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On the Heisenberg Commutation Relation II

By

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Abstract

We study canonical pairs of self-adjoint operators P and Q whose restrictions to a common invariant dense domain $\mathcal{D}$ of a Hilbert space are essentially self-adjoint and satisfy the Heisenberg commutation relation $PQ\varphi - QP\varphi = -i\varphi$ for $\varphi \in \mathcal{D}$. Under some additional assumption, we obtain a classification of such pairs. Moreover, we develop some methods for constructing canonical pairs of this class.

Introduction

This paper is devoted to the study of a class (denoted by $\mathscr{C}$) of representations of the Heisenberg commutation relation. To be precise, we investigate self-adjoint operators P and Q in a Hilbert space $\mathcal{H}$ such that their restrictions to a dense invariant domain $\mathcal{D}$ of $\mathcal{H}$ are essentially self-adjoint and satisfy the Heisenberg commutation relation

$$(1) \qquad PQ\varphi - QP\varphi = -i\varphi\,, \quad \varphi \in \mathcal{D}\,.$$

Moreover, we assume that the $Op*$-algebra generated by $P \upharpoonright \mathcal{D}$, $Q \upharpoonright \mathcal{D}$ and the identity is closed on $\mathcal{D}$ and that Q has a dense set of analytic vectors contained in $\mathcal{D}$. Let π be the $*$-representation of the Weyl algebra $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ defined by $\pi(\boldsymbol{p}) = P \upharpoonright \mathcal{D}$, $\pi(\boldsymbol{q}) = Q \upharpoonright \mathcal{D}$. We then write $(P, Q; \mathcal{D}) \in \mathscr{C}$ resp. $\pi \in \mathscr{C}$.

Clearly, the Schrödinger pair $P = -i\frac{d}{dx}$, $Q = x$ on $\mathcal{D} = \mathscr{S}(R_1)$ is in $\mathscr{C}$. Further, if self-adjoint operators P and Q fulfill the Weyl commutation relation

$$(2) \qquad e^{isP}e^{itQ} = e^{its}e^{itQ}e^{isP}, \quad s, t \in R_1\,,$$

then $(P, Q; \mathcal{D}) \in \mathscr{C}$ for an appropriately chosen domain $\mathcal{D}$. But we are

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mainly interested in canonical pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$ for which (2) does not hold in general.

The main purpose of this paper is to attempt a classification of canonical pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$. In doing this, we shall use another class, called $\mathcal{K}$, of representations of (1) which has been studied in a previous paper [17]. Moreover, we want to show how to construct "sufficiently many" irreducible inequivalent representations of the class $\mathscr{C}$.

Let us briefly describe the contents of the paper.

In Section 1 we collect some definitions and facts about unbounded operator algebras and prove some preliminary lemmas which are needed later. By the way we fix some notation.

In Section 2 we give the the precise definitions of the classes $\mathscr{C}$ and $\mathcal{K}$ and discuss some simple properties.

In Section 3 we construct some examples of pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$ which do not satisfy the Weyl relation.

In Section 4 we show that for given $(P, Q; \mathcal{D}) \in \mathscr{C}$ there is a largest pair $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$ such that $P_1 \subseteq P$ and $\mathcal{D}_1 \subseteq \mathcal{D}$. This is the starting point in our classification. The structure of canonical pairs in $\mathcal{K}$ for which Q has finite spectral multiplicity (and an additional assumption is satisfied) has been determined in [17]. Using this result, the problem of classifying the pairs in $\mathscr{C}$ reduces (under some assumptions) to knowing all pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$ for which $\overline{P_0} \subseteq P$ and $\mathcal{D}_0 \subseteq \mathcal{D}$. Here $\overline{P_0}$ is the differential operator $-i\dfrac{d}{dx}$ with boundary values zero, $Q = x$ and $\mathcal{D}_0 = \bigcap_{k,r=0}^{\infty} \mathcal{D}((\overline{P_0})^r Q^k)$ in a Hilbert space $\mathcal{H} = \sum_n^{\oplus} L_2(a_n, b_n)$. In Section 5 these pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$ are described in terms of certain unitary operators W, called weak intertwining operators, and vector spaces $\mathfrak{M}$ of boundary values, called admissible boundary spaces (Theorem 5.5).

In Section 6 we investigate the irreducibility and the unitary equivalence of these pairs.

In Section 7 we are dealing with the construction of weak intertwining operators. We prove that there are uncountably many inequivalent irreducible (self-adjoint) pairs $(P, Q; \mathcal{D}) \in \mathscr{C}$ which extend $(P_0, Q; \mathcal{D}_0) \in \mathcal{K}$ provided that the set of intervals (a_n, b_n) is linearly ordered and contains infinitely many finite intervals (Theorem 7.1).

In Section 8 we obtain some results that will be used in Section 9.

In Section 9 we construct canonical pairs of the class $\mathscr{C}$ by varying the admissible boundary space $\mathfrak{M}$. Some examples (and counter-examples) constructed there seem of some interest in representation theory of unbounded operator algebras as well.

§ 1. Preliminaries

1.1. Let $\mathscr{H}$ be a complex Hilbert space. Let T be a densely defined linear operator on $\mathscr{H}$. We always denote by $\mathscr{D}(T)$ the domain of T. We denote by $\overline{T}$ the closure and by T^* the adjoint of T. By definition, T^0 is the identity map of $\mathscr{H}$. $T\subseteq S$ means that $\mathscr{D}(T)\subseteq\mathscr{D}(S)$ and $T\varphi=S\varphi$ for $\varphi\in\mathscr{D}(T)$. Let $\mathscr{D}_\infty(T):=\bigcap_{n=1}^{\infty}\mathscr{D}(T^n)$. $\sigma(T)$ denotes the spectrum of T.

An *Op-algebra* $\mathscr{A}$ is an algebra over the complex numbers of linear operators on a common invariant dense linear subspace $\mathscr{D}=\mathscr{D}(\mathscr{A})$ of $\mathscr{H}$ (called the domain of $\mathscr{A}$) containing the identity map I of $\mathscr{D}$. $\mathscr{A}$ is an *Op*$*$-algebra if in addition $\mathscr{D}\subseteq\mathscr{D}(A^*)$ and $A^+:=A^*\restriction\mathscr{D}\in\mathscr{A}$ for all $A\in\mathscr{A}$. With the involution $A\rightarrow A^+$, $\mathscr{A}$ is a $*$-algebra. For an *Op*-algebra $\mathscr{A}$ on $\mathscr{D}$, the *graph topology* $t_{\mathscr{A}}$ is the locally convex topology on $\mathscr{D}$ generated by the seminorms $\|\varphi\|_A:=\|A\varphi\|$, $A\in\mathscr{A}$.

Now suppose that $\mathscr{A}$ is an *Op*$*$-algebra on $\mathscr{D}$. Then $\underline{\mathscr{D}}(\mathscr{A}):=\bigcap_{A\in\mathscr{A}}\mathscr{D}(\bar{A})$ is the completion of the locally convex space $\mathscr{D}[t_{\mathscr{A}}]$ and $\overline{\mathscr{A}}:=\{\bar{A}\restriction\underline{\mathscr{D}}(\mathscr{A})\,;\,A\in\mathscr{A}\}$ is an *Op*$*$-algebra on $\mathscr{D}(\overline{\mathscr{A}}):=\underline{\mathscr{D}}(\mathscr{A})$ which is called the *closure* of $\mathscr{A}$. $\mathscr{A}$ is said to be *closed* if $\mathscr{A}=\overline{\mathscr{A}}$, i.e., $\mathscr{D}=\underline{\mathscr{D}}(\mathscr{A})$. Let $\mathscr{D}_*(\mathscr{A}):=\bigcap_{A\in\mathscr{A}}\mathscr{D}(A^*)$. $\mathscr{A}^*:=\{A^*\restriction\mathscr{D}_*(\mathscr{A})\,;\,A\in\mathscr{A}\}$ is an *Op*-algebra on $\mathscr{D}(\mathscr{A}^*):=\mathscr{D}_*(\mathscr{A})$ called the *adjoint* of $\mathscr{A}$. $\mathscr{A}$ is said to be *self-adjoint* if $\mathscr{A}=\mathscr{A}^*$, i.e., $\mathscr{D}=\mathscr{D}_*(\mathscr{A})$.

Let $\boldsymbol{A}$ be a $*$-algebra with unit element 1. A *representation* [*$*$-representation*] π of $\boldsymbol{A}$ on $\mathscr{D}=\mathscr{D}(\pi)$ is a homomorphism [$*$-homomorphism] of $\boldsymbol{A}$ on an *Op*-algebra [*Op*$*$-algebra] $\pi(\boldsymbol{A})$ on $\mathscr{D}$ such that $\pi(1)=I$. Let $\mathscr{D}(\pi^*):=\mathscr{D}_*(\pi(\boldsymbol{A}))$ and $\pi^*(a):=\pi(a^+)^*\restriction\mathscr{D}(\pi^*)$, $a\in\boldsymbol{A}$, for a $*$-representation π of $\boldsymbol{A}$ on $\mathscr{D}$. Then π^* is a representation of $\boldsymbol{A}$ on $\mathscr{D}(\pi^*)$. π is called *self-adjoint* [*closed*] if $\pi(\boldsymbol{A})$ is a self-adjoint [closed] *Op*$*$-algebra on $\mathscr{D}$, i.e., $\mathscr{D}=\mathscr{D}_*(\pi(\boldsymbol{A}))\equiv\mathscr{D}(\pi^*)$ [$\mathscr{D}=\underline{\mathscr{D}}(\pi(\boldsymbol{A}))$]. If

π_1, π_2 are representations of A on $\mathcal{D}_1$, $\mathcal{D}_2$, we say π_1 is an *extension* of π_2 (denoted by $\pi_1 \supseteq \pi_2$) if $\mathcal{D}_1 \supseteq \mathcal{D}_2$ and $\pi_1(a) \upharpoonright \mathcal{D}_2 = \pi_2(a)$ for all $a \in A$. In particular, $\pi^* \supseteq \pi$ for each $*$-representation π. The proofs of all unproven facts mentioned above and more details can be found in [10] and [13].

