}_\gamma X_U/U\times_U\mathrm{Pic}_{\gamma'}X_U/U\to\mathrm{Pic}_{\gamma-\gamma'}X_U/U$) (cf. 1.3 c) v)) such that $\mu^*_{\gamma+\gamma'}\tilde{m}^*_{\gamma,\gamma'}=m^*_{\gamma,\gamma'}(\mu^*_\gamma\times_Y\mu^*_{\gamma'})$ where γ, $\gamma'\in\Gamma(f)$. Moreover there exists a meromorphic section $Y\to\mathrm{Pic}^*_0X/Y$ which is bimeromorphic to the identity section of $\mathrm{Pic}_0X_U/U\to U$.

6) Let $\nu: \tilde{Y}\to Y$ and $\tilde{X}$ be as in 2). Let $\mathcal{F}$ be any coherent analytic sheaf on $\tilde{X}$ which is invertible on $\tilde{X}_{\tilde{U}}$, $\tilde{U}=\nu^{-1}(U)$. Let $\tau: \tilde{U}\to\mathrm{Pic}_\gamma X_U/U$ be the universal U-morphism defined by $\mathcal{F}|_{\tilde{X}_{\tilde{U}}}$ for a unique $\gamma\in\Gamma(f)$. Then τ extends to a unique meromorphic Y-map $\tilde{\tau}: \tilde{Y}\to\mathrm{Pic}^*_\gamma X/Y$.

7) If f is Moishezon, then the structure morphism $\mathrm{Pic}^*_\gamma X/Y\to Y$ also is Moishezon.

8) $\mathrm{Pic}^*_\gamma X/Y$ is (up to bimeromorphic equivalences over Y) independent of the choice of U, so that in particular the above properties are valid for any U as above.

If Pic^*X/Y exists for f in the sense defined above, we set $\mathrm{Pic}^*X/Y=\coprod_\gamma\mathrm{Pic}^*_\gamma X/Y$ which is naturally a complex space over Y. In terms of Pic^*X/Y the above properties can informally be stated as follows. 1) Pic^*X/Y is bimeromorphic over U to $\mathrm{Pic}\, X_U/U$, 2) $\mathrm{Pic}^*X/Y\times_Y\tilde{Y}$ and $\mathrm{Pic}^*\tilde{X}/\tilde{Y}$ are naturally bimeromorphic over Y, 3) g induces the natural meromorphic Y-map $g^*: \mathrm{Pic}^*X/Y\to\mathrm{Pic}^*X'/Y$, 4) there exists a meromorphic Y-map $\mu^*_{X/Y}: \mathrm{Div}^*X/Y\to\mathrm{Pic}^*X/Y$ which is bimeromorphic to $\mu_{X_U/U}$ over U, 5) there exists a meromorphic Y-map $m^*_{X/Y}$ (resp. $a^*_{X/Y}$): $\mathrm{Pic}^*X/Y\times_Y\mathrm{Pic}^*X/Y\to\mathrm{Pic}^*X/Y$ which is bimeromorphic to $m_{X_U/U}$ (resp. $a_{X_U/U}$) over U, 6) there exists a meromorphic Y-map $\tau:\tilde{Y}\to\mathrm{Pic}^*X/Y$ defined by $\mathcal{F}$ which is bimeromorphic to the universal morphism $\tau:\tilde{U}\to\mathrm{Pic}\,X_U/U$ defined by $\mathcal{F}|_{\tilde{X}_{\tilde{U}}}$, 8) Pic^*X/Y is up to bimeromorphic equivalences over Y independent of the choice of U as above.

Then we prove the following:

Theorem 1. *Let $f: X\to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$ with $a(f)=\dim f$. Then* Pic^*X/Y *for f exists in the sense of Definition* 4.

3.3. *Proof of Theorem* 1. I. The case where f is generically locally projective.

Case 1. *γ is s. ample.* In this case recall that there exists a Zariski open subset $U_\gamma\subseteq Y$ such that $(\mu_\gamma)_{U_\gamma}:\mathrm{Div}_\gamma X_{U_\gamma}/U_\gamma\to\mathrm{Pic}_\gamma X_{U_\gamma}/U_\gamma$ is a holomorphic $\boldsymbol{P}^k$-bundle for some $k>0$. Recall also that $\mathrm{Div}_\gamma X_{U_\gamma}/U_\gamma$ admits a natural compactification $\mathrm{Div}_\gamma X_{U_\gamma}/U_\gamma\hookrightarrow\mathrm{Div}^*_\gamma X/Y$ with $\mathrm{Div}^*_\gamma X/Y$ a compact complex variety in $\mathcal{C}$ over Y. Then by Lemma 9 we get a Zariski open embedding $(\mathrm{Pic}_\gamma X_U/U)_{U_\gamma}=\mathrm{Pic}_\gamma X_{U_\gamma}/U_\gamma\hookrightarrow\mathrm{Pic}^*_\gamma X/Y$ (where $\mathrm{Pic}^*_\gamma X/Y$ is a comact complex variety in $\mathcal{C}$ over Y) such that $(\mu_\gamma)_{U_\gamma}:(\mathrm{Div}_\gamma X_U/U)_{U_\gamma}\to(\mathrm{Pic} X_U/U)_{U_\gamma}$ extends to a meromorphic Y-map $\mu^*_\gamma:\mathrm{Div}^*_\gamma X/Y\to\mathrm{Pic}^*_\gamma X/Y$ which is bimeromorphic to μ_γ over U. This is our definition of $\mathrm{Pic}^*_\gamma X/Y$. It is clear that $\mathrm{Pic}^*_\gamma X/Y$ is independent of the choice of U (cf. Lemma 9). Further since $(\mu_\gamma)_{U_\gamma}$ is a $\boldsymbol{P}^k$-bundle, μ^*_γ is Moishezon by Lemma 11. If f is Moishezon, then $\mathrm{Div}^*_\gamma X/Y\to Y$ is Moishezon by [6]. Hence by [6], Prop. 1, $\mathrm{Pic}^*_\gamma X/Y$ also is Moishezon over Y. Thus we have proved 1), 4), 7) and 8) in Case 1.

2) Consider the following diagram of meromorphic Y-maps (cf. 2.1 c)).

$$\begin{array}{ccc}
\mathrm{Div}^*_\gamma X/Y\times_Y\tilde{Y} & \overset{D^*_\gamma(\nu)}{\cong} & \bigcup_{\tilde{\gamma}}\mathrm{Div}^*_{\tilde{\gamma}}\tilde{X}/\tilde{Y} \\
\Big\downarrow{\scriptstyle \mu^*_\gamma\times_Y\tilde{Y}} & & \Big\downarrow{\scriptstyle \coprod_{\tilde{\gamma}}\mu^*_{\tilde{\gamma}}} \qquad \tilde{\gamma}\in\Gamma(\nu)^{-1}(\gamma) \\
\mathrm{Pic}^*_\gamma X/Y\times_Y\tilde{Y} & & \coprod_{\tilde{\gamma}}\mathrm{Pic}^*_{\tilde{\gamma}}\tilde{X}/\tilde{Y}
\end{array}$$

Restricting over $\tilde{U}_\gamma:=\nu^{-1}(U_\gamma)$ we get $P_\gamma(\nu_U)\mu^*_{\gamma,\tilde{Y}}=(\coprod\mu^*_{\tilde{\gamma}})D^*_\gamma(\nu)$ where $P_\gamma(\nu_U)$ is the isomorphism in 1.3 c) iii) associated to $\nu_U:\tilde{U}\to U$. Hence 2) follows from Lemma 10.

5) For $m^*_{\gamma,\gamma'}$. (The case where γ, γ' and $\gamma+\gamma'$ are all s. ample.) Write for simplicity $D^*_\gamma=\mathrm{Div}^*_\gamma X/Y$, $P^*_\gamma=\mathrm{Pic}^*_\gamma X/Y$ etc. Then consider the following diagram of meromorphic Y-maps.

$$\begin{array}{ccc} D^*_\gamma \dot{\times}_Y D^*_{\gamma'} & \xrightarrow{\tilde{m}^*_{\gamma,\gamma'}} & D^*_{\gamma+\gamma'} \\ {\scriptstyle \mu^*_\gamma \dot{\times}_Y \mu^*_{\gamma'}}\downarrow & & \downarrow{\scriptstyle \mu^*_{\gamma+\gamma'}} \\ P^*_\gamma \dot{\times}_Y P^*_{\gamma'} & & P^*_{\gamma+\gamma'} \end{array}$$

Restricting to $D^*_\gamma \dot{\times}_Y D^*_{\gamma'}|_{U_{\gamma,\gamma'}} = D_{\gamma,U_{\gamma,\gamma'}} \times_{U_{\gamma,\gamma'}} D_{\gamma',U_{\gamma,\gamma'}}$, $U_{\gamma,\gamma'} = U_\gamma \cap U_{\gamma'}$, we get $m_{\gamma,\gamma'}(\mu^*_\gamma \dot{\times}_Y \mu^*_{\gamma'}) = \mu^*_{\gamma+\gamma'} \tilde{m}^*_{\gamma,\gamma'}$. Hence by Lemma 10, 5) follows in our special case.

