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Preparatory Structure Theorem for Ideals Defining Space Curves

By

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Introduction

In the study of $\mathrm{Hilb}(\mathbf{P}_k^3)$ any knowledge of concrete structure of ideals defining the minimal cone of a curve in $\mathbf{P}_k^3$ would benefit one greatly. Here 'curve' means an equidimensional complete scheme over a field k of dimension one. Let $I \subset R := k[x_1, x_2, x_3, x_4]$ be the ideal defining the minimal cone of a curve $X \subset \mathbf{P}_k^3$. Then we know that $\dim R/I = 2$ and $\mathrm{depth}_{\mathfrak{m}} R/I \geqq 1$. The present paper is aimed at giving a way to describe all homogeneous ideals with this property. We show in Section 3 that any such ideal, with a free resolution for it, is determined by a matrix of special type which satisfies seemingly a simple relation (See Proposition 3.1, Corollary 3.5 and Theorem 3.7). We discuss briefly the easiest case in Section 4 to illustrate how the results of Section 3 work.

In order to provide necessary techniques for obtaining our main results, we describe in Section 2 a general method to compute a free resolution for any ideal in $k[[x_1, \cdots, x_n]]$. The free resolutions indicated there start from the generalised Weierstrass preparation theorem due to H. Grauert and H. Hironaka. The author borrowed this setting from [8].

Notation

1. k denotes an infinite field with arbitrary characteristic from Section 1 to Section 3.
2. Let A be $k[[t_1, \cdots, t_d]]$ or $k[t_1, \cdots, t_d]$, $\mathfrak{n}$ its maximal ideal gener-

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ated by $t_1, \cdots, t_d$, and $x_1, \cdots, x_n$ indeterminates over A. We set

$$A((i,n)) = A[[x_{i+1}, \cdots, x_n]]$$

$$A(i,n) = A[x_{i+1}, \cdots, x_n]$$

both for $0 \leqq i \leqq n$. In particular $A((n,n)) = A(n,n) = A$.

3. $A((i,n))^p$ or $A(i,n)^p$ denotes the set of column vectors unless otherwise specified.

4. Let $B_1, B_2, \cdots, B_s$ be arbitrary A-modules. We denote by $B_1 \oplus B_2 \oplus \cdots \oplus B_s$ or by $\bigoplus_{i=1}^{s} B_i$ the A-module

$$\left\{ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{pmatrix} \middle| b_i \in B_i \quad \text{for} \quad 1 \leqq i \leqq s \right\}.$$

5. For an $f \in A((i,n))^p$ (resp. $f \in A(i,n)^p$) $f(0)$ denotes f (mod $\mathfrak{n}$) which is naturally thought of as an element of $k((i,n))^p$ (resp. $k(i,n)^p$) via $k \hookrightarrow A$.

6. 1_p denotes $p \times p$ identity matrix.

7. For an $f = \sum_{|\nu|=0}^{\infty} a_\nu x^\nu \in A((i,n))$ with $a_\nu \in A$, $o(f) = \min\{|\nu| \mid a_\nu \neq 0\}$, and $\mathrm{in}(f) = \sum_{|\nu|=o(f)} a_\nu x^\nu$.

8. $\boldsymbol{Z}_0 = \{\alpha \in \boldsymbol{Z} \mid \alpha \geqq 0\}$.

§ 1. Preliminaries

Let A be a formal power series ring over a field k, $\mathfrak{n}$ its maximal ideal, and $x_1, \cdots, x_n$ indeterminates over A.

Definition 1.1. Let $\bar{a} = (a_1, \cdots, a_p)$ be a sequence of integers. For each $f = (f_i) \in A((0,n))^p$ we define

$$d_{\bar{a}}(f) = \min_{1 \leqq i \leqq p} (o(f_i) + a_i).$$

Suppose we are given a set of positive integers

$$\{m, l_1, \cdots, l_m, l = \sum_{i=1}^{m} l_i, \ \ s_\alpha = \sum_{i=\alpha+1}^{m} l_i \ \ (0 \leqq \alpha \leqq m-1)\},$$

a sequence of integers $\bar{q} = (q_1, \cdots, q_l)$, and $l \times s_\alpha$ matrices χ^α $(1 \leqq \alpha \leqq m-1)$

with entries in $A((0, n))$ satisfying the following two conditions:

(I) Write $\chi^{\alpha}=(\chi_1^{\alpha}, \cdots, \chi_{s_\alpha}^{\alpha})$ $(1 \leqq \alpha \leqq m-1)$, then each column vector χ_β^{α} is in $\bigoplus_{i=1}^{m} A((i-1, n))^{l_i}$.

(II) $d_{\bar{q}}(\chi_j^{\alpha}) \geqq 1+q_{l-s_\alpha+j}$ for $1 \leqq \alpha \leqq m-1,\ 1 \leqq j \leqq s_\alpha$.

We set

$$\psi^{\alpha}=(\psi_1^{\alpha}, \cdots, \psi_{s_\alpha}^{\alpha})=\begin{array}{c} \left(\begin{array}{c} 0 \\ \\ x_\alpha 1_{s_\alpha} \end{array}\right)\begin{array}{l} \updownarrow l-s_\alpha \\ \\ \\ \end{array} \\ \longleftarrow s_\alpha \longrightarrow \end{array} -\chi^{\alpha}$$

for $1 \leqq \alpha \leqq m-1$.

Proposition 1.2.

1) $A((0, n))^{l}=\{\bigoplus_{\alpha=1}^{m-1} \bigoplus_{j=1}^{s_\alpha} \psi_j^{\alpha} A((\alpha-1, n))\} \oplus \{\bigoplus_{i=1}^{m} A((i-1, n))^{l_i}\}$

(*direct sum as A-modules*).

2) *If* $f=\sum_{\alpha=1}^{m-1} \sum_{j=1}^{s_\alpha} \psi^{\alpha} g_j^{\alpha}+h$ *with* $g_j^{\alpha} \in A((\alpha-1, n))$ *and* $h \in \bigoplus_{l=1}^{m} A((i-1, n))^{l_i}$, *then*

$$\begin{cases} d_{\bar{q}}(f) \leqq d_{\bar{q}}(\psi^{\alpha})+o(g_j^{\alpha}) & \text{for} \quad 1 \leqq \alpha \leqq m-1, 1 \leqq j \leqq s_\alpha \\ d_{\bar{q}}(f) \leqq d_{\bar{q}}(h). & \end{cases}$$

Proof. 1) is a consequence of the following:

$$(1)_\mu \qquad A((0, n))^{l}=\{\bigoplus_{\alpha=1}^{\mu} \psi^{\alpha} A((\alpha-1, n))^{s_\alpha}\}$$
$$\oplus\{\bigoplus_{i=1}^{\mu+1} A((i-1, n))^{l_i} \oplus A((\mu, n))^{s_{\mu+1}}\}$$

for $1 \leqq \mu \leqq m-1$, *where* $s_m=0$.

Let $\psi^{\alpha 1}$ (resp. $\chi^{\alpha 1}$) be the $s_\alpha \times s_\alpha$ matrix consisting of lower s_α rows of ψ^{α} (resp. χ^{α}) and $\psi^{\alpha 0}$ the upper $l-s_\alpha$ rows of $\psi^{\alpha 1}$ for $1 \leqq \alpha \leqq m-1$. Then, to prove $(1)_\mu$, we need a lemma.

Lemma 1.3.

$$A((\alpha-1, n))^{s_\alpha}=\psi^{\alpha 1} A((\alpha-1, n))^{s_\alpha} \oplus A((\alpha, n))^{s_\alpha}$$

for $1 \leqq \alpha \leqq m-1$. *(direct sum as A-modules)*

Proof of Lemma 1.3. If $\psi^{\alpha 1} g + h = 0$ with $g \in A((\alpha-1, n))^{s_\alpha}$, $h \in A((\alpha, n))^{s_\alpha}$, then

$$(\det \psi^{\alpha 1}) g = -(\det \psi^{\alpha 1})(\psi^{\alpha 1})^{-1} h .$$

We observe that $\det \psi^{\alpha 1} = x_\alpha^{s_\alpha} + \sum_{i=1}^{s_\alpha - 1} \phi_i x_\alpha^i$, where $\phi_i \in A((\alpha, n))$ and $o(\phi_i x_\alpha^i) \geqq s_\alpha$ by (II), and that the degree of each component of $-(\det \psi^{\alpha 1})(\psi^{\alpha 1})^{-1} h$ with respect to x_α is strictly smaller than s_α. From this and Weierstrass division theorem we deduce $g = h = 0$. Thus the sum is direct. Next we show that each $f \in A((\alpha-1, n))^{s_\alpha}$ can be written $f = \psi^{\alpha 1} g + h$ with $g \in A((\alpha-1, n))^{s_\alpha}$ and $h \in A((\alpha, n))^{s_\alpha}$ for $1 \leqq \alpha \leqq m-1$. Write $f = \sum_{t=0}^{\infty} x_\alpha^t a_t$ where $a_t \in A((\alpha, n))^{s_\alpha}$. We claim

$$\left\{ \begin{aligned} g &= \sum_{t=1}^{\infty} \sum_{i=1}^{t} x_\alpha^{t-i} (\chi^{\alpha 1})^{i-1} a_t \\ h &= \sum_{t=0}^{\infty} (\chi^{\alpha 1})^t a_t \end{aligned} \right. \tag{1.4}$$

are both well defined.

Proof of (1.4) With each matrix X with entries in $A((0, n))$ we associate a matrix $\Delta(X)$ whose (i, j) component is the order $o(x_{ij})$ of the (i, j) component x_{ij} of X. Then we know by (II) that

$$(i, j) \text{ component of } \Delta(\chi^{\alpha 1}) \geqq 1 + q_{l-s_\alpha+j} - q_{l-s_\alpha+i}$$
$$\text{for} \quad 1 \leqq \alpha \leqq m-1, \quad 1 \leqq i, j \leqq s_\alpha .$$

From this we get

$$(i, j) \text{ component of } \Delta((\chi^{\alpha 1})^p) \geqq p + q_{l-s_\alpha+j} - q_{l-s_\alpha+i}$$
$$\text{for} \quad 1 \leqq \alpha \leqq m-1, \quad 1 \leqq i, j \leqq s .$$

Hence

$$d_{\bar{q}(\alpha)}((\chi^{\alpha 1})^p a_t) \geqq p + d_{\bar{q}(\alpha)}(a_t) \quad \text{for} \quad p \geqq 0, t \geqq 0 \tag{2}$$

where $\bar{q}(\alpha) = (q_{l-s_\alpha+1}, \cdots, q_l)$. Therefore the formal sums in (1.4) are well defined.

