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Justification of Partially-Multiplicative Averaging for a Class of Functional-Differential Equations with Impulses

By

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Summary

The authors obtain a justification of the method of partially-multiplicative averaging for a class of functional-differential equations with impulses and a transformed argument, dependent on the time and the unknown function.

§ 1. Introduction

The averaging method of Bogoljubov-Mitropol'skii is now recognized as one of the most efficient mathematical methods in the nonlinear mechanics. A detailed bibliography on this subject is given in [1]-[3].

In connection with some mathematical models arising in the theory of control systems the averaging method has been justified in [4]-[7] for certain classes of differential equations with impulse action. The generalization of the averaging method for asymptotic integration of systems of differential equations with impulses was substantiated by the following reasons:

—due to their complex structure, the qualitative investigation of the above systems is subject to great difficulties, while the averaged system introduced in the cited papers is without impulse action;

—the solution of the averaged system approximates the solution of the original system with any prescribed accuracy on an asymptotically large time-interval.

The present paper presents a justification of the method of partially-

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multiplicative averaging for a class of functional-differential equations with impulses and a transformed argument dependent on the time and the unknown function.

§ 2. Statement of the Problem

Let in the $(n+1)$-dimensional space (t, x), where x is an n-dimensional vector, the following hypersurfaces be given

$$\sigma_i: t=t_i(x), \quad i=1, 2, \cdots,$$

which for $x\in D\subset R^n$ lie in the half-space $t>0$ and satisfy the condition

$$t_i(x)<t_{i+1}(x), \quad i=1, 2, \cdots,$$

Let a mapping point P_t with current coordinates $(t, x(t))$ move in the domain $\{t\geq 0, x\in D\}$. We shall suppose that the motion of the point P_t is governed by a law characterized by:

a) the system of differential equations of a neutral type

$$\begin{aligned} \dot{x}(t) &= \varepsilon A(t, x(t), x(\Delta(t, x(t))), \dot{x}(\Delta(t, x(t))))X(t, x(t)), \\ &\qquad t>0, \quad t\neq t_i(x), \\ x(t) &= \varphi(t, \varepsilon), \quad t\in[-\delta, 0], \\ \dot{x}(t) &= \dot{\varphi}(t, \varepsilon), \quad t\in[-\delta, 0], \end{aligned} \tag{1}$$

where ε is a small parameter, $A(t, x, y, z) = (a_{ij}(t, x, y, z))_{nm}$, δ is a positive constant, $\Delta(t, x)$ is a transformed argument satisfying the condition

$$t-\delta\leq\Delta(t, x)\leq t \tag{2}$$

for $t\geq -\delta$ and $x\in D$, and $\varphi(t, \varepsilon)$ is an initial-value function defined together with its derivative $\dot{\varphi}(t, \varepsilon)$ with respect to t for $t\in[-\delta, 0]$ and $\varepsilon\in(0, \mathcal{E}]$, $\mathcal{E}=\text{const}>0$;

b) the set of hypersurfaces σ_i, $i=1, 2, \cdots$;

c) the set of vector-functions $I_i(x)$, $i=1, 2, \cdots$,

defined in D.

Note that in view of (2) the velocity of the point P_t at time t depends on the motion and velocity of P_t on the whole preceding interval $[t-\delta, t]$.

The motion itself can be described as follows. Departing from the point $(\tau_0=0, x_0=x(0)=\varphi(0,\varepsilon))$ the point P_t moves along the trajectory $(t, x(t))$, governed by the solution $x(t)$ of (1) until the moment $\tau_1>0$ at which the trajectory meets the hypersurface σ_1 at the point $(\tau_1, x_1^- = x(\tau_1))$. Then the point P_t instantly moves from the position (τ_1, x_1^-) to the position $(\tau_1, x_1^+ = x_1^- + \varepsilon I_1(x_1^-))$ and further on follows the trajectory $(t, x(t))$, described by the solution $x(t)$ of system (1) until it meets the hypersurface σ_2, etc.

The relations a), b), c) characterizing the motion of point P_t are said to be a system of functional-differential equations (1) with impulses. The curve described by the motion of point P_t is said to be the integral curve or the trajectory of this system in the space (t, x).

Thus the solution of the system of functional-differential equations (1) with impulses is a function satisfying (1) out of the hypersurfaces $\sigma_i, i=1, 2, \cdots$ and having instantaneous jumps

$$x_i^+ = x_i^- + \varepsilon I_i(x_i^-), \quad i=1, 2, \cdots \tag{3}$$

when meeting the hypersurfaces σ_i, $i=1, 2, \cdots$. Note that the point (τ_i, x_i^+) does not necessarily belong to the hypersurface $\sigma_i, i=1, 2, \cdots$.

Let the following limits exist

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} A(\theta, x, x, 0)\, d\theta = A_0(x) \tag{4}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t<t_i<t+T} I_i(x) = I_0(x).$$

Then we compare the system of functional-differential equations (1) with impulses to the averaged system of ordinary differential equations

$$\dot{\bar{x}}(t) = \varepsilon[A_0(\bar{x}(t)) X(t, \bar{x}(t)) + I_0(\bar{x}(t))] \tag{5}$$

with initial condition

$$\bar{x}(0) = x_0 . \tag{6}$$

Note that if $x=(x_1, \cdots, x_n)$, $A=(a_{ij})_{nm}$, then by definition

$$\|x\| = [\sum_{i=1}^n x_i^2]^{1/2}, \quad \|A\| = [\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2]^{1/2},$$

while by $\overline{1, n}$ we shall denote the set of positive integers $\{1, 2, \cdots, n\}$.

§ 3. Main Result

The following theorem for proximity between the solutions of the system of functional-differential equations (1) with impulses and the averaged system (5) with initial condition (6) holds true:

Theorem 1. *Let the following assumptions be fulfilled*:

1° *The functions* $A(t, x, y, z)$ *and* $X(t, x)$ *are continuous in the domain* $\{t\geq 0, x, y\in D, z\in D_1\subset R^n\}$. *The function* $\Delta(t, x)$ *is continuous and satisfies the condition* (2) *in the domain* $\{t\geq 0, x\in D\}$. *The functions* $\varphi(t, \varepsilon)$ *and* $\dot{\varphi}(t, \varepsilon)$ *are continuous in the domain* $\{t\in[-\delta, 0], \varepsilon\in(0, \mathcal{E}], \mathcal{E}=\text{const}>0\}$ *and* $\varphi(t, \varepsilon)\in D$, $\dot{\varphi}(t, \varepsilon)\in D_1$. *The functions* $I_i(x)$, $i=1, 2, \cdots$ *are continuous in* D. *The functions* $t_i(x)$, $i=1, 2, \cdots$ *are twice continuously differentiable in* D.

2° *There exist positive constants* M, K, C *and a function* $\gamma(\varepsilon)$ *such that*

$$\left\|\frac{\partial t_i(x)}{\partial x}\right\| + \|A(t, x, y, z)\| + \|X(t, x)\| + \|I_i(x)\| \leq M,$$

$$\|A(t, x, y, z) - A(t, x', y', z')\| \leq K(\|x-x'\| + \|y-y'\| + \|z-z'\|),$$

$$\|X(t, x) - X(t, x')\| + \|I_i(x) - I_i(x')\| \leq K\|x-x'\|, \left\|\frac{\partial^2 t_i(x)}{\partial x^2}\right\| \leq C$$

for all $t\geq 0$, $x, x', y, y'\in D$, $z, z'\in D_1$, $i=1, 2, \cdots$ *and* $\|\dot{\varphi}(t, \varepsilon)\|\leq\gamma(\varepsilon)$ *for* $\varepsilon\in(0, \mathcal{E}]$, *where* $\lim\limits_{\varepsilon\to 0}(\gamma(\varepsilon)/\varepsilon)=\text{const}>0$ *and* $\sup\limits_{\varepsilon\in(0, \mathcal{E}]}(\gamma(\varepsilon)/\varepsilon)=\text{const}>0$.

3° *Uniformly in* $t\geq 0$ *and* $x\in D$ *there exist the finite limits* (4) *and*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}1=d\,, \quad d=\text{const}>0\,.$$

4° *The functions* $a_{ij}(t, x, y, z) - a_{ij}^{(0)}(x)$, $i=\overline{1, n}, j=\overline{1, m}$, *where* $a_{ij}^{(0)}$ *are the elements of the matrix* $A_0(x)$, *do not change sign in the whole domain* $\{t\geq 0, x, y\in D, z\in D_1\}$, *i.e. either* $a_{ij}(t, x, y, z) - a_{ij}^{(0)}(x) \geq 0$ *or* $a_{ij}(t, x, y, z) - a_{ij}^{(0)}(x)\leq 0$ *in this domain.*

5° *For each* $\varepsilon\in(0, \mathcal{E}]$ *the system of functional-differential equations* (1) *with impulses has a continuous solution* $x(t)$ *for* $t\geq 0$,

$t \neq \tau_i$, $i=1,2,\cdots$ *which satisfies the matching conditions* $x(0+0) = \varphi(0, \varepsilon) = x_0$, $\dot{x}(0+0) = \dot{\varphi}(0, \varepsilon)$.

6° *For each* $\varepsilon \in (0, \mathcal{E}]$ the *averaged initial value problem* (5), (6) *has a solution* $\bar{x}(t)$, *which belongs to the domain* D *for* $t \geq 0$ *together with its neighbourhood of radius* $\rho = \text{const} > 0$, *and satisfies the inequalities* $\frac{\partial t_i(\bar{x}(t))}{\partial x} I_i(\bar{x}(t)) \leq \beta < 0$, $\beta = \text{const}$, $t \in (t_i', t_i'')$, $t_i' = \inf_{x \in D} t_i(x)$, $t_i'' = \sup_{x \in D} t_i(x)$, $i=1, 2, \cdots$, *or* $\frac{\partial t_i(x)}{\partial x} \equiv 0$, *when* σ_i *is a hyperplane.*

Then for each $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0 \in (0, \mathcal{E}]$ $(\varepsilon_0 = \varepsilon_0(\eta, L))$ *such that for* $\varepsilon \leq \varepsilon_0$ *the inequality* $\|x(t) - \bar{x}(t)\| < \eta$ *holds for* $0 \leq t \leq L\varepsilon^{-1}$.

We shall base the proof of Theorem 1 on the following lemma.