6) First we prove a lemma.

Lemma 12. *Let $f: X \to Y$ and U be as in Theorem* 1. *Let $\mathcal{F}_0$ be a coherent analytic sheaf on X which is invertible over U. Let $t: U \to \mathrm{Pic}\, X_U/U$ be the holomorphic section defined by $\mathcal{F}_0|_{X_U}$. Then t extends to a meromorphic section $t^*: Y \to \mathrm{Pic}^*_\gamma X/Y$ if $t(U) \subseteq \mathrm{Pic}_\gamma X_U/U$ with γ s. ample.*

Proof. Let $B = t(U)$ and $C = \mu_\gamma^{-1}(U) \subseteq \mathrm{Div}_\gamma X_U/U$. Since γ is s. ample, $\mu_\gamma(C) = B$. We show that the closure C^* of C in $\mathrm{Div}^*_\gamma X/Y$ is analytic. This would then show that the closure $B^* = \mu^*_\gamma(C^*)$ of B in $\mathrm{Pic}^*_\gamma X/Y$ is analytic, so that the lemma would follow. To show the analyticity of C take first a suitable proper modification $p: X_1 \to X$ so that the strict transform $\mathcal{F}_1$ of $\mathcal{F}_0$ to X_1 is invertible [21] and then take a proper modification $\varphi: Y' \to Y$ such that φ is isomorphic on $\varphi^{-1}(U)$ and the strict transform X' of X_1 in $X_1 \times_Y Y'$ is flat over Y' (cf. [15]). Let $\mathcal{F}'$ be the pull-back of $\mathcal{F}_1$ to X'. Let $E \to Y'$ be the linear fiber space in the sense of Fischer [4] representing the functor $F: (\mathrm{An}/Y') \to (\mathrm{Sets})$, $F(T) = \Gamma(X' \times_{Y'} T, \mathcal{F}'_T)$ where $\mathcal{F}'_T$ is the natural pull-back of $\mathcal{F}'$ to $X' \times_{Y'} T$. In fact, since $\mathcal{F}'$ is invertible, by Schuster [22] F is represented by $p_+ \mathrm{F}'$ where p_+ is the right adjoint functor of the base change functor $p^+(T) = X' \times_{Y'} T$ in the notation of [22], and F' is the line bundle corresponding to $\mathcal{F}'$. Then the associated projective fiber space $\boldsymbol{P}(E) \to Y'$ is naturally a subspace of $\mathrm{Div}_\gamma X'/Y'$ such that $\boldsymbol{P}(E)_y$ is the linear system associated to the line bundle F'_y, $y \in Y'$. Let $q: \mathrm{Div}^*_\gamma X'/Y' \to \mathrm{Div}^*_\gamma X_1/Y' \to \mathrm{Div}^*_\gamma X/Y$ be the natural bimeromorphic map which is an isomorphism over U if we identify U with $\varphi^{-1}(U)$ via φ. Then it is easy to see that C^* coincides with $q(\boldsymbol{P}(E))$ and hence is analytic. q. e. d.

Returning to the proof of 6) let $\tilde{t}: \tilde{U} \to \mathrm{Pic}_{\tilde{\gamma}} \tilde{X}_{\tilde{U}}/\tilde{U}$ be the holomorphic section defined by $\mathcal{F}|_{\tilde{X}_{\tilde{U}}}$ where $\tilde{\gamma} \in \Gamma(\nu)^{-1}(\gamma)$. Since γ is s. ample, $\tilde{\gamma}$ also is s. ample (cf. Lemma 4). Hence by the above lemma applied to $\tilde{f}$ and $\mathcal{F}$ instead of f and $\mathcal{F}_0$, $\tilde{t}$ extends to a meromorphic section $\tilde{t}^*: \tilde{Y} \to \mathrm{Pic}^* \tilde{X}/\tilde{Y}$. Then we define $\tilde{\tau}: = P_\gamma(\nu)\tilde{t}^*$ where $P_\gamma(\nu)$ is the natural meromorphic Y-map $\mathrm{Pic}^*_{\tilde{\gamma}} \tilde{X}/\tilde{Y} \to \mathrm{Pic}^*_\gamma X/Y$ (cf. 2)). The desired property is easily checked.

Case 2. *The general case.* Take s. ample α, β with $\alpha - \beta = \gamma$ as in Lemma 6. Let U_α, U_β be as in Case 1 defined respectively for α and β. Let $W = U_\alpha \cap U_\beta$.

Then by our construction in Case 1 we have the natural inclusion $(\mathrm{Pic}_\alpha X_U/U)_W \times_W (\mathrm{Pic}_\beta X_U/U)_W \subseteq \mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y \mathrm{Pic}_\beta^* X/Y$. Then, since $a_{\alpha\beta}$ is a fiber space, by Lemma 9 we can find a Zariski open embedding $(\mathrm{Pic}_\gamma X_U/U)_W \to \mathrm{Pic}_\gamma^* X/Y$ with $\mathrm{Pic}_\gamma^* X/Y$ a compact complex variety in $\mathcal{C}$ over Y such that $(a_{\alpha\beta})_W : (\mathrm{Pic}_\alpha X_U/U \times_U \mathrm{Pic}_\beta X_U/U)_W \to (\mathrm{Pic}_\gamma X_U/U)_W$ extends to a meromorphic Y-map $a_{\alpha\beta}^* : \mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y \mathrm{Pic}_\beta^* X/Y \to \mathrm{Pic}_\gamma^* X/Y$ which is bimeromorphic to $a_{\alpha\beta}$ over U. If f is Moishezon, then $\mathrm{Pic}_\alpha^* X/Y$ and $\mathrm{Pic}_\beta^* X/Y$ are Moishezon over Y by Case 1, and hence $\mathrm{Pic}_\gamma^* X/Y$ also is Moishezon over Y, $a_{\alpha\beta}^*$ being surjective. Moreover if $\gamma=0$ and $\alpha=\beta$, then $a_{\alpha\alpha}^*(\varDelta_\alpha) \subseteq \mathrm{Pic}_0^* X/Y$ ($\varDelta_\alpha$=the diagonal in $\mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y \mathrm{Pic}_\alpha^* X/Y$) defines the desired extension of the identity section of $\mathrm{Pic}_0 X_U/U \to U$. Thus we have proved the existence of $\mathrm{Pic}_\gamma^* X/Y$ satisfying 1) (set $U_\gamma=W$), part of 5) and 7). Before proceeding, however, it is reasonable to check that the above construction is independent of the chosen α and β.

Lemma 13. *Write* $\mathrm{Pic}_\gamma^* X/Y = \mathrm{Pic}_\gamma^* X/Y_{(\alpha,\beta)}$ *for the* $\mathrm{Pic}_\gamma^* X/Y$ *constructed above. Then* $\mathrm{Pic}_\gamma^* X/Y_{(\alpha,\beta)}$ *are naturally bimeromorphic to one another over* Y *for various choices of* s. *ample* α, β *with* $\alpha-\beta=\gamma$.