$g \in A((\alpha-1, n))^{s_\alpha}$, $h \in A((\alpha, n))^{s_\alpha}$ by the definition and we see by an easy computation that $f=\psi^{01}g+h$. Thus Lemma 1.3 is proved.

Proof of $(1)_\mu$. If $\mu=1$ we know from Lemma 1.3 that

$$A((0,n))^{s_1}=\psi^{11}A((0,n))^{s_1}\oplus A((1,n))^{s_1}.$$

From this

$$A((0,n))^l=\psi^1 A((0,n))^{s_1}$$
$$\oplus\{A((0,n))^{l_1}\oplus A((1,n))^{l_2}\oplus A((1,n))^{s_2}\}$$

is almost clear. So by induction we assume

$$(1)_{\mu_0}\qquad A((0,n))^l=\{\bigoplus_{\alpha=1}^{\mu_0}\psi^\alpha A((\alpha-1,n))^{s_\alpha}\}$$
$$\oplus\{\bigoplus_{i=1}^{\mu_0+1}A((i-1,n))^{l_i}\oplus A((\mu_0,n))^{s_{\mu_0+1}}\}$$

for some $\mu_0\geqq 1$. We have

$$(3)\qquad A((\mu_0,n))^{s_{\mu_0+1}}=\psi^{\mu_0+1\,1}A((\mu_0,n))^{s_{\mu_0+1}}\oplus A((\mu_0+1,n))^{s_{\mu_0+1}}$$

by Lemma 1.3. Since each column vector of $\psi^{\mu_0+1\,0}$ is in $\bigoplus_{i=1}^{\mu_0+1}A((i-1,n))^{l_i}$, (3) implies that

$$(4)\qquad \bigoplus_{i=1}^{\mu_0+1}A((i-1,n))^{l_i}\oplus A((\mu_0,n))^{s_{\mu_0+1}}$$
$$=\psi^{\mu_0+1}A((\mu_0,n))^{s_{\mu_0-1}}\oplus\{\bigoplus_{i=1}^{\mu_0+2}A((i-1,n))^{l_i}\oplus A((\mu_0+1,n))^{s_{\mu_0+2}}\}.$$

From $(1)_{\mu_0}$ and (4) we get $(1)_{\mu_0+1}$. Thus $(1)_\mu$ is obtained for $1\leqq\mu\leqq m-1$ and Proposition 1.2.1) is proved. Proposition 1.2.2) follows from (1.4), (2), and the proof of $(1)_\mu$. Q.E.D.

Let M be an $A((0,n))$-module which is a direct sum of A-modules

$$(*)\qquad M=\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{l_i}f_{l-s_{i-1}+j}A((i-1,n))$$

where $f_j\in M$ for $1\leqq j\leqq l$. We want to know the relation module

$$M_1=\{\psi={}^t(\psi_1,\cdots,\psi_l)\in A((0,n))^l\mid\sum_{i=1}^{l}\psi_i f_i=0\}.$$

Since M is an $A((0,n))$-module, $x_\alpha f_j\in M$ for any $1\leqq\alpha\leqq n$, $1\leqq j\leqq l$, whence we may write for $1 \leqq \alpha \leqq m-1$, $1 \leqq j \leqq s_\alpha$

$$(\ast)\qquad x_\alpha f_{l-s_\alpha+j} = \sum_{i=1}^{m} \sum_{\beta=1}^{l_i} f_{l-s_{i-1}+\beta} \tilde{\chi}^\alpha_{l-s_{i-1}+\beta,\, j}$$

with $\tilde{\chi}^\alpha_{l-s_{i-1}+\beta,\,j} \in A((i-1, n))$. We set

$$(\ast\ast)\qquad \begin{cases} \tilde{\chi}^\alpha_j = {}^t(\tilde{\chi}^\alpha_{1j}, \cdots, \tilde{\chi}^\alpha_{lj}) \\ \tilde{\chi}^\alpha = (\tilde{\chi}^\alpha_1, \cdots, \tilde{\chi}^\alpha_{s_\alpha}) \qquad \text{for } 1 \leqq \alpha \leqq m-1 . \end{cases}$$

Then $\tilde{\chi}^\alpha (1 \leqq \alpha \leqq m-1)$ satisfy the condition (I) by their construction but (II) may not be guaranteed. In the situations we shall later encounter the condition (II) is also satisfied by $\tilde{\chi}^\alpha(0)$. Therefore we assume that $\tilde{\chi}^\alpha(0)$ satisfies the condition (II).

Put

$$(\ast\ast\ast\ast)\qquad \begin{cases} \tilde{\psi}^\alpha = (\tilde{\psi}^\alpha_1, \cdots, \tilde{\psi}^\alpha_{s_\alpha}) = \begin{pmatrix} 0 \\ x_\alpha 1_{s_\alpha} \end{pmatrix} - \tilde{\chi}^\alpha \\ \\ \psi^\alpha = (\psi^\alpha_1, \cdots, \psi^\alpha_{s_\alpha}) = \begin{pmatrix} 0 \\ x_\alpha 1_{s_\alpha} \end{pmatrix} - \tilde{\chi}^\alpha(0) , \end{cases}$$

then $\tilde{\psi}^\alpha_j - \psi^\alpha_j \in \mathfrak{n} A((0, n))^l$ for $1 \leqq \alpha \leqq m-1$, $1 \leqq j \leqq s_\alpha$.

First we have

Corollary 1.5.

1) $A((0, n))^l = \{ \bigoplus_{\alpha=1}^{m-1} \bigoplus_{j=1}^{s_\alpha} \tilde{\psi}^\alpha_j A((\alpha-1, n)) \} \oplus \{ \bigoplus_{i=1}^{m} A((i-1, n))^{l_i} \}$

(direct sum as A-modules)

2) *If* $f = \sum_{\alpha=1}^{m-1} \sum_{j=1}^{s_\alpha} \tilde{\psi}^\alpha_j g^\alpha_j + h$ *with* $g^\alpha_j \in A((\alpha-1, n))$ *and* $h \in \bigoplus_{i=1}^{m} A((i-1, n))^{l_i}$, *then*

$$\begin{cases} d_{\bar{\mathfrak{q}}}(f(0)) \leqq d_{\bar{\mathfrak{q}}}(\tilde{\psi}^\alpha_j(0)) + o(g^\alpha_j(0)) \\ d_{\bar{\mathfrak{q}}}(f(0)) \leqq d_{\bar{\mathfrak{q}}}(h(0)) \end{cases}$$

Proof. Put $S=\bigoplus_{\alpha=1}^{m-1}\bigoplus_{j=1}^{s_\alpha}\psi_j^\alpha A((\alpha-1,n))$ and $H=\bigoplus_{i=1}^{m}A((i-1,n))^{l_i}$, then $A((0,n))^l=S\oplus H$ by Proposition 1. 2. 1). Consider the commutative diagrame

$$\begin{tikzcd} S\oplus H \arrow[r, "\tau"] \arrow[dr, "(p_1\circ\tau,\ p_2\circ\tau)"'] & A((0,n))^l \arrow[d, equal] & \text{Proposition 1. 2. 1)} \\ & S\oplus H & \end{tikzcd}$$

where τ is defined by $\tau(\sum\psi_j^\alpha g_j^\alpha+h)=\sum\tilde{\psi}_j^\alpha g_j^\alpha+h$ and p_1, p_2 are the projections to S, H respectively. Since $\tilde{\psi}_j^\alpha-\psi_j^\alpha\in\mathfrak{n}A((0,n))$ $id_S-p_1\circ\tau|_S$ maps $\mathfrak{n}^rS$ to $\mathfrak{n}^{r+1}S$ for any integer $r\geqq 0$, so that we can define $id_S+\sum_{i=1}^{\infty}\lambda^i$ with $\lambda=id_S-p_1\circ\tau|_S$ on S. We define $\kappa: S\oplus H\to S\oplus H$ to be the map

$$\kappa(a,b)=((id_S+\sum_{i=1}^{\infty}\lambda^i)(a),\ b-p_2\circ\tau\circ(id_S+\sum_{i=1}^{\infty}\lambda^i)(a)).$$

Then it is easily seen that

$$\kappa\circ(p_1\circ\tau,\ p_2\circ\tau)=id_{S\oplus H},\quad (p_1\circ\tau,\ \ p_2\circ\tau)\circ\kappa=id_{S\oplus H}\,.$$

Hence $(p_1\circ\tau, p_2\circ\tau)$ is an isomorphism, and so is τ. This proves 1). 2) is clear by Proposition 1. 2. 2). Q.E.D.

Theorem 1. 6. *Notations being as above, we have*

$$M_1=\bigoplus_{\alpha=1}^{m-1}\bigoplus_{j=1}^{s_\alpha}\tilde{\psi}_j^\alpha A((\alpha-1,n)).$$

Proof. $M_1\subset\bigoplus_{\alpha=1}^{m-1}\bigoplus_{j=1}^{s_\alpha}\tilde{\psi}_j^\alpha A((\alpha-1,n))$ is clear by ($\substack{*\\*}$), ($\substack{**\\*}$), and ($\substack{**\\**}$). So it is enough to show that $M_1\cap\bigoplus_{i=1}^{m}A((i-1,n))^{l_i}=0$. But this is just what the direct sum (*) means. Q.E.D.