Let $\mathcal{J}$ be an index set. For $\iota \in \mathcal{J}$, let π_ι be a representation of A on $\mathcal{D}_\iota$ in a Hilbert space $\mathcal{H}_\iota$. Define $\mathcal{H} = \sum_{\iota \in \mathcal{J}}^{\oplus} \mathcal{H}_\iota$. Let $\mathcal{D}$ be the set of all vectors $\varphi = (\varphi_\iota) \in \mathcal{H}$ for which $\varphi_\iota \in \mathcal{D}_\iota$ for all $\iota \in \mathcal{J}$ and $\pi(a)\varphi := (\pi_\iota(a)\varphi_\iota) \in \mathcal{H}$ for all $a \in A$. Then π is a representation of A on $\mathcal{D}$ called the *direct sum* of the representations π_ι. We denote this representation by $\sum_{\iota \in \mathcal{J}}^{\oplus} \pi_\iota$.

A representation π of A is called *irreducible* if π cannot be decomposed as a direct sum of non-trivial representations of A, that is, if $\pi = \pi_1 \oplus \pi_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $\mathcal{H}_1 = \{0\}$ or $\mathcal{H}_2 = \{0\}$. Further, π_1 and π_2 are called *unitarily equivalent* or simply *equivalent* (denoted by $\pi_1 \cong \pi_2$) if there is an isometry U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ so that $U\mathcal{D}_1 = \mathcal{D}_2$ and $U^*\pi_2(a)U = \pi_1(a)$ for all $a \in A$. Here $\mathcal{D}_1$ and $\mathcal{D}_2$ are the domains of π_1 and π_2 in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Both concepts fit together by introducing the (*strong*) *intertwining space* of two representations π_1 and π_2 of A:

$$\mathcal{J}(\pi_1, \pi_2)_s := \{C \in B(\mathcal{H}_1, \mathcal{H}_2) : C\mathcal{D}_1 \subseteq \mathcal{D}_2,\ C\pi_1(a)\varphi = \pi_2(a)C\varphi \text{ for all } \varphi \in \mathcal{D}_1 \text{ and } a \in A\}.$$

Moreover, we define the (*strong*) *commutant* of an *Op*-algebra $\mathcal{A}$ on $\mathcal{D}$ by $\mathcal{A}'_s = \{C \in B(\mathcal{H}) : C\mathcal{D} \subseteq \mathcal{D},\ CA\varphi = AC\varphi$ for all $\varphi \in \mathcal{D}$ and $A \in \mathcal{A}\}$. ($B(\mathcal{H}_1, \mathcal{H}_2)$ are the bounded linear operators of $\mathcal{H}_1$ into $\mathcal{H}_2$ and $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$.) It is obvious that $\pi_1 \cong \pi_2$ iff there is an isometry U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ so that $U \in \mathcal{J}(\pi_1, \pi_2)_s$ and $U^* \in \mathcal{J}(\pi_2, \pi_1)_s$. Clearly, $\mathcal{J}(\pi_1, \pi_1)_s = \pi_1(A)'_s$. It is easy to verify that π_1 is irreducible if and only if there is no projection $E \neq 0, I$ in $\pi_1(A)'_s = \mathcal{J}(\pi_1, \pi_1)_s$.

1.2. Suppose that S and T are linear symmetric operators defined on the dense domain $\mathcal{D}$ so that $S\mathcal{D} \subseteq \mathcal{D}$ and $T\mathcal{D} \subseteq \mathcal{D}$. We always write $\mathcal{A}(S, T)$ for the *Op*$*$-algebra generated by S, T and the identity I on $\mathcal{D}$.

Lemma 1. *Suppose that* $\mathcal{A}(S, T) = Lin\ \{T^r S^k; r, k \in N_0\}$[1]. *Then* $\mathcal{D}_*(\mathcal{A}) = \bigcap_{k,r=0}^{\infty} \mathcal{D}((S^k)^*(T^r)^*) = \bigcap_{k,r=0}^{\infty} \mathcal{D}((S^*)^k(T^*)^r)$.

Proof. Since $\mathcal{A} = \mathrm{Lin}\ \{T^r S^k; r, k \in N_0\}$, we have $\mathcal{D}_*(\mathcal{A}) = \bigcap_{k,r=0}^{\infty} \mathcal{D}((T^r S^k)^*)$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the domains as defined above. If A and B are linear operators in $\mathcal{H}$ such that $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(AB)$ are dense in $\mathcal{H}$, then $B^* A^* \subseteq (AB)^*$. This well-known fact implies that $\mathcal{D}_*(\mathcal{A}) \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_2$. Let $\varphi \in \mathcal{D}_*(\mathcal{A})$. Since the mapping $\mathcal{A} \ni A \to (A^+)^* \upharpoonright \mathcal{D}_*(\mathcal{A})$ is a homomorphism ([13], Lemma 4.1), $\mathcal{A} \ni B \to B^* \upharpoonright \mathcal{D}_*(\mathcal{A})$ is an antihomomorphism. Hence $(T^r S^k)^* \varphi = (S^*)^k (T^*)^r \varphi$ for k, $r \in N_0$ and thus $\varphi \in \mathcal{D}_2$. Therefore, $\mathcal{D}_2 \supseteq \mathcal{D}_*(\mathcal{A})$.

Lemma 2. *Let P and Q be closed symmetric operators in $\mathcal{H}$ such that* $\bigcap_{k,r=0}^{\infty} \mathcal{D}(Q^k P^r) = \bigcap_{k,r=0}^{\infty} \mathcal{D}(P^r Q^k) =: \mathcal{D}$. *Suppose $\mathcal{D}$ is dense in $\mathcal{H}$. Then $P\mathcal{D} \subseteq \mathcal{D}$ and $Q\mathcal{D} \subseteq \mathcal{D}$. The Op*-algebra $\mathcal{A} := \mathcal{A}(P \upharpoonright \mathcal{D}, Q \upharpoonright \mathcal{D})$ is closed on $\mathcal{D}$.*

Proof. The first equality for $\mathcal{D}$ shows that $P\mathcal{D} \subseteq \mathcal{D}$. The second one gives $Q\mathcal{D} \subseteq \mathcal{D}$. We prove that $\mathcal{A}$ is closed on $\mathcal{D}$. For let $\varphi \in \mathcal{D}(\bar{\mathcal{A}})$. Then there is a sequence $\{\varphi_n\}$ converging to φ relative to the graph topology $t_{\mathcal{A}}$. In particular, there are vectors $\varphi^{k,r} \in \mathcal{H}$, $k, r \in N_0$, so that $Q^k P^r \varphi_n \to \varphi^{k,r}$ in $\mathcal{H}$. Since P^r is closed, $\varphi_n \to \varphi$ and $P^r \varphi_n \to \varphi^{0,r}$ yield $P^r \varphi = \varphi^{0,r}$. From $P^r \varphi_n \to P^r \varphi$ and $Q^k P^r \varphi_n \to \varphi^{k,r}$ we obtain $P^r \varphi \in \mathcal{D}(Q^k)$ and $Q^k P^r \varphi = \varphi^{k,r}$ because Q^k is closed. Hence $\varphi \in \mathcal{D} = \bigcap_{k,r=0}^{\infty} \mathcal{D}(Q^k P^r)$ and $\mathcal{D} = \underline{\mathcal{D}}(\mathcal{A})$.

1.3. Some general notational conventions used in this paper are the following. Let $N_0 := \{0, 1, 2, \cdots\}$ and let $N := \{1, 2, \cdots\}$. For a (measurable) subset $\mathfrak{R}$ of R_n, $L_2(\mathfrak{R})$ is the L_2-space with respect to the Lebesgue measure. $\chi_{\mathfrak{R}}$ stands for the characteristic function of $\mathfrak{R}$. All Hilbert spaces are assumed to be complex and separable. We shall denote (general) Hilbert spaces by $\mathcal{G}, \mathcal{H}, \mathcal{H}_1$ etc. and dense domains by $\mathcal{D}$, $\mathcal{F}$, $\mathcal{F}_1$ etc. The norm and the scalar product of these spaces (and $L_2(\mathfrak{R})$)

[1] See the notations explained in 1.3.

are denoted by $\|\cdot\|$ resp. $\langle\cdot,\cdot\rangle$. If $\mathfrak{J}$ is an index set, then $|||\cdot|||$ denotes the norm of $l_2(\mathfrak{J})$ and $(\cdot,\cdot)$ denotes the scalar product of $l_2(\mathfrak{J})$. (The reason is that we want to distinguish between the (general) Hilbert space $\mathcal{H}$ and the Hilbert spaces $l_2(\mathfrak{J}^{\pm})$ of boundary values.) The abbreviation "e.s.a." stands for "essentially self-adjoint".

By the *Weyl algebra* $A(\boldsymbol{p}, \boldsymbol{q})$ we mean the associate algebra with unit element 1 (over the complex numbers) which is generated by two variables $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying $\boldsymbol{pq}-\boldsymbol{qp}=-i\cdot 1$. i always denotes the complex unit. Endowed with the involution induced by $\boldsymbol{p}^+=\boldsymbol{p}$, $\boldsymbol{q}^+=\boldsymbol{q}$, $1^+=1$, $A(\boldsymbol{p},\boldsymbol{q})$ becomes a $*$-algebra. Note that in [3], 4.6, the Weyl algebra is defined by means of the relation $\boldsymbol{pq}-\boldsymbol{qp}=1$.