Proof. For give α, β we take any s. ample δ such that both $\alpha+\delta$ and $\beta+\delta$ are s. ample and that $a_{\alpha+\delta,\beta+\delta}$ is a fiber space (cf. Lemma 6). We show that $\mathrm{Pic}_\gamma^* X/Y_{(\alpha,\beta)}$ and $\mathrm{Pic}_\gamma^* X/Y_{(\alpha+\delta,\beta+\delta)}$ are bimeromorphically equivalent over Y. (The general case follows from this special case readily.) By 2) in Case 1, together with Corollary to Lemma 10, replacing f by $f \dot{\times}_Y id_{\mathrm{Pic}_\delta^* X/Y}$ (id=identity) if necessary we may assume from the beginning that there exists a meromorphic section $s : Y \to \mathrm{Pic}_\delta^* X/Y$. Let $s(Y)=Y'$. Let $c_\alpha^* : \mathrm{Pic}_\alpha^* X/Y \to \mathrm{Pic}_{\alpha+\delta}^* X/Y$ be the bimeromorphic Y-map which is by definition the composite of the bimeromorphic Y-map $id \dot{\times}_Y s b_\alpha^* : \mathrm{Pic}_\alpha^* X/Y \to \mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y Y'$ ($b_\alpha^* : \mathrm{Pic}_\alpha^* X/Y \to Y$ being the natural map) and the restriction $(m_{\alpha,\delta}^*)_{Y'} : \mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y Y' \to \mathrm{Pic}_{\alpha+\delta}^* X/Y$ of $m_{\alpha,\delta}^*$ to $\mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y Y'$ where $m_{\alpha,\delta}^*$ is as in 5) in Case 1. Define $c_\beta^* : \mathrm{Pic}_\beta^* X/Y \to \mathrm{Pic}_{\beta+\delta}^* X/Y$ similarly. Then over U $a_{\alpha\beta} = a_{\alpha+\delta,\beta+\delta}(c_\alpha^* \dot{\times}_Y c_\beta^*)$ as a U-morphism $\mathrm{Pic}_\alpha X_U/U \times_U \mathrm{Pic}_\beta X_U/U \to \mathrm{Pic}_\gamma X_U/U$ where $(c_\alpha^* \dot{\times}_X c_\beta^*)_U$ gives a U-isomorphism of $\mathrm{Pic}_\alpha X_U/U \times_U \mathrm{Pic}_\beta X_U/U$ and $\mathrm{Pic}_{\alpha+\delta} X_U/U \times_U \mathrm{Pic}_{\beta+\delta} X_U/U$. Hence by Lemma 10 the identity $\mathrm{Pic}_\gamma^* X/Y_{(\alpha,\beta)}|_W = \mathrm{Pic}_\gamma X_W/W = \mathrm{Pic}_\gamma^* X/Y_{(a+\delta,\beta+\delta)}|_W$ extends to a desired bimeromorphic equivalence of $\mathrm{Pic}_\gamma^* X/Y_{(\alpha,\beta)}$ and $\mathrm{Pic}_\gamma^* X/Y_{(\alpha+\delta,\beta+\delta)}$.*)

2) Consider the following diagram

$$\begin{array}{ccc}
(\mathrm{Pic}_\alpha^* X/Y \dot{\times}_Y \tilde{Y}) \dot{\times}_{\tilde{Y}} (\mathrm{Pic}_\beta^* X/Y \dot{\times}_Y \tilde{Y}) & \xrightarrow{\;h\;} & (\coprod_{\tilde{\alpha}} \mathrm{Pic}_{\tilde{\alpha}}^* \tilde{X}/\tilde{Y}) \dot{\times}_{\tilde{Y}} (\coprod_{\tilde{\beta}} \mathrm{Pic}_{\tilde{\beta}}^* \tilde{X}/\tilde{Y}) \\
\Big\downarrow {\scriptstyle a_{\alpha\beta}^* \dot{\times}_Y \tilde{Y}} & & \Big\downarrow {\scriptstyle \coprod_{\tilde{\alpha},\tilde{\beta}} a_{\tilde{\alpha}\tilde{\beta}}} \qquad \tilde{\delta} \in \Gamma(\nu)^{-1}(\delta),\ \delta=\alpha,\ \beta,\ \gamma \\
\mathrm{Pic}_\gamma^* X/Y \dot{\times}_Y \tilde{Y} & & \coprod_{\tilde{\gamma}} \mathrm{Pic}_{\tilde{\gamma}}^* \tilde{X}/\tilde{Y}
\end{array}$$

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*) Here $W = U_\alpha \cap U_\beta \cap U_{\alpha+\delta} \cap U_{\beta+\delta}$.

where h is the bimeromorphic map given in Case 1. Then over $\tilde{U}_\gamma := \nu^{-1}(U_\gamma)$ $(U_\gamma = U_\alpha \cap U_\beta)$ the natural isomorphism $P_\gamma(\nu_{U_\gamma}): (\mathrm{Pic}^*_\gamma X/Y \dot{\times}_Y \tilde{Y})_{\tilde{U}_\gamma} = (\mathrm{Pic}_\gamma X_{U_\gamma}/U_\gamma) \times_{U_\gamma} \tilde{U}_\gamma \cong \coprod_{\tilde{\gamma}} \mathrm{Pic}_{\tilde{\gamma}} \tilde{X}_{\tilde{U}_{\tilde{\gamma}}}/\tilde{U}_{\tilde{\gamma}} = (\coprod_{\tilde{\gamma}} \mathrm{Pic}^*_{\tilde{\gamma}} \tilde{X}/\tilde{Y})_{\tilde{U}_\gamma}$ induced by $\nu_{U_\gamma}: \tilde{U}_\gamma \to U_\gamma$ (cf. 1.3 c) iii)) makes the above diagram commutative. Hence 2) follows from Lemma 10.

3) i) Assume that γ is s. ample. Consider the diagram

$$\begin{array}{ccc} \mathrm{Div}^*_\gamma X/Y & \xrightarrow{\tilde{g}^*_\gamma} & \mathrm{Div}^*_{\gamma'} X'/Y \\ \mu^*_\gamma \downarrow & & \downarrow \mu^*_{\gamma'} \\ \mathrm{Pic}^*_\gamma X/Y & & \mathrm{Pic}^*_{\gamma'} X'/Y \end{array} \qquad \gamma' = \Gamma(g)(\gamma)$$

Over $W = U_\gamma \cap U_{\gamma'}$ the natural morphism $g^*_{\gamma,W}: (\mathrm{Pic}^*_\gamma X/Y)_W = \mathrm{Pic}_\gamma X_W/W \to \mathrm{Pic}_{\gamma'} X'_W/W = (\mathrm{Pic}^*_{\gamma'} X'/Y)_W$ makes the above diagram commutative. Hence 3) follows from Lemma 10.

ii) In the general case we observe the following diagram

$$\begin{array}{ccc} \mathrm{Pic}^*_\alpha X/Y \dot{\times}_Y \mathrm{Pic}^*_\beta X/Y & \xrightarrow{g^*_\alpha \dot{\times}_Y g^*_\beta} & \mathrm{Pic}^*_{\alpha'} X'/Y \dot{\times}_Y \mathrm{Pic}^*_{\beta'} X'/Y \\ a^*_{\alpha\beta} \downarrow & & \downarrow a^*_{\alpha'\beta'} \\ \mathrm{Pic}^*_\gamma X/Y & & \mathrm{Pic}^*_{\gamma'} X'/Y \end{array} \qquad \begin{array}{l} \delta' = \Gamma(g)(\delta) \\ \delta = \alpha, \beta, \gamma \end{array}$$

Then by the same argument as in i), 3) follows.

4) Write $\gamma = \alpha - \beta$ with α, β s. ample. By 2), 2.1 c) i) and Corollary to Lemma 10, replacing f by $f \dot{\times}_Y id_{\mathrm{Div}^*_\beta X/Y}$ if necessary, we may assume that $\mathrm{Div}^*_\beta X/Y \to Y$ admits a meromorphic section $\tilde{s}_\beta$. This induces a meromorphic section s_β of $\mathrm{Pic}^*_\beta X/Y \to Y$ via μ^*_β which exists by Case 1. Identifying $\mathrm{Div}^*_\gamma X/Y$ and $\mathrm{Pic}^*_\gamma X/Y$ with $\mathrm{Div}^*_\gamma X/Y \dot{\times}_Y \tilde{s}_\beta(Y)$ and $\mathrm{Pic}^*_\gamma X/Y \dot{\times}_Y s_\beta(Y)$ respectively up to bimeromorphic equivalences over Y, these sections define meromorphic maps $\tilde{c}: \mathrm{Div}^*_\gamma X/Y \to \mathrm{Div}^*_\alpha X/Y$ and $c: \mathrm{Pic}^*_\gamma X/Y \to \mathrm{Pic}^*_\alpha X/Y$ respectively with c bimeromorphic such that $c\mu_\gamma = \mu^*_\alpha \tilde{c}$ over U_γ as a meromorphic map. Then μ^*_γ is given by $\mu^*_\gamma = c^{-1}\mu^*_\beta \tilde{c}$. The last assertion then follows from that for μ^*_β by the generic injectivity of $\tilde{c}$.

