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On the Completion of the G-Equivariant Unitary Cobordism Rings of G-Spaces

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday.

By

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§ 1. Introduction

Let G be a compact abelian Lie group. We treat G-equivariant unitary cobordism theories $U_G^*(X)$ and $U^*((EG\times X)/G)$. Denote by I_G the kernel of the forgetful homomorphism

$$\psi\colon U_G^*\to U^*$$

and by I the kernel of the augmentation

$$\varepsilon\colon U^*(BG)\to U^*.$$

A natural transformation introduced by tom-Dieck [4], [5], [6], [8]

$$\alpha\colon U_G^*(X)\to U^*((EG\times X)/G)$$

of multiplicative equivariant cohomology theories, which preserves Thom classes, derives a homomorphism

$$\hat{\alpha}:\widehat{U_G^*(X)}\to\widehat{U^*((EG\times X)/G)}$$

between the I_G-adic completion $\widehat{U_G^*(X)}$ and the I-adic completion $\widehat{U^*((EG\times X)/G)}$. When X is a point, it is shown by Löffler [14] that $\hat{\alpha}$ is isomorphic. On the other hand, the G-equivariant unitary cobordism is related to K_G-theory, [1], [17], by a natural transformation

$$\mu_G\colon U_G^*(X)\to K_G^*(X)$$

(cf. [2]). Taking up a multiplicative system T_K consisting of all one dimensional representations in the representation ring $R(G)\cong K_G(\mathrm{pt})$, the

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localization $T^{-1}U_G^*(X)$ by $T=\mu_G^{-1}(T_K)$ is also related to $K_G^*(X)$ by μ_G. In the present paper the authors determine a simple system of U_G^*-algebra $U_G^*(P(V))$, $P(V)$ the complex projective space in a complex G-module V, and observe the relation of $T^{-1}U_G^*(X)$ to $K_G^*(X)$ (cf. [15]). Furthermore we observe a natural transformation from $T^{-1}U_G^*(X)$ to $U^*((EG\times X)/G)$ for $G=S^1$ or Z_p, and obtain that if $G=S^1$ or Z_p, $U^*(X^G)$ is a free U^*-module and $T^{-1}U_G^*(X)$ is a free $T^{-1}U_G^*$-module, then $\hat{\alpha}\colon \widehat{U_G^*(X)}\to\widehat{U^*((EG\times X)/G)}$ is isomorphic.

§ 2. On a Simple System of the h_G^*-Algebra $h_G^*(P(V))$

Let h_G be a multiplicative G-equivariant cohomology theory equipped with the suspension isomorphism $\sigma_V\colon \tilde{h}_G^*(X)\to\tilde{h}_G^{*+|V|}(V^c\wedge X)$, $|V|=2\dim_C V$, for any complex G-module V. We assume that for any complex G-vector bundle $\xi\colon E\xrightarrow{\pi}X$ over a compact G-space X there exists a Thom class $t(\xi)$ in $\tilde{h}_G^{|\xi|}(T(\xi))$, where $T(\xi)$ denotes the Thom complex of ξ and $|\xi|=2\dim_C\xi$. Thom classes are defined as classes with the following properties:

(1) (naturality) $t(f^!\xi)=f^*t(\xi)$

(2) (multiplicativity) $t(\xi\times\eta)=t(\xi)\wedge t(\eta)$

(3) (normality) $t(\underline{\underline{V}})=\sigma_V(1)$ where $\underline{\underline{V}}\colon V\to\{a\ point\}$.

Then we obtain the Thom isomorphism for a complex G-vector bundle $\xi\colon E\xrightarrow{\pi}X$ over a compact G-space X:

$$\Phi: h_G^*(X)\to\tilde{h}_G^{*+|\xi|}(T(\xi))$$

which is defined by $\Phi(x)=\hat{d}^*(x\wedge t(\xi))$, where $\hat{d}$ is the map induced from a map $d\colon E\to X\times E$, $e\mapsto(\pi(e), e)$. The Euler class $e(\xi)$ of ξ is defined by

$$e(\xi)=s^*t(\xi)$$

where $s\colon X^+\to T(\xi)$ is the zero section.

The complex projective space $P(V)$ for a complex G-module V is the quotient space of the unit sphere $S(V)$ in V under the identification $v\equiv\lambda v$, $\lambda\in S^1$. Let $\rho_V\colon G\to U(n)$, $n=\dim V$, be the unitary representation corresponding to the complex G-module V. A G-action on $P(V)$ is given by letting φ take $[v]$ to $[\rho_V(\varphi)v]$. The fixed point set $P(V)^G$ is a disjoint union of complex projective spaces. Taking a complex G-module W, let $E(V;\ W)$ be a quotient space of $S(V)\times W$ under the equivalence relation which relates (v, w) to $(\lambda v, \lambda^{-1}w)$ for all $v\in S(V)$, $w\in W$ and $\lambda\in S^1$, which has a G-action given by $\varphi[v, w]=[\rho_V(\varphi)v, \rho_W(\varphi)w]$. We then have an equivariant complex G-vector bundle

$$\pi:\ E(V:W)\rightarrow P(V)$$

given by $\pi[v, w]=[v]$ which is denoted by $\eta(V;\ W)$. For complex G-modules W_1 and W_2 with the representations ρ_{W_1}: $G\rightarrow U(n_1)$ and ρ_{W_2}: $G\rightarrow U(n_2)$ respectively, one has the complex G-modules $\overline{W}_1$ for the representation given by $\rho_{\overline{W}_1}(\varphi)=\overline{\rho_{W_1}(\varphi)}$ and $W_1\otimes W_2$ for the representation given by $\rho_{W_1\otimes W_2}(\varphi)=\rho_{W_1}(\varphi)\otimes\rho_{W_2}(\varphi)$. The proof of the following proposition is clear.

Proposition 2.1. *If L_1 and L_2 are one dimensional complex G-modules, then $\eta(V;\ L_1)$ is isomorphic to $\eta(V\otimes L_2;\ L_1\otimes\overline{L}_2)$.*

The Thom complex $T(\xi)$ is a quotient space $D(\xi)/S(\xi)$ of the disk bundle $D(\xi)$ collapsing the sphere bundle $S(\xi)$. We obtain the following basic result which plays an important role in the computation of $U_G^*(P(V))$.

Proposition 2.2. (1) *The map ϕ: $P(W\oplus V)/P(W)\rightarrow T(\eta(V;\ \overline{W}))$ defined by*

$$\phi([w, v])=\begin{cases}\left[\dfrac{1}{\|v\|}v,\ \dfrac{1}{\|v\|}\overline{w}\right] & \textit{for}\quad v\neq 0\\ \textit{the base point} & \textit{for}\quad v=0\end{cases}$$

is a G-homeomorphism.

(2) *Suppose that L is a one dimensional complex G-module. Then $P(L\oplus V)$ is G-homeomorphic to $T(\eta(V;\overline{L}))$.*

We consider the injection i: $T(\eta(V;\ \overline{L}))\rightarrow T(\eta(L\oplus V;\ \overline{L}))$ induced from the bundle map $\eta(V;\ \overline{L})\rightarrow\eta(L\oplus V;\ \overline{L})$ taking $[v, z]$ to $[0, v, z]$, and the map j: $P(L\oplus V)^+\rightarrow T(\eta(V;\ \overline{L}))$ induced from the G-homeomorphism of Proposition 2. 2 (2). We give the following relation among i, s and j.