§ 2. A Method to Compute a Free Resolution

Let I be an ideal in $R=k[[x_1,\cdots,x_n]]$. The aim of this section is to give an algorithm to compute a free resolution for I. We begin by summarizing the generalised Weierstrass preparation theorem. Let $\mathfrak{m}=(x_1,\cdots,x_n)R$ be the maximal ideal of R and suppose $\operatorname{depth}_{\mathfrak{m}} R/I=d$. After a suitable linear coordinate transformation we may assume without loss of generality that $x_{n-d+1}, \cdots, x_n$ is a maximal R/I-regular sequence in $\mathfrak{m}$. Put $m=n-d$ and

$$(2.1.1) \qquad \bar{I}=\left\{\bar{f}\in k[[x_1, \cdots, x_m]] \,\middle|\, \begin{array}{l}\bar{f}=f(\mathrm{mod}\ (x_{m+1}, \cdots, x_n)R) \\ \text{for some } f\in I\end{array}\right\}.$$

Let u_{ij} $(1\leqq i, j\leqq m)$ be indeterminates over $k[[x_1, \cdots, x_m]]$ and K denote the field generated by u_{ij} $(1\leqq i, j\leqq m)$ over k. Define $z=(z_1, \cdots, z_m) \in K[[x_1, \cdots, x_m]]^m$ by the equations $x_i=\sum_{j=1}^{m} u_{ji}z_j$ $(1\leqq i\leqq m)$. Then $\bar{I}K[[x_1, \cdots, x_m]]=\bar{I}K[[z_1, \cdots, z_m]]$ and $E(z;\bar{I})$ is defined as a subset of $\boldsymbol{Z}_0^m$ by

$$(2.1.2) \qquad E(z;\bar{I})=\{\mathrm{lex}_z\, \mathrm{in}(\bar{F}) \mid \bar{F}\in \bar{I}K[[z_1, \cdots, z_m]]\}.$$

See [9; p. 280] for the definition of $\mathrm{lex}_z P$ where P is a polynomial. $E(z;\bar{I})$ has the following properties (see [8; Chap. 1]):

(2.1.3) *There exists a Zariski open set U in $GL(m, k)$ such that for every $a=(a_{ij})\in U$, $E(z;\bar{I})$ coincides with $\{\mathrm{lex}_{(y_1, \cdots, y_m)}\, \mathrm{in}(\bar{f}) \mid \bar{f}\in \bar{I}\}$, where $y_1, \cdots, y_m\in k[[x_1, \cdots, x_m]]$ are defined by the equations $x_i=\sum_{j=1}^{m} a_{ji}y_j$ $(1\leqq i\leqq m)$.*

$$(2.1.4) \qquad E(z;\bar{I})+\boldsymbol{Z}_0^m=E(z;\bar{I}),$$

(2.1.5) $(\nu_1, \cdots, \nu_m)\in E(z;\bar{I})$ *implies*

$$(\nu_1, \cdots, \nu_i, \sum_{j=i+1}^{m}\nu_j, 0, \cdots, 0)\in E(z;\bar{I}) \quad \textit{for} \quad 1\leqq i\leqq m-1.$$

Put $E=E(z;\bar{I})$. The structure of E is known in detail. Let us summarize the results we need later on.

First define $E_i\subset \boldsymbol{Z}_0^i$ by $E_i=\{\alpha\in \boldsymbol{Z}_0^i \mid (\alpha, 0, \cdots, 0)\in E\}$ for $1\leqq i\leqq m$ and then define $\Gamma_i', \Gamma_i, \Delta_i$ for $1\leqq i\leqq m-1$ inductively as follows (see [8; Chap. 1]):

$$\begin{cases} \Gamma_i'=\boldsymbol{Z}_0^i\setminus(E_i\cup \bigcup_{j=1}^{i-1}\Gamma_j\times \boldsymbol{Z}_0^{i-j}) \\ \Delta_i=\left\{\alpha\in\Gamma_i' \,\middle|\, \begin{array}{l}(\alpha, 0)\notin E_{i+1} \text{ and there exists a positive} \\ \text{integer } d \text{ such that } (\alpha, d)\in E_{i+1}\end{array}\right\} \\ \Gamma_i=\Gamma_i'\setminus\Delta_i. \end{cases}$$

We put $\Delta_0=\{\phi\}$ for convenience sake and further define

$$\Gamma_m=\boldsymbol{Z}_0^m\setminus(E\cup \bigcup_{j=1}^{m-1}\Gamma_j\times \boldsymbol{Z}_0^{m-j}).$$

For each $\delta\in\Delta_i$ let $d(\delta)$ be the minimum of d such that $(\delta, d)\in E_{i+1}$, in particular $d(\phi)$ is the smallest number of E_1. And put $A_{i\delta}=(\delta, d(\delta), 0, \cdots, 0)$. Then we have the following properties:

$$(2.1.6)\qquad \begin{cases} \boldsymbol{Z}_0^m=\bigcup_{i=0}^{m-1}\bigcup_{\delta\in\Delta_i}(A_{i\delta}+\boldsymbol{Z}_0(i))\cup\bigcup_{j=1}^{m}\Gamma_j\times\boldsymbol{Z}_0^{m-j} \\ \qquad\qquad (\textit{disjoint union}) \\ E=\bigcup_{i=0}^{m-1}\bigcup_{\delta\in\Delta_i}(A_{i\delta}+\boldsymbol{Z}_0(i)), \end{cases}$$

where $\boldsymbol{Z}_0(i)=\{\alpha=(\alpha_1, \cdots, \alpha_m)\in\boldsymbol{Z}_0^m\,|\,\alpha_1=\cdots=\alpha_i=0\}$.

(2.1.7) $\bigcup_{\delta\in\Delta_{i-1}}\delta\times[0, d(\delta))=\Delta_i\cup\Gamma_i$ *(disjoint) for* $1\leqq i\leqq m$ *and* Δ_m $=$ *empty.*

(2.1.8) *If* $(\nu_1, \cdots, \nu_i)\in\Delta_i$ *then* $(\nu_1, \cdots, \nu_{i_0})\in\Delta_{i_0}$ *for any* $i_0<i$.

The property (2.1.3) allows us to assume that $\{\mathrm{lex}_{(x_1,\cdots,x_m)}\mathrm{in}(\bar{f})\,|\,\bar{f}\in\bar{I}\}$ coincides with $E(z;\bar{I})$, so we shall continue the description with this assumption from now on.

Remark 2.2. Denote $\bigoplus_{j=1}^{m}\bigoplus_{\gamma\in\Gamma_j}x^{\gamma}k((j, n))$ by N_E. We deduce from (2.1.6)

1) $R=\bigoplus_{i=0}^{m-1}\bigoplus_{\delta\in\Delta_i}x^{A_{i\delta}}k((i, n))\oplus N_E$.

Let $x_t^{\nu_t}\cdots x_n^{\nu_n}$ be a monomial such that $\nu_t\neq 0$. For any monomial $x^{\alpha}\in R$ we can write uniquely

$$x_t^{\nu_t}\cdots x_n^{\nu_n}x^{\alpha}=\sum_{i=0}^{m-1}\sum_{\delta\in\Delta_i}g_{i\delta}x^{A_{i\delta}}+r$$

with $g_{i\delta}\in k((i, n))$ and $r\in N_E$ by 1). If $x^{\alpha}\in N_E$ or $x^{\alpha}=x^{A_{j\varepsilon}}$ for some $t\leqq j\leqq m-1$, $\varepsilon\in\Delta_j$, then we have

2) $g_{i\delta}=0$ for $i\leqq t-2$, $\delta\in\Delta_i$

3) $\deg_{x_t}g_{t-1\delta}\leqq\nu_t-1$ for $\delta\in\Delta_{t-1}$. In particular if $\nu_t=1$, $\deg_{x_t}g_{t-1\delta}=0$ i.e. $g_{t-1\delta}\in k((t, n))$.

These follow immediately from the definition of Δ_i and (2.1.8).

Put $A=k((m, n))$ and $\mathfrak{n}=(x_{m+1}, \cdots, x_n)A$. Then $R=A((0, m))$.

Theorem 2.3. (H. Grauert [5], H. Hironaka [7], [3]). *There exists* $f_{i\delta}\in I$ *such that* $f_{i\delta}-x^{A_{i\delta}}\in N_E$ *and* $o((f_{i\delta}(0))=o(x^{A_{i\delta}})$ *for each* $0\leqq i\leqq m-1$, $\delta\in\Delta_i$, *and we have the following*:

1) $R=I\oplus N_E$

2) $I=\bigoplus_{i=0}^{m-1}\bigoplus_{\delta\in\Delta_i} f_{i\delta}A((i,m))$

3) *If* $f=\sum_{i=0}^{m-1}\sum_{\delta\in\Delta_i} g_{i\delta}f_{i\delta}+r$ *with* $g_{i\delta}\in A((i,m))$ *and* $r\in N_E$, *then*

$$\begin{cases} o(f(0))\leqq o(f_{i\delta}(0))+o(g_{i\delta}(0)) & \text{for } 0\leqq i\leqq m-1,\ \delta\in\Delta_i \\ o(f(0))\leqq o(r(0)). \end{cases}$$

4) *If* $x_t f_{j\varepsilon}=\sum_{i=0}^{m-1}\sum_{\delta\in\Delta_i} g_{i\delta}f_{i\delta}+r$ *with* $g_{i\delta}\in A((i,m))$ *and* $r\in N_E$, *then* $r=0$, $g_{i\delta}=0$ *for* $i\leqq t-2$, $\delta\in\Delta_i$, *and* $g_{t-1\delta}\in A((t,m))$ *for* $\delta\in\Delta_{t-1}$, *provided* $t\leqq j$, $\varepsilon\in\Delta_j$.

5) *If for* $f\in N_E$ $x_t f=\sum_{i=0}^{m-1}\sum_{\delta\in\Delta_i} g_{i\delta}f_{i\delta}+r$ *with* $g_{i\delta}\in A((i,m))$ *and* $r\in N_E$, *then* $g_{i\delta}=0$ *for* $i\leqq t-2$, $\delta\in\Delta_i$, *and* $g_{t-1\delta}\in A((t,m))$ *for* $\delta\in\Delta_{t-1}$.

Proof. Note first that R/I is flat over A. Then the method of the proof of [4; Chap. 1 (1.2.7), (1.2.8)] is also applicable to our case, in which we do not have to care convergence, and we get 1), 2) and 3). Compare the argument of (1.5). 4), 5) follow easily from Remark 2.2 and the "division algorithm" since $f_{i\delta}-x^{A_{i\delta}}\in N_E$. Q.E.D.

Corollary 2.4. *Under the conditions of Theorem* 2.3 $\Delta_0, \Delta_1, \cdots, \Delta_{m-1}$ *are not empty.*

Proof. If Δ_i were empty for some $0\leqq i\leqq m-1$ then we would have $\Gamma_{i+1}=\Gamma_{i+2}=\cdots=\Gamma_m=\phi$ by (2.1.3). But then Theorem 2.3.1) would imply

$$R/I=N_E=\bigoplus_{j=1}^{i}\bigoplus_{\gamma\in\Gamma_j} x^{\gamma}A((j,m))$$

which means $\operatorname{depth}_{\mathfrak{m}} R/I\geqq d+1$. This contradicts the assumption that $\operatorname{depth}_{\mathfrak{m}} R/I=d$. Q.E.D.

Corollary 2.5. *Under the conditions of Theorem* 2.3 R/I *is*

Cohen-Macaulay if and only if $\Gamma_1 = \cdots = \Gamma_{m-1} = \phi$.

Proof. Easy and left to the reader.