§ 2. Definition of the Classes $\mathscr{C}$ and $\mathcal{K}$

2.1. From [17] we recall

Definition 1. *Let P be a symmetric operator defined on a dense domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Let Q be a self-adjoint operator in $\mathcal{H}$ so that $\mathcal{D}\subseteq\mathcal{D}(Q)$ and let $U(t)=e^{itQ}$, $t\in R_1$. We say that $(P,Q;\mathcal{D})$ is in the class $\mathcal{H}$ if the following conditions are true:*

(1.1) $P\mathcal{D}\subseteq\mathcal{D}$, $Q\mathcal{D}\subseteq\mathcal{D}$.

(1.2) *$\mathcal{A}(P, Q\restriction\mathcal{D})$ is a closed Op$*$-algebra on $\mathcal{D}$.*

(1.3) *$U(t)\varphi\in\mathcal{D}$ and $PU(t)\varphi=U(t)(P+t)\varphi$ for $\varphi\in\mathcal{D}$, $t\in R_1$.*

In this paper we are mainly dealing with the following class.

Definition 2. *Let $\mathcal{D}$ be a dense domain of a Hilbert space $\mathcal{H}$ and let P and Q be closed symmetric operators in $\mathcal{H}$ such that $\mathcal{D}\subseteq\mathcal{D}(P)$, $\mathcal{D}\subseteq\mathcal{D}(Q)$. We say $(P,Q;\mathcal{D})$ is a canonical pair of the class $\mathscr{C}$ if the following conditions hold:*

(2.1) $P\mathcal{D}\subseteq\mathcal{D}$, $Q\mathcal{D}\subseteq\mathcal{D}$.

(2.2) *$\mathcal{A}(P\restriction\mathcal{D}, Q\restriction\mathcal{D})$ is a closed Op$*$-algebra on $\mathcal{D}$.*

(2.3) *$PQ\varphi-QP\varphi=-i\varphi$ for $\varphi\in\mathcal{D}$.*

(2.4) $P \upharpoonright \mathscr{D}$ *is e.s.a..*

(2.5) *The set* $\mathscr{D}_a(Q) := \{\varphi \in \mathscr{D} : \varphi$ *is an analytic vector for* $Q\}$ *is dense in* $\mathscr{H}$.

Recall that a vector $\varphi \in \mathscr{D}_\infty(Q)$ is called *analytic* for Q ([11]) if there is a constant $M \in R_1$ such that $\|Q^n \varphi\| \leq M^n n!$ for all $n \in N_0$.

2.2. *Remarks* 1) By a theorem of Nelson [11], (2.5) implies

$$(2.6) \qquad Q \upharpoonright \mathscr{D} \quad \textit{is e.s.a..}$$

In Section 9 we shall see that (2.5) in Definition 2 cannot be replaced by (2.6) in general.

2) An equivalent definition of $\mathscr{K}$ is obtained if (1.3) is replaced by

$$(1.3)' \qquad f(Q)\varphi \in \mathscr{D} \quad \textit{and} \quad Pf(Q)\varphi - f(Q)P\varphi = -if'(Q)\varphi$$
$$\textit{for all } f \in C_0^\infty(R_1) \textit{ and } \varphi \in \mathscr{D}\,.$$

Proposition 3.1 in [17] shows that (1.3) implies (1.3)′. The converse direction follows from Lemma 3 below (applied in case $g(x) = e^{itx}$, $t \in R_1$).

3) Suppose that P and Q are closed symmetric linear operators satisfying (2.1) and (2.3). Because the Weyl algebra $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ is simple, $\pi(\boldsymbol{p}) := P \upharpoonright \mathscr{D}$ and $\pi(\boldsymbol{q}) := Q \upharpoonright \mathscr{D}$ define a $*$-representation π of the $*$-algebra $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathscr{D}$. We write $\pi \in \mathscr{K}$ and $\pi \in \mathscr{C}$ if and only if $(P, Q; \mathscr{D}) \in \mathscr{K}$ and $(P, Q; \mathscr{D}) \in \mathscr{C}$, respectively. (Note that (2.3) is fulfilled for $(P, Q; \mathscr{D}) \in \mathscr{K}$ as differentiation shows.) Thus, by definition, canonical pairs $(P, Q; \mathscr{D}) \in \mathscr{K}$ [resp. $\mathscr{C}$] and $*$-representations $\pi \in \mathscr{K}$ [resp. $\mathscr{C}$] are in one-to-one correspondence. We are mainly working with $\pi \in \mathscr{C}$, rather than $(P, Q; \mathscr{D}) \in \mathscr{C}$.

4) Suppose $(P, Q; \mathscr{D}) \in \mathscr{C}$. Then both operators P and Q are unbounded. We only sketch the proof. Assume that Q is bounded. Since $P = \overline{P \upharpoonright \mathscr{D}}$, (2.3) extends by continuity to $\mathscr{D}(P)$. Let $\mathscr{D}_a$ be the set of all analytic vectors for P in $\mathscr{D}_\infty(P)$. Let $\varphi \in \mathscr{D}_a$. By (2.3), we have $\|P^n Q \varphi\| \leq \|Q\| \|P^n \varphi\| + n \|P^{n-1} \varphi\|$ for $n \in N$. This implies $Q\varphi \in \mathscr{D}_a$.

Hence the power series expansions of $e^{itP}\varphi$ and $e^{itP}Q\varphi$ are converging for small $|t|$. Therefore, (2.3) yields $Qe^{itP}\varphi=e^{itP}(Q-t)\varphi$ for small $|t|$. Since $\mathcal{D}_a$ is dense (P is self-adjoint), this is true for all $\varphi\in\mathcal{H}$ and all $t\in R_1$. Taking analytic vectors for Q, we obtain the Weyl relation. This contradicts to the boundedness of Q. In case P the proof is similar.

5) Our notation is somewhat unsymmetric. For $(P, Q; \mathcal{D})\in\mathcal{K}$, we have $\mathcal{D}=\mathcal{D}(P)$ by Definition 1. That is, P is closable, but not closed. In case $(P, Q; \mathcal{D})\in\mathcal{C}$ P and Q denote closed operators.

6) A $*$-representation π of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}$ is called *integrable with respect to the Weyl relation* or briefly *integrable* if $P:=\overline{\pi(\boldsymbol{p})}$ and $Q:=\overline{\pi(\boldsymbol{q})}$ are self-adjoint operators in $\mathcal{H}$ satisfying the Weyl relation

$$e^{isP}e^{itQ}=e^{its}e^{itQ}e^{isP}, \quad s, t\in R_1, \tag{1}$$

and if

$$\mathcal{D}=\mathcal{D}_\infty(P)\cap\mathcal{D}_\infty(Q)\equiv\bigcap_{n=1}^{\infty}\mathcal{D}(P^n)\cap\mathcal{D}(Q^n). \tag{2}$$

The terminology comes from the representation theory of Lie groups (see [22], ch. 4, or [2], ch. 11). (1) means that the Lie algebra representation integrates to a unitary representation, say U, of the corresponding Lie group (the Heisenberg group). According to a result of Goodman ([7] or [22], p. 273), (2) says that the domain $\mathcal{D}$ is exactly the space $\mathcal{D}_\infty(U)$ of all C^∞-vectors for U.

It is well-known ([4], [15]) that each integrable $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ is a direct sum of Schrödinger pairs $P=-i\frac{d}{dx}$, $Q=x$, $\mathcal{D}=\mathcal{S}(R_1)$.

7) It is easy to see that an integrable representation is self-adjoint ([14]) and, moreover, in $\mathcal{K}$ and in $\mathcal{C}$. Conversely, if π is a self-adjoint $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ which is in $\mathcal{K}$ and in $\mathcal{C}$, then π is integrable.

We outline the proof of the second assertion. Since $\pi\in\mathcal{K}$ and $P:=\overline{\pi(\boldsymbol{p})}$, $Q:=\overline{\pi(\boldsymbol{q})}$ are self-adjoint by (2.4), (2.6), P and Q satisfy the Weyl relation (1) ([9] or [17], 2.2; Remark 4). Let U be the corresponding representation of the Heisenberg group. Let dU be the associated $*$-representation of $\mathcal{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}_\infty(U)$. Of course, $\pi \subseteq dU$. Because π is self-adjoint and hence maximal, we conclude that $\pi = dU$ and $\mathcal{D} = \mathcal{D}_\infty(U)$, i.e., π is integrable.

8) Fuglede [6] first constructed e.s.a. operators P and Q which satisfy the commutation relation (2.3), but not the Weyl relation. Another example of this kind can be found in [16], VIII, 5.

2.3. Lemma *Suppose that* $(P, Q; \mathcal{D}) \in \mathcal{K}$ *whereby* (1.3) *is replaced by* (1.3)′.

Then (1.3)′ *is true for all multipliers for* $\mathcal{S}(R_1)$. *That is, if* g *is a multiplier for* $\mathcal{S}(R_1)$ *and* $\varphi \in \mathcal{D}$, *then* $g(Q)\varphi \in \mathcal{D}$ *and* $Pg(Q)\varphi - g(Q)P\varphi = -ig'(Q)\varphi$. *Moreover,* $g(Q)\varphi$ *is in the* $t_{\mathcal{A}}$*-closure of* $\mathcal{D}_\varphi := \{f(Q)\varphi, f \in C_0^\infty(R_1)\}$.