Proposition 2. 3. *The following diagram is commutative up to G-homotopy:*

$$\begin{array}{ccc} & \overset{i}{\nearrow} & T(\eta(L\oplus V;\bar{L})) \\ & & \uparrow s \\ T(\eta(V;\bar{L})) & \overset{j}{\longleftarrow} & P(L\oplus V)^{+} \end{array} .$$

Proof. The homotopy H: $P(L\oplus V)^{+}\times I\rightarrow T(\eta(L\oplus V;\ \bar{L}))$ combining s and $i\circ j$ is given by

$$H([z,v],t)=\begin{cases}\left[\dfrac{t}{\|(tz,v)\|}z,\ \dfrac{1}{\|(tz,v)\|}v,\ \dfrac{1-t}{\|(tz,v)\|}\bar{z}\right] & \text{if } (v,t)\neq(0,0)\\ \text{the base point} & \text{if } (v,t)=(0,0)\end{cases}$$

and

H (*the base point*, t) $=$ *the base point*. Q.E.D.

The injection i of Proposition 2. 3 induces the homomorphism i^{*}: $\tilde{h}_G^{*}(T(\eta(L\oplus V;\ \bar{L}))\rightarrow\tilde{h}_G^{*}(T(\eta(V;\ \bar{L}))$ which takes $t(\eta(L\oplus V;\ \bar{L}))$ to $t(\eta(V;\ \bar{L}))$. For the map $\underline{i}$: $P(V)\rightarrow P(L\oplus V)$ given by $\underline{i}([v])=[0,v]$, we have the Gysin homomorphism (cf. [10])

$$\underline{i}_{!}:h_G^{*}(P(V))\longrightarrow h_G^{*}(P(L\oplus V))$$

defined by the following composition

$$\underline{i}_{!}:h_G^{*}(P(V))\xrightarrow{\ \Phi\ }\tilde{h}_G^{*}(T(\eta(V;\bar{L})))\xrightarrow{\ j^{*}\ }h_G^{*}(P(L\oplus V))$$

where Φ is the Thom isomorphism and j is the map of Proposition 2. 3. Then we obtain the following

Proposition 2. 4. *For any* $a\in h_G^{*}(P(L\oplus V))$

$$\underline{i}_{!}(\underline{i}^{*}(a))=e(\eta(L\oplus V;\bar{L}))\cdot a .$$

Proof. Using Proposition 2. 3, we calculate

$$\underline{i}_{!}(\underline{i}^{*}(a))=j^{*}\Phi\underline{i}^{*}(a)=j^{*}i^{*}\Phi(a)=s^{*}\Phi(a)=e(\eta(L\oplus V;\bar{L}))\cdot a .$$

Q.E.D.

We now determine the simple system of the h_G^*-algebra $h_G^*(P(V))$ as follows.

Theorem 2.5. *Suppose that V is G-isomorphic to a direct sum $L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of one dimensional complex G-modules L_i. Then $h_G^*(P(V))$ is a free h_G^*-module with the basis:*

$$1,\ x_1,\ x_1x_2,\ \cdots\cdots,\ x_1x_2\cdots x_{n-1}$$

where $x_j = e(\eta(V;\ \bar{L}_j))$.

Proof. We prove by induction on n. Since $P(L_1)$ is a point, the case of $n=1$ is clear. As an inductive hypothesis we assume that $h_G^*(P(V'))$, $V' = L_2 \oplus L_3 \oplus \cdots \oplus L_n$, is a free h_G^*-module with the basis 1, $x'_2, x'_2x'_3, \cdots, x'_2x'_3\cdots x'_{n-1}$, where $x'_j = e(\eta(V';\ \bar{L}_j))$. The short exact sequence for the pair $(P(V), P(L_1))$:

$$0 \longrightarrow \tilde{h}_G^*(P(V)/P(L_1)) \xrightarrow{\tilde{j}^*} h_G^*(P(V)) \xrightarrow{\tilde{i}^*} h_G^*(P(L_1)) \longrightarrow 0$$

implies that $h_G^*(P(V))$ is isomorphic to $h_G^*(P(L_1)) \oplus \tilde{j}^* \tilde{h}_G^*(P(V)/P(L_1))$. The following diagram is commutative by Proposition 2. 2.

$$\begin{array}{ccc}
\tilde{h}_G^*(P(V)/P(L_1)) & \xleftarrow[\cong]{\phi^*} & \tilde{h}_G^*(T(\eta(V';\bar{L}_1))) \\
\Big\downarrow \tilde{j}^* & \swarrow j^* & \Big\uparrow \Phi : \text{Thom isomorphism} \\
h_G^*(P(V)) & & h_G^*(P(V'))
\end{array}$$

Hence, $h_G^*(P(V))$ is isomorphic to $h_G^*(P(L_1)) \oplus j^*\Phi h_G^*(P(V'))$. Proposition 2. 4 implies that

$$j^*\Phi(x'_2\cdots x'_k) = x_1x_2\cdots x_k\,.$$

This completes the proof.

We shall now proceed to analyze the relations among the x_i's.

Proposition 2.6. *In the situation of Theorem 2.5, the following relation holds:*

$$x_1x_2\cdots x_n = 0$$

where $x_j = e(\eta(V;\ \bar{L}_j))$.

Proof. We prove by induction on n. At first we consider the exact sequence for the pair $(P(L_1 \oplus V),\ P(L_1))$

$$\tilde{h}_G^*(P(L_1 \oplus V)/P(L_1)) \xrightarrow{\tilde{j}^*} h_G^*(P(L_1 \oplus V)) \xrightarrow{\tilde{i}^*} h_G^*(P(L_1)).$$

Then, by making use of Proposition 2. 3 we have

$$\begin{aligned} e(\eta(L_1;\ \bar{L}_1)) &= \tilde{i}^* e(\eta(L_1 \oplus V;\ \bar{L}_1)) \\ &= \tilde{i}^* j^* t(\eta(V;\ \bar{L}_1)) \\ &= \tilde{i}^* \tilde{j}^* \phi^* t(\eta(V;\ \bar{L}_1)). \end{aligned}$$

This implies that $x_1 = 0$ in $h_G^*(P(L_1))$. Next, suppose that $x_2' \cdots x_n' = 0$ in $h_G^*(P(V'))$, where $V' = L_2 \oplus \cdots \oplus L_n$ and $x_j' = e(\eta(V';\ \bar{L}_j))$. It follows from Proposition 2. 4 that

$$x_1 x_2 \cdots x_n = \underline{i}_! \underline{i}^* (x_2 \cdots x_n) = 0\,. \qquad \text{Q.E.D.}$$

Proposition 2. 7. *In the situation of Theorem* 2. 5, *the following relation holds:*

$$(x - e(L_1))(x - e(L_2)) \cdots (x - e(L_n)) = 0$$

where $x = e(\eta(V;\ C))$ *and* $e(L_j)$ *denotes the Euler class for a G-vector bundle* $L_j \to \{a\ point\}$.