5) Write $\gamma = \alpha - \beta$ and $\gamma' = \alpha' - \beta'$ with $\alpha, \beta, \alpha', \beta'$ s. ample. Taking these suitably we may assume that $\alpha + \alpha'$ and $\beta + \beta'$ are also s. ample. Write for simplicity $D^*_\alpha = \mathrm{Div}^*_\alpha X/Y$, $P^*_\alpha = \mathrm{Pic}^*_\alpha X/Y$ etc. Then consider the following diagram of meromorphic Y-maps.

$$\begin{array}{ccc}(P_\alpha^*\dot{\times}_Y P_\beta^*)\dot{\times}_Y(P_{\alpha'}^*\dot{\times}_Y P_{\beta'}^*)\cong(P_\alpha^*\dot{\times}_Y P_{\alpha'}^*)\dot{\times}_Y(P_\beta^*\dot{\times}_Y P_{\beta'}^*) & \xrightarrow{m_{\alpha\alpha'}^*\dot{\times}_Y m_{\beta\beta'}^*} & P_{\alpha+\alpha'}^*\dot{\times}_Y P_{\beta+\beta'}^* \\ \downarrow a_{\alpha\beta}^*\dot{\times}_Y a_{\alpha',\beta'}^* & & \downarrow a_{\alpha+\alpha',\beta+\beta'}^* \\ P_\gamma^*\dot{\times}_Y P_{\gamma'}^* & & P_{\gamma-\gamma'}^*\end{array}$$

Since over a small Zariski open subset we get a morphism $a_{\gamma,\gamma'}: P_\gamma^*\dot{\times}_Y P_{\gamma'}^*\to P_{\gamma-\gamma'}^*$ making the above diagram commutative 5) follows from Lemma 10. The proof for $m_{\gamma\gamma'}^*$ is similar.

6) Let $\gamma=\alpha-\beta$ with α and β s. ample. i) Suppose first that there exists a holomorphic section $s: Y\to\mathrm{Div}_\beta^* X/Y$. Consider the coherent analytic sheaf $\mathcal{L}_\beta:=(id_X\dot{\times}_Y s\nu)^*(\mathcal{H}om_{\mathcal{O}}(\mathcal{I}_\beta, \mathcal{O}))$, $\mathcal{O}=\mathcal{O}_{X\dot{\times}_Y\mathrm{Div}_\beta^* X/Y}$, on $\tilde{X}$ where $\mathcal{I}_\beta$ is the ideal sheaf of Z_β^* in $X\dot{\times}_Y\mathrm{Div}_\beta^* X/Y$. Let $\tilde{\mathcal{L}}_\beta^\vee=\mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathcal{L}}_\beta, \mathcal{O}_{\tilde{X}})$ and $\mathcal{F}_\beta=\mathcal{F}\otimes\tilde{\mathcal{L}}_\beta^\vee$. Then $\mathcal{F}_\beta|_{\tilde{X}_{\tilde{U}}}$ is invertible and induces the universal U-morphism $\tau': \tilde{U}\to\mathrm{Pic}\, X_U/U$, with $\tau'(\tilde{U})\subseteqq\mathrm{Pic}_\alpha X_U/U$, as follows from the relation $\alpha=\beta+\gamma$. By what we have proved in Case 1 there exists a meromorphic Y-map $\tau'^*: \tilde{Y}\to\mathrm{Pic}_\alpha^* X/Y$ which is bimeromorphic to τ' on U. On the other hand, the surjective meromorphic Y-map $a_{\alpha\beta}^*: \mathrm{Pic}_\alpha^* X/Y\times_Y\mathrm{Pic}_\beta^* X/Y\to\mathrm{Pic}_\gamma^* X/Y$ restricted to $\mathrm{Pic}_\alpha^* X/Y\dot{\times}_Y s(Y)$ defines a bimeromorphic Y-map $\varphi_{\alpha\gamma}: \mathrm{Pic}_\alpha^* X/Y\to\mathrm{Pic}_\gamma^* X/Y$. Further we infer readily that $\tau'=\varphi_{\alpha\gamma}|_U\cdot\tau$. Hence $\tau^*=\varphi_{\alpha\gamma}^{*-1}\tau'^*$ is a desired meromorphic map.

ii) Next we consider the general case. For simplicity of notation, however, we consider only the case where $\tilde{Y}=Y$ and leave the general case to the reader. Let $\xi: Y_1\to Y$ be the natural proper surjective morphism where $Y_1=\mathrm{Div}_\beta^* X/Y$, so that $Y_1\dot{\times}_Y\mathrm{Div}_\beta^* X/Y\to Y_1$ admits a holomorphic section. Let $U_1=\xi^{-1}(U)$ and $X_1=X\dot{\times}_Y Y_1$. Let $\mathcal{F}_1$ be the pull-back of $\mathcal{F}$ to X_1 so that $\mathcal{F}_1|_{X\times_U U_1}$ defines a holomorphic section $\tau_1: U_1\to\mathrm{Pic}\,(X_{1,U_1}/U_1)$. Let $p_2: \mathrm{Pic}^* X_1/Y_1\to\mathrm{Pic}^* X/Y$ be the meromorphic Y-map which is bimeromorphic over U_1 to the natural projection $p_2: \mathrm{Pic}\,(X_{1,U_1}/U_1)\to\mathrm{Pic}\, X_U/U$ (cf. 2)). Then we have $p_2\tau_1=\tau(\xi|_{U_1})$. Since there exists a meromorphic Y_1-map $\tau_1^*: Y_1\to\mathrm{Pic}^* X_1/Y_1$ which is bimeromorphic to τ_1 over U_1 by i), it follows that there exists also a meromorphic Y-map $\tau^*: Y\to\mathrm{Pic}^* X/Y$ which is bimeromorphic to τ over U by Lemma 10. From this 6) follows.

II. *The general case.* By Proposition 4 (cf. Remark 1) we can find a bimeromorphic Y-morphism $\sigma: X'\to X$ of compact complex varieties such that the induced morphism $f'=f\sigma: X'\to Y$ is generically smooth and generically locally projective. Then σ induces a natural injection $\Gamma(\sigma): \Gamma(f)\to\Gamma(f')$ such that $\mathrm{Pic}_\gamma X_U/U\cong\mathrm{Pic}_{\gamma'} X'_U/U$, $\gamma'=\Gamma(\sigma)\gamma$, over any Zariski open subset U of Y over which both f and f' are smooth. Then we set $\mathrm{Pic}_\gamma^* X/Y=\mathrm{Pic}_{\gamma'}^* X'/Y$ where $\mathrm{Pic}_{\gamma'}^* X'/Y$ is constructed in I. The independence of $\mathrm{Pic}_\gamma^* X/Y$ (up to bimeromorphic equivalences over Y) of the choice of f' as above follows immediately from the property 3) in I together with the following fact; given two bimeromorphic fiber spaces $f_i: X_i \to Y$, $i=1, 2$, we can always find another fiber space $f_3: X_3 \to Y$ which is generically locally projective and which dominates holomorphically and bimeromorphically both f_1 and f_2.

Form the definition the properties 1), 2), 3), 5), 7), 8) follow immediately from the case I. 4) Let $\tilde{g}^*: \mathrm{Div}_\gamma^* X/Y \to \mathrm{Div}_{\gamma'}^* X'/Y$ be induced by g with X' as above (cf. 2.1 c) ii)). Then we have only to set $\mu_{\gamma'}^* = \mu_\gamma^* g^*$. 6) Let $\tilde{\sigma} := \sigma \dot{\times}_Y \tilde{Y}$: $X' \dot{\times}_Y \tilde{Y} \to \tilde{X} = X \dot{\times}_Y \tilde{Y}$. Then we have only to define $\tilde{\tau}: \tilde{Y} \to \mathrm{Pic}_{\gamma'}^* X'/Y = \mathrm{Pic}_\gamma^* X/Y$ to be the universal meromophic map defined by $\tilde{\sigma}^* F$.

3.4. *Meromorphic Poincare sheaf.* a) In the proof of the next proposition and also in Section 4 we adopt the following convention. Let $f: X \to Y$ and f': $X' \to Y$ be proper morphisms of complex spaces. Let $\varphi: X \to X'$ be a meromorphic Y-map. Let $\Gamma \subseteq X \times_Y X'$ be the graph of φ and let $q: \Gamma \to X$ and $q': \Gamma \to X'$ be the natural projections. Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then we write for simplicity $\varphi_* \mathcal{F} = q'_* q^* \mathcal{F}$.

b) Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties with $a(f) = \dim f$.