Proof. (See the proof of Lemma 11 in [17]) Let g be a multiplier for $\mathcal{S}(R_1)$, i.e., $g \in C^\infty(R_1)$ and g and each of its derivatives is polynomially bounded ([23], p. 90). Let $k, r \in N_0$. We write Q' instead of $Q \upharpoonright \mathcal{D}$. There are numbers $C_r \in R_1$ and $s_r \in N$ such that $|g^{(j)}(x)| \leq C_r(1+x^2)^{s_r}$ for all $x \in R_1$ and $j = 0, \cdots, r$. In particular, $\mathcal{D} \subseteq \mathcal{D}(Q^k g^{(j)}(Q))$ for $j = 0, \cdots, r$. We choose a function $\omega \in C_0^\infty(R_1)$ so that $\omega(x) \equiv 1$ on $[-1, 1]$ and $\operatorname{supp} \omega \subseteq [-2, 2]$. Put $f_\delta(x) := \omega(x\delta)$ for $0 < \delta < 1$ and $M := \sup\{|\omega^{(j)}(x)|; x \in R_1, j = 0, \cdots, r\}$. Fix a vector $\varphi \in \mathcal{D}$. Combining (1.3)′ with the Leibniz rule, we obtain

$$(3) \qquad Q'^k P^r (gf_\delta)(Q)\varphi = \sum_{n=0}^{r} \binom{r}{n} (-i)^m f_\delta(Q) Q^k g^{(n)}(Q) P^{r-n}\varphi$$
$$+ \sum_{n=1}^{r} \sum_{j=0}^{n-1} \binom{r}{n}\binom{n}{j} (-i)^n f_\delta^{(n-j)}(Q) Q^k g^{(j)}(Q) P^{r-n}\varphi.$$

Let ψ_δ and ξ_δ denote the vectors on the right-hand side of (3). Since $|f_\delta^{(n-j)}(x)| \leq \delta M$ for $j = 0, \cdots, n-1$ and $n = 1, \cdots, r$, it follows that $\|\xi_\delta\| \leq \text{const.} \sum_{n=1}^{r} M\delta C_r \|Q^k(I+Q^2)^{s_r} P^{r-n}\varphi\|$ and hence $\lim_{\delta \to +0} \xi_\delta = 0$. On the other side, we obviously have $\lim_{\delta \to +0} \psi_\delta = \sum_{n=1}^{r} \binom{r}{n} (-i)^m Q^k g^{(n)}(Q) P^{r-n}\varphi$ and $\lim_{\delta \to +0} (gf_\delta)(Q)\varphi = g(Q)\varphi$. Since $Q'^k P^r$ is closable, we obtain $g(Q)\varphi \in \mathcal{D}(\overline{Q'^k P^r})$ and

$$(4) \qquad \overline{Q'^k P^r}\, g(Q)\varphi = \sum_{n=0}^{r} \binom{r}{n} (-i)^n Q^k g^{(n)}(Q) P^{r-n}\varphi$$
$$= \lim_{\delta \to +0} Q'^k P^r (gf_\delta)(Q)\varphi .$$

Setting $k=0$, $r=1$ and $g(x)\equiv x$, we see that $\bar{P}Q\psi = PQ\psi = QP\psi - i\psi$ for all $\psi \in \mathcal{D}$. Therefore, $\mathcal{A} = \mathrm{Lin}\{Q'^k P^r;\ k, r \in N_0\}$. Since $\mathcal{A} = \mathcal{A}(P, Q')$ is closed on $\mathcal{D}$ by (1.2), this implies $g(Q)\varphi \in \mathcal{D}$ and $Pg(Q)\varphi - g(Q)P\varphi = -ig'(Q)\varphi$. (4) exactly means that $g(Q)\varphi = t_{\mathcal{A}} - \lim_{\delta \to +0} (gf_\delta)(Q)\varphi$ which completes the proof.

§ 3. Examples

3.1. The example is in the spirit of the famous example constructed by Nelson [11] (see also ([13]).

Example 1. We consider the following one-parameter unitary groups in $\mathcal{H} = L_2(R_2)$:

$$(U(t)\varphi)(x, y) = e^{itx}\varphi(x, y+t), \quad t \in R_1,$$

$$(V(s)\varphi)(x, y) = \begin{cases} \varphi(x+s, y) & \text{for } y>0,\ x\geq 0 \text{ and } y>0,\ x+s<0 \\ z\varphi(x+s, y) & \text{for } y>0,\ x<0 \text{ and } x+s\geq 0 \\ \varphi(x+s, y) & \text{for } y\leq 0,\ x\in R_1 \end{cases}$$

for $s>0$ and similarly for $s<0$. z is a fixed complex number so that $|z|=1$ and $z\neq 1$. In other words, $V(s)$ is the translation in x-direction with the following modification: If the positive y-axis is crossed, then the function is multiplied by z.

The infinitesimal generators of $U(t)$ and $V(s)$ are given by

$$iQ = ix + \frac{\partial}{\partial y} \quad \text{and} \quad iP = \frac{\partial}{\partial x}$$

where the functions in $\mathcal{D}(P)$ satisfy the boundary condition $\varphi(-0, y) = z\varphi(+0, y)$ for $y>0$. Let $\mathcal{D}$ be the set of all $\varphi \in L_2(R_2)$ such that φ is a C^∞-function on the manifold with boundary obtained from $R_2 \backslash (0, 0)$ by cutting up along the positive y-axis and

$$\frac{\partial^n \varphi}{\partial x^n}(-0, y) = z\frac{\partial^n \varphi}{\partial x^n}(+0, y) \quad \text{for all } n \in N_0 \text{ and } y>0.$$

Then we have

(i) $(P, Q; \mathcal{D}) \in \mathscr{C}$.

Proof. (2.1)-(2.3) are obvious. To prove (2.4), it suffices (by Nelson's theorem) to show that $\mathcal{D}_a(P)$ is dense in $\mathcal{H}$. Let $\eta(x, y)$ be the function on R_2 which is z for $x>0$, $y>0$ and $+1$ otherwise. Each finite sum $\sum_j \eta(x, y) f_j(x) g_j(y)$ where $f_j \in \mathscr{S}(R_1)$ are analytic vectors for $-i\frac{d}{dx}$ and $g_j \in C_0^\infty(R_1 \setminus \{0\})$ is in $\mathcal{D}$ and an analytic vector for P. Because this set is dense in $\mathcal{H}$, $\mathcal{D}_a(P)$ is dense in $\mathcal{H}$. In a similar way we see that $\mathcal{D}_a(Q)$ is dense in $\mathcal{H}$.

Moreover, the operators $P^n \restriction \mathcal{D}$ and $Q^n \restriction \mathcal{D}$ are e.s.a. for all $n \in N$. Given $s \geq 0$, $t \geq 0$, let $\mathcal{R}(s, t) := \{(x, y) \in R_2 : 0 < x \leq s, 0 < y \leq t\}$. From the definition of $U(t)$, $V(s)$ it follows that

(ii) $W_{s,t}\varphi := (I - e^{-its} V(-s) U(-t) V(s) U(t))\varphi = (1-z)\chi_{\mathcal{R}(s,t)}\varphi$ *for* $\varphi \in \mathcal{H}$ *and* $s \geq 0$, $t \geq 0$ *and similarly in the other cases.*

Because $1-z \neq 0$, (ii) shows that P and Q do not satisfy the Weyl relation. Therefore, the corresponding representation $\pi \in \mathscr{C}$ of $\mathcal{A}(\boldsymbol{p}, \boldsymbol{q})$ is not integrable.

(iii) π *is irreducible.*

Proof. Suppose $C \in \mathcal{A}(P \restriction \mathcal{D}, Q \restriction \mathcal{D})'_s$. Since $P \restriction \mathcal{D}$ and $Q \restriction \mathcal{D}$ are e.s.a., C commutes with $U(t)$, $V(s)$ and hence with $W_{s,t}$ for all $s, t \in R_1$. From (ii) we conclude that $L_\infty(R_2) = \{W_{s,t};\ s, t \in R_1\}''$. Hence $C \in L_\infty(R_2)'$. Since $L_\infty(R_2)$ is maximal commutative, there is a function $\psi \in L_\infty(R_2)$ so that $C\varphi = \psi\varphi$ for all $\varphi \in \mathcal{H}$. $CU(t) = U(t)C$ and $CV(s) = V(s)C$ for $s, t \in R_1$ imply that the function ψ is a constant almost everywhere. This completes the proof.

Using the idea from Example 1 it is not difficult to construct many inequivalent irreducible canonical pairs of the class $\mathscr{C}$. Because we later prove a quite stronger result (Theorem 7.1), we only indicate the construction and omit the proofs.

Example 1′. Let $\mathbf{I}$ be a non-empty denumerable index set. Let

$z_n, n \in \mathbb{I}$, be complex numbers so that $|z_n|=1$ and $z_n \neq 1$ for all $n \in \mathbb{I}$, and let $\mathfrak{T}_n, n \in \mathbb{I}$, be non-closed Jordan arcs in R_2 from $A_n=(a_n, b_n)$ to $B_n=(c_n, d_n)$ satisfying some (rather general) technical conditions. (For instance, it suffices that they lie "discrete" in some sense.) The unitary groups $U(t)$ and $V(s)$ are defined similarly as in Example 1. That, is, $U(t)$ is the translation in y-direction and multiplication by e^{itx} and $V(s)$ is the translation in x-direction both of which combined with the following rule: In crossing a curve $\mathfrak{T}_n, n \in \mathbb{I}$, the function will be multiplied by z_n. Similarly as in Example 1, we obtain a representation, say π, of the class $\mathscr{C}$. It is clear that $P:=\overline{\pi(\boldsymbol{p})}$ and $Q:=\overline{\pi(\boldsymbol{q})}$ do not fulfill the Weyl relation, since $z_n \neq 1$ for $n \in \mathbb{I}$.