Let l_i $(1 \leqq i \leqq m)$ be the number of elements of Δ_{i-1}, and we set $l = \sum_{i=1}^{m} l_i$, $s_\alpha = \sum_{i=\alpha+1}^{m} l_i$ $(0 \leqq \alpha \leqq m-1)$. For each $1 \leqq i \leqq m$ put $f_{i-1\delta}(\delta \in \Delta_{i-1})$ in a suitable order and write them, say, $f_{l-s_{i-1}+1}, f_{l-s_{i-1}+2}, \cdots, f_{l-s_i}$. Then Theorem 2.3.2) becomes

$$(2.6.I) \qquad I = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{l_i} f_{l-s_{i-1}+j} A((i-1, m)).$$

We can compute

$$M_1 = \{\psi = {}^t(\psi_1, \cdots, \psi_l) \in R^l = A((0, m))^l \mid \sum_{i=1}^{l} \psi_i f_i = 0\}$$

by Theorem 1.6. Let $\tilde{\psi}_j^\alpha$, $\tilde{\chi}_j^\alpha$ $(1 \leqq \alpha \leqq m-1,\ 1 \leqq j \leqq s_\alpha)$ be defined as in Section 1 $(^*_*)$, $(^{**}_{*})$, and $(^{**}_{**})$, then we have

(I-0) $\quad \tilde{\chi}_j^\alpha(0) \in \bigoplus_{i=1}^{m} k((i-1, \mathrm{m}))^{l_i}$,

(II-0) $\quad d_{\bar{q}}(\tilde{\chi}_j^\alpha(0)) \geqq 1 + q_{l-s_\alpha+j}$ *for* $1 \leqq \alpha \leqq m-1$, $1 \leqq j \leqq s$ *where* $q_i = o(f_i(0))$ *and* $\bar{q} = (q_1, \cdots, q_l)$.

(I-0) is trivial and (II-0) is deduced from the defining equations $(^*_*)$ and Theorem 2.3.3). Hence we get by Theorem 1.6

$$(2.6.M_1)^* \qquad M_1 = \bigoplus_{\alpha=1}^{m-1} \bigoplus_{j=1}^{s_\alpha} \tilde{\psi}_j^\alpha A((\alpha-1, m)).$$

Put $l'_i = s_i$ $(1 \leqq i \leqq m-1)$, $m' = m-1$, $s'_\alpha = \sum_{i=\alpha+1}^{m'} l'_i$ $(0 \leqq \alpha \leqq m'-1)$, $l' = \sum_{i=1}^{m'} l'_i$ and $A' = k((m', n))$. We set $f'_{l'-s_{\alpha-1}'+j} = \tilde{\psi}_j^\alpha$ for $1 \leqq \alpha \leqq m'$, $1 \leqq j \leqq l'_\alpha$, then $(2.6.M_1)^*$ becomes

$$(2.6.M_1) \qquad M_1 = \bigoplus_{i=1}^{m'} \bigoplus_{j=1}^{l_i'} f'_{l'-s_{i-1}'+j} A'((i-1, m')).$$

Thus we are in the same situation as before. Let

$$M_2 = \{\psi = {}^t(\psi_1, \cdots, \psi_{l'}) \in R^{l'} = A'((0, m'))^{l'} \mid \sum_{i=1}^{l'} \psi_i f'_i = 0\}.$$

If $m' = 1$ then M_1 is a free R-module and $M_2 = 0$. If $m' \geqq 2$ then we can compute M_2 defining $\tilde{\phi}'^{\alpha}_{j}$, $\tilde{\chi}'^{\alpha}_{j}$ $(1 \leqq \alpha \leqq m'-1,\ 1 \leqq j \leqq s'_{\alpha})$ by the formulae (⁑), (⁎⁎), and (⁑⁑) of Section 1 using $(2.6.M_1)$, and obtain

$$(2.6.M_2)^* \qquad M_2 = \bigoplus_{\alpha=1}^{m'-1} \bigoplus_{j=1}^{s_{\alpha}'} \tilde{\phi}'^{\alpha}_{j} A'((\alpha-1, m')).$$

Note that in this case the condition corresponding to (II-0) above, namely

(II-0)′ $d_{\bar{q}'}(\tilde{\chi}'^{\alpha}_{j}(0)) \geqq 1 + q'_{l'-s_{\alpha}'+j}$ for $1 \leqq \alpha \leqq m'-1$, $1 \leqq j \leqq s'_{\alpha}$ where $q'_i = d_{\bar{q}}(f'_i(0))$ and $\bar{q}' = (q'_1, \cdots, q'_{l'})$

is deduced from Corollary 1.5.2). Continuing this procedure we can compute a free resolution for R/I of length $m = n - \operatorname{depth}_{\mathfrak{m}} R/I$ on and on.

Example 2.7. When $n - \operatorname{depth}_{\mathfrak{m}} R/I = 2$ the results of this section appear essentially in [2]. If, in this case, I is generated by homogeneous polynomials and R/I is Cohen-Macaulay, then Theorem 2.3.2) becomes

$$I = f_1 k((0, n)) \oplus \bigoplus_{i=1}^{l_2} f_{1+i} k((1, n))$$

where f_i $(1 \leqq i \leqq 1 + l_2)$ are homogeneous polynomials in I such that $\deg f_1 \leqq \deg f_{1+i}$ for $1 \leqq i \leqq l_2$ and $l_2 = \deg f_1$. We may assume without loss of generality that $\deg f_i \leqq \deg f_{i+1}$ for $2 \leqq i \leqq l_2$. The sequence of integers $(\nu_{l_2}, \nu_{l_2-1}, \cdots, \nu_1)$ with $\nu_i = \deg f_{1+i}$ $(1 \leqq i \leqq l_2)$ is the "*caractère numérique*" appeared in [6].

Example 2.8. When $n - \operatorname{depth}_{\mathfrak{m}} R/I = 3$, R/I has a free resolution

$$0 \longrightarrow R^{l_3} \xrightarrow{\lambda_3} R^{l_2+2l_3} \xrightarrow{\lambda_2} R^{1+l_2+l_3} \xrightarrow{\lambda_1} R \xrightarrow{\lambda_0} R/I \longrightarrow 0$$

where the matrices λ_1, λ_2, λ_3 enjoy the properties:

1) $\lambda_1 = (f_1, f_2, \cdots, f_{l_2+1}, f_{l_2+2}, \cdots, f_{l_2+l_3+1})$

2)
$$\lambda_2 = \begin{array}{c} \left(\begin{array}{ccc} U_{01} & U_{02} & 0 \\ U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{array}\right) \begin{array}{l} \}1 \\ \}l_2 \\ \}l_3 \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{l_2} & \underbrace{\qquad}_{l_3} & \underbrace{\qquad}_{l_3} \end{array} \end{array}$$

i) Each entry of U_{01}, U_{02}, U_{12}, and $U_{11}-x_1\cdot 1_{l_2}$ is in $k((1, n))$.

ii) Each entry of U_{21}, U_{13}, $U_{22}-X_1\cdot 1_{l_3}$, and $U_{23}-x_2\cdot 1_{l_3}$ is in $k((2, n))$.

3) $$\lambda_3=\begin{pmatrix} -U_{13} \\ -U_{23} \\ U_{22} \end{pmatrix} \quad \text{and} \quad \lambda_2\cdot\lambda_3=0\,.$$

1) and 2) follow directly from the argument of this section while 3) holds by exactly the same reason as that of Corollaries 3. 5. 3)–3. 5. 4). Observe that one does not have to do any further computation to determine λ_3 if λ_2 is already known.

§ 3. Main Results

In this section we present a method to handle the ideal defining the minimal cone of a curve in $\mathbf{P}_k^3$ as an application of the results of the previous sections. As in the introduction 'curve' means an equidimensional complete scheme over a field k of dimension 1. We state the results in a slightly general situation which includes the case of our interest. Let $x_1, \cdots, x_n$ be indeterminates, $R=k[x_1, \cdots, x_n]$, and $\mathfrak{m}=(x_1, \cdots, x_n)R$. For any matrix ϕ with entries in R we define $I(\phi)$ to be the ideal generated by $s\times s$ minors of ϕ where s is the rank of ϕ (see [1]).

Proposition 3. 1. *Let I be a homogeneous ideal in R such that* $\dim R/I\leqq n-2$ *and* $\operatorname{depth}_{\mathfrak{m}} R/I\geqq n-3$, *and let J be any homogeneous subideal of I such that* $\dim R/J=\operatorname{depth}_{\mathfrak{m}} R/J=n-2$. *Then, for a suitable choice of homogeneous coordinates, there exist homogeneous polynomials $f_0, f_1, \cdots, f_a\in J$ ($a=\deg f_0$) and $f_{a+1}, \cdots, f_{a+b}\in I$ ($b\geqq 0$) such that*

1) $$J=f_0k(0, n)\oplus\bigoplus_{i=1}^{a} f_ik(1, n),$$
$$I=f_0k(0, n)\oplus\bigoplus_{i=1}^{a} f_ik(1, n)\oplus\bigoplus_{i=1}^{b} f_{a+i}k(2, n).$$

2) *if for* $1\leqq j\leqq a+b$, $x_1f_j=\sum_{i=0}^{a+b} g_if_i$ *with*
$${}^t(g_0, \cdots, g_{a+b})\in k(0, n)\oplus k(1, n)^a\oplus k(2, n)^b, \text{ then } g_0\in k(1, n).$$

3) *if for* $a+1\leqq j\leqq a+b$, $x_2f_j=\sum_{i=0}^{a+b}g_if_i$ ${}^t(g_0, \cdots, g_{a+b})\in k(0, n)\oplus k(1, n)^a\oplus k(2, n)^b$, *then* $g_0=0$ *and* $g_i\in k(2, n)$ $(1\leqq i\leqq a+b)$.

Before proving the proposition we make a remark.

Remark 3.2. In Example 2.8 it is not always true that $U_{21}=0$. This implies that $f_1k((0, n))\oplus(\bigoplus_{i=1}^{l_2}f_{1+i}k((1, n))$ is not always an ideal of R. Thus Proposition 3.1 is somewhat different from Example 2.8.