Proposition 5. *Let $W \subseteq Y$ be an open subset. Suppose that f admits a meromorphic section $s: W \to X_W$. Let U_0 be a Zariski open subset of W on which s is defined. Then there exists a coherent analytic sheaf $\mathcal{L}$ on $X_W \dot{\times}_W (\mathrm{Pic}^* X/Y)_W$ such that for any $\gamma \in \Gamma(f)$ if $\mathcal{L}_\gamma$ is the restriction of $\mathcal{L}$ to $X_W \dot{\times}_W (\mathrm{Pic}_\gamma^* X/Y)_W$, then on $X_{V_\gamma} \dot{\times}_{V_\gamma} \mathrm{Pic}_\gamma (X_{V_\gamma}/V_\gamma)$ where $V_\gamma = U_\gamma \cap U_0$, $\mathcal{L}_\gamma$ is invertible and coincides with the relative Poincare sheaf for the smooth map f_{V_γ} associated to the section $s|_{V_\gamma}$.*

Proof. First we assume that f is generically locally projective. For simplicity of notation we only consider the case $W=Y$. (The proof is completely the same in the general case.) It suffices to construct $\mathcal{L}_\gamma$ on each $X \dot{\times}_Y P_\gamma^*$ with the desired property. For simplicity we write $D_\gamma^* = \mathrm{Div}_\gamma^* X/Y$, $P_\gamma^* = \mathrm{Pic}_\gamma^* X/Y$ etc.

Case 1. Assume that γ is s. ample, so that $\mu_\gamma^*: D_\gamma^* \to P_\gamma^*$ is a holomorphic $\mathbf{P}^k$-bundle over $(P_\gamma^*)_{U_\gamma}$ for some $k>0$. Let $Z_\gamma^* \subseteq X \dot{\times}_Y D_\gamma^*$ be the associated meromorphic universal divisor. Let $\mathcal{I}_\gamma$ be the ideal sheaf of Z_γ^* and let $\mathcal{F}_\gamma = \mathcal{H}om_{\mathcal{O}_\gamma}(\mathcal{I}_\gamma, \mathcal{O}_\gamma)$ where $\mathcal{O}_\gamma = \mathcal{O}_{X \dot{\times}_Y D_\gamma^*}$. Set $S = s(Y)$ and let $q: X \dot{\times}_Y D_\gamma^* \to D_\gamma^*$ be the natural projection. Let $Z_\gamma^{*\prime} = q^{-1} q((S \dot{\times}_Y D_\gamma^*) \cap Z_\gamma^*) \subseteq X \dot{\times}_Y D_\gamma^*$. $Z_\gamma^{*\prime}$ is a relative divisor over D_{γ, V_γ}^*, γ being s. ample, and in fact $(Z_\gamma^{*\prime})_u = Z_\gamma^* \cap (s(u) \times D_\gamma^*)$ for $u \in V_\gamma$. Let $\mathcal{I}_\gamma'$ be the ideal sheaf of $Z_\gamma^{*\prime}$, and set $\mathcal{E}_\gamma = \mathcal{F}_\gamma \otimes_{\mathcal{O}_\gamma} \mathcal{I}_\gamma'$. Then $\mathcal{L}_\gamma := (id_X \dot{\times}_Y \mu_\gamma)_* (\mathcal{E}_\gamma)$ is a coherent analytic sheaf on $X \dot{\times}_Y P_\gamma^*$ (cf. a)). We claim that this $\mathcal{L}_\gamma$ has the desired property. In fact, by the definition $\mathcal{E}_\gamma$ is invertible on $(X \dot{\times}_Y D_\gamma^*)_{U_\gamma}$ and trivial when restricted to each fiber $\{x\} \times D_{\gamma, p}^*$, $(x, p) \in X \dot{\times}_Y P_\gamma^*$, of $id_X \dot{\times}_Y \mu_\gamma^*$ over U_γ. Hence, $id_X \dot{\times}_Y \mu_\gamma^*$ being a holomorphic $\mathbf{P}^k$-bundle over U_γ, $\mathcal{L}_\gamma$ also is invertible over U_γ. Further since $H^1(D_{\gamma, p}^*, \mathcal{O}_{D_{\gamma, p}^*}) = 0$, $\mathcal{E}_\gamma$ is cohomologically flat (in dimension zero) with respect to $id_X \dot{\times}_Y \mu_\gamma^* |_{(X \dot{\times}_Y D_\gamma^*)_{U_\gamma}}$ (cf. [1]).

Hence over V_γ we have $\mathcal{L}_\gamma \otimes_{\mathcal{O}_{X \dot{\times}_Y P_\gamma^*}} \mathcal{O}_{S \dot{\times}_Y P_\gamma^*} \cong ((id_X \dot{\times}_Y \mu_\gamma^*)_* \mathcal{E}_\gamma) \otimes_{\mathcal{O}_{X \dot{\times}_Y P_\gamma^*}} \mathcal{O}_{S \dot{\times}_Y P_\gamma^*}$ $\cong (id_S \dot{\times}_Y \mu_\gamma^*)_* (\mathcal{E}_\gamma \otimes_{\mathcal{O}_{X \dot{\times}_Y D_\gamma^*}} \mathcal{O}_{S \dot{\times}_Y D_\gamma^*}) \cong (id_S \dot{\times}_Y \mu_\gamma^*)_* \mathcal{O}_{S \dot{\times}_Y D_\gamma^*} \cong \mathcal{O}_{S \dot{\times}_Y P_\gamma^*}$. Thus it suffices to show that for $y \in V_\gamma$ the restriction $\mathcal{L}_{\gamma,y}$ of $\mathcal{L}_\gamma$ to $(X \dot{\times}_Y P_\gamma^*)_y \cong X_y \times P_{\gamma,y}^*$ is the normalized Poincaré sheaf (restricted) on $X_y \times P_{\gamma,y}^*$ associated to the base point $s(y) \in X_y$. ($P_{\gamma,y}^*$ is a union of connected components of Pic X_y.) In fact, by the cohomological flatness of $\mathcal{E}_\gamma$, $\mathcal{L}_{\gamma,y} = (id_X \times \mu_{\gamma,y}^*)_* (\mathcal{E}_\gamma \otimes_{\mathcal{O}} \mathcal{O}_{X_y \times D_{\gamma,y}^*})$, $\mathcal{O} = \mathcal{O}_{X_y \dot{\times} D_{\gamma,y}^*}$ for $y \in V_\gamma$ where $\mu_{\gamma,y}^* : D_{\gamma,y}^* \to P_{\gamma,y}^*$, and then the result follows from the absolute case (Lemma 1).

Case 2. Write $\gamma = \alpha - \beta$ with α, β s. ample as in Lemma 6. Let $q_\alpha : P_\alpha^* \dot{\times}_Y P_\beta^* \to P_\alpha^*$, $q_\beta : P_\alpha^* \dot{\times}_Y P_\beta^* \to P_\beta^*$ be the natural projections and let $\tilde{\mathcal{L}}_\alpha = (id_X \times_Y q_\alpha)^* \mathcal{L}_\alpha$, $\tilde{\mathcal{L}}_\beta = (id_X \times_Y q_\beta)^* \mathcal{L}_\beta$ where $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are constructed in Case 1 for $\gamma = \alpha$ and β respectively. Then we set $\mathcal{L}_\gamma = (id_X \dot{\times}_Y a_{\alpha\beta}^*)_* (\tilde{\mathcal{L}}_\alpha \otimes_{\mathcal{O}} \tilde{\mathcal{L}}_\beta^{\vee})$ where $\mathcal{O} = \mathcal{O}_{X \dot{\times}_Y P_\alpha^* \dot{\times}_Y P_\beta^*}$ and $a_{\alpha\beta}^* : P_\alpha^* \dot{\times}_Y P_\beta^* \to P_\gamma^*$ is as in Case 2 of the construction of P_γ^* (cf. a)). Then by 1.1 d) $\mathcal{L}_\gamma$ is invertible over U_γ and is the relative Poincare sheaf over V_γ associated to $s|_{V_\gamma}$, as was desired.

In the general case, take a proper modification $\sigma : X' \to X$ as in 3.3 II. Let $\mathcal{L}_{\gamma'}$, $\gamma' = \Gamma(\sigma)\gamma$, be an extension of the relative Poincaré sheaf on $X' \dot{\times}_Y \mathrm{Pic}_\gamma^* X'/Y$ constructed above for X' and $\sigma^{-1}s$. Let $r' : C' \to X' \dot{\times}_Y \mathrm{Pic}_{\gamma'}^* X'/Y$, $r : C \to X \dot{\times}_Y \mathrm{Pic}_\gamma^* X/Y$ be resolutions of respective spaces such that the strict transform $\mathcal{L}_{C'}$ of $\mathcal{L}_{\gamma'}$ on C' is invertible and there exists a morphism $\Sigma : C' \to C$ such that $r\Sigma = (\sigma \times_Y id_{\mathrm{Pic}_{\gamma'}^* X'/Y}) r'$. Let $L_{C'}$ be the line bundle corresponding to $\mathcal{L}_{C'}$. Let $L_C = \Sigma_* L_{C'}$ be the direct image of $L_{C'}$ as a line bundle [7]. Then we set $\mathcal{L}_\gamma = r_* \mathcal{L}_C$, and it is easy to see that $\mathcal{L}_\gamma$ meet the requirement of the proposition.

q. e. d.