Lemma 1. *Let the conditions of Theorem* 1 *be fulfilled. Let* $T > \delta$ *be a sufficiently large and fixed number. Then for each positive integer* $p \geq 1$ *the following inequality holds*

$$\|x(pT) - \bar{x}(pT)\| \leq \varepsilon \sum_{i=0}^{p-1} [1 + \varepsilon(3M+d)KT]^i [\alpha(T)T + \varepsilon \overline{M}], \tag{7}$$

where $\overline{M} = (M+d)(3M+d)KMT^2 + \max_{i=\overline{1,p}} M_i$ *and* $M_i = M_i(T, d_1, \cdots, d_i)$ *are constants depending on* T *and on the constants* $d_j > 0$, $j = \overline{1, i}$.

Proof of Lemma 1. The condition 3° of Theorem 1 guarantees the existence of a function $\alpha(t)$, monotonously decreasing towards zero as t tends to infinity, such that for each $t \geq 0$ and $x \in D$ the following inequalities hold

$$\left\| \int_t^{t+T} [A(\theta, x, x, 0) - A_0(x)] d\theta \right\| \leq \alpha(T) T / 2Mm\sqrt{n}, \tag{8}$$

$$\Big\| \sum_{t < t_i < t+T} I_i(x) - I_0(x) T \Big\| \leq \alpha(T) T/2.$$

We shall carry out the proof of Lemma 1 by the method of complete mathematical induction.

First we shall prove the inequality (7) for $p=1$.

We consider the system of functional-differential equations (1) with impulses on the interval $[0, T]$.

Let d_1 points lie on the interval $(0, T)$

$$t_1(x_0) = t_1^{(0)}, \cdots, t_{d_1}(x_0) = t_{d_1}^{(0)} \tag{9}$$

in which case $t_i^{(0)} < t_{i+1}^{(0)}$, $i = \overline{1, (d_1 - 1)}$.

We denote by $x_1^{(0)}(t, 0, x_0)$ the solution of the system

$$x_1^{(0)}(t, 0, x_0) = \begin{cases} x_0 + \varepsilon \int_0^t A_0(\theta, x_1^{(0)}(\theta, 0, x_0), x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0), \\ \dot{x}_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0)) X(\theta, x_1^{(0)}(\theta, 0, x_0)) d\theta, \quad t > 0. \\ \varphi(t, \varepsilon), \quad -\delta \le t \le 0, \end{cases} \tag{10}$$

$$\dot{x}_1^{(0)}(t, 0, x_0) = \dot{\varphi}(t, \varepsilon), \quad -\delta \le t \le 0,$$

where $\Delta_1^{(0)}(t) = \Delta(t, x_1^{(0)}(t, 0, x_0))$.

Obviously, the solution of (10) coincides with the solution $x(t)$ of the system of functional-differential equations (1) with impulses until the moment τ_1 at which the trajectory $(t, x(t))$ of this system meets the hypersurface σ_1, i.e. $x(t) = x_1^{(0)}(t, 0, x_0)$, $t \in [-\delta, \tau_1]$.

Let us consider the function

$$\tilde{x}_1^{(0)}(t, 0, x_0) = x_0 + \varepsilon \int_0^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$

and estimate in terms of norm the difference

$$R_1^{(0)}(t, 0, x_0, \varepsilon) = x_1^{(0)}(t, 0, x_0) - \tilde{x}_1^{(0)}(t, 0, x_0).$$

For $0 < t \le T$ we have

$$\begin{aligned} \| R_1^{(0)}(t, 0, x_0, \varepsilon) \| &\le \varepsilon \int_0^t \| A(\theta, x_1^{(0)}(\theta, 0, x_0), x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0), \\ &\quad \dot{x}_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0)) X(\theta, x_1^{(0)}(\theta, 0, x_0)) - A(\theta, x_0, x_0, 0) X(\theta, x_0) \| d\theta \\ &\le \varepsilon \int_0^t \{ \| A(\theta, x_1^{(0)}(\theta, 0, x_0), x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0), \dot{x}_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0)) \\ &\quad - A(\theta, x_0, x_0, 0) \| \cdot \| X(\theta, x_1^{(0)}(\theta, 0, x_0)) \| + \| A(\theta, x_0, x_0, 0) \| \\ &\quad \cdot \| X(\theta, x_1^{(0)}(\theta, 0, x_0)) - X(\theta, x_0) \| \} d\theta \le \varepsilon K M \int_0^t \{ 2 \| x_1^{(0)}(\theta, 0, x_0) - x_0 \| \\ &\quad + \| x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0) - x_0 \| + \| \dot{x}_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0) \| \} d\theta \end{aligned}$$

$$\leq 2\varepsilon^2 KM \int_0^t d\theta \int_0^\theta \| A(l, x_1^{(0)}(l, 0, x_0), x_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0),$$

$$\dot{x}_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0)) \| \cdot \| X(l, x_1^{(0)}(l, 0, x_0)) \| dl$$

$$+ \varepsilon KM \Big\{ \int_{J_{0,t}^-} \| \varphi(\Delta_1^{(0)}(\theta), \varepsilon) - \varphi(0, \varepsilon) \| d\theta$$

$$+ \varepsilon \int_{J_{0,t}^+} d\theta \int_0^{\Delta_1^{(0)}(\theta)} \| A(l, x_1^{(0)}(l, 0, x_0),$$

$$x_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0), \dot{x}_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0)) \| \cdot \| X(l, x_1^{(0)}(l, 0, x_0)) \| dl \Big\}$$

$$+ \varepsilon KM \Big\{ \int_{J_{0,t}^-} \| \dot{\varphi}(\Delta_1^{(0)}(\theta), \varepsilon) \| d\theta$$

$$+ \varepsilon \int_{J_{0,t}^+} \| A(\Delta_1^{(0)}(\theta), x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0),$$

$$x_1^{(0)}(\Delta_1^{(0)}(\Delta_1^{(0)}(\theta)), 0, x_0), \dot{x}_1^{(0)}(\Delta_1^{(0)}(\Delta_1^{(0)}(\theta)), 0, x_0)) \| \cdot \| X(\Delta_1^{(0)}(\theta),$$

$$x_1^{(0)}(\Delta_1^{(0)}(\theta), 0, x_0)) \| d\theta \Big\} \leq 2\varepsilon^2 KM^3 \int_0^t d\theta \int_0^\theta dl$$

$$+ \varepsilon\gamma(\varepsilon)\sqrt{n} KM \int_{J_{0,t}^-} |\Delta_1^{(0)}(\theta)| d\theta + \varepsilon^2 KM^3 \int_{J_{0,t}^+} \Delta_1^{(0)}(\theta) d\theta$$

$$+ \varepsilon\gamma(\varepsilon) KM \int_{J_{0,t}^-} d\theta + \varepsilon^2 KM^3 \int_{J_{0,t}^+} d\theta \leq \varepsilon^2 KM^3 T^2$$

$$+ \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1) KM \int_0^t d\theta + \varepsilon^2 KM^3 \int_0^t \theta d\theta + \varepsilon^2 KM^3 \int_0^t d\theta$$

$$\leq 3\varepsilon^2 KM^3 T^2/2 + \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1) KMT + \varepsilon^2 KM^3 T \equiv \omega_1^{(0)}(\varepsilon^2, T),$$

where

$$J_{0,t}^- \cup J_{0,t}^+ = (0, t],$$

$$J_{0,t}^- = \{\theta : \theta \in (0, t] \wedge \Delta_1^{(0)}(\theta) \in [-\delta, 0]\},$$

$$J_{0,t}^+ = (0, t] \setminus J_{0,t}^- .$$

The obtained estimate shows that the function $\tilde{x}_1^{(0)}(t, 0, x_0)$ approximates the solution $x_1^{(0)}(t, 0, x_0)$ of the system (10) on the interval $(0, T]$ to a precision of order ε^2.

The moment τ_1, at which the trajectory $(t, x(t))$ meets the hypersurface σ_1, is a solution of the equation

$$t = t_1(x_1^{(0)}(t, 0, x_0)). \tag{11}$$

Since

$$(12)\qquad t_1(x_1^{(0)}(t, 0, x_0)) = t_1(\tilde{x}_1^{(0)}(t, 0, x_0) + R_1^{(0)}(t, 0, x_0, \varepsilon))$$

$$= t_1\left(x_0 + \varepsilon \int_0^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + O(\varepsilon^2)\right)$$

$$= t_1(x_0) + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + O(\varepsilon^2)$$

$$= t_1^{(0)} + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^{(0)}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$

$$+ \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_{t_1^{(0)}}^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + O(\varepsilon^2)$$

$$= t_1^{(0)} + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^{(0)}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$

$$+ \varepsilon \frac{\partial t_1(x_0)}{\partial x} (t - t_1^{(0)}) A(\tilde{t}, x_0, x_0, 0) X(\tilde{t}, x_0) + O(\varepsilon^2),$$

$$\tilde{t} = t_1^{(0)} + \mu(t - t_1^{(0)}), \quad 0 \leq \mu \leq 1,$$

then from (11) it follows that $\tau_1 = t_1^{(0)} + \varepsilon\Theta_1^{(0)} + O(\varepsilon^2)$, where

$$\Theta_1^{(0)} = \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^{(0)}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta .$$

We shall note that in (12) the values of the constant μ in the different components of the vector $A(\tilde{t}, x_0, x_0, 0) X(\tilde{t}, x_0)$ are, generally speaking, different.

The inequality $t_1^{(0)} > 0$ implies that $\tau_1 > \tau_0$ if ε is sufficiently small.

Thus

$$x(t) = x_1^{(0)}(t, 0, x_0) = \tilde{x}_1^{(0)}(t, 0, x_0) + R_1^{(0)}(t, 0, x_0, \varepsilon)$$

for $\tau_0 < t \leq \tau_1 = t_1^{(0)} + \varepsilon\Theta_1^{(0)} + O(\varepsilon^2)$.

Henceforth we find

$$x_1^+ = x_1^{(0)}(\tau_1, 0, x_0) + \varepsilon I_1(x_1^{(0)}(\tau_1, 0, x_0))$$

i.e.

$$x_1^+ = \tilde{x}_1^{(0)}(\tau_1, 0, x_0) + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon)$$

$$= x_0 + \varepsilon \int_0^{\tau_1} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon),$$

where $I_1^{(0)} \equiv I_1(x_1^{(0)}(\tau_1, 0, x_0))$.