Proof of Proposition 3.1. Let R^*, I^* and J^* be the $\mathfrak{m}$-adic completion of R, I and J, respectively. After a suitable linear coordinate transformation we may assume that $x_4, \cdots, x_n$ (resp. $x_3, x_4, \cdots, x_n$) is an R^*/I^*-regular sequence (resp. a maximal R^*/J^*-regular sequence) in $\mathfrak{m}$. Put $\bar{R}^*=k((0,3))$, $\bar{I}^*=I^*$ $(\mathrm{mod}(x_4, \cdots, x_n)R^*)$, and $\bar{J}^*=J^*$ $(\mathrm{mod}$ $(x_4, \cdots, x_n)R^*)$. Then x_3 becomes a maximal R^*/J^*-regular sequence. So we deduce from Theorem 2.3 that there exist homogeneous polynomials $\bar{f}_0, \cdots, \bar{f}_a$ $(a=\deg\bar{f}_0$, see Example 2.7 also) such that

$$(1)\qquad \bar{R}^*=\bar{J}^*\oplus\bigoplus_{\gamma\in\Gamma_2}x^\gamma k((2,3)),$$

$$(2)\qquad \bar{J}^*=\bar{f}_0k((0,3))\oplus\bigoplus_{i=1}^{a}\bar{f}_ik((1,3)).$$

We see from (1) that $\bar{I}^*/\bar{J}^*\cong\bar{I}^*\cap\bigoplus_{\gamma\in\Gamma_2}x^\gamma k((2,3))$ is a $k[[x_3]]$ submodule of $\bigoplus_{\gamma\in\Gamma_2}x^\gamma k((2,3))=\bigoplus_{\gamma\in\Gamma_2}x^\gamma k[[x_3]]$, so that there exist homogeneous polynomials $\bar{f}_{a+1}, \cdots, \bar{f}_{a+b}\in\bar{I}^*\cap\bigoplus_{\gamma\in\Gamma_2}x^\gamma k((2,3))$ such that

$$(3)\qquad \bar{I}^*\cap\bigoplus_{\gamma\in\Gamma_2}x^\gamma k((2,3))=\bigoplus_{i=1}^{b}\bar{f}_{a+i}k[[x_3]]$$

by elementary linear algebra over the principal ideal domain $k[[x_3]]$. Further there exist a subset $\Gamma\subset\Gamma_2$ and a nonnegative integer $e(\gamma)$ for each $\gamma\in\Gamma$ such that

$$(4)\qquad \bigoplus_{\gamma\in\Gamma_2}x^\gamma k[[x_3]]=\{I^*\cap\bigoplus_{\gamma\in\Gamma_2}x^\gamma k[[x_3]]\}\oplus\{\bigoplus_{\gamma\in\Gamma}\bigoplus_{0\leqq i<e(\gamma)}x^\gamma x_3^i\cdot k\}\oplus\{\bigoplus_{\gamma\in\Gamma_2\backslash\Gamma}x^\gamma k[[x_3]]\}.$$

It follows from (1), (2), (3), and (4) that

$$(5)\qquad \begin{cases} \bar{I}^* = \bar{f}_0 k((0,3)) \oplus \bigoplus_{i=1}^{a} \bar{f}_i k((1,3)) \oplus \bigoplus_{i=1}^{b} \bar{f}_{a+i} k((2,3)), \\ \bar{R}^* = \bar{I}^* \oplus \{ \bigoplus_{\gamma\in\Gamma} \bigoplus_{0\leqq j<e(\gamma)} x^{\gamma} x_3^j k \oplus \bigoplus_{\gamma\in\Gamma_2\setminus\Gamma} x^{\gamma} k((2,3))\}. \end{cases}$$

Put $A^* = k((3,n))$. Let f'_{a+i} $(1\leqq i\leqq b)$ be homogeneous polynomials of I^* such that $f'_{a+i}(0) = \bar{f}_{a+i}$, and let f_i $(0\leqq i\leqq a)$ be those homogeneous polynomials of J^* described in (2.3). Then $f_i(0) = \bar{f}_i$ $(0\leqq i\leqq a)$ and

$$(6)\qquad \begin{cases} J^* = f_0 A^*((0,3)) \oplus \bigoplus_{i=1}^{a} f_i A^*((1,3)), \\ R^* = J^* \oplus \bigoplus_{\gamma\in\Gamma_2} x^{\gamma} A^*((2,3)). \end{cases}$$

Using (5) and noting that R^*/I^* is flat over A^* we deduce

$$(7)\qquad \begin{cases} I^* = f_0 A^*((0,3)) \oplus \bigoplus_{i=1}^{a} f_i A^*((1,3)) \oplus \bigoplus_{i=1}^{b} f'_{a+i} A^*((2,3)), \\ R^* = I^* \oplus N^*, \end{cases}$$

where $N^* = \bigoplus_{\gamma\in\Gamma} \bigoplus_{0\leqq j<e(\gamma)} x^{\gamma} x_3^j A^* \oplus \bigoplus_{\gamma\in\Gamma_2\setminus\Gamma} x^{\gamma} A^*((2,3))$.

See the proof of Corollary 1.5 and [4; (1, 2, 8)].

(7) enables us to find homogeneous polynomials $\tilde{f}_{a+i}$ in N^* $(1\leqq i \leqq b)$ such that $\tilde{f}_{a+i} \equiv f'_{a+i} - f'_{a+i}(0)$ (mod I^*). Put $f_{a+i} = f'_{a+i}(0) + \tilde{f}_{a+i}$ $(1\leqq i\leqq b)$, then $f_{a+i} \in I^* \cap \bigoplus_{\gamma\in\Gamma_2} x^{\gamma} A^*((2,3))$, and we again get (7) with $(f'_{a+1}, \cdots, f'_{a+b})$ replaced by $(f_{a+1}, \cdots, f_{a+b})$ since $f_{a+i}(0) = f'_{a+i}(0) = \bar{f}_{a+i}$ for $1\leqq i\leqq b$. I and J being homogeneous this proves 1). 2) and 3) follow from Theorem 2.3.4)–2.3.5), and from the fact that $f_{a+i} \in \bigoplus_{\gamma\in\Gamma_2} x^{\gamma} A^*((2,3))$ for $1\leqq i\leqq b$.

Corollary 3.3. *In Proposition* 3.1

1) $0\leqq b\leqq \sum_{i=1}^{a} (\deg f_i + i - a)$.

2) $$\dim R/I = \begin{cases} n-2 & \text{if } \; b< \sum_{i=1}^{a} (\deg f_i + i - a), \\ n-3 & \text{if } \; b= \sum_{i=1}^{a} (\deg f_i + i - a). \end{cases}$$

Proof. Let $F(\nu)$ be the Hilbert function of R/I. One can compute

$F(\nu)$ using Proposition 3.1.1) and get

$$(1)\quad F(\nu)=\binom{n-1+\nu}{n-1}-\binom{n-1+\nu-a}{n-1}-\sum_{i=1}^{a}\binom{n-2+\nu-\deg f_i}{n-2}-\sum_{i=1}^{b}\binom{n-3+\nu-\deg f_{a+i}}{n-3}$$

for $\nu\gg 0$.

We deduce from (1)

$$F(\nu)=\frac{1}{(n-3)!}\{\sum_{i=1}^{a}(\deg f_i+i-a)-b\}\nu^{n-3}+(\text{terms of degree}<n-3)$$

for $\nu\gg 0$. Hence 1) follows. 2) is obvious since $\dim R/I\geqq \operatorname{depth}_{\mathfrak{m}} R/I\geqq n-3$ by hypothesis. Q.E.D.

In the situation of Proposition 3.1 we set $\nu_j=\deg f_j$ $(0\leqq j\leqq a+b)$, $\mu_{ij}=\nu_j+1-\nu_i$ $(0\leqq i\leqq a+b,\ 1\leqq j\leqq a+b)$, and $\mu_{i,a+b+j}=\mu_{i,a+j}$ $(1\leqq j\leqq b)$. Then ν_j, μ_{ij} enjoy the properties:

$$(3.4)\quad\begin{cases} 1) & \mu_{i_1j_1}-\mu_{i_2j_1}=\mu_{i_1j_2}-\mu_{i_2j_2} \\ & \text{for}\quad 0\leqq i_1,\, i_2\leqq a+b,\, 1\leqq j_1,\, j_2\leqq a+2b\,. \\ 2) & \mu_{ii}=1,\quad \text{for}\quad 0\leqq i\leqq a+b\,. \\ 3) & \mu_{i,j+a+b}=\mu_{i,a+j},\quad \text{for}\quad 1\leqq j\leqq b\,. \\ 4) & \nu_j=\sum_{i=j+1}^{a}\mu_{ii}+\sum_{i=1}^{j-1}\mu_{i,i+1},\quad \text{for}\quad 0\leqq j\leqq a\,. \end{cases}$$

Corollary 3.5. *In the situation of Proposition* 3.1 *R/I has a free resolution*

$$(\mathrm{A})\qquad 0\longrightarrow R^b\xrightarrow{\lambda_3}R^{a+2b}\xrightarrow{\lambda_2}R^{a+b+1}\xrightarrow{\lambda_1}R\xrightarrow{\lambda_0}R/I\longrightarrow 0$$

such that the matrices λ_1, λ_2, λ_3 have the following properties:

1) $\lambda_1=(f_0, f_1, \cdots, f_a, f_{a+1}, \cdots, f_{a+b})$.

2)
$$\lambda_2=\begin{array}{c} \begin{pmatrix} U_{01} & U_{02} & 0 \\ U_1 & U_2 & U_4 \\ 0 & U_3 & U_5 \end{pmatrix} \begin{array}{l} \}1 \\ \}a \\ \}b \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{a} & \underbrace{\quad}_{b} & \underbrace{\quad}_{b} \end{array} \end{array}.$$

α) *Each nonzero* (i, j) *component of* λ_2 *is homogeneous of degree* μ_{ij}, *where* $0 \leqq i \leqq a+b$, $1 \leqq j \leqq a+2b$.

β) U_{01}, U_{02}, U_2, *and* $U_1 - x_1 \cdot 1_a$ *take entries in* $k(1, n)$.

γ) U_4, $U_3 - x_1 \cdot 1_b$, *and* $U_5 - x_2 \cdot 1_b$ *take entries in* $k(2, n)$.

3)
$$\lambda_3 = \begin{pmatrix} -U_4 \\ -U_5 \\ U_3 \end{pmatrix}.$$

4) $\lambda_2 \cdot \lambda_3 = 0$.

5) $R/I\begin{pmatrix} U_{01} \\ U_1 \end{pmatrix}$ *is a Cohen-Macaulay ring of dimension* $n-2$, *and* $I(\lambda_3)$ *contains an R-sequence of length* 3 *or* $I(\lambda_3) = R$.