Proof. We prove this by induction on n. For $n = 1$, we consider the bundle map:

$$\begin{array}{ccc} E(L_1;C) & \xrightarrow{\tilde{c}} & L_1 \\ {\scriptstyle \eta(L_1;C)}\Big\downarrow & & \Big\downarrow{\scriptstyle L_1} \\ P(L_1) & \xrightarrow{c} & \{\mathrm{pt}\} \end{array}$$

where $\tilde{c}([v, z]) = zv$. Since $e(\eta(L_1;\ C)) = c^*(e(L_1))$ which is denoted by $e(L_1)$, one has $e(\eta(L_1;\ C)) - e(L_1) = 0$. Suppose that in $h_G^*(P(V'))$, $V' = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$,

$$(x' - e(L_1))(x' - e(L_2)) \cdots (x' - e(L_{n-1})) = 0$$

where $x' = e(\eta(V';\ C))$. Take G-invariant subspaces in $P(V)$

$$P_0=\{[z_1,\cdots,z_n]\,|\,\|z_n\|<1\}\xrightarrow{i_0}P(V)$$

and

$$P_1=\{[z_1,\cdots,z_n]\,|\,\|z_n\|>0\}\xrightarrow{i_1}P(V)$$

where each i_s denotes the natural inclusion. The injections $\tilde{i}_0$: $P(L_1\oplus L_2\oplus\cdots\oplus L_{n-1})\to P_0$ defined by $\tilde{i}_0([z_1,z_2,\cdots,z_{n-1}])=[z_1,z_2,\cdots,z_{n-1},0]$ and $\tilde{i}_1$: $P(L_n)\to P_1$ by $\tilde{i}_1([z])=[0,\cdots,0,z]$ give G-equivariant homotopy equivalences, respectively. Then one has the following:

$$i_0^*((x-e(L_1))(x-e(L_2))\cdots(x-e(L_{n-1}))=0$$

$$i_1^*((x-e(L_n))=0.$$

Here, we consider the following commutative diagram

$$\begin{array}{ccc}
h_G^*(P(V),P_0)\otimes h_G^*(P(V),P_1) & \xrightarrow{j_0^*\otimes j_1^*} & h_G^*(P(V))\otimes h_G^*(P(V)) \\
\downarrow \times & & \downarrow \times \\
h_G^*(P(V)\times P(V),P(V)\times P_1\smile P_0\times P(V)) & \xrightarrow{(j_0\times j_1)^*} & h_G^*(P(V)\times P(V)) \\
\downarrow d^* & & \downarrow d^* \\
0=h_G^*(P(V),P(V)) & \xrightarrow{j^*} & h_G^*(P(V))
\end{array}$$

where j_0, j_1 and j are natural injections and d is the diagonal map. Since there are elements a in $h_G^*(P(V),P_0)$ and b in $h_G^*(P(V),P_1)$ such that

$$j_0^*(a)=(x-e(L_1))(x-e(L_2))\cdots(x-e(L_{n-1}))$$

$$j_1^*(b)=x-e(L_n),$$

it follows that

$$(x-e(L_1))(x-e(L_2))\cdots(x-e(L_n))=0\,. \qquad \text{Q.E.D.}$$

Here we shall observe the ring structure of $K_G^*(P(V))$, where $V=L_1\oplus L_2\oplus\cdots\oplus L_n$ and $\dim L_j=1$. We can see that in K_G-theory

$$x_j=1-\bar{L}_j\cdot\eta(V;C) \quad \text{and} \quad e(L_j)=1-L_j\,.$$

Then Proposition 2.6 implies that

$$(1-L_1\cdot\overline{\eta(V;C)})(1-L_2\cdot\overline{\eta(V;C)})\cdots(1-L_n\cdot\overline{\eta(V;C)})=0$$

and

$$\sum(-1)^i\lambda_i(V)\{\overline{\eta(V;C)}\}^i=0.$$

Therefore it follows from Theorem 2.5 that

Proposition 2.8. *Suppose that $V=L_1\oplus L_2\oplus\cdots\oplus L_n$ where L_j is a one dimensional complex G-module. Then $K_G^*(P(V))$ is isomorphic to*

$$K_G^*(\mathrm{pt})[\overline{\eta(V;C)}]/(\sum(-1)^i\lambda_i(V)\{\overline{\eta(V;C)}\}^i).$$

Let $P(\xi)\xrightarrow{\pi}X$ be the projective space bundle associated with a complex G-vector bundle ξ and let η_P be the canonical line bundle over $P(\xi)$. Making use of the local triviality for complex G-vector bundles, G a compact abelian Lie group (cf. [9]), and the Mayer-Vietoris argument, we obtain

Theorem 2.9 (Segal [17]). *Suppose that ξ is an n-dimensional complex G-vector bundle over a compact G-space X. Then $K_G^*(P(\xi))$ is isomorphic to*

$$K_G^*(X)[\overline{\eta}_P]/(\sum(-1)^i\pi^*\lambda_i(\xi)\overline{\eta}_P^i).$$

§ 3. On the Natural Transformation α: $U_G^*(X)\to U^*((EG\times X)/G)$

Let X be a compact Hausdorff G-space and let h_G^* be the equivariant cohomology theory treated in section 2. For a complex G-module V, we consider the G-vector bundle $\underline{\underline{V}}$: $X\times V\to X$ and denote by $e(\underline{\underline{V}})$ the Euler class in the h_G^*-theory. When we discuss the $U^*((EG\times -)/G)$-theory, where $EG\to BG$ is the universal G-principal bundle, the Euler class $e(\underline{\underline{V}})$ is interpreted as the Euler class for the complex vector bundle $EG\times_G\underline{\underline{V}}$: $(EG\times X\times V)/G\to(EG\times X)/G$ in the U^*-theory. In particuler, regarding EG as the direct limit space $\lim EG^{(n)}$ of G-invariant n-connected finite CW-complexes $EG^{(n)}$, one has that if X is a finite G-CW-complex, then

$$U^*((EG\times X)/G)=\varprojlim U^*((EG^{(n)}\times X)/G)$$

(cf. [1], [12], [19]). And we see that there holds the Thom isomorphism in the theory $h_G^*(-)=U^*((EG\times -)/G)$ for any finite dimensional complex G-vector bundle over a finite G-CW-complex.

Let M and N be closed G-manifolds. For a G-map $f\colon M\to N$ with a complex orientation which is compatible with the G-action [10], [16], we obtain a Gysin homomorphism

$$f_!\colon \tilde{h}_G^*(M^+)\longrightarrow \tilde{h}_G^{*+\dim N-\dim M}(N^+).$$

The Gysin homomorphisms satisfy the following properties:

(3.1) $\qquad (gf)_!=g_!f_!$

(3.2) $\qquad f_!(f^*(x)\cup y)=x\cup f_!(y).$

The exact sequence of the pair $(D(V),S(V))$ of the unit disk $D(V)$ and the unit sphere $S(V)$ in a complex G-module V and the Thom isomorphism imply the following result.

Proposition 3.3. *There exists an exact sequence*

$$\longrightarrow \tilde{h}_G^*(\mathrm{pt}^+)\xrightarrow{\cdot e(V)}\tilde{h}_G^{*+|V|}(\mathrm{pt}^+)\xrightarrow{\pi^*}\tilde{h}_G^{*+|V|}(S(V)^+)\xrightarrow{\pi_!}\tilde{h}_G^{*+1}(\mathrm{pt}^+)\longrightarrow$$

where $\pi\colon S(V)\to \mathrm{pt}=\{a\ point\}$ *and* $|V|=2\dim_C V$.

Let $\mathcal{V}$ be a set consisting of all finite dimensional complex G-modules which have no trivial summand and let

$$S_{h_G}=\{e(V)\,|\,V\in\mathcal{V}\}.$$

We denote by $S_{h_G}^{-1}h_G^*(X)$ the localized module of the h_G^*-module $h_G^*(X)$ consisting of all fractions $\{x/e(V)\,|\,x\in h_G^*(X)$ and $e(V)\in S_{h_G}\}$. For complex G-modules V and W we consider the natural injection $j_{V,V\oplus W}\colon S(V)\to S(V\oplus W)$ defined by $j(v)=(v,0)$ and the direct limit

$$\varinjlim \tilde{h}_G^{*+|V|}(S(V)^+)$$

with respect to the direct system $\{\tilde{h}_G^{*+|V|}(S(V)^+),\ j_{V,V\oplus W!}\,|\,V,\ W\in\mathcal{V}\}$. Then one has the following result, which is applied to $\tilde{h}_G^*(-)=\tilde{U}_G^*(X^+\wedge -)$, $\tilde{U}^*((EG^+\wedge X^+\wedge -)/G)$, X a finite G-CW-complex.