We call any $\mathcal{L}$ with the property of the above proposition a *meromorphic relative Poincaré sheaf associated to s.*

§ 4. Relative Albanese Variety

4.1. *Statement of the theorem.*

Definition 5. Let $f : X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Then a *relative Albanese map for f* in $\mathcal{C}$ is a commutative diagram

(*)
$$\begin{array}{ccc} X & \xrightarrow{\ \psi\ } & \mathrm{Alb}^* X/Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \eta} & \\ & Y & \end{array}$$

where $\mathrm{Alb}^* X/Y$ is a compact complex variety in $\mathcal{C}$, η is a generically smooth fiber space with any smooth fiber a complex torus and ϕ is a meromorphic Y-map (which is necessarily holomorphic over some Zariski open subset of Y) with the following universal property: Let $\nu: \tilde{Y} \to Y$ be any proper surjective morphism with $\tilde{Y}$ a variety. Then for any commutative diagram

$$\begin{array}{ccc} X\dot{\times}_Y\tilde{Y} & \xrightarrow{\phi'} & A \\ & {\scriptstyle f\times_Y\tilde{Y}}\searrow \quad \swarrow{\scriptstyle \eta'} & \\ & \tilde{Y} & \end{array}$$

where ϕ' is a meromorphic $\tilde{Y}$-map, A is a compact complex variety in $\mathcal{C}$ and η' is a generically smooth fiber space with any smooth fiber a complex torus, there exists a unique meromorphic Y-map $b: (\mathrm{Alb}^*X/Y)\dot{\times}_Y\tilde{Y} \to A$ such that $\phi' = b(\phi\dot{\times}_Y\tilde{Y})$. We also call ϕ itself a *relative Albanese map for* f. We call Alb^*X/Y a *relative Albanese variety* associated to f. Clearly Alb^*X/Y is unique up to bimeromorphic equivalences over Y if one exists.

Theorem 2. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$ with $a(f) = \dim f$. Then there exists a relative Albanese map* (*) *for f with the following additional properties.* 1) *There exists a Zariski open subset $V \subseteq Y$ such that both X and Alb^*X/Y are smooth over V and the induced map $\phi_V: X_V \to (\mathrm{Alb}^*X/Y)_V$ is holomorphic and isomorphic to the Albanese map for the smooth morphism f_V, and* 2) *the map $\phi: X \to \mathrm{Alb}^*X/Y$ is Moishezon (i.e., any of its holomorphic model is Moishezon). Moreover if f is Moishezon, $\eta: \mathrm{Alb}^*X/Y \to Y$ also is Moishezon.*

Corollary. *A meromorphic Y-map $\phi': X \to A$ of X into a compact complex variety A in $\mathcal{C}$ over Y is a relative Albanese map for f if there exists a Zariski open subset $U \subseteq Y$ such that for $y \in U$, X_y, A_y are smooth and the induced map $\phi'_y: X_y \to A_y$ is holomorphic and isomorphic to an Albanese map of X.*

Proof. By the universality of $\phi: X \to \mathrm{Alb}^*X/Y$ there exists a unique meromorphic Y-map $u: \mathrm{Alb}^*X/Y \to A$ such that $u\phi = \phi'$. On the other hand, from our assumption it follows that u must give an isomorphism of any fiber over $y \in V$. Hence u is bimeromorphic and ϕ' is a relative Albanese map. q.e.d.

4.2. *Proof of Theorem* 2. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth.

I. *Construction of* Alb^*X/Y. a) First we assume that there exists a meromorphic section $s: Y \to X$. We then set $\mathrm{Alb}^*X/Y := \mathrm{Pic}_0^*((\mathrm{Pic}_0^*X/Y)/Y)$. Since Pic_0^*X/Y is in $\mathcal{C}$ and generically smooth over Y, this makes sense. Note that by the property 7) of Pic^*X/Y Alb^*X/Y is Moishezon over Y if f is. Let $\mathcal{L}_0$ be a meromorphic relative Poincaré sheaf on $X\dot{\times}_Y\mathrm{Pic}_0^*X/Y$ constructed in Proposition 5 with $W = Y$ there. Let $\phi = \phi_X: X \to \mathrm{Alb}^*X/Y$ be the universal meromorphic Y-map defined by $\mathcal{L}_0$. (See Definition 4 6) applied to $f: X\to Y$ and $\mathcal{L}_0$ instead of to $\nu: \tilde{Y}\to Y$ and $\mathcal{F}$ respectively.) We claim that ψ is a desired relative Albanese map for f. For any $y\in U_0$ (cf. Definition 4 1)) ψ induces a map ψ_y: $X_y\to \mathrm{Pic}_0(\mathrm{Pic}_0 X_y)$, and this coincides with the Albanese map of X_y by the construction of ψ in view of 1.6 a). Hence by Proposition 1 the additional property 1) of ψ in the above proposition is checked. In particular, if the general fiber of f is an abelian variety, ψ is bimeromorphic. We shall next prove the universality of ψ. Let $g: A\to Y$ be any generically smooth fiber space of compact complex varieties in $\mathcal{C}$ whose general fiber is a complex torus. Let $\psi': X\to A$ be an arbitrary meromorphic Y-map, which is necessarily holomorphic over some Zariski open subset of Y. (For simplicity of notation we consider only the case $Y=\tilde{Y}$ in Definition 5.)

a1) First we assume that the general fiber of g is an abelian variety, i. e., $a(g)=\dim g$. Then by the property 3) of Pic^*X/Y, ψ' induces a meromorphic Y-map $\mathrm{Pic}_0^*A/Y\to \mathrm{Pic}_0^*X/Y$ which in turn induces a meromorphic Y-map $\alpha: \mathrm{Alb}^*X/Y\to \mathrm{Alb}^*A/Y$ again by the property 3) where Alb^*A/Y $=\mathrm{Pic}_0^*((\mathrm{Pic}_0^*A/Y)/Y)$ as above. Since g admits the meromorphic section $\psi's$ we get a meromorphic Y-map $\psi_A: A\to \mathrm{Alb}^*A/Y$ as above, which is in fact bimeromorphic as we have remarked above. Then setting $\alpha'=\psi_A^{-1}\alpha$, we claim that $\psi'=\alpha'\psi$. In fact, it is enough to check this on the general fiber of f and hence to check this in the absolute case. And in the absolute case this is true in view of 1.6 a).

a2) It remains to consider the case where $a(g)<\dim g$. In this case, by what we have proved above, it suffices to show that ψ' factors through a subvariety $A_1\subseteq A$ whose general fiber $A_{1,y}$ over Y is an abelian subvariety of A_y. By Proposition 1, over U we have a natural morphism $a(U): \mathrm{Alb}\, X_U/U\to A_U$ such that $\psi'_U=a(U)\psi_U$. Moreover, the image $A_1(U):=a(U)(\mathrm{Alb}\, X_U/U)$ contains $\psi's(U)$ and its fiber $A_1(U)_y$ over $y\in U$ is an abelian subvariety of A_y. We show that the closure A_1 of $A_1(U)$ in A is analytic. Let $S'=\psi's(Y)\subseteq A$. Let $D_{A/Y}(S')$ $=\{d\in D_{A/Y,red};\ Z_{A/Y,d}\ni S'_d:=\psi's(d)\}$ where $D_{A/Y}$ is the relative Douady space for g. Then $D_{A/Y}(S')$ is an analytic subset of $D_{A/Y,red}$. Let $\tau(U): U\to D_{A/Y}(S')_U$ be the universal U-morphism associated to the inclusion $A_1(U)\subseteq A_U$. Let D_α be the irreducible component of $D_{A/Y}(S')$ which contains $\tau(U)(U)$. Since, for any $y\in U$, $D_{\alpha,y}$ contains the point $d(y)$ corresponding to $A_1(U)_y$ as an isolated point, $\tau(U)(U)$ must be Zariski open in D_α. This implies that $D_{\alpha,U}=\tau(U)(U)$. Hence the natural image A_1 of the universal subspace $Z_\alpha\subseteq A\times_Y D_\alpha$ in A is the desired subspace of A which is the closure of $A_1(U)$.