Proof. Let $\tilde{\chi}^1 = (\tilde{\chi}^1_{ij})$ be the matrix defined by the equations $x_1 f_j = \sum_{i=0}^{a+b} \tilde{\chi}^1_{ij} f_i$ with ${}^t(\tilde{\chi}^1_{0j}, \cdots, \tilde{\chi}^1_{a+b,j}) \in k(0, n) \oplus k(1, n)^a \oplus k(2, n)^b$ for $1 \leqq j \leqq a+b$, and $\tilde{\chi}^2 = (\tilde{\chi}^2_{ij})$ the matrix defined by the equations $x_2 f_{a+j} = \sum_{i=0}^{a+b} \tilde{\chi}^2_{ij} f_i$ with ${}^t(\tilde{\chi}^2_{0j}, \cdots, \tilde{\chi}^2_{a+b,j}) \in k(0, n) \oplus k(1, n)^a \oplus k(2, n)^b$ for $1 \leqq j \leqq b$. Put

$$(\tilde{\varphi}^1_{ij}) = (\tilde{\varphi}^1_1, \cdots, \tilde{\varphi}^1_{a+b}) = \begin{pmatrix} 0 \cdots 0 \\ x_1 1_{a+b} \end{pmatrix} - \tilde{\chi}^1,$$

$$(\tilde{\varphi}^2_{ij}) = (\tilde{\varphi}^2_1, \cdots, \tilde{\varphi}^2_b) = \begin{pmatrix} 0 \\ x_2 1_b \end{pmatrix} - \tilde{\chi}^2,$$

$$\lambda_2 = (\tilde{\varphi}^1_{ij} | \tilde{\varphi}^2_{ij}) = \begin{array}{c} \left(\begin{array}{ccc} U_{01} & U_{02} & U_{03} \\ U_1 & U_2 & U_4 \\ U_1' & U_3 & U_5 \end{array} \right) \begin{array}{l} \}1 \\ \}a \\ \}b \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{a} & \underbrace{\quad}_{b} & \underbrace{\quad}_{b} \end{array} \end{array}$$

and $\lambda_1 = (f_0, f_1, \cdots, f_{a+b})$. Then $\tilde{\chi}^1_{ij} = 0$ for $a+1 \leqq i \leqq a+b$, $1 \leqq j \leqq a$ since $J = f_0 k(0, n) \oplus \bigoplus_{i=1}^{a} f_i k(1, n)$ is an ideal of R. This implies $U_1' = 0$. $U_{03} = 0$ by Proposition 3.1.3). 2.β), 2.γ) follow from 2) and 3) of Proposition 3.1. 2.α) is obvious.

Now we verify by 2) that $\tilde{\chi}^\alpha$ $(\alpha = 1, 2)$ satisfy conditions (I) and (II) of Section 1 with $\bar{q} = (\deg f_0, \cdots, \deg f_{a+b})$. Hence we deduce from Proposition 3.1.1) and Theorem 1.6 that

$$\text{(1)} \qquad \operatorname{Ker} \lambda_1 = \bigoplus_{i=1}^{a+b} \tilde{\varphi}^1_i k(0, n) \oplus \bigoplus_{i=1}^{b} \tilde{\varphi}^2_i k(1, n) = \operatorname{Im} \lambda_2.$$

Let λ_3 be the matrix defined by the formula 3), and let $\tilde{\phi}_1, \cdots, \tilde{\phi}_b$ be its column vectors. We must show that $\operatorname{Ker} \lambda_2 = \operatorname{Im} \lambda_3$. First observe that each column vector of $\lambda_3 - \begin{pmatrix} 0 \\ x_1 1_b \end{pmatrix}$ is in $k(0, n)^{a+b} \oplus k(1, n)^b$, so that if we have $\lambda_2 \cdot \lambda_3 = 0$, $\operatorname{Ker} \lambda_2$ must be equal to $\operatorname{Im} \lambda_3 = \bigoplus_{i=1}^{b} \tilde{\phi}_i k(0, n)$ by Theorem 1.6. But it is easily seen that each column vector of $\lambda_2 \cdot \lambda_3$ is in $k(0, n) \oplus k(1, n)^a \oplus k(1, n)^b$ by 2) and that $\lambda_1 \cdot (\lambda_2 \cdot \lambda_3) = (\lambda_1 \cdot \lambda_2) \cdot \lambda_3 = 0$. Hence $\lambda_2 \cdot \lambda_3 = 0$ by Proposition 3.1.1), and $\operatorname{Ker} \lambda_2 = \operatorname{Im} \lambda_3$. Thus (A) is exact.

$I\begin{pmatrix} U_{01} \\ U_1 \end{pmatrix} = J$ since

$$(\mathrm{A}_J) \qquad 0 \longrightarrow R^a \xrightarrow{\begin{pmatrix} U_{01} \\ U_1 \end{pmatrix}} R^{a+1} \xrightarrow{(f_0, \cdots, f_a)} R \xrightarrow{\lambda_0} R/J \longrightarrow 0$$

is exact by (A) applied to J. So the first part of 5) follows. The last part of 5) is merely the criterion of [1; Corollary 1]. Q.E.D.

Corollary 3.6. *In Corollary* 3.5 *we set*

$$W_1 = \begin{pmatrix} U_{01} & U_{02} \\ U_1 & U_2 \\ 0 & U_3 \end{pmatrix}, \quad W_2 = \begin{pmatrix} U_{01} & 0 \\ U_1 & U_4 \\ 0 & U_5 \end{pmatrix},$$

and let $W_i^{(j)}$ $(0 \leqq j \leqq a+b, i=1, 2)$ *denote the square matrix obtained by leaving out the j-th row from* W_i. *Then we have for some* $\varepsilon (\neq 0) \in k$

1) $(\det U_3) f_j = (-1)^j \cdot \varepsilon \cdot \det W_1^{(j)}$ *for* $0 \leqq j \leqq a+b$,

2) $(\det U_5) f_j = (-1)^j \cdot \varepsilon \cdot \det W_2^{(j)}$ *for* $0 \leqq j \leqq a+b$.

Proof. Put $G_i = (\det W_i^{(0)}, -\det W_i^{(1)}, \cdots, (-1)^{a+b} \det W_i^{(a+b)})$ $(i=1, 2)$. Since $\lambda_0 W_i = 0$, $G_i W_i = 0$, $\operatorname{rank} W_i = a+b$ for $i=1, 2$, and $\operatorname{ht} I \geqq 2$ we find that $u_i \lambda_0 = G_i$ for some $u_i \in R$, so that $u_i f_0 = \det W_i^{(0)} = \det U_1 \cdot \det U_{2i+1}$ $(i=1, 2)$. But we know that $f_0 = \varepsilon \cdot \det U_1$ for some $\varepsilon (\neq 0) \in k$, thus $\varepsilon u_i = \det U_{2i+1}$ $(i=1, 2)$ which implies 1) and 2). Q.E.D.

Next theorem is a converse version of Proposition 3.1 and Corollaries 3.3, 3.5 and 3.6.

Theorem 3.7. *Let* μ_{ij} $(0 \leqq i \leqq a+b, 1 \leqq j \leqq a+2b)$, ν_j *be integers* *satisfying* (3.4) *and* $0 \leqq b \geqq \sum_{i=1}^{a} (\nu_i + i - a)$. *Let* λ_2 *and* λ_3 *be any matrix satisfying the conditions* 2), 3), 4) *and* 5) *of Corollary* 3.5 *and set* W_1, W_2 *as in Corollary* 3.6. *Then we have*

1) $\det W_1^{(j)}$ (resp. $\det W_2^{(j)}$) *is divisible by* $\det U_3$ (*resp.* $\det U_5$).

2) *Put* $f_j = (-1)^j \det W_1^{(j)} / \det U_3$, *and let* I (*resp.* J) *be the homogeneous ideal in* R *generated by* $f_0, \cdots, f_{a+b}$ (*resp.* $f_0, \cdots, f_a$), *then*

i) $J = f_0 k(0, n) \oplus \bigoplus_{i=1}^{a} f_i k(1, n)$,

ii) $I = f_0 k(0, n) \oplus \bigoplus_{i=1}^{a} f_i k(1, n) \oplus \bigoplus_{i=1}^{b} f_{a+i} k(2, n)$ *and* R/I *has a free resolution of the form* (A).

3) $\dim R/I \leqq n-2$ *and*

$$\operatorname{depth}_{\mathfrak{m}} R/I = \begin{cases} n-2 & \text{if } I(\lambda_3) = R, \\ n-3 & \text{if } I(\lambda_3) \neq R. \end{cases}$$

Proof of 1). Note first that $\det W_1^{(j)}$ (resp. $\det W_2^{(j)}$) is evidently divisible by $\det U_3$ (resp. $\det U_5$) for $1 \leqq j \leqq a$. Put

$$G_i = (\det W_i^{(0)}, -\det W_i^{(1)}, \cdots, (-1)^{a+b} \det W_i^{(a+b)})$$

and $f_j = (-1)^j \det \begin{pmatrix} U_{10} \\ U_1 \end{pmatrix}^{(j)}$ for $0 \leqq j \leqq a$, where $\begin{pmatrix} U_{01} \\ U_1 \end{pmatrix}^{(j)}$ denotes the matrix obtained by leaving out the j-th row from $\begin{pmatrix} U_{01} \\ U_1 \end{pmatrix}$. Obviously

$$\text{(1)} \qquad \begin{cases} \det W_1^{(j)} = (-1)^j (\det U_3) f_j, \\ \det W_2^{(j)} = (-1)^j (\det U_5) f_j \end{cases}$$

for $0 \leqq j \leqq a$, and $f_0, \cdots, f_a$ have no common factor other than units by the condition 3.5.5). This enables us to write $G_i = h_i K_i$ $(i = 1, 2)$, where K_i ia a row vector in R^{a+b+1} without any common factor except units among the entries, and $h_i \in R$ divides $\det U_{2i+1}$ for $i = 1, 2$. Put $u_i = \det U_{2i+1} / h_i$ $(i = 1, 2)$. Then, for $i = 1, 2$, u_i is a homogenous polynomial of $k[x_i, x_3, x_4]$ which is monic in x_i. Observe that $K_1 = (u_1 f_0, -u_1 f_1, \cdots, (-1)^a u_1 f_a, \cdots)$ and $K_2 = (u_2 f_0, -u_2 f_1, \cdots, (-1)^a u_2 f_a, \cdots)$ by (1). We want to show that h_1 (resp. h_2) is in fact equal to $\det U_3$ (resp. $\det U_5$)

up to units. It is enough to show that both u_1 and u_2 are units. The condition $\lambda_2\cdot\lambda_3=0$ can be expressed in the following form:

$$\begin{pmatrix} U_{02} \\ U_2 \\ U_3 \end{pmatrix} U_5 = \begin{pmatrix} U_{01} & 0 \\ U_1 & U_4 \\ 0 & U_5 \end{pmatrix} \begin{pmatrix} -U_4 \\ U_3 \end{pmatrix}. \tag{2}$$

$G_2\begin{pmatrix} U_{01} \\ U_1 \\ 0 \end{pmatrix}=0$ and $G_2\begin{pmatrix} 0 \\ U_4 \\ U_5 \end{pmatrix}=0$ by the definition of G_2, so that we get $G_2W_1=0$ by (2). On the other hand $G_1W_1=0$ by the definition of G_1, thus we obtain $K_1W_1=K_2W_1=0$. Since $\det W_1^{(0)}$ is a non-zero polynomial monic in x_1, W_1 has the maximal rank $a+b$. We have therefore that $AK_1=AK_2$ for some relatively prime polynomials $A, B\in R$. But A and B must be units, since the entries of K_i have no common factor other than units for $i=1, 2$. Thus $K_1=sK_2$ with $s\in k$, and hence $u_1f_j=su_2f_j$ for $0\leqq j\leqq a$. This implies $u_1=su_2$, and we conclude that both u_1 and u_2 must be units, because u_i is a homogeneous polynomial of $k[x_i, x_3, x_4]$ which is monic in x_i for $i=1, 2$.