We denote by $x_2^{(0)}(t, \tau_1, x_1^+)$ the solution of the system

$$(13)\quad x_2^{(0)}(t,\tau_1,x_1^+)=\begin{cases} x_1^+ + \varepsilon\displaystyle\int_{\tau_1}^t A(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+), x_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+), \\ \qquad \dot{x}_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+))X(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+))d\theta, \quad t>\tau_1, \\ x_1^{(0)}(t,0,x_0), \quad -\delta\le t\le\tau_1, \end{cases}$$

$$\dot{x}_2^{(0)}(t,\tau_1,x_1^+)=\dot{x}_1^{(0)}(t,0,x_0), \quad -\delta\le t\le\tau_1,$$

where $\Delta_2^{(0)}(t)=\Delta(t, x_2^{(0)}(t,\tau_1,x_1^+))$.

The solution of (13) coincides with the solution $x(t)$ of the system of functional-differential equations (1) with impulses until the moment τ_2, at which the trajectory $(t, x(t))$ meets the hypersurface σ_2, i.e.

$$x(t)=x_2^{(0)}(t,\tau_1,x_1^+) \qquad \text{for} \quad t\in[-\delta,\tau_2].$$

Let us consider the function

$$\tilde{x}_2^{(0)}(t,\tau_1,x_1^+)=x_1^+ + \varepsilon\int_{\tau_1}^t A(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta$$

and estimate in terms of norm the difference

$$R_2^{(0)}(t,\tau_1,x_1^+,\varepsilon)=x_2^{(0)}(t,\tau_1,x_1^+)-\tilde{x}_2^{(0)}(t,\tau_1,x_1^+).$$

For $0<\tau_1<t\le T$ we obtain

$$\begin{aligned}\|R_2^{(0)}(t,\tau_1,x_1^+,\varepsilon)\| &\le \varepsilon\int_{\tau_1}^t \|A(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+), x_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+), \\ &\quad \dot{x}_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+))X(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+)) - A(\theta,x_0,x_0,0)X(\theta,x_0)\|d\theta \\ &\le \varepsilon\int_{\tau_1}^t \{\|A(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+), x_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+), \dot{x}_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+)) \\ &\quad - A(\theta,x_0,x_0,0)\|\cdot\|X(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+))\| + \|A(\theta,x_0,x_0,0)\| \\ &\quad \cdot\|X(\theta, x_2^{(0)}(\theta,\tau_1,x_1^+)) - X(\theta,x_0)\|\}d\theta \\ &\le \varepsilon KM\int_{\tau_1}^t \{2\|x_2^{(0)}(\theta,\tau_1,x_1^+)-x_0\| + \|x_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+)-x_0\| \\ &\quad + \|\dot{x}_2^{(0)}(\Delta_2^{(0)}(\theta),\tau_1,x_1^+))\|\}d\theta \le 2\varepsilon^2 KM\int_{\tau_1}^t \Big\{\int_0^{\tau_1}\|A(l,x_0,x_0,0)\| \\ &\quad \cdot\|X(l,x_0)\|dl + \|I_1^{(0)}\| + \varepsilon^{-1}\|R_1^{(0)}(\tau_1,0,x_0,\varepsilon)\|\end{aligned}$$

$$+\int_{\tau_1}^{\theta}\|A(l, x_2^{(0)}(l, \tau_1, x_1^{+}), x_2^{(0)}(\Delta_2^{(0)}(l), \tau_1, x_1^{+}), \dot{x}_2^{(0)}(\Delta_2^{(0)}(l), \tau_1, x_1^{+}))\|$$

$$\cdot\|X(l, x_2^{(0)}(l, \tau_1, x_1^{+}))\|dl\Big\}d\theta+\varepsilon KM\Big\{\int_{J_{\tau_1,t}^{-}}\|\varphi(\Delta_2^{(0)}(\theta), \varepsilon)-\varphi(0, \varepsilon)\|d\theta$$

$$+\int_{J_{\tau_1,t}^{1}}\|x_1^{(0)}(\Delta_2^{(0)}(\theta), 0, x_0)-x_0\|d\theta+\varepsilon\int_{J_{\tau_1,t}^{+}}\Big[\int_0^{\tau_1}\|A(l, x_0, x_0, 0)\|$$

$$\cdot\|X(l, x_0)\|dl+\|I_1^{(0)}\|+\varepsilon^{-1}\|R_1^{(0)}(\tau_1, 0, x_0, \varepsilon)\|$$

$$+\int_{\tau_1}^{\Delta_2^{(0)}(\theta)}\|A(l, x_2^{(0)}(l, \tau_1, x_1^{+}), x_2^{(0)}(\Delta_2^{(0)}(l), \tau_1, x_1^{+}),$$

$$\dot{x}_2^{(0)}(\Delta_2^{(0)}(l), \tau_1, x_1^{+}))\|\cdot\|X(l, x_2^{(0)}(l, \tau_1, x_1^{+}))\|dl\Big]d\theta\Big\}$$

$$+\varepsilon KM\Big\{\int_{J_{\tau_1,t}^{-}}\|\dot{\varphi}(\Delta_2^{(0)}(\theta))\|d\theta+\varepsilon\int_{J_{\tau_1,t}^{1}}\|A(\Delta_2^{(0)}(\theta), x_1^{(0)}(\Delta_2^{(0)}(\theta), 0, x_0),$$

$$x_1^{(0)}(\Delta_1^{(0)}(\Delta_2^{(0)}(\theta)), 0, x_0), \dot{x}_1^{(0)}(\Delta_1^{(0)}(\Delta_2^{(0)}(\theta)), 0, x_0)\|$$

$$\cdot\|X(\Delta_2^{(0)}(\theta), x_1^{(0)}(\Delta_2^{(0)}(\theta), 0, x_0))\|d\theta$$

$$+\varepsilon\int_{J_{\tau_1,t}^{+}}\|A(\Delta_2^{(0)}(\theta), x_2^{(0)}(\Delta_2^{(0)}(\theta), \tau_1, x_1^{+}),$$

$$x_2^{(0)}(\Delta_2^{(0)}(\Delta_2^{(0)}(\theta)), \tau_1, x_1^{+}), \dot{x}_2^{(0)}(\Delta_2^{(0)}(\Delta_2^{(0)}(\theta)), \tau_1, x_1^{+}))\|$$

$$\cdot\|X(\Delta_2^{(0)}(\theta), x_2^{(0)}(\Delta_2^{(0)}(\theta), \tau_1, x_1^{+}))\|d\theta\Big\}\leq 2\varepsilon^2KM^2\int_{\tau_1}^{t}(M\theta+1)d\theta$$

$$+2\varepsilon KM\|R_1^{(0)}(\tau_1, 0, x_0, \varepsilon)\|\int_{\tau_1}^{t}d\theta+\varepsilon\gamma(\varepsilon)\sqrt{n}KM\int_{J_{\tau_1,t}^{-}}|\Delta_2^{(0)}(\theta)|d\theta$$

$$+\varepsilon^2KM\int_{J_{\tau_1,t}^{1}}d\theta\int_0^{\Delta_2^{(0)}(\theta)}\|A(l, x_1^{(0)}(l, 0, x_0), x_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0),$$

$$\dot{x}_1^{(0)}(\Delta_1^{(0)}(l), 0, x_0))\|\cdot\|X(l, x_1^{(0)}(l, 0, x_0))\|dl$$

$$+\varepsilon^2KM^2\int_{J_{\tau_1,t}^{+}}(M\Delta_2^{(0)}(\theta)+1)d\theta$$

$$+\varepsilon KM\|R_1^{(0)}(\tau_1, 0, x_0, \varepsilon)\|\int_{J_{\tau_1,t}^{+}}d\theta+\varepsilon\gamma(\varepsilon)KM\int_{J_{\tau_1,t}^{-}}d\theta$$

$$+\varepsilon^2KM^3\Big(\int_{J_{\tau_1,t}^{1}}d\theta+\int_{J_{\tau_1,t}^{+}}d\theta\Big)\leq\varepsilon^2KM(MT+1)^2+\varepsilon\gamma(\varepsilon)\delta\sqrt{n}KMT$$

$$+\varepsilon^2KM^2\int_{J_{\tau_1,t}^{1}}M\Delta_2^{(0)}(\theta)d\theta+\varepsilon^2KM^2\int_{J_{\tau_1,t}^{+}}(M\Delta_2^{(0)}(\theta)+1)d\theta$$

$$+\varepsilon\omega_1^{(0)}(\varepsilon^2, T)KM\int_{J_{\tau_1,t}^{+}}d\theta+2\varepsilon\omega_1^{(0)}(\varepsilon^2, T)KMT+\varepsilon\gamma(\varepsilon)KM\int_{\tau_1}^{t}d\theta$$

$$+\varepsilon^2 KM^3 \int_{\tau_1}^{t} d\theta \leq \varepsilon^2 KM(MT+1)^2 + \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KMT$$

$$+\varepsilon^2 KM^2 \int_{\tau_1}^{t} (M\theta+1)\,d\theta + 3\varepsilon\omega_1^{(0)}(\varepsilon^2, T)KMT + \varepsilon^2 KM^3 T$$

$$\leq 3\varepsilon^2 KM(MT+1)^2/2 + \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KMT$$

$$+3\varepsilon\omega_1^{(0)}(\varepsilon^2, T)KMT + \varepsilon^2 KM^3 T \equiv \omega_2^{(0)}(\varepsilon^2, T),$$

where

$$J^-_{\tau_1,t} \cup J^1_{\tau_1,t} \cup J^+_{\tau_1,t} = (\tau_1, t],$$

$$J^-_{\tau_1,t} = \{\theta : \theta \in (\tau_1, t] \wedge \Delta_2^{(0)}(\theta) \in [-\delta, 0]\},$$

$$J^1_{\tau_1,t} = \{\theta : \theta \in (\tau_1, t] \wedge \Delta_2^{(0)}(\theta) \in (0, \tau_1]\},$$

$$J^+_{\tau_1,t} = (\tau_1, t] \setminus (J^-_{\tau_1,t} \cup J^1_{\tau_1,t}).$$

Therefore, the function $\tilde{x}_2^{(0)}(t, \tau_1, x_1^+)$ approximates the solution $x_2^{(0)}(t, \tau_1, x_1^+)$ of the system (13) on the interval $(\tau_1, t] \subset (0, T]$ to a precision of order ε^2.

It can be shown that after the moment τ_1 the trajectory $(t, x(t))$ does not again meet the hypersurface σ_1.

Indeed, solving the equation

$$t = t_1(x_2^{(0)}(t, \tau_1, x_1^+)),$$

we obtain its root

$$\bar{t}_1 = \tau_1 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} I_1^{(0)} + O(\varepsilon^2).$$

Whence, and from the condition 6° of Theorem 1 and the continuity of the vector-function $I_1(x)$ it follows that the inequality $\bar{t}_1 < \tau_1$ is fulfilled for sufficiently small values of ε. Thus, we showed that the trajectory $(t, x(t))$ for $t > \tau_1$ does not again meet the hypersurface σ_1.