b) We consider the general case. Let $\tilde{Y}=X$, $\tilde{X}=X\dot{\times}_Y\tilde{Y}$. Let $\tilde{f}: \tilde{X}\to\tilde{Y}$ be the natural morphism. We set $\nu=f: \tilde{Y}\to Y$. Let $\tilde{U}=\nu^{-1}(U)$. Since $\tilde{f}$ admits a holomorphic section, by what we have proved in a) we have the relative Albanese map $\tilde{\psi}: \tilde{X}\to \mathrm{Alb}^*\tilde{X}/\tilde{Y}$ for $\tilde{f}$. On the other hand, by our construction of $\mathrm{Alb}^*\tilde{X}/\tilde{Y}$ we see readily that if we restrict U, $(\mathrm{Alb}^*\tilde{X}/\tilde{Y})_{\tilde{U}}$ is smooth over $\tilde{U}$, and then, it is isomorphic to $\mathrm{Alb}\,\tilde{X}_{\tilde{U}}/\tilde{U}\cong(\mathrm{Alb}\,X_U/U)\times_U\tilde{U}$. Let $u:(\mathrm{Alb}^*\tilde{X}/\tilde{Y})_{\tilde{U}}\to\mathrm{Alb}\,X_U/U$ be the induced morphism. Then u is smooth and hence by Lemma 9 there exists a Zariski open embedding $\mathrm{Alb}\,X_U/U\to\mathrm{Alb}^*X/Y$ with Alb^*X/Y a compact complex variety in $\mathcal{C}$ over Y such that u extends to a meromorphic Y-map $u^*:\mathrm{Alb}^*\tilde{X}/\tilde{Y}\to\mathrm{Alb}^*X/Y$. Then, since $u\tilde{\phi}=\phi_{X_U}\tilde{\nu}$ on $\tilde{X}_{\tilde{U}}$ where $\tilde{\nu}:\tilde{X}\to X$ is the natural map and $\phi_{X_U}:X_U\to\mathrm{Alb}\,X_U/U$ is the relative Albanese map for the smooth morphism f_U, by Lemma 10 $\tilde{\phi}$ induces a meromorphic extension $\phi:X\to\mathrm{Alb}^*X/Y$ of ϕ_{X_U}. The universality can be seen in a similar way by reducing to the absolute case as in a). If f is Moishezon, then $\tilde{f}$, and hence $\mathrm{Alb}^*\tilde{X}/\tilde{Y}\to\tilde{Y}$ also, is Moishezon. Hence $\mathrm{Alb}^*X/Y\to Y$ is Moishezon by ([6], Prop. 1).*)

II. *Moishezonness of* ϕ. By Proposition 4 passing to another bimeromorphic model we may assume that f is generically locally projective. (By the property 1) and the universality, the relative Albanese map is bimeromorphically invariant.) Take and fix an s. ample $\alpha\in\Gamma(f)$ such that $\mathrm{Pic}^*_\alpha X/Y\to Y$ is a fiber space (Lemma 5). Since α is s. ample, $\mathrm{Div}^*_\alpha X/Y$ is smooth over U_α, and $(\mathrm{Div}^*_\alpha X/Y)_y$ is connected for any $y\in Y$. In particular for $y\in U_\alpha$, there exists a unique $\alpha_y\in NS(X_y)$ such that $(\mathrm{Div}^*_\alpha X/Y)_y=\mathrm{Div}_{\alpha_y}X_y$. Let $Z^*_\alpha\subseteq X\times_Y\mathrm{Div}^*_\alpha X/Y$ be the meromorphic universal divisor. Considering X as a parameter space we have the universal meromorphic Y-map $\varphi_\alpha:X\to\mathrm{Div}^*((\mathrm{Div}^*_\alpha X/Y)/Y)$. For simplicity write $D^*_\alpha=\mathrm{Div}^*_\alpha X/Y$. Let $\mathrm{Pic}^*_\gamma(D^*_\alpha/Y)$ be the unique irreducible component of $\mathrm{Pic}^*(D^*_\alpha/Y)$ containing the image of X under the composite meromorphic map $\mu_{D^*_\alpha/Y}\varphi_\alpha$: $X\to\mathrm{Pic}^*(D^*_\alpha/Y)$. Let $\phi_\alpha:X\to\mathrm{Pic}^*_\gamma(D^*_\alpha/Y)$ be the induced map.

We claim that ϕ_α is a relative Albanese map for f. For this, it suffices by Corollary (which depends only on the Property 1) of (*)) to show that for general $y\in U$ the induced morphism $\phi_{\alpha,y}:X_y\to\mathrm{Pic}^*_\gamma(D^*_\alpha/Y)_y$ is isomorphic to the Albanese map of X_y. We first note that $\mathrm{Pic}^*_\gamma(D^*_\alpha/Y)_y$ is connected. In fact, let $\beta^*_{\gamma,1}$: $\mathrm{Pic}^*_\gamma D^*_\alpha/Y\to\bar{P}^*_\gamma$, $\beta^*_{\gamma,2}:\bar{P}^*_\gamma\to Y$ be the Stein factorization of $\beta^*_\gamma:\mathrm{Pic}^*_\gamma D^*_\alpha/Y\to Y$. Then $\beta^*_{\gamma,1}\phi_\alpha(X)\subseteq\bar{P}^*_\gamma$ gives a meromorphic section to $\beta^*_{\gamma,2}$ since f is a fiber space. Hence $\beta^*_{\gamma,2}$ is bimeromorphic and β^*_γ is a fiber space as was desired. Thus there exists a unique $\gamma_y\in NS(D^*_{\alpha,y})$ such that $(\mathrm{Pic}^*_\gamma D^*_\alpha)_y=\mathrm{Pic}_{\gamma_y}(\mathrm{Div}_{\alpha_y}X_y)$ for $y\in U_\alpha$. Moreover $\phi_{\alpha,y}:X_y\to\mathrm{Pic}_{\gamma_y}(\mathrm{Div}_{\alpha_y}X_y)$ is precisely the morphism defined from the inclusion $Z^*_{\alpha,y}=Z_{\alpha_y}\subseteq X_y\times\mathrm{Div}_{\alpha_y}X_y$ as in Lemma 2. Hence by that lemma, $\phi_{\alpha,y}$ is an Albanese map of X_y. Finally since α is s. ample, X is bimeromorphic over Y to the image of X in $\mathrm{Div}^*_\gamma(D^*_\alpha/Y)$ via φ_α, and hence, ϕ_α is Moishezon since μ^*_γ is Moishezon by the property 4) of $\mathrm{Pic}^*_\gamma X/Y$. q. e. d.

4.3. *Some applications.* Let $g:Z\to W$ be a fiber space of complex varieties. A *meromorphic multi-section* to g is an analytic subvariety $B\subseteq Z$ such that the restriction $g|_B:B\to W$ is surjective and generically finite.

*) From our construction it follows that Alb^*X/Y is bimeromorphic over U to $\mathrm{Alb}\,X_U/U$ for any U as above.

Proposition 6. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$ with $a(f)=\dim f$. Let $W \subseteq Y$ be an open subset. Suppose that there exists a meromorphic multi-section to f defined in a neighborhood $\overline{W}$, the closure of W. Then $f_W: X_W \to W$ is Moishezon.*