Proposition 3.4. *There exists an exact sequence:*

$$\longrightarrow \tilde{h}_G^*(\mathrm{pt}^+)\longrightarrow S_{h_G}^{-1}\tilde{h}_G^*(\mathrm{pt}^+)\longrightarrow \varinjlim \tilde{h}_G^{*+|V|}(S(V)^+)\longrightarrow \tilde{h}_G^{*+1}(\mathrm{pt}^+)\longrightarrow.$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
\to \tilde{h}_G^*(\mathrm{pt}^+) & \xrightarrow{\cdot e(V)} & \tilde{h}_G^{*+|V|}(\mathrm{pt}^+) & \xrightarrow{\pi_V^*} & \tilde{h}_G^{*+|V|}(S(V)^+) & \xrightarrow{\pi_!} & \tilde{h}_G^{*+1}(\mathrm{pt}^+)\to \\
(*)\quad \downarrow = & (1) & \downarrow \cdot e(W) & (2) & \downarrow j_{V,V\oplus W!} & (3) & \downarrow = \\
\to \tilde{h}_G^*(\mathrm{pt}^+) & \xrightarrow{\cdot e(V\oplus W)} & \tilde{h}_G^{*+|V\oplus W|}(\mathrm{pt}^+) & \xrightarrow{\pi_{V\oplus W}^*} & \tilde{h}_G^{*+|V\oplus W|}(S(V\oplus W)^+) & \xrightarrow{\pi_!} & \tilde{h}_G^{*+1}(\mathrm{pt}^+)\to .
\end{array}$$

The multiplicativity of the Euler classes and (3.1) imply the commutativity of (1) and (3), respectively. Let $0<\varepsilon<1$. For the disk $D(W;\varepsilon)=\{w\in W \mid \|w\|\leq\varepsilon\}$ and the sphere $S(W;\varepsilon)=\{w\in W \mid \|w\|=\varepsilon\}$, a map $j_1\colon D(W)\to D(W;\varepsilon)$ given by $j_1(w)=\varepsilon w$ induces a map $\tilde{j}_1\colon D(W)/S(W)\to D(W;\varepsilon)/S(W;\varepsilon)$. We define a map $j\colon S(V\oplus W)\to(S(V)\times D(W;\varepsilon))/(S(V)\times S(W;\varepsilon))$ by

$$j(v,w)=\begin{cases}\left[\dfrac{v}{\|v\|},\ w\right] & \text{if } \|w\|<\varepsilon \\ \textit{the base point} & \text{if } \|w\|\geq\varepsilon\end{cases}$$

and define maps $\pi_1\colon (S(V)\times D(W;\varepsilon))/(S(V)\times S(W;\varepsilon))\to D(W;\varepsilon)/S(W;\varepsilon)$ and $\pi_2\colon S(V\oplus W)\to D(W)$ by $\pi_1([v,w])=[w]$ and $\pi_2(v,w)=w$, respectively. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{h}_G^{*+|V|}(\mathrm{pt}^+) & \xrightarrow{\Phi_\varepsilon} & \tilde{h}_G^{*+|V\oplus W|}(D(W;\varepsilon)/S(W;\varepsilon)) & \xrightarrow{\tilde{j}_1^*} & \tilde{h}_G^{*+|V\oplus W|}(D(W)/S(W)) \\
\downarrow \pi_V^* & & \downarrow \pi_1^* & & \\
\tilde{h}_G^{*+|V|}(S(V)^+) & \xrightarrow{\Phi} & \tilde{h}_G^{*+|V\oplus W|}((S(V)\times D(W;\varepsilon))/(S(V)\times S(W;\varepsilon))) & & \downarrow p^* \\
& \searrow j_{V,V\oplus W!} & \downarrow j^* & & \\
& & \tilde{h}_G^{*+|V\oplus W|}(S(V\oplus W)^+) & \xleftarrow{\pi_2^*} & \tilde{h}_G^{*+|V\oplus W|}(D(W)^+) \\
& & & \nwarrow \pi_{V\oplus W}^* & \downarrow s^* \\
& & & & \tilde{h}_G^{*+|V\oplus W|}(\mathrm{pt}^+)
\end{array}$$

where Φ and Φ_ε denote the Thom isomorphisms, p the projection, and s the zero section. We can see that $j_{V,V\oplus W}\colon S(V)\to S(V\oplus W)$ is the G-embedding and the tubular neighborhood of $S(V)$ in $S(V\oplus W)$ is G-homeomorphic to $S(V)\times\mathring{D}(W;\varepsilon)$ by j, where $\mathring{D}(W;\varepsilon)=\{w\in W \mid \|w\|<\varepsilon\}$. Hence we obtain that $j^*\Phi=j_{V,V\oplus W!}$. It is easy to see the commutativity of the others. Noting that $\tilde{j}_1^*\Phi_\varepsilon$ is the Thom isomorphism Φ:

$\tilde{h}_G^{*+|V|}(pt^+) \to \tilde{h}_G^{*+|V\oplus W|}(D(W)/S(W))$, we have

$$\begin{aligned} j_{V,V\oplus W!}\pi_V^*(x) &= \pi_{V\oplus W}^* s^* p^* \tilde{j}_1^* \Phi_\varepsilon(x) \\ &= \pi_{V\oplus W}^* s^* p^* \Phi(x) \\ &= \pi_{V\oplus W}^*(x\cdot e(W)). \end{aligned}$$

Thus the square (2) in the diagram (*) is commutative. Taking the direct limit for the diagram, we have the proposition.

Let X^G be the fixed point set of a G-space X. T. tom-Dieck [7] proved the following proposition for equivariant cohomology theories equipped with the continuity axiom discussed in [7].

Proposition 3.5. $S_{h_G}^{-1}\tilde{h}_G^*(X^+) \cong S_{h_G}^{-1}\tilde{h}_G^*((X^G)^+)$.

Now let us summarize some basic properties of the natural transformation

$$\alpha\colon U_G^*(X) \to U^*((EG\times X)/G)$$

of equivariant cohomology theories which is introduced by tom-Dieck (cf. [4], [5], [6], [8], [13]):

(3.6) *α is a U^*-homomorphism.*

(3.7) *α is multiplicative.*

(3.8) *If X is a compact free G-space, then α is isomorphic.*

(3.9) *α preserves the Thom classes.*

(3.10) *For $G=Z_p$ or S^1, $\alpha\colon U_G^* \to U^*(BG)$ is injective.*

For a trivial G-space X, one has a natural monomorphism

$$\iota\colon U^*(X) \to U_G^*(X)$$

by taking $x=[f\colon S^{2n-k}\wedge X^+ \to MU(n)]$ to an element of $U_G^*(X)$ with the representative f. For any G-space Y, $U_G^*(Y)$ is a U^*-module by the homomorphism

$$m: U^*\otimes U_G^*(Y) \xrightarrow{\iota\otimes id} U_G^*\otimes U_G^*(Y) \xrightarrow{m_G} U_G^*(Y)$$

where m_G is the multiplication in U_G^*-theory. We now obtain

Proposition 3.11 ([4], [14]). *For $G=S^1$ or Z_p, U_G^* is a flat U^*-module.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} U^{k+|V|}((EG\times S(V))/G) & \cong & U^{k+|V|}(S(V)/G) & \overset{D}{\cong} & U_{-k-1-\dim G}(S(V)/G) \\ \downarrow j_{V,V\oplus W!} & & \downarrow j_{V,V\oplus W!} & & \downarrow j_{V,V\oplus W*} \\ U^{k+|V\oplus W|}((EG\times S(V\oplus W))/G) & \cong & U^{k+|V\oplus W|}(S(V\oplus W)/G) & \overset{D}{\cong} & U_{-k-1-\dim G}(S(V\oplus W)/G) \end{array}$$

where D denotes the Atiyah-Poincaré duality isomorphism, we have an isomorphism

$$\varinjlim U^{*+|V|}((EG\times S(V))/G)\cong U_{-*-1-\dim G}(BG).$$

Therefore it follows from Proposition 3.4 that there exists an exact sequence:

$$\cdots\to U_{-*-\dim G}(BG)\to U_G^*\xrightarrow{\lambda} S_{U_G^*}^{-1}U_G^*\to U_{-*-1-\dim G}(BG)\to U_G^{*+1}\to\cdots.$$