The moment at which the trajectory $(t, x(t))$ meets the hypersurface σ_2 is

$$\tau_2 = t_2^{(0)} + \varepsilon\Theta_2^{(0)} + O(\varepsilon^2),$$

where

$$\Theta_2^{(0)} = \frac{\partial t_2(x_0)}{\partial x}\left[\int_0^{t_2^{(0)}} A(\theta, x_0, x_0, 0)\,X(\theta, x_0)\,d\theta + I_1^{(0)}\right].$$

$t_2^{(0)} > t_1^{(0)}$ implies that $\tau_2 > \tau_1$ when ε is sufficiently small.

Thus

$$x(t) = x_2^{(0)}(t, \tau_1, x_1^+) = \tilde{x}_2^{(0)}(t, \tau_1, x_1^+) + R_2^{(0)}(t, \tau_1, x_1^+, \varepsilon)$$
$$= \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) + R_2^{(0)}(t, \tau_1, x_1^+, \varepsilon)$$

for $\tau_1 < t \leq \tau_2 = t_2^{(0)} + \varepsilon \Theta_2^{(0)} + O(\varepsilon^2)$ and

$$x_2^+ = x_2^{(0)}(\tau_2, \tau_1, x_1^+) + \varepsilon I_2(x_2^{(0)}(\tau_2, \tau_1, x_1^+)),$$

i.e.

$$x_2^+ = \tilde{x}_1^{(0)}(\tau_2, 0, x_0) + \varepsilon(I_1^{(0)} + I_2^{(0)}) + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon)$$
$$+ R_2^{(0)}(\tau_2, \tau_1, x_1^+, \varepsilon) = x_0 + \varepsilon \int_0^{\tau_2} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$
$$+ \varepsilon(I_1^{(0)} + I_2^{(0)}) + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) + R_2^{(0)}(\tau_2, \tau_1, x_1^+, \varepsilon),$$

where $I_2^{(0)} \equiv I_2(x_2^{(0)}(\tau_2, \tau_1, x_1^+))$.

In the general case $(s = \overline{2, (d_1+1)})$ we denote by $x_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+)$ the solution of the system

(14) $$x_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+)$$
$$= \begin{cases} x_{s-1}^+ + \varepsilon \int_{\tau_{s-1}}^t A(\theta, x_s^{(0)}(\theta, \tau_{s-1}, x_{s-1}^+), x_s^{(0)}(\Delta_s^{(0)}(\theta), \tau_{s-1}, x_{s-1}^+), \\ \quad \dot{x}_s^{(0)}(\Delta_s^{(0)}(\theta), \tau_{s-1}, x_{s-1}^+)) X(\theta, x_s^{(0)}(\theta, \tau_{s-1}, x_{s-1}^+)) d\theta, \quad t > \tau_{s-1} \\ x_{s-1}^{(0)}(t, \tau_{s-2}, x_{s-2}^+), \quad -\delta \leq t \leq \tau_{s-1}, \end{cases}$$

$$\dot{x}_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+) = \dot{x}_{s-1}^{(0)}(t, \tau_{s-2}, x_{s-2}^+), \quad -\delta \leq t \leq \tau_{s-1},$$

where $\Delta_s^{(0)}(t) = \Delta(t, x_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+))$,

$$x_{s-1}^+ = x_{s-1}^{(0)}(\tau_{s-1}, \tau_{s-2}, x_{s-2}^+) + \varepsilon I_{s-1}(x_{s-1}^{(0)}(\tau_{s-1}, \tau_{s-2}, x_{s-2}^+))$$
$$= x_0 + \varepsilon \int_0^{\tau_{s-1}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + \varepsilon \sum_{i=1}^{s-1} I_i^{(0)}$$
$$+ \sum_{i=1}^{s-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon),$$

$$I_{s-1}^{(0)} \equiv I_{s-1}(x_{s-1}^{(0)}(\tau_{s-1}, \tau_{s-2}, x_{s-2}^+)), \quad x_0^+ = x_0.$$

The solution of (14) coincides with the solution of the system of functional-differential equations (1) with impulses on the interval $[-\delta,$

$\tau_s]$, where τ_s is the moment at which the trajectory $(t, x(t))$ meets the hypersurface σ_s.

If

$$\tilde{x}_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+) = x_{s-1}^{(0)} + \varepsilon \int_{\tau_{s-1}}^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$

it can be shown, as we did in the cases $s=1$ and $s=2$ that the difference

$$R_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+, \varepsilon) = x_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+) - \tilde{x}_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+)$$

on the interval $0<\tau_{s-1}<t\leq T$ satisfies the inequality

$$\begin{aligned}\|R_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+, \varepsilon)\| &\leq 3\varepsilon^2 KM(MT+s-1)^2/2 \\ &+ \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KMT + 3\varepsilon\sum_{i=1}^{s-1}\omega_i^{(0)}(\varepsilon^2, T)KMT + \varepsilon^2 KM^3 T \\ &\equiv \omega_s^{(0)}(\varepsilon^2, T).\end{aligned}$$

Therefore, the function $\tilde{x}_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+)$ approximates the solution $x(t)$ of the system of functional-differential equations (1) with impulses on the interval $(\tau_{s-1}, t] \subset (0, T]$ to a precision of order ε^2, etc.

Since for $s=\overline{2, (d_1+1)}$ we have

$$\begin{aligned}\tilde{x}_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+) &= x_{s-1}^+ + \varepsilon \int_{\tau_{s-1}}^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta \\ &= x_0 + \varepsilon \int_0^{\tau_{s-1}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + \varepsilon \sum_{i=1}^{s-1} I_i^{(0)} \\ &\quad + \sum_{i=1}^{s-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + \varepsilon \int_{\tau_{s-1}}^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta \\ &= \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon \sum_{i=1}^{s-1} I_i^{(0)} + \sum_{i=1}^{s-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon),\end{aligned}$$

then

$$\begin{aligned}x(t) = x_s^{(0)}(t, \tau_{s-1}, x_{s-1}^+) &= \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon \sum_{i=0}^{s-1} I_i^{(0)} \\ &+ \sum_{i=0}^{s-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{s-1}^+, \varepsilon) + R_3^{(0)}(t, \tau_{s-1}, x_{s-1}^+, \varepsilon)\end{aligned} \tag{15}$$

for

$$t_{s-1}^{(0)} + \varepsilon\Theta_{s-1}^{(0)} + \gamma_{s-1}O(\varepsilon^2) = \tau_{s-1} < t \leq \tau_s = t_s^{(0)} + \varepsilon\Theta_s^{(0)} + O(\varepsilon^2),$$

where

$$\Theta_s^{(0)} = \frac{\partial t_s(x_0)}{\partial x}\left[\int_0^{t_s^{(0)}} A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta + \sum_{i=1}^{s-1} I_i^{(0)}\right],$$

$$t_0^{(0)} = \Theta_0^{(0)} = \gamma_0 = 0\ ,\ I_0^{(0)} = R_0^{(0)}(\tau_0, \tau_{-1}, x_{-1}^{+}, \varepsilon) = 0\ ,\ \gamma_s = 1\ ,\ s = \overline{1, d_1}\ ,$$

as well as for

$$t_{d_1}^{(0)} + \varepsilon\Theta_{d_1}^{(0)} + O(\varepsilon^2) = \tau_{d_1} < t \leq T\ ,\quad s = d_1 + 1\ .$$

Therefore

$$x(T) = x_{d_1+1}^{(0)}(T, \tau_{d_1}, x_{d_1}^{+}) = x_0 + \varepsilon\int_0^t A(\theta, x_0, x_0, 0) X(\theta, x_0) d\theta$$

$$+ \varepsilon\sum_{i=0}^{d_1} I_i^{(0)} + \sum_{i=0}^{d_1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^{+}, \varepsilon) + R_{d_1+1}^{(0)}(T, \tau_{d_1}, x_{d_1}^{+}, \varepsilon)$$

Let $\overline{x}(t)$ be the solution of the averaged system (5) with initial condition (6). Then for $t \geq 0$

$$\overline{x}(t) = x_0 + \varepsilon\int_0^t [A_0(\overline{x}(\theta)) X(\theta, \overline{x}(\theta)) + I_0(\overline{x}(\theta))] d\theta$$

and

$$\overline{x}(T) = x_0 + \varepsilon\int_0^T [A_0(\overline{x}(\theta)) X(\theta, \overline{x}(\theta)) + I_0(\overline{x}(\theta))] d\theta\ .$$

We shall estimate the difference $x(T) - \overline{x}(T)$. For the purpose, taking into account (8), we write down $x(T)$ in the form

(16) $$x(T) = x_0 + \varepsilon I_0(x_0) T + \varepsilon A_0(x_0)\int_0^T X(\theta, x_0) d\theta$$

$$+ \varepsilon\int_0^T [A(\theta, x_0, x_0, 0) - A_0(x_0)] X(\theta, x_0) d\theta$$

$$+ \varepsilon\left[\sum_{i=0}^{d_1} I_i^{(0)} - I_0(x_0) T\right] + \sum_{i=0}^{d_1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^{+}, \varepsilon)$$

$$+ R_{d_1+1}^{(0)}(T, \tau_{d_1}, x_{d_1}^{+}, \varepsilon)\ .$$

For each $x \in D$ we define the operator B_p $(p = 1, 2, \cdots)$ in the following way

$$B_p x = x + \varepsilon I_0(x) T + \varepsilon A_0(x)\int_{(p-1)T}^{pT} X(\theta, x) d\theta\ .$$

From (16), in virtue of (8), the conditions of Theorem 1, the generalized theorem for the mean values in Integral Calculus, and the Cauchy inequality in the discrete case we obtain