Let π and π' be two representations of $\mathcal{A}(\boldsymbol{p}, \boldsymbol{q})$ which are defined as indicated above by means of sequences $A_n=(a_n, b_n)$, $B_n=(c_n, d_n)$, $\mathfrak{T}_n, z_n, n \in \mathbb{I}$, resp. $A'_n=(a'_n, b'_n)$, $B'_n=(c'_n, d'_n)$, $\mathfrak{T}'_n, z'_n, n \in \mathbb{I}'$. Then it can be shown that:

(iv) *π is not irreducible if and only if there is a permutation τ of $\mathbb{I}$ and a positive number a so that $a_n=a+a_{\tau(n)}$, $b_n=b_{\tau(n)}$, $c_n=a+c_{\tau(n)}$, $d_n=d_{\tau(n)}$, $z_n=z_{\tau(n)}$ for all $n \in \mathbb{I}$.*

(v) *π is unitarily equivalent to π' if and only if there is a one-to-one map τ of $\mathbb{I}$ on $\mathbb{I}'$ and a number $a \in R_1$ so that $a_n=a+a'_{\tau(n)}$, $b_n=b'_{\tau(n)}$, $c_n=a+c'_{\tau(n)}$, $d_n=d'_{\tau(n)}$, $z_n=z'_{\tau(n)}$) for all $n \in \mathbb{I}$.*

Roughly speaking, (v) means that $\pi \cong \pi'$ iff after new enumeration and translation in x-direction the "sequence" $\{A_n, B_n, z_n, n \in \mathbb{I}\}$ coincides with $\{A'_n, B'_n, z'_n, n \in \mathbb{I}'\}$. In particular, the path $\mathfrak{T}_n$ from A_n to B_n occurs neither in (iv) nor in (v).

3.2. The next example is based on a different idea.

Example 2. Let $\mathcal{R}_{\alpha\beta}:=\{(x, y) \in R_2: 0 \leq x \leq \alpha, 0 \leq y \leq \beta\} \setminus (0, 0) \cup (\alpha, 0) \cup (0, \beta) \cup (\alpha, \beta)$ where $\alpha>0$ and $\beta>0$ are fixed. Again we consider two one-parameter unitary groups in the Hilbert space $\mathcal{H}=L_2(\mathcal{R}_{\alpha\beta})$ defined by

$$(U(t)\varphi)(x, y)=e^{itx}\varphi(x, (y-t)^{\sim})$$

$$(V(s)\varphi)(x, y) = \left\{ \begin{array}{lll} \varphi(x+s, y) & \text{for} & 0 \leq x+s < \alpha \\ e^{i\alpha y}\varphi(x+s-\alpha, y) & \text{for} & x+s \geq \alpha \end{array} \right\}$$

for $0 < s < \alpha$ and similarly for all real s. Here $(y-t)^{\sim}$ is determined by $(y-t)^{\sim} \in [0, \beta)$ and $(y-t)^{\sim} \equiv y-t \mod \beta$.

The generators of $U(t)$ and $V(s)$ are $iQ = ix - \frac{\partial}{\partial y}$ with boundary condition $\varphi(x, 0) = \varphi(x, \beta)$ and $iP = \frac{\partial}{\partial x}$ with boundary condition $e^{i\alpha y}\varphi(0, y) = \varphi(\alpha, y)$.

Let $\mathcal{D}$ be all $\varphi \in \mathcal{H}$ such that $\varphi \in C^{\infty}(\mathcal{R}_{\alpha,\beta})$,

$$\frac{\partial^n \varphi}{\partial y^n}(x, 0) = \frac{\partial^n \varphi}{\partial y^n}(x, \beta) \quad \text{for} \quad 0 < x < \alpha,\ n \in N_0,$$

and

$$e^{i\alpha y}\frac{\partial^n \varphi}{\partial x^n}(0, y) = \frac{\partial^n \varphi}{\partial x^n}(\alpha, y) \quad \text{for} \quad 0 < y < \beta,\ n \in N_0.$$

We then have

(i) $(P, Q; \mathcal{D}) \in \mathscr{C}$.

Proof. Using the boundary condition for P we conclude that $Q\mathcal{D} \subseteq \mathcal{D}$ and $P\mathcal{D} \subseteq \mathcal{D}$. (2.2) and (2.3) are clear. As in Example 1, the subspaces $\mathcal{D}_a(P)$ and $\mathcal{D}_a(Q)$ of $\mathcal{D}$ are dense in $\mathcal{H}$. We only carry out the proof for $\mathcal{D}_a(P)$. Each function $\varphi = \psi(y)\exp(ixy + 2\pi kx/\alpha)$ where $k \in Z$ and $\psi \in C_0^{\infty}(0, \beta)$ is in $\mathcal{D}$ and an analytic vector for P. The linear span of these functions is dense in $\mathcal{H}$ and still contained in $\mathcal{D}_a(P)$. Therefore, $\mathcal{D}_a(P)$ is dense in $\mathcal{H}$ and $(P, Q; \mathcal{D}) \in \mathscr{C}$.

The corresponding $*$-representation of $\mathcal{A}(\boldsymbol{p}, \boldsymbol{q})$ is denoted by $\pi_{\alpha,\beta}$. As in Example 1, the operators $P^n \restriction \mathcal{D}$ and $Q^n \restriction \mathcal{D}$ are e.s.a. for $n \in N$. From the definition of $U(t)$, $V(s)$ it follows that for $0 \leq s \leq \alpha$, $0 \leq t \leq \beta$

(ii) $W_{s,t}\varphi \equiv (I - e^{-its}V(-s)U(-t)V(s)U(t))\varphi = (1 - e^{-i\alpha\beta})\chi_{\mathcal{N}(s,t)}\varphi$ *where* $\mathcal{N}(s, t) := \{(x, y) \in \mathcal{R}_{\alpha\beta}\colon 0 \leq x < s, \beta - t \leq y < \beta\}$.

Now we are able to discuss the integrability of $\pi_{\alpha,\beta}$. First suppose that $\alpha\beta = 2\pi k$ for all $k \in N$. Then, by (ii), P and Q do not satisfy the Weyl relation and hence $\pi_{\alpha,\beta}$ is not integrable. Suppose now that $\alpha\beta = 2\pi k$ for some $k \in N$. Then $\pi_{\alpha,\beta}$ is integrable and unitarily equivalent to a direct sum of k Schrödinger pairs. We verify this in case $k=1$. Let U be the isometry from $L_2(R_1)$ on $L_2(\mathcal{R}_{\alpha,\beta})$ defined by

$$(U\varphi)(x,y)\equiv\sum_{n=-\infty}^{+\infty}(U\chi_{[n,n+1)}\varphi)(x,y)$$

$$=\sum_{n=-\infty}^{+\infty}\alpha^{-1/2}(\chi_{[n,n+1)}\varphi)(x-n\cdot\alpha)\exp(2\pi iny/\alpha).$$

It is easy to check that $(P,Q;\mathcal{D})$ and the Schrödinger pair on $\mathcal{S}(R_1)$ are unitarily equivalent via U. Thus we obtained a new realization $\pi_{\alpha,2\pi/\alpha}$ of the Schrödinger pair in $L_2(\mathcal{R}_{\alpha,2\pi/\alpha})$.

Next we decide on the irreducibility and the unitary equivalence.

(iii) *$\pi_{\alpha,\beta}$ is irreducible iff $\alpha\beta\neq 2\pi k$ for all $k\in N$, $k\geq 2$.*

Proof. By the preceding discussion, it suffices to show that $\pi_{\alpha,\beta}$ is irreducible in case $\alpha\beta\neq 2\pi k$ for $k\in N$. For let $C\in\mathcal{A}(P\upharpoonright\mathcal{D},Q\upharpoonright\mathcal{D})'_s$. As in Example 1, C commutes with all operators $W_{s,t}$ and because $1-e^{-i\alpha\beta}\neq 0$ with all multiplication operators $\chi_{\mathcal{N}(s,t)}$. Now the proof is the same as in Example 1.

(iv) *If $\alpha\beta=2\pi k$ for some $k\in N$, then $\pi_{\alpha,\beta}\cong\pi_{\alpha',\beta'}$ iff $\alpha'\beta'=2\pi k$. If $\alpha\beta\neq 2\pi k$ for all $k\in N$, then $\pi_{\alpha,\beta}\cong\pi_{\alpha',\beta'}$ iff $\alpha=\alpha'$ and $\beta=\beta'$.*

Proof. First let $\alpha\beta=2\pi k$, $k\in N$. As mentioned above, $\pi_{\alpha,\beta}$ is unitarily equivalent to a direct sum of k Schrödinger pairs. Hence $\pi_{\alpha,\beta}\cong\pi_{\alpha',\beta'}$ iff $\pi_{\alpha',\beta'}$ is unitarily equivalent to a direct sum of k Schrödinger pairs, that is, $\alpha'\beta'=2\pi k$. Now we treat the case in which $\alpha\beta=2\pi k$ for all $k\in N$. Suppose that $\pi_{\alpha,\beta}\cong\pi_{\alpha',\beta'}$ and the equivalence is implemented by a unitary operator U. Our aim is to show that $\alpha=\alpha'$ and $\beta=\beta'$. For suppose this were not the case. We will restrict ourselves to the case $\alpha'<\alpha$. (The case $\beta\neq\beta'$ is similar.) Since $\pi_{\alpha,\beta}(\boldsymbol{p})$ is e.s.a., $\pi_{\alpha,\beta}(\boldsymbol{p})=U^*\pi_{\alpha',\beta'}(\boldsymbol{p})U$ implies that $U(t)=U^*U'(t)U$ for $t\in R_1$. We will denote by $U'(t)$, $V'(s)$ and $W'_{s,t}$ the corresponding operators for $\pi_{\alpha',\beta'}$. Similarly, $V(s)=U^*V'(s)U$ for $s\in R_1$. Therefore, $W_{s,t}=U^*W'_{s,t}U$ for $s,t\in R_1$. Since $\alpha\beta\neq 2\pi k$ for all $k\in N$, we have $1-e^{-i\alpha\beta}\neq 0$ which gives $1-e^{-i\alpha'\beta'}\neq 0$ by (ii). Let $\beta''=\min(\beta,\beta')$. Put $\mathcal{N}=\{(x,y)\in R_2:$

$\alpha' \leq x \leq \alpha,\ 0 \leq y \leq \beta\}$. Take a non-zero vector φ in $L_2(\mathcal{R}_{\alpha,\beta})$ such that $\chi_{\mathcal{N}}\varphi = \varphi$. Then, by (ii), $U(t)\varphi$ is contained in the kernel of the operator $W_{\alpha',\beta''}$ for all $t \in R_1$. But, since $1 - e^{-i\alpha'\beta'} \neq 0$, there is no non-zero vector in $L_2(\mathcal{R}_{\alpha',\beta'})$ having this property. This contradiction ends the proof of (iv).