Suppose that $G=Z_p$. Then we have an exact sequence (cf. [5])

$$0\longrightarrow U^*\longrightarrow U_G^*\xrightarrow{\lambda} S_{U_G^*}^{-1}U_G^*\longrightarrow U_{-*-1}(BG)\longrightarrow 0.$$

Let $\mathfrak{U}_*^G$ denote the bordism algebra of G-actions with unrestricted isotropy groups on closed U-manifolds. Let $\mathfrak{M}_*^U(G)$ denote the bordism algebra of pairs (T, W), where T is a smooth G-action on the compact U-manifold W with no fixed points in the boundary of W. Then we have the following exact sequence [3]:

$$0\longrightarrow U_*\xrightarrow{\alpha}\mathfrak{U}_*^G\xrightarrow{\beta}\mathfrak{M}_*^U(G)\longrightarrow U_{*-1}(BG)\longrightarrow 0.$$

In [18], R. E. Stong shows that $\mathfrak{U}_*^G$ is a free U^*-module on even dimensional generators and $\mathfrak{U}_0^G$ is a free abelian group on the actions $[G/H, m]$, where m is the multiplication and H runs through all subgroups of G. Furthermore the image of α is then generated by $[G, m]$. Therefore the cokernel of α is a free U^*-module, and there exists a short exact sequence

$$0 \longrightarrow \operatorname{Coker} \alpha \longrightarrow \mathfrak{M}_*^U(G) \longrightarrow U_{*-1}(BG) \longrightarrow 0 .$$

Since $\mathfrak{M}_*^U(G)$ is a free U_*-module [3], the projective dimension of the U_*-module $U_*(BG)$ is less than or equal to 1. Consider now the exact sequence

$$0 \longrightarrow \operatorname{Image} \lambda \longrightarrow S_{U_G^*}^{-1} U_G^* \longrightarrow U_{-*-1}(BG) \longrightarrow 0 .$$

Noting that $S_{U_G^*}^{-1} U_G^*$ is a free U^*-module (cf. [5]), we have that for any U^*-module R, $\operatorname{Tor}^1_{U^*}(\operatorname{Image} \lambda, R) = 0$. Making use of the exact sequence

$$0 \longrightarrow U^* \longrightarrow U_G^* \longrightarrow \operatorname{Image} \lambda \longrightarrow 0 ,$$

we have that $\operatorname{Tor}^1_{U^*}(U_G^*, R) = 0$ and U_G^* is a flat U^*-module.

Suppose that $G = S^1$. Then we have a short exact sequence:

$$0 \longrightarrow U_G^* \longrightarrow S_{U_G^*}^{-1} U_G^* \longrightarrow U_{-*-2}(BG) \longrightarrow 0 .$$

Since $S_{U_G^*}^{-1} U_G^*$ and $U_*(BG)$ are free U^*-modules (cf. [3], [5]), U_G^* is a projective U^*-module. Q.E.D.

As described in [4], we obtain

Proposition 3.12. *Let G be Z_p or S^1. If X is a finite CW-complex with the trivial G-action, then there is a U^*-isomorphism:*

$$m_G : U_G^* \otimes_{U^*} U^*(X) \longrightarrow U_G^*(X) .$$

Proposition 3.13. *Let G be Z_p or S^1. If X is a finite CW-complex with the trivial G-action and $U^*(X)$ is a free U^*-module,*

$$\alpha\colon\ U_G^*(X) \to U^*(BG \times X)$$

is injective.

Proof. (3.7) derives the following commutative diagram:

$$\begin{array}{ccc} U_G^* \otimes_{U^*} U^*(X) & \xrightarrow{\alpha \otimes 1} & U^*(BG) \otimes_{U^*} U^*(X) \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle m} \\ U_G^*(X) & \xrightarrow{\alpha} & U^*(BG \times X) \end{array} .$$

Since $U^*(X)$ is the free U^*-module, (3.10) implies that $\alpha \otimes 1$ is injective.

By [11] $\mathfrak{m}$ is isomorphic. Hence α is injective. Q.E.D.

Remark. If X is a finite CW-complex and the integral cohomology $H^*(X)$ has no torsion, we use the Atiyah-Hirzebruch spectral sequence for $U^*(X)$ to obtain that $U^*(X)$ is a free U^*-module (cf. [3]) and we can apply Proposition 3.13 to this case.

Denoting by S_G or S the multiplicative system S_{h_G} according as $h_G^*(-) = U_G^*(-)$ or $U^*((EG\times -)/G)$, one has

$$\alpha(S_G) = S.$$

Therefore Proposition 3.13 implies the following result.

Proposition 3.14. *In the situation of Proposition* 3.13, *the localized map*

$$S_{h_G}^{-1}\alpha : S_G^{-1}U_G^*(X) \longrightarrow S^{-1}U^*(BG\times X)$$

is injective.

Here we shall prove the following

Theorem 3.15. *Let G be Z_p or S^1. Let X be a finite G-CW-complex. Suppose that the integral cohomology groups of the fixed point set $H^{\text{even}}(X^G)$ has no torsion elements and $H^{\text{odd}}(X^G) = 0$. Then*

$$\alpha : U_G^{\text{even}}(X) \longrightarrow U^{\text{even}}((EG\times X)/G)$$

is injective.

Proof. We consider the following commutative diagram with respect to the sphere bundle π: $S(\underline{\underline{V}}) \to X$ of a complex G-bundle $\underline{\underline{V}}$: $X\times V \to X$, $V\in \mathcal{V}$:

$$\begin{array}{ccccccccc}
\to & U_G^*(X) & \xrightarrow{\cdot e_1} & U_G^{*+|V|}(X) & \xrightarrow{\pi^*} & U_G^{*+|V|}(S(\underline{\underline{V}})) & \xrightarrow{\pi_!} & U_G^{*+1}(X) & \to \\
 & \downarrow \alpha_1 & (1) & \downarrow \alpha_2 & (2) & \downarrow \alpha_3 & (3) & \downarrow \alpha_1 & \\
\to & U^*((EG\times X)/G) & \xrightarrow{\cdot e_2} & U^{*+|V|}((EG\times X)/G) & \xrightarrow{\pi^*} & U^{*+|V|}((EG\times S(\underline{\underline{V}}))/G) & \xrightarrow{\pi_!} & U^{*+1}((EG\times X)/G) & \to
\end{array}$$

where $e_1 = e(\underline{\underline{V}})$, $e_2 = e((EG\times \underline{\underline{V}})/G)$ and α_i's denote the natural transformations. The commutativity of the above diagram is shown by the naturality, (3.7) and (3.9). α_3 is isomorphic by (3.8). Taking the direct limit, we have the commutative diagram:

$$\begin{array}{ccccccccc}
\to & S_G^{-1}U_G^{*-1}(X) & \longrightarrow & \varinjlim U_G^{*-1+|V|}(S(\underline{V})) & \longrightarrow & U_G^*(X) & \longrightarrow & S_G^{-1}U_G^*(X) \to \\
& \downarrow S_{h_G}^{-1}\alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow S_{h_G}^{-1}\alpha \\
\to & S^{-1}U^{*-1}((EG\times X)/G) & \to & \varinjlim U^{*-1+|V|}((EG\times S(\underline{V}))/G) & \to & U^*((EG\times X)/G) & \to & S^{-1}U^*((EG\times X)/G).
\end{array}$$

It follows from Propositions 3.5 and 3.14 that the localized map $S_{h_G}^{-1}\alpha$ is injective. The condition $H^{\mathrm{odd}}(X^G)=0$ derives $U^{\mathrm{odd}}(X^G)=0$ and since $U^*(BG\times X^G)\cong U^*(BG)\otimes_{U^*}U^*(X^G)$ [11], it follows that $U^{\mathrm{odd}}(BG\times X^G)=0$. Hence Proposition 3.5 implies that $(S^{-1}U^*((EG\times X)/G))^{\mathrm{odd}}=0$ and $(S_G^{-1}U_G^*(X))^{\mathrm{odd}}=0$. Therefore the theorem follows.