(17)

$$\|x(T)-B_1x_0\|\leq\varepsilon\left\|\int_0^T[A(\theta,x_0,x_0,0)-A_0(x_0)]X(\theta,x_0)d\theta\right\|$$

$$+\varepsilon\|\sum_{i=0}^{d_1}I_i^{(0)}-I_0(x_0)T\|+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$\leq\varepsilon\alpha(T)T/2+\varepsilon\|\sum_{i=1}^{d_1}I_i(x_0)-I_0(x_0)T\|+\varepsilon\|\sum_{i=1}^{d_1}(I_i^{(0)}-I_i(x_0))\|$$

$$+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$\leq\varepsilon\alpha(T)T+\varepsilon\sum_{i=1}^{d_1}\|I_i(x_i^{(0)}(\tau_i,\tau_{i-1},x_{i-1}^+)-I_i(x_0)\|+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$\leq\varepsilon\alpha(T)T+\varepsilon K\sum_{i=1}^{d_1}\|x_i^{(0)}(\tau_i,\tau_{i-1},x_{i-1}^+)-x_0\|+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$=\varepsilon\alpha(T)T+\varepsilon K\sum_{i=1}^{d_1}\|x_0+\varepsilon\int_0^{\tau_i}A(\theta,x_0,x_0,0)X(\theta,x_0)d\theta$$

$$+\varepsilon\sum_{l=0}^{i-1}I_l^{(0)}+\sum_{l=0}^{i}R_l^{(0)}(\tau_l,\tau_{l-1},x_{l-1}^+,\varepsilon)-x_0\|+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$\leq\varepsilon\alpha(T)T+\varepsilon^2KM^2Td_1+\varepsilon^2K\sum_{i=1}^{d_1}\sum_{l=0}^{i-1}\|I_l^{(0)}\|$$

$$+\varepsilon K\sum_{i=1}^{d_1}\sum_{l=0}^{i}\|R_l^{(0)}(\tau_l,\tau_{l-1},x_{l-1}^+,\varepsilon)\|+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$\leq\varepsilon\alpha(T)T+\varepsilon^2KMd_1(2MT+d_1-1)/2+\sum_{i=0}^{d_1+1}\omega_i^{(0)}(\varepsilon^2,T)$$

$$+\varepsilon K\sum_{i=1}^{d_1}\sum_{l=0}^{i}\omega_l^{(0)}(\varepsilon^2,T)\leq\varepsilon\alpha(T)T+\varepsilon^2M_1,$$

where $\omega_0^{(0)}(\varepsilon^2,T)\equiv0$, $M_1=M_1(T,d_1)$ is a constant. For $t\geq0,\tau\in[0,T)$ and $x\in D$ we have

$$\|A_0(x)\|=\left\|\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}A(\theta,x,x,0)d\theta\right\|\leq M;$$

$$\|I_0(x)\|\leq\lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}\|I_i(x)\|\leq Md;$$

$$\|\bar{x}(\tau)-x_0\|\leq\varepsilon M(M+d)T;$$

$$\|A_0(\bar{x}(\tau))-A_0(x_0)\|\leq\lim_{T\to\infty}\frac{1}{T}\left\|\int_t^{t+T}[A(\theta,\bar{x}(\tau),\bar{x}(\tau),0)\right.$$

$$\left.-A(\theta,x_0,x_0,0)]d\theta\right\|\leq 2\varepsilon(M+d)KMT;$$

$$\|X(\tau,\bar{x}(\tau))-X(\tau,x_0)\|\leq\varepsilon(M+d)KMT;$$

$$\|I_0(\bar{x}(\tau))-I_0(x_0)\|\leq\lim_{T\to\infty}\frac{1}{T}\|\sum_{t<t_i<t+T}I_i(\bar{x}(\tau))-I_i(x_0)\|$$

$$\leq\varepsilon d(M+d)KMT.$$

Making use of these estimates, we get

$$\|\bar{x}(T)-B_1x_0\|=\left\|x_0+\varepsilon\int_0^T[A_0(\bar{x}(\theta))X(\theta,\bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta\right.\tag{18}$$

$$\left.-x_0-\varepsilon I_0(x_0)T-\varepsilon A_0(x_0)\int_0^T X(\theta,x_0)d\theta\right\|$$

$$\leq\varepsilon\int_0^T\{\|A_0(\bar{x}(\theta))-A_0(x_0)\|\cdot\|X(\theta,\bar{x}(\theta))\|$$

$$+\|A_0(x_0)\|\cdot\|X(\theta,\bar{x}(\theta))-X(\theta,x_0)\|$$

$$+\|I_0(\bar{x}(\theta))-I_0(x_0)\|\}d\theta\leq\varepsilon^2(M+d)(3M+d)KMT^2.$$

(17) and (18) yield the inequality

$$\|x(T)-\bar{x}(T)\|\leq\|x(T)-B_1x_0\|+\|\bar{x}(T)-B_1x_0\|\tag{19}$$

$$\leq\varepsilon\alpha(T)T+\varepsilon^2\bar{M},$$

where $\bar{M}=(M+d)(3M+d)KMT^2+M_1$.

Thus we obtained an estimate for $\|x(T)-\bar{x}(T)\|$ and established the proximity of the points $x(T)$ and $\bar{x}(T)$.

Since $\bar{x}(T)$ belongs to the domain D with its neighbourhood of radius ρ, then (18) and (19) imply that the points B_1x_0 and $x(T)$ also belong to the domain D.

Thus inequality (7) is substantiated for $p=1$.

We introduce the notations

$$\tau_i^{(r-1)}\equiv\tau_{d_0+d_1+\cdots+d_{r-1}+i},$$

$$x_i^{(r-1)+}\equiv x^+_{d_0+d_1+\cdots+d_{r-1}+i},\quad d_0=0,\quad i=\overline{1,d_r}$$

$$\tau_0^{(r-1)}\equiv(r-1)T,\quad\tau_{d_r+1}^{(r-1)}\equiv rT,$$

$$x_0^{(r-1)+} \equiv x((r-1)T), \quad x_{d_r+1}^{(r-1)+} \equiv x(rT), \quad r=1, 2, \cdots .$$

Note that with the notations thus introduced we have

$$\tau_0^{(r-1)} \equiv \tau_{d_{r-1}+1}^{(r-2)} \quad \text{and} \quad x_0^{(r-1)+} \equiv x_{d_{r-1}+1}^{(r-2)+}, \quad r=2, 3, \cdots .$$

Let us assume that for $p=r$, $r \geq 2$ inequality (7) is fulfilled and we have results of the type of (15) and (17)-(19), i.e. we have

$$x(t) = x_s^{(r-1)}(t, \tau_{s-1}^{(r-1)}, x_{s-1}^{(r-1)+}) = \tilde{x}_1^{(r-1)}(t, (r-1)T, x((r-1)T))$$
$$+ \varepsilon \sum_{i=0}^{s-1} I_i^{(r-1)} + \sum_{i=0}^{s-1} R_i^{(r-1)}(\tau_i^{(r-1)}, \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}, \varepsilon)$$
$$+ R_s^{(r-1)}(t, \tau_{s-1}^{(r-1)} x_{s-1}^{(r-1)+}, \varepsilon)$$

for

$$t_{s-1}^{(r-1)} + \varepsilon \Theta_{s-1}^{(r-1)} + \gamma_{s-1} O(\varepsilon^2) = \tau_{s-1}^{(r-1)} < t \leq \tau_s^{(r-1)} = t_s^{(r-1)} + \varepsilon \Theta_s^{(r-1)} + O(\varepsilon^2),$$

where

$$\Theta_s^{(r-1)} = \frac{\partial t_{d_1+\cdots+d_{r-1}+s}(x((r-1)T))}{\partial x}$$
$$\cdot \left[\int_{(r-1)T}^{t_s^{(r-1)}} A(\theta, x((r-1)T), x((r-1)T), 0) \right.$$
$$\left. \cdot X(\theta, x((r-1)T)) d\theta + \sum_{i=0}^{s-1} I_i^{(r-1)} \right],$$

$$t_0^{(r-1)} = rT, \quad \Theta_0^{(r-1)} = \gamma_0 = 0,$$

$$I_0^{(r-1)} = 0, \quad R_0^{(r-1)}(\tau_0^{(r-1)}, \tau_{-1}^{(r-1)}, x_{-1}^{(r-1)+}, \varepsilon) = 0, \quad \gamma_s = 1, \quad s = \overline{1, d_r}$$

as well as for

$$t_{d_r}^{(r-1)} + \varepsilon \Theta_{d_r}^{(r-1)} + O(\varepsilon^2) = \tau_{d_r}^{(r-1)} < t \leq \tau_{d_r+1}^{(r-1)} = rT, \quad s = d_r + 1;$$

$$\|x(rT) - B_r x((r-1)T)\| \leq \varepsilon \alpha(T) T + \varepsilon^2 K M d_r (2MT + d_r - 1)/2$$
$$+ \sum_{i=0}^{d_r+1} \omega_i^{(r-1)}(\varepsilon^2, T) + \varepsilon K \sum_{i=1}^{d_r} \sum_{l=0}^{i} \omega_l^{(r-1)}(\varepsilon^2, T)$$
$$= \varepsilon \alpha(T) T + \varepsilon^2 M_r,$$

where $\omega_0^{(r-1)}(\varepsilon^2, T) \equiv 0$, $M_r = M_r(T, d_1, \cdots, d_r)$ is a constant;

$$\|B_r x((r-1)T) - B_r \bar{x}((r-1)T)\|$$
$$\leq [1 + \varepsilon(3M + d) KT] \sum_{i=0}^{r-2} [1 + \varepsilon(3M + d) KT]^i [\varepsilon \alpha(T) T + \varepsilon^2 \overline{M}],$$

where $\overline{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,(r-1)}} M_i$;

$$\|B_r\overline{x}((r-1)T)-\overline{x}(rT)\|\leq\varepsilon^2(M+d)(3M+d)KMT^2;$$

$$\|x(rT)-\overline{x}(rT)\|\leq\|x(rT)-B_rx((r-1)T)\|$$

$$+\|B_rx((r-1)T)-B_r\overline{x}((r-1)T)\|+\|B_r\overline{x}((r-1)T)-\overline{x}(rT)\|$$

$$\leqq\sum_{i=0}^{r-1}[1+\varepsilon(3M+d)KT]^i[\varepsilon\alpha(T)T+\varepsilon^2\overline{M}],$$

where $\overline{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,r}} M_i$.