Finally, we state without proof

(v) *The operators* $P := \overline{\pi_{\alpha,\beta}(\boldsymbol{p})}$ *and* $Q := \overline{\pi_{\alpha,\beta}(\boldsymbol{q})}$ *have absolutely continuous spectra which are given by*

$$\sigma(P) = \bigcup_{n=-\infty}^{+\infty} [2\pi n/\alpha, \beta + 2\pi n/\alpha],$$

$$\sigma(Q) = \bigcup_{n=-\infty}^{+\infty} [2\pi n/\beta, \alpha + 2\pi n/\beta].$$

Having Example 2 we can easily construct new canonical pairs of the class $\mathscr{C}$ by gluing together finitely or infinitely many rectangles. We outline this method in case of two rectangles.

Example 2′. Let $\alpha, \beta, \gamma, \delta$ be positive numbers so that $\alpha > \gamma$ and $\beta > \delta$. Let $\mathcal{R} \equiv \mathcal{R}_{\alpha,\beta,\gamma,\delta}$ be the set $\{(x, y) \in R_2 :\ 0 \leq x \leq \alpha,\ 0 \leq y \leq \delta$ or $0 \leq x \leq \gamma,\ \delta \leq y \leq \beta\}$ without the 8 points $(0, 0)$, $(\gamma, 0)$, $(\alpha, 0)$, $(0, \delta)$, (γ, δ), (α, δ), $(0, \beta)$, (γ, β). The unitary groups $U(t)$, $V(s)$ in the Hilbert space $\mathcal{H} = L_2(\mathcal{R})$ are now defined by

$$(U(t)\varphi)(x, y) = \left\{ \begin{array}{ll} e^{itx}\varphi(x, (y-t)^{\sim}) & \text{for} \quad 0 \leq x \leq \gamma \\ e^{itx}\varphi(x, (y-t)^{\approx}) & \text{for} \quad \gamma < x \leq \alpha \end{array} \right\}$$

where $(y-t)^{\sim}$ and $(y-t)^{\approx}$ mean calculation modulo β resp. δ,

$$(V(s)\varphi)(x, y) = \left\{ \begin{array}{ll} \varphi(x+s, y) & \text{for} \quad 0 \leq x+s < \alpha,\ 0 \leq y \leq \delta \\ e^{i\alpha y}\varphi(x+s-\alpha, y) & \text{for} \quad \alpha \leq x+s,\ 0 \leq y \leq \delta \\ \varphi(x+s, y) & \text{for} \quad 0 \leq x+s < \gamma,\ \delta < y \leq \beta \\ e^{i\gamma y}\varphi(x+s-\gamma, y) & \text{for} \quad \gamma \leq x+s,\ \delta < y \leq \beta \end{array} \right\}$$

for $0 < s < \gamma$ and similarly for all $s \in R_1$. The infinitesimal generators of $U(t)$, $V(s)$ are $iQ = ix - \dfrac{\partial}{\partial y}$ and $iP = \dfrac{\partial}{\partial x}$ (of course, with corresponding boundary conditions). Let $\mathcal{D}$ be the set of all $\varphi \in L_2(\mathcal{R}) \cap C^{\infty}(\mathcal{R})$ satisfying the boundary conditions

$$\frac{\partial^n \varphi}{\partial y^n}(x, 0) = \frac{\partial^n \varphi}{\partial y^n}(x, \beta) \qquad \text{for} \quad 0 < x < \gamma, \ n \in N_0,$$

$$\frac{\partial^n \varphi}{\partial y^n}(x, 0) = \frac{\partial^n \varphi}{\partial y^n}(x, \delta) \qquad \text{for} \quad \gamma < x < \alpha, \ n \in N_0,$$

$$e^{i\alpha y} \frac{\partial^n \varphi}{\partial x^n}(0, y) = \frac{\partial^n \varphi}{\partial x^n}(\alpha, y) \qquad \text{for} \quad 0 < y < \delta, \ n \in N_0,$$

$$e^{i\gamma y} \frac{\partial^n \varphi}{\partial x^n}(0, y) = \frac{\partial^n \varphi}{\partial x^n}(\gamma, y) \qquad \text{for} \quad \delta < y < \beta, \ n \in N_0.$$

Again we obtain a canonical pair $(P, Q; \mathcal{D}) \in \mathcal{C}$. Independently of α, β, γ, δ the Weyl relation is not satisfied provided that $\alpha > \gamma$ and $\beta > \delta$. It should be noted that in this case the spectrum of $W_{s,t}$, $0 < s < \gamma$, $0 < t < \delta$, has a non-trivial absolutely continuous part.

Conculding remarks: 1) Let π_1 and π_2 denote arbitrary representations of the class $\mathcal{C}$ as defined in 3.1 and 3.2, respectively. It can be shown that π_1 and π_2 are not unitarily equivalent.

2) If the plane is replaced by a rectangle, then the construction in 3.1 works as well. Thus it is possible to combine the methods of 3.1 with that of 3.2 and to construct new canonical pairs $(P, Q; \mathcal{D}) \in \mathcal{C}$.

§ 4. Associated Canonical Pairs of the Class $\mathcal{K}$

In this section we take up the classification of the canonical pairs $(P, Q; \mathcal{D}) \in \mathcal{C}$. An important (but not very difficult) step is to show that for each $(P, Q; \mathcal{D}) \in \mathcal{C}$ there is a largest pair $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$ which is a restriction of $(P, Q; \mathcal{D})$.

Proposition 4.1. *Suppose that $(P, Q; \mathcal{D}) \in \mathcal{C}$. Let $\mathcal{A} = \mathcal{A}(P \upharpoonright \mathcal{D}, Q \upharpoonright \mathcal{D})$. Let $\mathcal{D}_1$ be the closure of $\mathcal{D}_a(Q)$ in $\mathcal{D}[t_{\mathcal{A}}]$ and let $P_1 := P \upharpoonright \mathcal{D}_1$. Then $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$. If $(P_2, Q_2; \mathcal{D}_2) \in \mathcal{K}$ is another pair in the same Hilbert space such that $P_2 \subseteq P$, $Q_2 \subseteq Q$, $\mathcal{D}_2 \subseteq \mathcal{D}$, then $\mathcal{D}_2 \subseteq \mathcal{D}_1$ (and of course $P_2 \subseteq P_1$ and $Q_2 = Q$).*

Proof. We first prove that $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$.

Let $\varphi \in \mathcal{D}_a(Q)$. Let $U(t)=e^{itQ}$, $t\in R_1$. By definition, there is a positive constant M (depending on φ!) such that $\|Q^n\varphi\| \leq M^n n!$ for all $n\in N_0$. We can assume, without loss of generality, that $M\geq 1$. Let $k\in N_0$. By the commutation relation (2.3), we obtain for $n\in N$, $n\geq 2k$,

$$\begin{aligned}\|P^kQ^n\varphi\|^2 &= \langle Q^nP^{2k}Q^n\varphi, \varphi\rangle \\ &= \sum_{j=0}^{2k}\binom{2k}{j}(-i)^j n(n-1)\cdots(n-j+1)\langle P^{2k-j}\varphi, Q^{2n-j}\varphi\rangle \\ &\leq \sum_{j=0}^{2k}\binom{2k}{j}\|P^{2k-j}\varphi\| n(n-1)\cdots(n-j+1)M^{2n-j}(2n-j)! \\ &\leq \rho_k M^{2n}(2n)! \leq \rho_k(2M)^{2n}(n!)^2,\end{aligned}$$

that is,

$$\|P^kQ^n\varphi\| \leq \rho_k^{1/2}(2M)^n n!, \tag{1}$$

where $\rho_k := \sum_{j=0}^{2k}\binom{2k}{j}\|P^{2k-j}\varphi\|$. Furthermore,

$$\|Q^kQ^n\varphi\| \leq M^k(n+k)\cdots(n+1)M^n n! \quad \text{for} \quad n\in N. \tag{2}$$

Now fix a $t\in R_1$ so that $|t|2M<1$. By Lemma 1.1 in [17], the graph topology $t_{\mathcal{A}}$ is generated by the seminorms $\|\cdot\|_{P^k} := \|P^k\cdot\|$ and $\|\cdot\|_{Q^k} := \|Q^k\cdot\|$, $k\in N_0$. From (1) and (2) we therefore conclude that the sequence $\left\{S_m\varphi := \sum_{n=0}^{m}\frac{1}{n!}(itQ)^n\varphi;\ m\in N\right\}$ is a Cauchy sequence in $\mathcal{D}[t_{\mathcal{A}}]$. By (2.2), $\mathcal{D}[t_{\mathcal{A}}]$ is complete. Since $U(t)\varphi = \lim_m S_m\varphi$ in the Hilbert space norm of $\mathcal{H}$ (because $|t|M<1$), we obtain $U(t)\varphi = t_{\mathcal{A}} - \lim_m S_m\varphi$ and $U(t)\varphi\in\mathcal{D}$. Since $U(t)\varphi$ is trivially an analytic vector for Q, the latter gives $U(t)\varphi\in\mathcal{D}_a(Q)$. Obviously, by (2), $Q\varphi\in\mathcal{D}_a(Q)$. Using (1) and (2.3), we get

$$\|Q^nP\varphi\| \leq \|PQ^n\varphi\| + n\|Q^{n-1}\varphi\| \leq \rho_1^{1/2}(2M)^n n! + M^{n-1}n! \quad \text{for } n\in N. \tag{3}$$

Hence $P\varphi\in\mathcal{D}_a(Q)$. Moreover, (3) implies that $U(t)P\varphi = \lim_m S_mP\varphi$ in $\mathcal{H}$. Since $PS_m\varphi = S_mP\varphi + tS_{m-1}\varphi$ by the commutation rule, we therefore obtain $PU(t)\varphi = U(t)P\varphi + tU(t)\varphi$. Thus we have shown that

$$U(t)\varphi\in\mathcal{D}_a(Q) \quad \text{and} \quad PU(t)\varphi = U(t)(P+t)\varphi, \tag{4}$$

whenever $|t|<(2M)^{-1}$. Since $\varphi' := U(t)\varphi$ is an analytic vector for Q

with the same constant M, we can replace φ by φ' and obtain $U(t'+t)\varphi \in \mathcal{D}_a(Q)$ and $PU(t'+t)\varphi = U(t'+t)(P+t'+t)\varphi$ for $|t|<(2M)^{-1}$ and $|t'|<(2M)^{-1}$. Proceeding in this manner, (4) follows for all $t \in R_1$.