Furthermore we have

Proposition 3.16. *Let G be Z_p or S^1. Let X be a finite G-CW-complex. If $U_G^*(X)$ is a free U_G^*-module and $U^*(X^G)$ is a free U^*-module, then*

$$\alpha\colon U_G^*(X)\to U^*((EG\times X)/G)$$

is injective.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc}
\tilde{U}_G^*(X/X^G) & \xrightarrow{j^*} & U_G^*(X) \\
\downarrow & & \downarrow \lambda \\
S_G^{-1}\tilde{U}_G^*(X/X^G) & \longrightarrow & S_G^{-1}U_G^*(X).
\end{array}$$

Since $U_G^*(X)$ is a free U_G^*-module, λ is injective. And it follows from $S_G^{-1}\tilde{U}_G^*(X/X^G)=0$ (cf. [7]) that j^* is a zero homomorphism. Hence the long exact sequence of the pair (X, X^G) becomes a short exact sequence:

$$0\longrightarrow U_G^*(X)\longrightarrow U_G^*(X^G)\longrightarrow \tilde{U}_G^{*+1}(X/X^G)\longrightarrow 0.$$

Proposition 3.13 completes the proof.

§ 4. On the Localization $T^{-1}U_G^*(X)$

Let γ_G^n be the universal complex G-vector bundle and denote by MG_n the Thom complex. Let $x\in\widetilde{U}_G^{2k}(X)$ be represented by f: $V^C\wedge X\to MG_{\|V\|+k}$ where $\|V\|=\dim_C V$. Let $\mu_G(x)$ be the image of the Thom class $t_K(\gamma_G^{\|V\|+k})$ of K_G-theory in the composition

$$\widetilde{K}_G^{|V|+2k}(MG_{\|V\|+k})\xrightarrow{f^!}\widetilde{K}_G^{|V|+2k}(V^C\wedge X)\overset{\sigma_V^{-1}}{\cong}\widetilde{K}_G^{2k}(X).$$

If $x\in\widetilde{U}_G^{2k+1}(X)$ be represented by f: $V^C\wedge S^1\wedge X\to MG_{\|V\|+k+1}$, $\mu_G(x)$ is defined by $\sigma_V^{-1}\sigma_{S^1}^{-1}f^!t_K(\gamma_G^{\|V\|+k+1})$. Thus we have a multiplicative natural transformation

$$\mu_G: U_G^*(-)\longrightarrow K_G^*(-)$$

of cohomology theories which preserves the Thom classes and the Euler classes. We take up a multiplicative set T_K in a representation ring $R(G)\equiv K_G^0(\mathrm{pt})$ which consists of all one dimensional representation spaces and we consider a multiplicative system $T=\mu_G^{-1}(T_K)$ in U_G^0. Since each element of T_K is invertible, the localization $T_K^{-1}K_G^*(X)$ is isomorphic to $K_G^*(X)$, and the natural transformation μ_G induces a natural transformation

$$T^{-1}\mu_G: T^{-1}U_G^*(X)\longrightarrow K_G^*(X).$$

Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
U_G^0 & \xrightarrow{\alpha} & U^0(BG) & \xrightarrow{\varepsilon} & U^0\cong Z \\
\downarrow{\mu_G} & & \downarrow{\mu} & & \downarrow{\mu} \\
K_G^0 & \xrightarrow{\alpha_K} & K^0(BG) & \xrightarrow{\varepsilon} & K^0\cong Z
\end{array}$$

where α_K is defined by mapping each complex G-vector bundle ξ to a complex vector bundle $(EG\times\xi)/G\to BG$, ε the augmentation and μ the natural transformation of Conner-Floyd [2]. Let y be in T_K, then $\alpha_K(y)$ is a one dimensional vector bundle, so $\varepsilon\alpha_K(y)=1$. Taking an element x in T, we can see that $\varepsilon\alpha(x)=1$ and $\alpha(x)$ is invertible. Therefore the natural transformation α: $U_G^*(X)\to U^*((EG\times X)/G)$ induces a natural transformation $T^{-1}\alpha$: $T^{-1}U_G^*(X)\to U^*((EG\times X)/G)$. Then we shall verify the following

Proposition 4.1. *Let G be Z_p or S^1 and let X be a finite G-CW-complex. Suppose that $U^*(X^G)$ is a free U^*-module and $T^{-1}U_G^*(X)$ is a free $T^{-1}U_G^*$-module. Then*

$$T^{-1}\alpha : T^{-1}U_G^*(X) \longrightarrow U^*((EG\times X)/G)$$

is injective.

Proof. Proposition 3.5 implies that $S^{-1}T^{-1}U_G^*(X, X^G)=0$. Therefore the proof is quite similar to that of Proposition 3.16.

We shall compute the ring $T^{-1}U_G^*(P(V))$.

Theorem 4.2. *Let $V=L_1\oplus L_2\oplus\cdots\oplus L_n$ where L_j is a one dimensional complex G-module. Then there exists an isomorphism*

$$T^{-1}U_G^*(P(V))\cong T^{-1}U_G^*[y]/((y-e(L_1))\cdots(y-e(L_n)))$$

where $y=e(\eta(V;\ C))/1$.

Proof. Let $x=e(\eta(V;\ C))$ and $x_i=e(\eta(V;\ \bar{L}_i))$ $(i=1,\cdots,n)$ in U_G^*-theory. Then $y=x/1$ and $y_i=x_i/1$ are the Euler classes of $\eta(V;\ C)$ and $\eta(V;\ \bar{L}_i)$ in $T^{-1}U_G^*$-theory. Using Theorem 2.5, we can uniquely express $1, x, \cdots, x^{n-1}$ as linear combinations of $1, x_1, x_1x_2, \cdots\cdots, x_1x_2\cdots x_{n-1}$ over U_G^*:

$$x^k=c_{k,0}1+c_{k,1}x_1+\cdots+c_{k,n-1}x_1x_2\cdots x_{n-1}\quad (k=0, 1, \cdots, n-1).$$

Then $1, y, \cdots, y^{n-1}$ can be uniquely described as linear combinations of $1, y_1, y_1y_2, \cdots, y_1y_2\cdots y_{n-1}$ over $T^{-1}U_G^*$ as follows:

$$(4.3)\qquad y^k=d_{k,0}1+d_{k,1}y_1+\cdots+d_{k,n-1}y_1y_2\cdots y_{n-1}\quad (k=0, 1, \cdots, n-1)$$

where $d_{k,j}=c_{k,j}/1$. For simplicity we put

$$\eta=\eta(V;\ C)\quad\text{and}\quad \eta_i=\eta(V;\ \bar{L}_i).$$

Applying the homomorphism $T^{-1}\mu_G$: $T^{-1}U_G^*(X)\to K_G^*(X)$, we have

$$(1-\eta)^k=a_{k,0}1+a_{k,1}(1-\eta_1)+\cdots+a_{k,n-1}(1-\eta_1)(1-\eta_2)\cdots(1-\eta_{n-1})$$

where $a_{k,j}=T^{-1}\mu_G(d_{k,j})=\mu_G(c_{k,j})$. Noting that $\eta_i=\bar{L}_i\eta$ and $1, \eta, \cdots, \eta^{n-1}$ are linearly independent, we have

$$a_{k,j}=\begin{cases} 0 & \text{if } k<j \\ L_1L_2\cdots L_k & \text{if } k=j\,. \end{cases}$$