Let d_{r+1} points lie on the interval $(rT, (r+1)T)$

$$t_{d_1+\cdots+d_r+1}(\overline{x}(rT)),\cdots,t_{d_1+\cdots+d_r+d_{r+1}}(\overline{x}(rT)),$$

in which case

$$t_{d_1+\cdots+d_r+i}(\overline{x}(rT))<t_{d_1+\cdots+d_r+i+1}(\overline{x}(rT)),\quad i=\overline{1,(d_{r+1}-1)}$$

Then from (7) for $p=r$ and from the continuity of the functions $t_i(x), i=1, 2, \cdots$ it follows that if ε is sufficiently small, d_{r+1} points lie on the interval $(rT, (r+1)T)$

$$t_{d_1+\cdots+d_r+1}(x(rT))=t_1^{(r)},\cdots, \tag{20}$$

$$t_{d_1+\cdots+d_r+d_{r+1}}(x(rT))=t_{d_{r+1}}^{(r)},$$

where

$$t_i^{(r)}<t_{i+1}^{(r)},\quad i=\overline{1,(d_{r+1}-1)}.$$

The conditions of Lemma 1 and (7) for $p=r$ imply that if ε is sufficiently small there exists a constant $\beta_r\in[-\beta,0)$ such that for $i=\overline{1,d_{r+1}}$ the inequality

$$\frac{\partial t_{d_1+\cdots+d_r+i}(x(rT))}{\partial x}I_{d_1+\cdots+d_r+i}(x(rT))\leq\beta_r<0 \tag{21}$$

holds.

We shall prove the validity of (7) for $p=r+1$.

The solution of the system of functional-differential equations (1) with impulses, which we accept to be constructed on the intervals $((p-1)T, pT], p=\overline{1,r}$ will be continued onto the next interval $(rT, (r+1)T]$, denoting for the sake of brevity $x(pT)$ by x_{pT}.

Let $x_1^{(r)}(t, rT, x_{rT})$ be a solution of the system

(22) $x_1^{(r)}(t, rT, x_{rT})$

$$=\begin{cases} x_{rT}+\varepsilon\displaystyle\int_{rT}^{t} A(\theta, x_1^{(r)}(\theta, rT, x_{rT}), x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT})), \\ \quad \dot{x}_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}))X(\theta, x_1^{(r)}(\theta, rT, x_{rT}))d\theta\,, \quad t>rT\,, \\ x_{d_r+1}^{(r-1)}(t, \tau_{d_r}^{(r-1)}, x_{d_r}^{(r-1)+}), \quad -\delta\leq t\leq rT\,, \end{cases}$$

$$\dot{x}_1^{(r)}(t, rT, x_{rT})=\dot{x}_{d_r+1}^{(r-1)}(t, \tau_{d_r}^{(r-1)}, x_{d_r}^{(r-1)+}), \quad -\delta\leq t\leq rT\,,$$

where

$$\Delta_1^{(r)}(t)=\Delta(t, x_1^{(r)}(t, rT, x_{rT}))\,.$$

The solution of (22) coincides with the solution of the system of functional-differential equations (1) with impulses until the moment $\tau_1^{(r)}$ at which the trajectory $(t, x(t))$ meets the hypersurface $\sigma_{d_1+\dots+d_r+1}$, i.e. for

$$t\in[-\delta, \tau_1^{(r)}], \quad x(t)=x_1^{(r)}(t, rT, x_{rT})\,.$$

We consider the function

$$\tilde{x}_1^{(r)}(t, rT, x_{rT})=x_{rT}+\varepsilon\int_{rT}^{t} A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})d\theta\,.$$

For $rT<t\leq(r+1)T$ we have

$$\begin{aligned}
&\|R_1^{(r)}(t, rT, x_{rT}, \varepsilon)\|=\|x_1^{(r)}(t, rT, x_{rT})-\tilde{x}_1^{(r)}(t, rT, x_{rT})\| \\
&\quad\leq\varepsilon\int_{rT}^{t}\|A(\theta, x_1^{(r)}(\theta, rT, x_{rT}), x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}), \\
&\qquad\qquad \dot{x}_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}))X(\theta, x_1^{(r)}(\theta, rT, x_{rT})) \\
&\qquad -A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})\|d\theta \\
&\quad\leq\varepsilon\int_{rT}^{t}\{\|A(\theta, x_1^{(r)}(\theta, rT, x_{rT}), x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}), \\
&\qquad\qquad \dot{x}_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}))-A(\theta, x_{rT}, x_{rT}, 0)\| \\
&\qquad \cdot\|X(\theta, x_1^{(r)}(\theta, rT, x_{rT}))\|+\|A(\theta, x_{rT}, x_{rT}, 0)\| \\
&\qquad \cdot\|X(\theta, x_1^{(r)}(\theta, rT, x_{rT}))-X(\theta, x_{rT})\|\}d\theta \\
&\quad\leq\varepsilon KM\int_{rT}^{t}\{2\|x_1^{(r)}(\theta, rT, x_{rT})-x_{rT}\|+\|x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT})
\end{aligned}$$

$$
\begin{aligned}
&-x_{rT}\| + \|\dot{x}_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT})\|\} d\theta \\
&\leq 2\varepsilon^2 KM \int_{rT}^{t} d\theta \int_{rT}^{\theta} \|A(l, x_1^{(r)}(l, rT, x_{rT}), x_1^{(r)}(\Delta_1^{(r)}(l), rT, x_{rT}) \\
&\qquad \dot{x}_1^{(r)}(\Delta_1^{(r)}(l), rT, x_{rT}))\| \cdot \|X(l, x_1^{(r)}(l, rT, x_{rT}))\| dl \\
&+ \varepsilon KM \Big\{ \sum_{i=1}^{d_r+1} \int_{J_{rT,t}^i} [\|x_i^{(r-1)}(\Delta_1^{(r)}(\theta), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}) - x_{(r-1)T}\| \\
&+ \|x_{rT} - x_{(r-1)T}\|] d\theta + \varepsilon \int_{\tilde{J}_{rT,t}^+} d\theta \int_{rT}^{\Delta_1^{(r)}(\theta)} \|A(l, x_1^{(r)}(l, rT, x_{rT}), \\
&\qquad x_1^{(r)}(\Delta_1^{(r)}(l), rT, x_{rT}), \dot{x}_1^{(r)}(\Delta_1^{(r)}(l), rT, x_{rT}))\| \\
&\cdot \|X(l, x_1^{(r)}(l, rT, x_{rT}))\| dl \Big\} \\
&+ \varepsilon KM \Big\{ \sum_{i=1}^{d_r+1} \int_{J_{rT,t}^i} \|\dot{x}_i^{(r-1)}(\Delta_1^{(r)}(\theta), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+})\| d\theta \\
&+ \int_{\tilde{J}_{rT,t}^+} \|\dot{x}_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT})\| d\theta \Big\} \leq 2\varepsilon^2 KM^3 \int_{rT}^{t} d\theta \int_{rT}^{\theta} dl \\
&+ \varepsilon^2 KM \sum_{i=1}^{d_r+1} \int_{J_{rT,t}^i} \Big\{ \int_{(r-1)T}^{\tau_{i-1}^{(r-1)}} \|A(l, x_{(r-1)T}, x_{(r-1)T}, 0)\| \cdot \|X(l, x_{(r-1)T})\| dl \\
&+ \sum_{j=0}^{i-1} \|I_j^{(r-1)}\| + \varepsilon^{-1} \sum_{j=1}^{i-1} \|R_j^{(r-1)}(\tau_j^{(r-1)}, \tau_{j-1}^{(r-1)}, x_{j-1}^{(r-1)+}, \varepsilon)\| \\
&+ \int_{\tau_{i-1}^{(r-1)}}^{\Delta_1^{(r)}(\theta)} \|A(l, x_i^{(r-1)}(l, \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}), x_i^{(r-1)}(\Delta_i^{(r-1)}(l), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}), \\
&\qquad \dot{x}_i^{(r-1)}(\Delta_i^{(r-1)}(l), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}))\| \cdot \|X(l, x_i^{(r-1)}(l, \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}))\| dl \Big\} d\theta \\
&+ \varepsilon KMT [\|x_{rT} - B_r x_{(r-1)T}\| + \|B_r x_{(r-1)T} - x_{(r-1)T}\|] \\
&+ \varepsilon^2 KM^3 \int_{\tilde{J}_{rT,t}^+} (\Delta_1^{(r)}(\theta) - rT) d\theta + \varepsilon^2 KM \Big\{ \sum_{i=1}^{d_r+1} \int_{J_{rT,t}^i} \|A(\Delta_1^{(r)}(\theta), \\
&\qquad x_i^{(r-1)}(\Delta_1^{(r)}(\theta), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}), x_i^{(r-1)}(\Delta_i^{(r-1)}(\Delta_1^{(r)}(\theta)), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}), \\
&\qquad \dot{x}_i^{(r-1)}(\Delta_i^{(r-1)}(\Delta_1^{(r)}(\theta)), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}))\| \cdot \|X(\Delta_1^{(r)}(\theta), \\
&x_i^{(r-1)}(\Delta_1^{(r)}(\theta), \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)+}))\| d\theta + \int_{\tilde{J}_{rT,t}^+} \|A(\Delta_1^{(r)}(\theta), x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}), \\
&\qquad x_1^{(r)}(\Delta_1^{(r)}(\Delta_1^{(r)}(\theta)), rT, x_{rT}), \dot{x}_1^{(r)}(\Delta_1^{(r)}(\Delta_1^{(r)}(\theta)), rT, x_{rT}))\| \\
&\cdot \|X(\Delta_1^{(r)}(\theta), x_1^{(r)}(\Delta_1^{(r)}(\theta), rT, x_{rT}))\| d\theta \Big\} \leq \varepsilon^2 KM^3 T^2
\end{aligned}
$$

$$+\varepsilon^2 KM^2 \sum_{i=1}^{d_r+1} \int_{J^i_{rT,t}} \{M[\varDelta_1^{(r)}(\theta)-(r-1)T]+(i-1)\}d\theta$$

$$+\varepsilon KM \sum_{i=1}^{d_r+1}\Big(\sum_{j=0}^{i-1}\omega_j^{(r-1)}(\varepsilon^2, T)\int_{J^i_{rT,t}} d\theta\Big)+\varepsilon^2 KMT[\alpha(T)T+\varepsilon M_r$$

$$+M(M+d)T]+\varepsilon^2 KM^3\int_{rT}^{t}(\theta-rT)d\theta+\varepsilon^2 KM^3\Big(\sum_{i=1}^{d_r+1}\int_{J^i_{rT,t}} d\theta$$

$$+\int_{\tilde{J}^+_{rT,t}} d\theta\Big)\leq \varepsilon^2 KM^3T^2+\varepsilon^2 KM^2\int_{rT}^{t}\{M[\theta-(r-1)T]+d_r\}d\theta$$

$$+\varepsilon KMT\sum_{j=0}^{d_r}\omega_j^{(r-1)}(\varepsilon^2, T)+\varepsilon^2 KMT[\alpha(T)T+\varepsilon M_r$$

$$+M(M+d)T]+\varepsilon^2 KM^3T^2/2+\varepsilon^2 KM^3T\leq 3\varepsilon^2 KM^3T^2/2$$

$$+\varepsilon^2 KM^2T(2MT+d_r)+\varepsilon^2 KM^2T^2(M+d)$$

$$+\varepsilon^2 KMT[\alpha(T)T+\varepsilon M_r]+\varepsilon^2 KM^3T$$

$$+\varepsilon KMT\sum_{i=0}^{d_r}\omega_i^{(r-1)}(\varepsilon^2, T)\equiv\omega_1^{(r)}(\varepsilon^2, T),$$

where

$$\Big(\bigcup_{i=1}^{d_r+1} J^i_{rT,t}\Big)\cup\tilde{J}^+_{rT,t}=(rT, t],$$

$$J^i_{rT,t}=\{\theta:\theta\in(rT, t]\wedge\varDelta_1^{(r)}(\theta)\in(\tau_{i-1}^{(r-1)}, \tau_i^{(r-1)}]\},\quad i=\overline{1, (d_r+1)},$$

$$\tilde{J}^+_{rT,t}=(rT, t]\setminus\Big(\bigcup_{i=1}^{d_r+1} J^i_{rT,t}\Big),$$

$$(\tau_i^{(0)}\equiv\tau_i,\quad i=\overline{0, d_1},\quad \tau_{d_1+1}^{(0)}\equiv T).$$