Proof. For simplicity of notation we consider only the case $W=Y$. The general case can be treated completely in the same way. Let $B \subseteq X$ be a meromorphic multisection to f. Since $B \to Y$ is generically finite it is Moishezon. So it suffices to show that $f \times_Y B: X \times_Y B \to B$ is Moishezon; we may assume from the beginning that f admits a meromorphic section $s: Y \to X$. Now by Theorem 2 it suffices to show that $\mathrm{Alb}^* X/Y \to Y$ is Moishezon, so that (considering $\mathrm{Alb}^* X/Y$ instead of X) we may assume that the general fiber of f is an abelian variety. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth and on which s is defined. Then there exists on X_U a unique structure of a relative complex Lie group over U (cf. [10]). Then by Mumford [20] we can construct a line bundle on X_U which is relatively ample with respect to f_U. Our idea is then nothing but to check that his construction extends 'meromorphically' to the whole X. First, by [10] Prop. 7, the relative group multiplication $X_U \times_U X_U \to X_U$ of X_U extends to a meromorphic Y-map $b^*: X \dot{\times}_Y X \to X$. Take an s. ample component $P_\gamma^* := \mathrm{Pic}_\gamma^* X/Y$ which is a fiber space over Y (Lemma 5). Let $\mathcal{L}_\gamma$ be a meromorphic relative Poincaré sheaf on $X_\gamma := X \dot{\times}_Y P_\gamma^*$ associated to f and s (Proposition 5). Let $b_\gamma^*: X_\gamma \dot{\times}_{P_\gamma^*} X_\gamma \to X_\gamma$ be induced by b^*. Let $p_i: X_\gamma \dot{\times}_{P_\gamma^*} X_\gamma \to X_\gamma$ be the projections to the i-th factors. Set $\mathcal{M}_\gamma = b_\gamma^* \mathcal{L}_\gamma \otimes p_1^* \mathcal{L}_\gamma^\vee \otimes p_2^* \mathcal{L}_\gamma^\vee$ (cf. 3.4 a)) which is a coherent analytic sheaf on $X_\gamma \dot{\times}_{P_\gamma^*} X_\gamma$ and is invertible on some Zariski open subset of Y. Consider $X_\gamma \dot{\times}_{P_\gamma^*} X_\gamma$ as a complex space over X_γ via p_1. Then by the property 6) of $\mathrm{Pic}^* X/Y$, $\mathcal{M}_\gamma$ defines the universal meromorphic P_γ^*-map $X_\gamma \to \mathrm{Pic}_0^*(X_\gamma/P_\gamma^*)$, denoted by $\Lambda(\mathcal{L}_\gamma)$, which is holomorphic over $P_{\gamma,U}^* := P_\gamma^* \times_Y U$ (cf. [20], p. 120). Define $\Lambda(\mathcal{L}_\gamma)': X_\gamma \to (\mathrm{Pic}_0^* X/Y) \dot{\times}_Y P_\gamma^*$ by the composition of $\Lambda(\mathcal{L}_\gamma)$ and the natural bimeromorphic Y-map $\mathrm{Pic}_0^*(X_\gamma/P_\gamma^*) \to (\mathrm{Pic}_0^* X/Y) \dot{\times}_Y P_\gamma^*$ (cf. Def. 4, 2)). Then by a theorem of Weil $\Lambda(\mathcal{L}_\gamma)'_{U_0}$ descends to a U-morphism $\overline{\Lambda(\mathcal{L}_\gamma)}_{U_0}: X_{U_0} \to \mathrm{Pic}_0^* X_{U_0}/U_0$ (cf. [20], p. 120, Def. 6.2), where U_0 is as in 1) of Definition 4. Then by Lemma 10 $\overline{\Lambda(\mathcal{L}_\gamma)}_{U_0}$ extends to a meromorphic Y-map $\overline{\Lambda(\mathcal{L}_\gamma)}$: $X \to \mathrm{Pic}_0^* X/Y$. We set $\mathcal{F} = j^*(id_X \times_Y \overline{\Lambda(\mathcal{L}_\gamma)})^* \mathcal{L}_0$ where $\mathcal{L}_0$ is the meromorphic relative Poincaré sheaf on $X \dot{\times}_Y \mathrm{Pic}_0^* X/Y$ and $j: X \to X \times_Y X$ is the embedding as the diagonal. Then by [20] Prop. 6.10, the restriction $\mathcal{F}_y$ of $\mathcal{F}$ to X_y, $y \in U_0$, is an ample invertible sheaf. It follows that f is Moishezon. q.e.d.

Proposition 7. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$ with $a(f)=\dim f$. Suppose that $q(X_y)=0$ for a general fiber X_y of f where $q(X_y) := \dim H^1(X_y, \mathcal{O}_{X_y})$ is the irregularity of X_y. Then f is Moishezon.*

Proof. Since $q(X_y)=0$, $\mathrm{Alb}^* X/Y \to Y$ is bimeromorphic. Hence f is bimeromorphic to its Albanese map $\psi: X \to \mathrm{Alb}^*X/Y$ which is Moishezon by Theorem 2. q. e. d.

A proper morphism $f: X \to Y$ of complex spaces is called *locally Moishezon* if for any $y \in Y$ there exists a neighborhood $y \in V$ such that $f_V: X_V \to V$ is Moishezon. By Chow lemma [15] it is immediate to see that if f is locally Moishezon, every fiber of f is Moishezon.

Proposition 8. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. Then the following conditions are equivalent.* 1) $a(f) = \dim f$, 2) f_U: $X_U \to U$ *is locally Moishezon, and* 3) *there exists a bimeromorphic model $f^*: X^* \to Y^*$ of f which is locally Moishezon.*

Proof. By the remark preceding the proposition it is clear the 2) or 3) implies 1). So we show that 1) implies 2) and 3). 1)→2): Since $\eta_U: \mathrm{Alb} X_U/U \to U$ is smooth, we can get a holomorphic section to η_U at any point of U. Since Alb^*X/Y is bimeromorphic over U to $\mathrm{Alb}\, X_U/U$, Alb^*X/Y then admits a meromorphic section locally at any point of U. Hence by Proposition 6 f is locally Moishezon. 1)→3): Let $p: \tilde{Y} \to Y$ be a proper modification such that the strict transform $(\mathrm{Alb}^*X/Y)^{\sim}$ in $(\mathrm{Alb}^*X/Y) \times_Y \tilde{Y}$ is flat over $\tilde{Y}$. Since $(\mathrm{Alb}^*X/Y)^{\sim}$ is bimeromorphic to $\mathrm{Alb}^*(X \times_Y \tilde{Y}/\tilde{Y})$, it follows that $\mathrm{Alb}^*(X \times_Y \tilde{Y}/Y)$ admits a meromorphic multi-section at any point of $\tilde{Y}$. Hence $f_{\tilde{Y}}: X \times_Y \tilde{Y} \to \tilde{Y}$ is locally Moishezon by Proposition 6. Take $f^* = f_{\tilde{Y}}$. q. e. d.

Remark 2. In general even if $a(f) = \dim f$, f may not be locally Moishezon unless we take a flattening of f. In fact, let $f: X \to S$ be a flat elliptic fiber space such that f is an algebraic reduction of X, where $\dim X = 3$ and $\dim S = 2$. Suppose that there exists an irreducible exceptional curve of the first kind C on S such that $f: X_C \to C$ is an algebraic reduction of X_C. Let $\phi: S \to S'$ be the contraction of C to a smooth point $p \in S'$. Then $a(f') = \dim f' = 1$ for $f' = \phi f$: $X \to S'$ while $f^{-1}(p) = X_C$ is not Moishezon. Further it is easy to find an actual example of such.

Proposition 9. *Let $f: X \to Y$ be a generically smooth fiber space of compact complex varieties in $\mathcal{C}$. Suppose that* $\dim X = 3$. *Then the relative Albanese map for f exists except possibly the case where the general fiber of f is an elliptic surface with trivial homological invariant* (*cf.* [18]).

Proof. If $a(f) = \dim f$, this follows from Theorem 2. If $\dim f = 0$ or 3, then the proposition is clearly true. So we may assume that $\dim f = 2$.

If $a(f) = 1$, then the general fiber of f is an elliptic surface. Let $\psi(U): X_U \to \mathrm{Alb}\, X_U/U$ be the relative Albanese map for f_U where U is a Zariski open subset of Y over which f is smooth. Let $\psi(U)(X_U) = C(U)$. Suppose that X_y has non-trivial homological invariant. Then we have $\dim C(U) = 2$ and the induced map $\varphi(U): X_U \to C(U)$ is a flat fiber space (cf. [18]). Hence by Lemma 9 there exists a Zariski open embedding $C(U) \subseteq C$ with C a compact complex variety in C over Y such that $\varphi(U)$ extends to a meromorphic Y-map $\varphi: X \to C$. Since $b: C \to Y$ has relative dimension 1 and hence $a(b)=1$, by Theorem 2 we have the relative Albanese map $\psi_C: C \to \mathrm{Alb}^* C/Y$ for b. Then it is immediate to see that $\psi_C \varphi: X \to \mathrm{Alb}^* C/Y$ is the desired Albanese map for X (cf. Corollary to Theorem 2).

Finally suppose that $a(f)=0$. Then X_y is either bimeromorphic to a complex torus or a K3 surface. In the latter case there is nothing to prove since $q(X_y)=0$. In the former case we use [11] § 1 Theorem; according to it either X_y is isomorphic to a complex torus or f is bimeromorphic to a morphism $(S \times E)/G \to E/G$ where E is a compact Riemann surface, S is a complex torus, and G is a finite group acting on both E and T. In the first case we may set $X = \mathrm{Alb}^* X/Y$, and in the second case we can take $(S \times E)/G$ as the relative Albanese variety by Corollary to Theorem 2. q. e. d.

Final Remark. Let $f: X \to Y$ be a fiber space of compact complex varieties. We say that $f \in \mathcal{C}/Y$ if there exist a proper locally Kähler morphism $g: Z \to Y$ and a surjective meromorphic Y-map $\varphi: Z \to X$ (cf. [6]). Then the results of this paper are true even if the condition $X \in \mathcal{C}$ is replaced by a weaker one $f \in \mathcal{C}/Y$ if in the statements everything is restricted to an arbitrary relatively compact open subset of Y. (In particular if Y is compact no restriction is needed.)

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Note added in Proof. The relative Albanese map with the property 1) of Theorem 2 has recently been constructed by F. Campana without the assumption that $\dim f = a(f)$, by quite a different method.