Consider matrices $C=(c_{k,j})$ and $D=(d_{k,j})$ with the elements $c_{k,j}$ and $d_{k,j}$ respectively. Then we have $\mu_G(\det C)=L_1\cdot L_1L_2\cdots L_1L_2\cdots L_{n-1}$. Thus $\det D=(\det C)/1$ is invertible in $T^{-1}U_G^*$. Therefore there is an inverse matrix of D. Hence $1, y, \cdots, y^{n-1}$ is a free basis of $T^{-1}U_G^*$-module $T^{-1}U_G^*(P(V))$, because by Theorem 2.5 $1, y_1, y_1y_2, \cdots, y_1y_2\cdots y_{n-1}$ is the free basis.

The relation follows from Proposition 2.7. Q.E.D.

We can use Theorem 4.2 to compute $U^*((EG\times P(V))/G)$.

Proposition 4.4. *Let $V=L_1\oplus L_2\oplus\cdots\oplus L_n$ where L_j is a one dimensional complex G-module. Then there exists an isomorphism*

$$U^*((EG\times P(V))/G)\cong U^*(BG)[y']/((y'-e_1)\cdots(y'-e_n))$$

where $y'=e((EG\times\eta(V;C))/G)$ and $e_j=\pi^*(e((EG\times L_j)/G))$, π: $(EG\times P(V))/G\to BG$ the projection.

Proof. We now note that $\alpha(e(\eta(V;L)))=e((EG\times\eta(V;L))/G)$. Let $y'_j=e((EG\times\eta(V;\bar{L}_j))/G)$. Applying $T^{-1}\alpha$ to (4.3), we have

$$(y')^k=d'_{k,0}1+d'_{k,1}y'_1+\cdots+d'_{k,n-1}y'_1y'_2\cdots y'_{n-1}\quad (k=0, 1, \cdots, n-1),$$

where $d'_{k,j}=T^{-1}\alpha(d_{k,j})$. Let D' be a matrix consisting of the elements $d'_{k,j}$. Then, in virtue of Theorem 4.2 D' has an inverse matrix. Therefore Theorem 2.5 completes the proof.

Using Theorem 4.2 and the local triviality of a complex G-vector bundle [9], G a compact abelian Lie group, the Mayer-Vietoris argument establishes the following

Theorem 4.5 (cf. [15]). *Let ξ be an n-dimensional complex G-vector bundle over a compact G-space X, and π: $P(\xi)\to X$ the pro-*

jective space bundle associated with ξ. *Then* $T^{-1}U_G^*(P(\xi))$ *is a free* $T^{-1}U_G^*(X)$*-module on the generators* $1, x_P, x_P^2, \cdots, x_P^{n-1}$, *where* x_P *is the Euler class of the canonical line bundle over* $P(\xi)$.

Thus we can obtain characteristic classes $c_i^G(\xi)\in T^{-1}U_G^{2i}(X)$, $0\leq i\leq n$ $(c_0^G(\xi)=1)$, of an n-dimensional complex G-vector bundle ξ over a compact G-space X defined by the following

$$x_P^n=\pi^*c_1^G(\xi)x_P^{n-1}-\pi^*c_2^G(\xi)x_P^{n-2}+\cdots+(-1)^{n-1}\pi^*c_n^G(\xi),$$

which satisfy

(1) $c_i^G(f^!\xi)=f^*c_i^G(\xi)$ for any G-map f,

(2) $c_i^G(\xi\oplus\eta)=\sum_{l+k=i}c_l^G(\xi)c_k^G(\eta)$,

(3) $c_1^G(\eta(V;\ C))=e(\eta(V;\ C))$.

As usual we can prove the following

Proposition 4.6. *If* ξ *is an n-dimensional complex G-vector bundle over a compact G-space* X, $\xi_1, \xi_2, \cdots, \xi_n$ *the usual line bundles over the flag bundle* $F(\xi)$ *of* ξ, *then the map defined by* $t_i\to c_1^G(\xi_i)$ *defines an isomorphism of* $T^{-1}U_G^*(X)$*-modules*

$$T^{-1}U_G^*(X)[t_1, t_2, \cdots, t_n]/I\to T^{-1}U_G^*(F(\xi))$$

where I *is the ideal generated by the elements*

$$\mathfrak{S}^i(t_1, t_2, \cdots, t_n)-c_i^G(\xi),\quad i=1, 2, \cdots, n,$$

$\mathfrak{S}^i$ *being the i-th elementary symmetric function.*

Proposition 4.7. *Let* $\pi\colon E(\xi)\to X$ *be an n-dimensional complex G-vector bundle over a compact G-space and* $G_k(\xi)$ *the Grassmann bundle of k-dimensional subspaces of* $E(\xi)$. *Let* η *be the canonical k-dimensional bundle over* $G_k(\xi)$, η' *the quotient bundle* $\pi^*\xi/\eta$. *Then the map defined by* $t_i\to c_i^G(\eta)$, $s_j\to c_j^G(\eta')$ *defines an isomorphism of* $T^{-1}U_G^*(X)$*-module*

$$T^{-1}U_G^*(X)[t_1, t_2, \cdots, t_k, s_1, s_2, \cdots, s_{n-k}]/I\to T^{-1}U_G^*(G_k(\xi))$$

where I *is the ideal generated by the elements*

$$\sum_{i+j=l} t_i s_j - c_l^G(\xi) \quad \textit{for all } l .$$

Then we have the Conner-Floyd isomorphism

Theorem 4.8. *For any compact G-space X*

$$K_G^*(X) \cong U_G^*(X) \otimes_{U_G^*} K_G^*(\mathrm{pt}) .$$

In the description $EG = \lim EG^{(n)}$, we can take $EG^{(n)} \to EG^{(n)}/G$ as a G-principal bundle. Then $(EG^{(n)} \times P(V))/G \to EG^{(n)}/G$ is a complex projective space bundle and $U^*((EG^{(n)} \times P(V))/G)$ is a free $U^*(EG^{(n)}/G)$-module on the generators $1, x_P, x_P^2, \cdots, x_p^{n-1}$, where V is an n-dimensional complex G-module and x_P denotes the first Chern class of the canonical line bundle over $(EG^{(n)} \times P(V))/G$ [2]. This result and Proposition 4.4 give rise to a similar discussion to $T^{-1}U_G^*$-theory for the theory $h_G^*(-) = U^*((EG \times -)/G)$ or $U^*((EG^{(n)} \times -)/G)$. Then we have characteristic classes $c_i^{h_G}(\xi)$ in the h_G^*-theory for any finite dimensional complex G-vector bundle over a finite G-CW-complex. Hence we obtain the following

Proposition 4.9. *Suppose that X is a finite G-CW-complex in the situation of Proposition* 4.6, *then the map defined by* $t_i \to c_1^{h_G}(\xi_i)$ *defines an isomorphism of* $h_G^*(X)$*-modules*

$$h_G^*(X)[t_1, t_2, \cdots, t_n]/I \to h_G^*(F(\xi))$$

where I is the ideal generated by the elements

$$\mathfrak{S}^i(t_1, t_2, \cdots, t_n) - c_i^{h_G}(\xi), \quad i = 1, 2, \cdots, n ,$$