Except (1.2) all conditions of Definition 2.1 are satisfied for $\mathcal{D}_a(Q)$. Closing up in the graph topology $t_{\mathcal{A}}$ (recall that $U(t)$ is $t_{\mathcal{A}}$ -continuous because of (1.3)), we obtain a canonical pair $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$.

To prove the second part, suppose that $(P_2, Q_2; \mathcal{D}_2) \in \mathcal{K}$ and $P_2 \subseteq P$, $Q_2 \subseteq Q$, $\mathcal{D}_2 \subseteq \mathcal{D}$. Since Q_2 is self-adjoint in $\mathcal{H}$, $Q_2 = Q$. Let $\varphi \in \mathcal{D}_2$ and let $f \in C_0^\infty(R_1)$. By Proposition 3.1 in [17], we have $f(Q)\varphi \in \mathcal{D}_2$ and $P_2 f(Q)\varphi - f(Q)P_2\varphi = -if'(Q)\varphi$. Applying Lemma 2.3 to $(P_2, Q; \mathcal{D}_2) \in \mathcal{K}$ in case $g(x) \equiv 1$, we conclude that φ is in the $t_{\mathcal{A}_2}$-closure of $\mathcal{D}_\varphi := \{f(Q)\varphi; f \in C_0^\infty(R_1)\}$, where $\mathcal{A}_2 := \mathcal{A}(P_2 \restriction \mathcal{D}_2, Q \restriction \mathcal{D}_2)$. Since $P_2 \subseteq P$ and $\mathcal{D}_2 \subseteq \mathcal{D}$, we have $t_{\mathcal{A}} \restriction \mathcal{D}_2 = t_{\mathcal{A}_2}$. Clearly, $\mathcal{D}_\varphi \subseteq \mathcal{D}_a(Q) \subseteq \mathcal{D}_1$. Hence φ is in the $t_{\mathcal{A}}$ -closure of $\mathcal{D}_1$. Since $\mathcal{D}_1$ is $t_{\mathcal{A}}$-closed in $\mathcal{D}$, $\varphi \in \mathcal{D}_1$. This ends the proof.

4.2. Now we essentially use a theorem from our previous paper [17]. For precise statements in what follows we refer to [17].

Let $(P, Q; \mathcal{D})$ and $(P_1, Q; \mathcal{D}_1)$ be as in Proposition 1. Let $\{\Delta_n, n \in N\}$ denote the supporting sequence of $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$ as defined in [17], Ch. 4. Recall from [17] that Δ_n is an open subset of R_1 and that $\Delta_n \supseteq \Delta_{n+1}$ for each $n \in N$. We now assume that Q has a finite spectral multiplicity, say m, and that the set $\mathcal{J}$ of all end points of connected components of Δ_j for $j \in N$ has no finite limit point. Then, by Theorem 5.3 in [17], $(P_1, Q; \mathcal{D}_1) \in \mathcal{K}$ is unitarily equivalent to a $(P_3 \restriction \mathcal{D}_3; Q_3; \mathcal{D}_3) \in \mathcal{K}$ in the Hilbert space $\mathcal{H}_3 = \sum_{j=1}^{m}{}^{\oplus} L_2(\Delta_j)$, where $\mathcal{D}_3 := \bigcap_{k,r=0}^{\infty} \mathcal{D}(P_3^r Q_3^k)$, Q_3 is the multiplication operator by x and P_3 is the (closed symmetric) differential operator $-i\dfrac{d}{dx}$ with some boundary conditions at a certain subset of $\mathcal{J}$ and boundary values zero otherwise (see [17], ch. 5, for a precise definition of P_3). (The notation above differs slightly from that used in [17].)

Let $\overline{P_0}$ denotes the closed symmetric operator $-i\dfrac{d}{dx}$ in $\mathcal{H}_3$ with boundary values zero at all end points of connected components Δ_j, $j=1, \cdots, m$. Set $\mathcal{D}_0 := \bigcap_{k,r=0}^{\infty} \mathcal{D}((\overline{P_0})^r Q_3^k)$ and $P_0 := \overline{P_0} \restriction \mathcal{D}_0$. (The latter is justified because the closure of $\overline{P_0}\upharpoonright \mathcal{D}_0$ is indeed $\overline{P_0}$.) Then, $(P_0, Q_3; \mathcal{D}_0) \in \mathcal{K}$. [Indeed, Lemma 1.2 gives (1.1) and (1.2). (1.3) follows easily from $e^{itx}\mathcal{D}(\overline{P_0}) = \mathcal{D}(\overline{P_0})$ and $\overline{P_0}e^{itx}\varphi = e^{itx}(\overline{P_0}+t)\varphi$ for $\varphi \in \mathcal{D}(\overline{P_0})$ and $t \in R_1$.] Moreover, we have $P_0 \subseteq P_3$ and $\mathcal{D}_0 \subseteq \mathcal{D}_3$. In other words, under the above assumptions concerning Q and $\mathcal{J}$, we know all pairs $(P, Q; \mathcal{D}) \in \mathcal{C}$ if we can describe all canonical pairs in $\mathcal{C}$ which are extensions of $(P_0, Q_3; \mathcal{D}_0)$. The rest of the paper is devoted to the study of these pairs.

4.3. We now fix some notations which will be freely used in the remainder of this paper.

Let $\mathfrak{J}$ be a (non-empty) denumerable index set. For $n \in \mathfrak{J}$, let $a_n \in R_1 \cup \{-\infty\}$, $b_n \in R_1 \cup \{+\infty\}$, $a_n < b_n$. Throughout the paper, we always assume that the set of intervals (a_n, b_n), $n \in \mathfrak{J}$, satisfy the following condition:

$$(+) \qquad \inf_{n \in \mathfrak{J}} b_n - a_n > 0 .$$

Let $c \leq 1$ be a fixed positive number such that $b_n - a_n \geq c$ for all $n \in \mathfrak{J}$. Let $\mathcal{H} = \sum_{n \in \mathfrak{J}}^{\oplus} L_2(a_n, b_n)$. The elements of $\mathcal{H}$ are written as $\varphi = (\varphi_n) = (\varphi_n, n \in \mathfrak{J})$. Q and $\overline{P_0}$ denote the (self-adjoint) multiplication operator by x and the (closed symmetric) differential operator $-i\dfrac{d}{dx}$ with boundary values zero at all a_n and b_n, $n \in \mathfrak{J}$, in $\mathcal{H}$, respectively. Let $\mathcal{D}_0 := \bigcap_{k,r=0}^{\infty} \mathcal{D}((\overline{P_0})^r Q^k)$ and $P_0 := \overline{P_0} \upharpoonright \mathcal{D}_0$. Arguing as above, we conclude that $(P_0, Q; \mathcal{D}_0) \in \mathcal{K}$. Denote by π_0 the corresponding $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}_0$. Let $\mathcal{A}_0$ be the $Op*$-algebra $\mathcal{A}(P_0, Q \upharpoonright \mathcal{D}_0)$. Let $\mathcal{P}_0$ be the $Op*$-algebra of all polynomials in $\overline{P_0} \upharpoonright \mathcal{D}_\infty(\overline{P_0})$ on the domain $\mathcal{D}_\infty(\overline{P_0})$. As mentioned already, in the remainder of the paper we are mainly concerned with canonical pairs $(P, Q; \mathcal{D}) \in \mathcal{C}$ which extend the fixed pair $(P_0, Q; \mathcal{D}_0) \in \mathcal{K}$. A complete description of these pairs will be given in Section 5. Sections 7 and 9 are devoted to the construction of such pairs.

It should be noted that the assumtions in 4.2 and in 4.3 are not the same. If the set $\mathcal{J}$ has no finite limit point, then $(+)$ is not fulfilled in general (Example: $\mathfrak{J} = N$, $a_n = \sum_{j=1}^{\infty} 1/j$, $b_n = a_{n+1}$). On the other hand, the situation described in 4.3 is more general. It includes pairs $(P_0, Q; \mathcal{D}_0)$

where the spectral multiplicity of Q is infinite.