Proof of 2). It is trivial that

$$0\longrightarrow R^b\xrightarrow{\lambda_3}R^{a+2b}\xrightarrow{\lambda_2}R^{a+b+1}\xrightarrow{\lambda_1}R\xrightarrow{\lambda_0}R/I\longrightarrow 0$$

is a complex. To prove exactness we need only verify the conditions of [1; Corollary 1]. The condition on ranks is obviously satisfied. Let f, g be an R-sequence in J, and let H be the ideal

$$(f\cdot\det U_3,\ f\cdot\det U_5,\ g\cdot\det U_3,\ g\cdot\det U_5)R\,.$$

Then the height of H is equal to or larger than 2 since $\det U_3$ and $\det U_5$ are relatively prime. In addition, H is contained in $I(\lambda_2)$, because both f and g are linear combinations of $f_j=(-1)^j\det W_1^{(j)}/\det U_3=(-1)^j\det W_2^{(j)}/\det U_5$ $(0\leqq j\leqq a)$. Hence $I(\lambda_2)$ contains an R-sequence of length 2. $\operatorname{ht} I(\lambda_2)\geqq\operatorname{ht} J=2\geqq 1$ and $I(\lambda_3)$ contains an R-sequence of length 3 or $I(\lambda_3)=R$ by assumption, thus the complex above is exact.

Set $(\tilde{\phi}_1, \cdots, \tilde{\phi}_b)=\lambda_3$ and $(\tilde{\psi}_1, \cdots, \tilde{\psi}_{a+2b})=\lambda_2$. We know by Corollary 1.5 that

$$R^{a+2b} = \bigoplus_{i=1}^{b} \tilde{\phi}_i k(0, n)\oplus\{k(0, n)^{a+b}\oplus k(1, n)^b\}, \tag{3}$$

$$(4)\qquad R^{a+b+1}=\{\bigoplus_{i=1}^{a+b}\tilde{\varphi}_i k(0,n)\oplus\bigoplus_{i=1}^{b}\tilde{\varphi}_{a+b+i}k(1,n)\}$$

$$\oplus\{k(0,n)\oplus k(1,n)^a\oplus k(2,n)^b\}.$$

We have $\tilde{\varphi}_i\in\operatorname{Im}\lambda_3=\operatorname{Ker}\lambda_2$ for $1\leqq i\leqq b$, so that we deduce from (3) and (4) that

$$(5)\qquad \operatorname{Ker}\lambda_1=\operatorname{Im}\lambda_2=\bigoplus_{i=1}^{a+b}\tilde{\varphi}_i k(0,n)\oplus\bigoplus_{i=1}^{b}\tilde{\varphi}_{a+b+i}k(1,n).$$

Using (4) and (5) we find that any element of $\operatorname{Im}\lambda_1$ can be written $\sum_{i=0}^{a+b}g_if_i$ with ${}^t(g_0,\cdots,g_{a+b})\in k(0,n)\oplus k(1,n)^a\oplus k(2,n)^b$, and that $k(0,n)\oplus k(1,n)^a\oplus k(2,n)^b\cap\operatorname{Ker}\lambda_1=0$. Thus we obtain

$$\operatorname{Im}\lambda_1=f_0k(0,n)\oplus\bigoplus_{i=1}^{a}f_ik(1,n)\oplus\bigoplus_{i=1}^{b}f_{a+i}k(2,n)$$

This proves 2–ii). 2–i) is proved similarly. 3) is obvious. Q.E.D.

Remark 3.8. In the case $n=4$, if one wishes to deal with the ideal in R defining the minimal cone of a curve in $\mathbb{P}_k^3$, b must be taken to be strictly smaller than $\sum_{i=1}^{a}(\nu_i+i-a)$ and the condition 5) of Corollary 3.5 should be altered as follows:

(3.5.5)′ $R/I\begin{pmatrix}U_{01}\\U_1\end{pmatrix}$ is a Cohen-Macaulay ring of dimension 2 and $I(\lambda_3)$ contains an R-sequence of length 4 or $I(\lambda_3)=R$.

Remark 3.9. The conclusions from Proposition 3.1 to Corollary 3.6 are also valid for any ideal $I^*\subset R^*=k[[x_1,\cdots,x_n]]$ such that depth $R^*/I^*\geqq n-3$ and dim $R^*/I^*\leqq n-2$.

§ 4. Discussions in the Case $b=1$

In Theorem 3.7 the relation $\lambda_2\cdot\lambda_3=0$ is essential. When $b=1$ this relation is rather easy to solve provided that $n=4$ and $I(\lambda_3)$ contains an R-sequence of length 4 or $I(\lambda_3)=R$. The aim of this section is to illustrate how Theorem 3.7 works in this special case.

We assume the field k to be algebraically closed with characteristic

0 throughout this section. We begin with a remark.

Remark 4.1. Let λ_2, λ_3 be as in Corollary 3.5.2) and 3). Using Lemma 1.3 twice with $\alpha=1, 2$ and $s_1=s_2=b$, we get

1) $k(0, n)^b = k(0, n)^b U_3 \oplus k(1, n)^b U_5 \oplus k(2, n)^b$

where $k(i, n)^b$ $(i=0, 1, 2)$ denote the sets of row vectors.

Set

$$2)\quad \begin{cases} U_3 - x_1 1_b = -\overset{\circ}{U}_3, \quad U_5 - x_2 1_b = -\overset{\circ}{U}_5, \\ \begin{pmatrix} U_{01} \\ U_1 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 1_a \end{pmatrix} + \sum_{r\geqq 0} x_2^r V^{(r)}, \end{cases}$$

where $V^{(r)}$ are matrices with entries in $k(2, n)$. Then we see by 1) and 2) that $\lambda_2 \cdot \lambda_3 = 0$ is equivalent to

$$3)\quad \begin{cases} \begin{pmatrix} 0 \\ 1_a \end{pmatrix} U_4 \overset{\circ}{U}_3 + \sum_{r\geqq 0} V^{(r)} U_4 \overset{\circ}{U}{}_5^r = 0 \\ \overset{\circ}{U}_3 \overset{\circ}{U}_5 = \overset{\circ}{U}_5 \overset{\circ}{U}_3 \\ \begin{pmatrix} U_{02} \\ U_2 \end{pmatrix} = -\sum_{r\geqq 1} \sum_{i=0}^{r-1} x_2^{r-i-1} V^{(r)} U_4 \overset{\circ}{U}{}_5^i. \end{cases}$$

Now we restrict ourselves to the case where $n=4$ and $b=1$. Let Λ_2 be a matrix (μ_{ij}) satisfying (3.4), and let $S(\Lambda_2)$ be the set of subschemes of $\mathbf{P}_k^3$ defined by

$$S(\Lambda_2) = \left\{ \operatorname{Proj} R/I \;\middle|\; \begin{array}{l} I \text{ is defined as in 3.7.2) by} \\ \text{a matrix } \lambda_2 \text{ satisfying the} \\ \text{conditions of Theorem 3.7.} \end{array} \right\}.$$

Let $I(X)$ denote the ideal $f_0 k(0, 4) \oplus \bigoplus_{i=1}^{a} f_i k(1, 4) \oplus f_{a+1} k(2, 4)$ defining $X \in S(\Lambda_2)$. We may assume without loss of generality that $\nu_0 \leqq \nu_1 \leqq \cdots \leqq \nu_a$ (see Example 2.7). After the change of variables $(x_1 - \overset{\circ}{U}_3, x_2 - \overset{\circ}{U}_5, x_3, x_4) \to (x_1', x_2', x_3', x_4')$ we may assume that $\overset{\circ}{U}_3 = \overset{\circ}{U}_5 = 0$. Then 4.1.3) becomes

$$(4.1.3)' \qquad \begin{cases} V^{(0)} U_4 = 0, \\ \begin{pmatrix} U_{02} \\ U_2 \end{pmatrix} = -\sum_{r\geqq 1} x_2^{r-1} V^{(r)} U_4. \end{cases}$$

Consider the problem "When does there exist an integral curve in $S(\Lambda_2)$?" The answer is known if $\mu_{a,a+2}\geqq 1$. Before stating the results let us make preparations first.

Set $U_4={}^t(h_1, \cdots, h_a)$, $\mathfrak{a}=(h_1, \cdots, h_a)k(2,4)\subset k[x_3, x_4]$. If $I(\lambda_3)=R$ then $\mathfrak{a}=k(2,4)$; that is one of h_i $(1\leqq i\leqq a)$ is a unit, so that (4.1.3)$'$ can be solved easily. If $I(\lambda_3)\neq R$ and contains an R-sequence of length 4, then $\mathfrak{a}$ contains a $k(2,4)$-sequence of length 2, that is $k(2,4)/\mathfrak{a}$ is Cohen-Macaulay of dimention 0. Hence, the $k(2,4)$-module $M=\{(v_1, \cdots, v_a)\in k(2,4)^a \mid \sum_{i=1}^{a} v_i h_i=0\}$, which makes the sequence

$$0\longrightarrow M\longrightarrow k(2,4)^a \xrightarrow{{}^tU_4} k(2,4)\longrightarrow k(2,4)/\mathfrak{a}\longrightarrow 0$$

exact, is free of rank $a-1$ over $k(2,4)$ by Auslander-Buchsbaum's theorem. And each row vector of $V^{(0)}$ satisfying (4.1.3)$'$ is in M. Write M_ν for $\{v\in M \mid d_{\bar{e}}(v)=\nu\}$ where $\bar{e}=(\deg h_1, \cdots, \deg h_a)=(\mu_{1,a+2}, \cdots, \mu_{a,a+2})$, and let N_p be the submodule of M generated by $\bigoplus_{\nu\leqq p} M_\nu$. Put $\omega_i=(\mu_{i1}, \cdots, \mu_{ia})$ and $c_i=\deg h_j+\mu_{ij}$ (independent of j) for $0\leqq i\leqq a$. We see $c_0\geqq c_1\geqq\cdots\geqq c_a$.