$\mathfrak{S}^i$ *being the i-th elementary symmetric function.*

Proposition 4.10. *Suppose that X is a finite G-CW-complex in the situation of Proposition* 4.7, *then the map defined by* $t_i \to c_i^{h_G}(\eta)$, $s_j \to c_j^{h_G}(\eta')$ *defines an isomorphism of* $h_G^*(X)$*-modules*

$$h_G^*(X)[t_1, t_2, \cdots, t_k, s_1, s_2, \cdots, s_{n-k}]/I \to h_G^*(G_k(\xi))$$

where I is the ideal generated by the elements

$$\sum_{i+j=l} t_i s_j - c_l^{h_G}(\xi) \quad \textit{for all } l\,.$$

§ 5. On the Completion of $U_G^*(X)$, $G=S^1$ or Z_p

Let ψ: $U_G^* \to U^*$ be the forgetful homomorphism and ε: $U^*(BG) \to U^*$ the augmentation homomorphism. We consider an ideal $I=\ker\varepsilon$ of $U^*(BG)$ and an ideal $I_G=\ker\psi$ of U_G^*. In discussion on I (resp. I_G)-adic completion of $U^*((EG\times X)/G)$ (*resp.* $U_G^*(X)$) the following fact is useful.

Proposition 5.1. *Let L be the one dimensional canonical complex G-module which is the generator of the Lie ring $R(G)$. Let $x=e((EG\times L)/G)$ and $x_G=e(L)$. Then, for a finite G-CW-complex X*

(i) *$I^n\cdot U^*((EG\times X)/G)$ is an ideal generated by x^n.*

(ii) *$I_G^n\cdot U_G^*(X)$ is an ideal generated by x_G^n.*

Proof. (i) $U^*(BS^1)=U^*[[x]]$ and $U^*(BZ_p)$ is a U^*-algebra of formal power series of x with a relation $e((EZ_p\times L^p)/Z_p)=0$. Since $\varepsilon(x)=0$, it follows that $(x^n)=I^n\cdot U^*((EG\times X)/G)$.

(ii) The commutative diagram

$$\begin{array}{ccc} U_G^* & \xrightarrow{\psi} & U^* \\ \alpha\downarrow & \nearrow \varepsilon & \\ U^*(BG) & & \end{array}$$

and $\alpha(x_G)=x$ imply that $x_G\in I_G$ and $(x_G^n)\subset I_G^n\cdot U_G^*(X)$. Let $V=nL$. Consider the Gysin exact sequence with respect to $\underline{\underline{V}}$: $X\times V\to X$ (cf. Proposition 3.3). Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} \to & U_G^*(X) & \xrightarrow{\cdot x_G^n} & U_G^*(X) & \xrightarrow{\pi_G^*} & U_G^*(S(\underline{\underline{V}})) & \xrightarrow{\pi_{G!}} & U_G^*(X) & \to \\ (5.2) & \alpha\downarrow & & \alpha\downarrow & & \alpha_1\downarrow & & \alpha_2\downarrow & \\ \to & U^*((EG\times X)/G) & \xrightarrow{\cdot x^n} & U^*((EG\times X)/G) & \xrightarrow{\pi^*} & U^*((EG\times S(\underline{\underline{V}}))/G) & \xrightarrow{\pi_!} & U^*((EG\times X)/G) & \to. \end{array}$$

If $y\in I_G^n\cdot U_G^*(X)$, $\alpha(y)\in I^n\cdot U^*((EG\times X)/G)=\ker\pi^*$. Since α_1 is isomorphic, y belongs to (x_G^n). Q.E.D.

Theorem 5.3. *Let X be a finite G-CW-complex. Then*

(i) *α: $U_G^*(X) \to U^*((EG\times X)/G)$ induces a monomorphism*

$$\hat{\alpha}: U_G^*(X)/I_G^n \cdot U_G^*(X) \to U^*((EG\times X)/G)/I^n \cdot U^*((EG\times X)/G).$$

(ii) *If $U^*(X^G)$ is a free U^*-module and $T^{-1}U_G^*(X)$ is a free $T^{-1}U_G^*$-module, then $\hat{\alpha}$ is isomorphic.*

Proof. Proposition 5.1 shows that α induces the monomorphism $\hat{\alpha}$. We give a proof of (ii). Suppose that $[b] \in U^*((EG\times X)/G)/I^n \cdot U^*((EG\times X)/G)$. Consider the diagram (5.2), in which α_2 coincides with the composition $U_G^*(X) \xrightarrow{\lambda} T^{-1}U_G^*(X) \xrightarrow{T^{-1}\alpha} U^*((EG\times X)/G)$. There exists an element c in $U_G^*(S(\underline{\underline{V}}))$ such that $\alpha_1(c) = \pi^*(b)$. Since $\pi_! \alpha_1(c) = 0$ and $T^{-1}\alpha$ is injective by Proposition 4.1, $\lambda(\pi_{G!}(c)) = 0$ and there exists an element t in $T \subset U_G^0$ such that

$$t\pi_{G!}(c) = 0.$$

Here we note that $\psi(t) = 1 = \psi(1)$ and $1-t \in I_G$. We put $u = 1-t$. Then we see that

$$\pi_{G!}((1-u^n)c) = (1+u+\cdots+u^{n-1})t\pi_{G!}(c) = 0$$

and get an element d in $U_G^*(X)$ such that

$$\pi_G^*(d) = (1-u^n)c.$$

Now we calculate

$$\pi^*\alpha(d) = \alpha_1((1-u^n)c) = (1-\alpha(u^n))\pi^*(b) = \pi^*(b-\alpha(u^n)b),$$

then we see that $\alpha(d) - b + \alpha(u^n)b$ belongs to (x^n). Since $\alpha(u^n)b \in (x^n)$, we obtain

$$\hat{\alpha}([d]) = [b].$$

Hence $\hat{\alpha}$ is surjective. Q.E.D.

By an elementary observation of the I (resp. I_G)-adic topology for $U^*((EG\times X)/G)$ (resp. $U_G^*(X)$), we obtain the following

Theorem 5.4. *Let X be a finite G-CW-complex. If $U^*(X^G)$ is a free U^*-module and $T^{-1}U_G^*(X)$ is a free $T^{-1}U_G^*$-module, then α induces a topological isomorphism*

$$\widehat{U_G^*(X)} \cong \overline{U^*((EG\times X)/G)}.$$

Let G $(=Z_p$ or $S^1)$ act on S^{2n+1} by $\varphi(z_0, z_1, \cdots, z_n) = (\varphi z_0, \varphi z_1, \cdots, \varphi z_n)$, $\varphi\in G$. We describe $EG^{(2n)}$ as S^{2n+1}. Then it follows from [1], [12], [19] that for a finite G-CW-complex X, there exists an isomorphism

$$U^*((EG\times X/G) \cong \varprojlim U^*((EG^{(2n)}\times X)/G).$$

If X is the projective space $P(V)$, the Grassmann manifold $G_k(V)$ or the flag manifold $F(V)$, V a finite dimensional complex G-module, then Propositions 4.4, 4.9 and 4.10 imply that

$$\mathrm{Ker}\{i^*\colon U^*((EG\times X)/G) \to U^*((EG^{(2n)}\times X)/G)\},$$

i the natural injection, is an ideal generated by $\{e((EG\times L)/G)\}^{n+1}$, L the canonical one dimensional complex G-module. Therefore Proposition 5.1 and Theorem 5.4 imply

Corollary 5.5. *Let $X=P(V)$, $G_k(V)$ or $F(V)$, V a finite dimensional complex G-module. Then there exists a topological isomorphism*

$$\widehat{U_G^*(X)} \cong U^*((EG\times X)/G).$$

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