Therefore, the function $\tilde{x}_1^{(r)}(t, rT, x_{rT})$ approximates the solution of (22) on the interval $(rT, (r+1)T]$ to a precision of order ε^2.

For the root $\tau_1^{(r)}$ of the equation

$$t=t_{d_1+\cdots+d_r+1}(x_1^{(r)}(t, rT, x_{rT}))$$

we obtain

$$\tau_1^{(r)}=t_1^{(r)}+\varepsilon\Theta_1^{(r)}+O(\varepsilon^2), \tag{23}$$

where

$$\Theta_1^{(r)}=\frac{\partial t_{d_1+\cdots+d_r+1}(x_{rT})}{\partial x}\int_{rT}^{t_1^{(r)}} A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})d\theta\,.$$

(20) and (23) imply that for ε sufficiently small the inequality $\tau_1^{(r)}>rT$ holds.

Thus

$$x(t)=x_1^{(r)}(t, rT, x_{rT})=\tilde{x}_1^{(r)}(t, rT, x_{rT})+R_1^{(r)}(t, rT, x_{rT}, \varepsilon)$$

for $rT<t\leq\tau_1^{(r)}$.

Further on we obtain

$$\begin{aligned} x_1^{(r)+} &= x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT})+\varepsilon I_{d_1+\cdots+d_r+1}(x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT})) \\ &= \tilde{x}_1^{(r)}(\tau_1^{(r)}, rT, x_{rT})+\varepsilon I_1^{(r)}+R_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}, \varepsilon) \\ &= x_{rT}+\varepsilon\int_{rT}^{\tau_1^{(r)}} A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})d\theta \\ &\quad +\varepsilon I_1^{(r)}+R_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}, \varepsilon), \end{aligned}$$

where $I_1^{(r)}\equiv I_{d_1+\cdots+d_r+1}(x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}))$

In the general case $s=\overline{2, (d_{r+1}+1)}$ we denote by $x_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+})$ the solution of the system

$$(24)\quad x_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}) = \begin{cases} x_{s-1}^{(r)+}+\varepsilon\displaystyle\int_{\tau_{s-1}^{(r)}}^{t} A(\theta, x_s^{(r)}(\theta, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}), \\ \qquad x_s^{(r)}(\varDelta_s^{(r)}(\theta), \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}), \dot{x}_s^{(r)}(\varDelta_s^{(r)}(\theta), \tau_{s-1}^{(r)}, x_{s-1}^{(r)+})) \\ \cdot X(\theta, x_s^{(r)}(\theta, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}))d\theta, \quad t>\tau_{s-1}^{(r)}, \\ x_{s-1}^{(r)}(t, \tau_{s-2}^{(r)}, x_{s-2}^{(r)+}), \quad -\delta\leq t\leq\tau_{s-1}^{(r)}, \end{cases}$$

$$\dot{x}_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+})=\dot{x}_{s-1}^{(r)}(t, \tau_{s-2}^{(r)}, x_{s-2}^{(r)+}), \quad -\delta\leq t\leq\tau_{s-1}^{(r)},$$

where

$$\varDelta_s^{(r)}(t)=\varDelta(t, x_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}))$$

and

$$\begin{aligned} x_{s-1}^{(r)+} &= x_{s-1}^{(r)}(\tau_{s-1}^{(r)}, \tau_{s-2}^{(r)}, x_{s-2}^{(r)+})+\varepsilon I_{d_1+\cdots+d_r+s-1}(x_{s-1}^{(r)}(\tau_{s-1}^{(r)}, \tau_{s-2}^{(r)}, x_{s-2}^{(r)+})) \\ &= x_{rT}+\varepsilon\int_{rT}^{\tau_{s-1}^{(r)}} A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})d\theta \\ &\quad +\varepsilon\sum_{i=1}^{s-1} I_i^{(r)}+\sum_{i=1}^{s-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon), \end{aligned}$$

$$I_{s-1}^{(r)}\equiv I_{d_1+\cdots+d_r+s-1}(x_{s-1}^{(r)}(\tau_{s-1}^{(r)}, \tau_{s-2}^{(r)}, x_{s-2}^{(r)+})).$$

The solution of (24) coincides with the solution of the system of functional-differential equations (1) with impulses on the interval $[-\delta, \tau_s^{(r)}]$.

We consider the function

$$\tilde{x}_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}) = x_{s-1}^{(r)+} + \varepsilon \int_{\tau_{s-1}^{(r)}}^{t} A(\theta, x_{rT}, x_{rT}, 0) X(\theta, x_{rT}) d\theta .$$

It can be shown that on the interval $rT < \tau_{s-1}^{(r)} < t \leq (r+1)T$ the following estimate holds

$$\begin{aligned}(25)\quad & \|R_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}, \varepsilon)\| \\ & = \|x_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}) - \tilde{x}_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+})\| \\ & \leq 3\varepsilon^2 KM[MT + (s-1)]^2/2 + \varepsilon^2 KM^2 T(2MT + d_r) \\ & \quad + \varepsilon^2 KM^2 T^2 (M+d) + \varepsilon^2 KMT[\alpha(T)T + \varepsilon M_r] \\ & \quad + \varepsilon^2 KM^3 T + \varepsilon KMT \sum_{i=0}^{d_r} \omega_i^{(r-1)}(\varepsilon^2, T) \\ & \quad + 3\varepsilon KMT \sum_{i=1}^{s-1} \omega_i^{(r)}(\varepsilon^2, T) \equiv \omega_s^{(r)}(\varepsilon^2, T).\end{aligned}$$

Since

$$\begin{aligned}\tilde{x}_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}) &= x_{rT} + \varepsilon \int_{rT}^{t} A(\theta, x_{rT}, x_{rT}, 0) X(\theta, x_{rT}) d\theta \\ & \quad + \varepsilon \sum_{i=1}^{s-1} I_i^{(r)} + \sum_{i=1}^{s-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon) \\ &= \tilde{x}_1^{(r)}(t, rT, x_{rT}) + \varepsilon \sum_{i=1}^{s-1} I_i^{(r)} + \sum_{i=1}^{s-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)}, \varepsilon),\end{aligned}$$

then we obtain

$$\begin{aligned}(26)\quad x(t) &= x_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}) = \tilde{x}_1^{(r)}(t, rT, x_{rT}) \\ & \quad + \varepsilon \sum_{i=0}^{s-1} I_i^{(r)} + \sum_{i=0}^{s-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon) + R_s^{(r)}(t, \tau_{s-1}^{(r)}, x_{s-1}^{(r)+}, \varepsilon)\end{aligned}$$

for

$$t_{s-1}^{(r)} + \varepsilon \Theta_{s-1}^{(r)} + \gamma_{s-1} O(\varepsilon^2) = \tau_{s-1}^{(r)} < t \leq \tau_s^{(r)} + \varepsilon \Theta_s^{(r)} + O(\varepsilon^2),$$

where

$$\Theta_s^{(r)} = \frac{\partial t_{d_1 + \cdots + d_r + s}(x_{rT})}{\partial x} \left[\int_{rT}^{t_s^{(r)}} A(\theta, x_{rT}, x_{rT}, 0) X(\theta, x_{rT}) d\theta \right.$$

$$+\sum_{i=0}^{s-1} I_i^{(r)}\Big], \quad t_0^{(r)}=rT, \quad \Theta_0^{(r)}=\gamma_0=0, \quad I_0^{(r)}=0,$$

$$R_0^{(r)}(\tau_0^{(r)}, \tau_{-1}^{(r)}, x_{-1}^{(r)+}, \varepsilon)=0, \quad \gamma_s=1, \quad s=\overline{1, d_{r+1}},$$

as well as for

$$t_{d_{r+1}}^{(r)}+\varepsilon\Theta_{d_{r+1}}^{(r)}+O(\varepsilon^2)=\tau_{d_{r+1}}^{(r)}<t\leq\tau_{d_{r+1}+1}^{(r)}=(r+1)T,$$

$$s=d_{r+1}+1.$$

We work out $x((r+1)T)$ and $\bar{x}((r+1)T)$

$$x((r+1)T)=x_{d_{r+1}+1}^{(r)}((r+1)T, \tau_{d_{r+1}}^{(r)}, x_{d_{r+1}}^{(r)+})$$

$$=x_{rT}+\varepsilon\int_{rT}^{(r+1)T} A(\theta, x_{rT}, x_{rT}, 0)X(\theta, x_{rT})d\theta+\varepsilon\sum_{i=0}^{d_{r+1}} I_i^{(r)}$$

$$+\sum_{i=0}^{d_{r+1}+1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon)=x_{rT}+\varepsilon I_0(x_{rT})T$$

$$+\varepsilon A_0(x_{rT})\int_{rT}^{(r+1)T} X(\theta, x_{rT})d\theta$$

$$+\varepsilon\int_{rT}^{(r+1)T}[A(\theta, x_{rT}, x_{rT}, 0)-A_0(x_{rT})]X(\theta, x_{rT})d\theta$$

$$+\varepsilon\Big[\sum_{i=0}^{d_{r+1}} I_i^{(r)}-I_0(x_{rT})T\Big]+\sum_{i=0}^{d_{r+1}+1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon),$$

$$\bar{x}((r+1)T)=x_0+\varepsilon\int_0^{(r+1)T}[A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta$$

$$=\bar{x}(rT)+\varepsilon\int_{rT}^{(r+1)T}[A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta.$$