Suppose $$\begin{cases} c_1=\cdots=c_{t_1}=\varepsilon_1\,, \\ c_{t_1+1}=\cdots=c_{t_1+t_2}=\varepsilon_2\,, \quad \varepsilon_1>\varepsilon_2>\cdots>\varepsilon_p\,, \\ \quad\cdots\cdots\,, \\ c_{t_1+\cdots+t_{p-1}+1}=\cdots=c_{t_1+\cdots+t_p}=\varepsilon_p\,, \\ t_1+\cdots+t_p=a\,. \end{cases}$$

Then

$$(\mu_{ij})_{\substack{0\leqq i\leqq a\\ 1\leqq j\leqq a}}=\begin{pmatrix} \mu_{01} & \cdots\cdots\cdots\cdots\cdots & & \mu_{0a} \\ \hline A_1 & & & \\ B_2 & A_2 & & * \\ & B_3 & \ddots & \\ & & B_p & A_p \end{pmatrix}$$

where A_i is the $t_i\times t_i$ matrix with all entries 1 for $1\leqq i\leqq p$ and B_i is the $t_i\times t_{i-1}$ matrix with all entries 0 for $2\leqq i\leqq p$. Let $\mathcal{M}$ denote the above matrix. We form a new $(a-1)\times a$ matrix $D=(d_{ij})$ $0\leqq i\leqq a-2$, $1\leqq j\leqq a$ with entries in $\boldsymbol{Z}$ in the following way. First put $\xi_i=\omega_{t_{i-1}+1}$ for

$2\leqq i\leqq p$, and

$$D'=\left(\begin{array}{ll} & \leftarrow \\ \xi_2 & \leftarrow \\ \xi_3 & \leftarrow \\ \xi_q & \leftarrow \\ p-q\left\{\begin{array}{l} \xi_{q+1} \\ \xi_{q+2} \\ \vdots \\ \xi_p \end{array}\right. & \begin{array}{l} \\ \\ \\ \leftarrow \end{array} \end{array}\right) \begin{array}{l} \cdots 0\text{-th row} \\ \cdots t_1+1\text{-th row} \\ \cdots t_1+t_2+1\text{-th row} \\ \sum_{i=1}^{q-1} t_i+1\text{-th row} \\ \\ \\ \\ (a-2)\text{-th row}\,, \end{array}$$

where q is the largest number of i such that $t_1+\cdots+t_{i-1}+1\leqq(a-2)-(p-i)$. Next fill each blank row of D' with the corresponding row of $\mathcal{M}$. Let D be the matrix thus obtained.

Put

$$\rho(A_2)=\sum_{i=0}^{a-2} d_{i\ i+1}\,.$$

Lemma 4.2. *Suppose $b=1$, $a\geqq 2$. If $S(A_2)$ contains an integral curve which is not projectively Cohen-Macaulay, then*

1) $\deg f_0\leqq\deg f_i$ *for* $1\leqq i\leqq a+1$

2) $0\leqq\deg f_{i+1}-\deg f_i\leqq 1$ *for* $1\leqq i\leqq a-1$.

Proof. First note that $\deg f_0\leqq\deg f_i$ for $1\leqq i\leqq a$ by assumption or rather by Example 2.7. Therefore, if $\deg f_{a+1}<\deg f_0$ we must have $\mu_{i,a+2}\leqq 0$ for $1\leqq i\leqq a$. This implies that every nonzero h_i must be in k. Thus $R/I(X)$ turns out to be Cohen-Macaulay, which contradicts the assumption. Hence $\deg f_0\leqq\deg f_{a+1}$. Since $I(X)$ is prime f_0 must be irreducible, from which 2) follows. See [6; Proposition 2.1].

Lemma 4.3. *Suppose $b=1$, $a\geqq 2$. There exists a scheme X in $S(A_2)$ which does not contain $L=\{(x_1:x_2:x_3:x_4)\in\mathbf{P}_k^3\,|\,x_1=x_2=0\}$ as an irreducible component if and only if* $\operatorname{rank} N_{c_i}\geqq a-1-i$ *for all* $0\leqq i\leqq a-2$.

Proof. Let $J\subset I$ be as in Theorem 3.7. We see easily that Proj R/J contains L as an irreducible component. Hence Proj R/I does not contain L as an irreducible component if and only if $f_{a+1}(0, 0, x_3, x_4)\neq 0$. This is possible if and only if rank $N_{c_i}\geqq a-1-i$ for all $0\leqq i\leqq a-2$. With Lemmas 4.2 and 4.3 in mind we get

Proposition 4.4.

1) *Suppose* $b=1$, $a\geqq 3$, $\nu_0\leqq\nu_1\leqq\cdots\leqq\nu_a$, $\nu_{i+1}-\nu_i\leqq 1$ *for* $1\leqq i\leqq a-1$, *and* $\mu_{a,a+2}\geqq 1$. *Then* $S(A_2)$ *contains an integral curve if and only if*

$$\mu_{a,a+2}\leqq\rho(A_2).$$

2) *Suppose* $b=1$, $a=2$, *and* $\mu_{24}\geqq 1$. *Then* $S(A_2)$ *contains an integral curve if and only if*

$$\mu_{14}=\mu_{24}=1 \text{ and } \mu_{01}=\mu_{02}\geqq 2.$$

For the proof we use only Bertini's Theorem and elementary properties of determinants. Details are omitted.

Example 4.5. Suppose $r\leqq n$, $2\leqq n$, and put

$$A_2=\begin{pmatrix} n & n & n & n+r & n+r \\ 1 & 1 & 1 & r+1 & r+1 \\ 1 & 1 & 1 & r+1 & r+1 \\ 1 & 1 & 1 & r+1 & r+1 \\ -r+1 & -r+1 & -r+1 & 1 & 1 \end{pmatrix}$$

$$\lambda_2=$$

$$\left(\begin{array}{c|c|c|c|c} -x_2^n+x_3^n & -x_3^{n-r}x_4^r & (sx_3+tx_4)x_2^{n-1}+ux_2^n & \begin{array}{c} x_4^{r+1}x_2^{n-1}-ux_3^{r+1}x_2^{n-1} \\ -x_3^{r+1}(sx_3+tx_4)x_2^{n-2} \end{array} & 0 \\ \hline x_1 & -x_2+x_3 & -x_4 & x_3^r x_4 & x_4^{r+1} \\ \hline x_2 & x_1 & -x_2 & x_3^{r+1}-x_4^{r+1} & x_3^r x_4 \\ \hline 0 & x_2 & x_1 & -x_3^r x_4 & x_3^{r+1} \\ \hline 0 & 0 & 0 & x_1 & x_2 \end{array}\right).$$

Then $\rho(A_2)=n+1\geqq\mu_{a,a+2}=\mu_{35}=r+1$, $\lambda_2\cdot\lambda_3=0$, and

$$f_0 = x_1^3 + x_1x_2(2x_2 - x_3) - x_2^2x_4\,,$$

$$f_1 = x_1^2(-x_2^n + x_3^n) + x_1x_2x_3^{n-r}x_4^r + x_2^2x_3^n - x_2^{n+2} + sx_2^{n+1}x_3$$
$$+ tx_2^{n+1}x_4 + ux_2^{n+2}\,,$$

$$f_4(0, 0, x_3, x_4) = -x_3^{n+r}x_4^2 - x_3^{n+r+2} - x_3^{n+1}x_4^{r+1}\,.$$

One verifies directly that

$$\operatorname{Spec} k[z_1, z_3, z_4]/(f_0(z_1, 1, z_3, z_4), f_1(z_1, 1, z_3, z_4))$$

is irreducible reduced for a suitable choice of $s, t, u \in k$, and that $\operatorname{Proj} R/I = X$ does not have any irreducible component in $\{(x_1: x_2: x_3: x_4) \in \mathbf{P}_k^3 \mid x_2 = 0\}$. Thus x is an integral curve for a suitable choice of $s, t, u \in k$.

Remark 4.6. The curves obtained in Proposition 4.4 have singularities in many cases. In fact we can prove the following:

(#) Let $g_0, \cdots, g_{a-2}$ be a free basis for M, and suppose $d_{\bar{e}}(g_0) \geqq d_{\bar{e}}(g_1) \geqq \cdots \geqq d_{\bar{e}}(g_{a-2})$. If $d_{\bar{e}}(g_0) < c_0$ and $g_0 \notin N_{c_1}$, then the integral curves in $S(\Lambda_2)$ must have singularities.

References

[1] Buchsbaum, D. and Eisenbud, D., What makes a complex exact? *Journal of Algebra*, **25** (1973), 259-268.

[2] Briançon, J. et Galligo, A., Déformations de points $\boldsymbol{R}^2$ ou $\boldsymbol{C}^2$, *Astérisque*, **7** et **8** (1973), 129-138.

[3] Galligo, A., A propos du théorèm de préparation, *Springer Lecture Notes*, **409** (1973), 543-579.

[4] ———, Théorèm de division et stabilité en géometrie analytique locale, *Ann. Inst. Fourier, Grenoble* **29** (1979), 107-184.

[5] Grauert, H., Über die Deformation isolierter Singularitäten analytischer Mengen, *Inventiones Math.*, **15** (1972), 171-198.

[6] Gruson, L. et Peskine, C., Genre des courbes de l'espace projectif, *Proceedings, Tromsφ, Norway* 1977, *Springer Lecture Notes* **687**.

[7] Hironaka, H., Bimeromorphic smoothing of complex analytic spaces, *Preprint, Warwick University*, 1971.

[8] Lectured by Hironaka, H., notes by Urabe, T., "*Introduction to analytic spaces*" (In Japanese) Asakura Publ. Comp. 1982.

[9] Urabe, T., On Hirona's monoideal. *Publ. RIMS, Kyoto Univ.*, **15** (1979), 279-287.