Taking into consideration the definition of the operator B_{r+1} and the results for $x((r+1)T)$ and $\bar{x}((r+1)T)$ we can write

$$\|x((r+1)T)-\bar{x}((r+1)T)\|\leq\|x((r+1)T)-B_{r+1}x_{rT}\| \tag{27}$$

$$+\|B_{r+1}x_{rT}-B_{r+1}\bar{x}(rT)\|+\|B_{r+1}\bar{x}(rT)-\bar{x}((r+1)T)\|.$$

Dealing in a similar way as in (17), for the first addend on the right-hand side of (27) we get

$$\|x((r+1)T)-B_{r+1}x_{rT}\| \tag{28}$$

$$\leq\varepsilon\alpha(T)T+\varepsilon^2KMd_{r+1}(2MT+d_{r+1}-1)/2$$

$$+\sum_{i=0}^{d_{r+1}+1}\omega_i^{(r)}(\varepsilon^2, T)+\varepsilon K\sum_{i=1}^{d_{r+1}}\sum_{l=0}^{i}\omega_l^{(r)}(\varepsilon^2, T)$$

$$=\varepsilon\alpha(T)T+\varepsilon^2 M_{r+1},$$

where $\omega_0^{(r)}(\varepsilon^2, T)\equiv 0$, $M_{r+1}=M_{r+1}(T, d_1, \cdots, d_{r+1})$ is a constant.

For the second addend on the right-hand side of (27) we have

$$(29)\quad \|B_{r+1}x_{rT}-B_{r+1}\bar{x}(rT)\|=\|x_{rT}+\varepsilon I_0(x_{rT})T$$
$$+\varepsilon A_0(x_{rT})\int_{rT}^{(r+1)T}X(\theta, x_{rT})d\theta-\bar{x}(rT)-\varepsilon I_0(\bar{x}(rT))T$$
$$+\varepsilon A_0(\bar{x}(rT))\int_{rT}^{(r+1)T}X(\theta, \bar{x}(rT))d\theta\|\leq\|x_{rT}-\bar{x}(rT)\|$$
$$+\varepsilon T\|I_0(x_{rT})-I_0(\bar{x}(rT))\|+\varepsilon\|A_0(x_{rT})-A_0(\bar{x}(rT))\|.$$
$$\int_{rT}^{(r+1)T}\|X(\theta, x_{rT})\|d\theta$$
$$+\varepsilon\|A_0(\bar{x}(rT))\|\int_{rT}^{(r+1)T}\|X(\theta, x_{rT})-X(\theta, \bar{x}(rT))\|d\theta$$
$$\leq[1+\varepsilon(3M+d)KT]\|x_{rT}-\bar{x}(rT)\|$$
$$\leq[1+\varepsilon(3M+d)KT]\sum_{i=0}^{r-1}[1+\varepsilon(3M+d)KT]^i$$
$$\cdot[\varepsilon\alpha(T)T+\varepsilon^2\bar{M}].$$

where $\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,r}} M_i$

Since for $t\in(rT, (r+1)T]$ the inequality

$$\|\bar{x}(t)-\bar{x}(rT)\|\leq\varepsilon\int_{rT}^{t}[\|A_0(\bar{x}(\theta))\|\cdot\|X(\theta, \bar{x}(\theta))\|$$
$$+\|I_0(\bar{x}(\theta))\|]d\theta\leq\varepsilon(M+d)MT,$$

holds, then for the third addend on the right-hand side of (27) we obtain

$$(30)\quad \|B_{r+1}\bar{x}(rT)-\bar{x}((r+1)T)\|=\Big\|\bar{x}(rT)+\varepsilon I_0(\bar{x}(rT))T$$
$$+\varepsilon A_0(\bar{x}(rT))\int_{rT}^{(r+1)T}X(\theta, \bar{x}(rT))d\theta-\bar{x}(rT)$$
$$-\varepsilon\int_{rT}^{(r+1)T}[A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta\Big\|$$
$$\leq\varepsilon\|A_0(\bar{x}(rT))\|\int_{rT}^{(r+1)T}\|X(\theta, \bar{x}(rT))-X(\theta, \bar{x}(\theta))\|d\theta$$
$$+\varepsilon\int_{rT}^{(r+1)T}\|A_0(\bar{x}(rT))-A_0(\bar{x}(\theta))\|\cdot\|X(\theta, \bar{x}(\theta))\|d\theta$$

$$+\varepsilon\int_{rT}^{(r+1)T}\|I_0(\bar{x}(rT))-I_0(\bar{x}(\theta))\|d\theta$$

$$\leq\varepsilon(3M+d)K\int_{rT}^{(r+1)T}\|\bar{x}(rT)-\bar{x}(\theta)\|d\theta$$

$$\leq\varepsilon^2(M+d)(3M+d)KMT^2.$$

(27) – (30) imply the inequality

$$\|x((r+1)T)-\bar{x}((r+1)T)\|$$

$$\leq\sum_{i=0}^{r}[1+\varepsilon(3M+d)KT]^i\cdot[\varepsilon\alpha(T)T+\varepsilon^2\bar{M}],$$

where $\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,(r+1)}}M_i$

The last inequality shows that (7) is fulfilled for $p=r+1$ and that $x((r+1)T)$ belongs to the domain D.

Thus Lemma 1 is proved.

Proof of Theorem 1. By virtue of the condition 3° of Theorem there exists a constant $C(T)<\infty$ such that for each $i=1,2,\cdots$ the inequality $d_i<C(T)$ holds. Hence, there also exists a constant $M_0(T)$ $<\infty$ such that

$$\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=1,2,\cdots}M_i\leq M_0(T). \tag{31}$$

Let q be equal to the whole part of the number $L/\varepsilon T$. Then for each $p\in\overline{1,q}$, by virtue of (31) and Lemma 1, we have

$$\|x(pT)-\bar{x}(pT)\|$$

$$\leq\varepsilon\sum_{i=0}^{p-1}[1+\varepsilon(3M+d)KT]^i[\alpha(T)T+\varepsilon M_0(T)]$$

$$\leq[\alpha(T)T+\varepsilon M_0(T)][1+\varepsilon(3M+d)KT]^p/(3M+d)KT$$

$$\leq[e^{(3M+d)KL}+O(\varepsilon)][\alpha(T)T+\varepsilon M_0(T)]/(3M+d)KT.$$

We choose T sufficiently large, so that

$$e^{(3M+d)KL}\alpha(T)/(3M+d)K<\eta/4$$

and then we choose ε sufficiently small, so that

$$\{O(\varepsilon)[\alpha(T)T+\varepsilon M_0(T)]+\varepsilon e^{(3M+d)KL}M_0(T)\}/(3M+d)KT<\eta/4.$$

Then for each $p \in \overline{1, q}$ the following inequality will hold

$$\|x(pT) - \bar{x}(pT)\| < \eta/2 . \tag{32}$$

Further on we estimate $\|\bar{x}(t) - \bar{x}((p-1)T)\|$ and $\|x(t) - x((p-1)T)\|$ on the interval $(p-1)T \leq t \leq pT$.

We have

$$\begin{aligned} &\|\bar{x}(t) - \bar{x}((p-1)T)\| \\ &\leq \varepsilon \int_{(p-1)T}^{t} \|A_0(\bar{x}(\theta)) X(\theta, \bar{x}(\theta)) + I_0(\bar{x}(\theta))\| d\theta \\ &\leq \varepsilon (M+d) MT , \end{aligned} \tag{33}$$

$$\begin{aligned} &\|x(t) - x((p-1)T)\| = \|x_s^{(p-1)}(t, \tau_{s-1}^{(p-1)}, x_{s-1}^{(p-1)+}) - x((p-1)T)\| \\ &= \|\tilde{x}_1^{(p-1)}(t, (p-1)T, x_{(p-1)T}) + \varepsilon \sum_{i=0}^{s-1} I_i^{(p-1)} \\ &\quad + \sum_{i=0}^{s-1} R_i^{(p-1)}(\tau_i^{(p-1)}, \tau_{i-1}^{(p-1)}, x_{i-1}^{(p-1)+}, \varepsilon) + R_s^{(p-1)}(t, \tau_{s-1}^{(p-1)}, x_{s-1}^{(p-1)+}, \varepsilon) \\ &\quad - x((p-1)T)\| \\ &\leq \varepsilon \int_{(p-1)T}^{t} \|A(\theta, x_{(p-1)T}, x_{(p-1)T}, 0)\| \|X(\theta, x_{(p-1)T})\| d\theta \\ &\quad + \varepsilon \sum_{i=0}^{s-1} \|I_i^{(p-1)}\| + \sum_{i=0}^{s-1} \|R_i^{(p-1)}(\tau_i, \tau_{i-1}^{(p-1)}, x_{i-1}^{(p-1)+}, \varepsilon)\| \\ &\quad + \|R_s^{(p-1)}(t, \tau_{s-1}^{(p-1)}, x_{s-1}^{(p-1)+}, \varepsilon)\| \leq \varepsilon M[MT + (s-1)] \\ &\quad + \sum_{i=0}^{s} \omega_i^{(p-1)}(\varepsilon^2, T) \leq \varepsilon M[MT + C(T)] + \varepsilon^2 M_0(T) \equiv \Psi(\varepsilon, T) \end{aligned} \tag{34}$$

We see that for T chosen as it was, if ε is sufficiently small, we shall have

$$\Psi(\varepsilon, T) < \eta/2 . \tag{35}$$

It follows from (32)-(35) that for T chosen as it was, if ε is sufficiently small, for $p = 1, 2, \cdots, q$ on the interval $(p-1)T \leq t \leq pT$ the following inequality will hold

$$\begin{aligned} &\|x(t) - \bar{x}(t)\| \leq \|x(t) - x((p-1)T)\| \\ &\quad + \|x((p-1)T) - \bar{x}((p-1)T)\| + \|\bar{x}((p-1)T) - \bar{x}(t)\| < \eta . \end{aligned}$$

Therefore, for T chosen as it was, if ε is sufficiently small $(0<\varepsilon \leq \varepsilon_0 \leq \mathscr{E})$, the inequality $\|x(t)-\bar{x}(t)\|<\eta$ will hold on the whole interval $0\leq t\leq L\varepsilon^{-1}$.

Thus Theorem 1 is proved.

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