§ 5. Classification of Canonical Pairs of the Class $\mathscr{C}$

5.1. In this subsection the domains $\mathscr{D}(\mathscr{P}_0^*)$ and $\mathscr{D}(\mathscr{A}_0^*)$ will be studied.

Since $P_0^* = (\overline{P_0} \upharpoonright \mathscr{D}_\infty(\overline{P_0}))^*$, we have $\mathscr{D}(\mathscr{P}_0^*) = \bigcap_{r=1}^{\infty} \mathscr{D}((P_0^*)^r)$. By Lemma 1.1, $\mathscr{D}(\mathscr{A}_0^*) = \bigcap_{k,r=0}^{\infty} \mathscr{D}(Q^k(P_0^*)^r)$, because obviously $Q = (Q \upharpoonright \mathscr{D}_0)^*$. It is clear that $\mathscr{D}((P_0^*)^r)$ is the set of all $\varphi = (\varphi_n) \in \mathscr{H}$ for which the distributive derivatives $\varphi^{(j)} \equiv (\varphi_n^{(j)})$, $j = 1, \cdots, r$, are in $\mathscr{H}$. Therefore,

$$\mathscr{D}(\mathscr{P}_0^*) = \{\varphi = (\varphi_n) \in \mathscr{H} : \varphi^{(r)} \in \mathscr{H} \text{ for all } r \in N\} \tag{1}$$

and

$$\mathscr{D}(\mathscr{A}_0^*) = \{\varphi = (\varphi_n) \in \mathscr{H} : x^k \varphi^{(r)} \in \mathscr{H} \text{ for all } k, r \in N_0\}. \tag{2}$$

Now suppose that $\varphi = (\varphi_n) \in \mathscr{D}((P_0^*)^r)$. It is well-known that the limits $\varphi_n^{(j)}(a_n+)$ and $\varphi_n^{(j)}(b_n-)$ exist for all $n \in \mathfrak{J}$ and $j \in N_0$, $j \leq r-1$. Moreover, if $a_n = -\infty$ resp. $b_n = +\infty$, then $\varphi_n^{(j)}(a_n+) = 0$ resp. $\varphi_n^{(j)}(b_n-) = 0$ for $n \in \mathfrak{J}$ and $j \in N_0$, $j \leq r-1$. If $\varphi \in \mathscr{D}(\mathscr{P}_0^*)$, then these assertions are true for all $j \in N_0$. In particular, we then have $\varphi_n \in C^\infty[a_n, b_n]$ for all $n \in \mathfrak{J}$. We need some more notation.

Let $\mathfrak{J}^+ = \{n \in \mathfrak{J} : a_n \neq -\infty\}$, $\mathfrak{J}^- = \{n \in \mathfrak{J} : b_n \neq +\infty\}$. Let $B_j^+(\varphi) = \{\varphi_n^{(j)}(a_n+), n \in \mathfrak{J}^+\}$, $B_j^-(\varphi) = \{\varphi_n^{(j)}(b_n-), n \in \mathfrak{J}^-\}$ for $\varphi \in \mathscr{D}((P_0^*)^r)$ and $j = 0, \cdots, r-1$ and let $B^\pm(\varphi) = (B_j^\pm(\varphi), j \in N_0)$ for $\varphi \in \mathscr{D}(\mathscr{P}_0^*)$. Suppose that $\mathscr{D}$ is a linear subspace of $\mathscr{H}$. For $\mathscr{D} \subseteq \mathscr{D}((P_0^*)^r)$, let $\mathfrak{B}_r^\pm(\mathscr{D})$ denote the set of all r-tuple $(B_0^\pm(\varphi), \cdots, B_{r-1}^\pm(\varphi))$, $\varphi \in \mathscr{D}$. For $\mathscr{D} \subseteq \mathscr{D}(\mathscr{P}_0^*)$, $\mathfrak{B}^\pm(\mathscr{D})$ is the set of all $B^\pm(\varphi)$, $\varphi \in \mathscr{D}$. $\mathfrak{B}_r^+(\mathscr{D})$, $\mathfrak{B}_r^-(\mathscr{D})$, $\mathfrak{B}^+(\mathscr{D})$ and $\mathfrak{B}^-(\mathscr{D})$ are vector spaces under point-wise addition and multiplication with complex numbers. Let $\mathfrak{B}(\mathscr{D})$ be the algebraic direct sum of the vector spaces $\mathfrak{B}^+(\mathscr{D})$ and $\mathfrak{B}^-(\mathscr{D})$. Let B be the mapping $\mathscr{D} \ni \varphi \to (B^+(\varphi), B^-(\varphi)) \in \mathfrak{B}(\mathscr{D})$.

$l_2(\mathfrak{J}^\pm)$ is the l_2-space of the index set $\mathfrak{J}^\pm$ with the Hilbert space norm $|||\cdot|||$ and the scalar product $(\cdot, \cdot)$. Let $l_2^n(\mathfrak{J}^\pm)$, $n \in N$, be the orthogonal direct sum $l_2(\mathfrak{J}^\pm) \oplus \cdots \oplus l_2(\mathfrak{J}^\pm)$ (n times). We let $\mathfrak{a}$ denote the diagonal operator in $l_2(\mathfrak{J}^\pm)$ acting on the standard orthobase $\mathfrak{e}_k := \{\delta_{kn}, n \in \mathfrak{J}^+\}$ by $\mathfrak{a}\mathfrak{e}_k = a_k \mathfrak{e}_k$. $\mathfrak{t}_a$ denotes the locally convex topology on

$\mathcal{D}_\infty(\mathfrak{a})$ generated by the seminorms $|||\varphi|||_{\mathfrak{a}^n} := |||\mathfrak{a}^n\varphi|||$, $n\in N_0$. Similarly, $\mathfrak{b}$ and $\mathfrak{t}_\mathfrak{b}$ are defined in $l_2(\mathfrak{J}^-)$. Let $\mathfrak{L}^+$, $\mathfrak{L}^-$, $\mathfrak{L}^+_\infty$ and $\mathfrak{L}^-_\infty$ be the vector spaces (with pointwise operations) of all sequences $(\mathfrak{x}_0, \mathfrak{x}_1, \cdots)$, where $\mathfrak{x}_j\in l_2(\mathfrak{J}^+)$, $\mathfrak{x}_j\in l_2(\mathfrak{J}^-)$, $\mathfrak{x}_j\in \mathcal{D}_\infty(\mathfrak{a})$ and $\mathfrak{x}_j\in\mathcal{D}_\infty(\mathfrak{b})$, respectively, for all $j\in N_0$. We denote by $\mathfrak{L}$ (resp. $\mathfrak{L}_\infty$) the direct sum of the vector spaces $\mathfrak{L}^+$ and $\mathfrak{L}^-$ (resp. $\mathfrak{L}^+_\infty$ and $\mathfrak{L}^-_\infty$). $\mathfrak{L}$ and $\mathfrak{L}_\infty$ will be endowed with the product topologies $\mathfrak{t}$ and $\mathfrak{t}_\infty$ with respect to the spaces $l_2(\mathfrak{J}^+)$, $l_2(\mathfrak{J}^-)$ and $\mathcal{D}_\infty(\mathfrak{a})[\mathfrak{t}_\mathfrak{a}]$, $\mathcal{D}_\infty(\mathfrak{b})[\mathfrak{t}_\mathfrak{b}]$, respectively.

Lemma 1. (i) $\mathfrak{B}(\mathcal{D}(\mathcal{P}_0^*))\subseteq\mathfrak{L}$, $\mathfrak{B}(\mathcal{D}(\mathcal{A}_0^*))\subseteq\mathfrak{L}_\infty$.
(ii) *The linear mappings* B: $\mathcal{D}(\mathcal{P}_0^*)[t_{\mathcal{P}_0^*}]\to\mathfrak{L}[\mathfrak{t}]$ *and* B: $\mathcal{D}(\mathcal{A}_0^*)[t_{\mathcal{A}_0^*}]\to\mathfrak{L}_\infty[\mathfrak{t}_\infty]$ *are continuous.*

Proof. Suppose $\varphi=(\varphi_n)\in\mathcal{D}(\mathcal{P}_0^*)$. Let $j\in N_0$. For this proof, let $\|\cdot\|_n$ denote the norm of the Hilbert space $L_2(a_n, a_n+c)$, $n\in\mathfrak{J}^+$. For $t\in(a_n, a_n+c)$ and $n\in\mathfrak{J}^+$, we have

$$|\varphi_n^{(j)}(a_n+)|^2=\left|-\varphi_n^{(j)}(t)+\int_{a_n}^t\varphi_n^{(j+1)}(x)\,dx\right|^2$$
$$\leq 2|\varphi_n^{(j)}(t)|^2+2c\|\varphi_n^{(j+1)}\|_n^2$$

and by integration on (a_n, a_n+c)

$$|\varphi_n^{(j)}(a_n+)|^2c\leq 2\|\varphi_n^{(j)}\|_n^2+2c^2\|\varphi_n^{(j+1)}\|_n^2\,.$$

Since $b_n-a_n\geq c$ for $n\in\mathfrak{J}$ and $\varphi\in\mathcal{D}((P_0^*)^{j+1})$ by (1), we obtain

$$|||B_j^+(\varphi)|||^2=\sum_{n\in\mathfrak{J}^+}|\varphi_n^{(j)}(a_n+)|^2\leq 2c^{-1}\|(P_0^*)^j\varphi\|^2+2c\|(P_0^*)^{j+1}\varphi\|^2 \tag{3}$$

and similarly

$$|||B_j^-(\varphi)|||^2\leq 2c^{-1}\|(P_0^*)^j\varphi\|^2+2c\|(P_0^*)^{j+1}\varphi\|^2\,. \tag{4}$$

Therefore, $B_j^\pm(\varphi)\in l_2(\mathfrak{J}^\pm)$ for each $j\in N_0$, that is, $B(\varphi)\in\mathfrak{L}$. Moreover, (3) and (4) show the continuity of B: $\mathcal{D}(\mathcal{P}_0^*)[t_{\mathcal{P}_0^*}]\to\mathfrak{L}[\mathfrak{t}]$.

Now suppose that $\varphi\in\mathcal{D}(\mathcal{A}_0^*)$. Let $k, n\in N_0$. Since $Q^k(P_0^*)^n\varphi\in\mathcal{D}(\mathcal{P}_0^*)$ we can replace φ by $Q^k(P_0^*)^n\varphi$ in (3), (4). Setting $j=0$ and using $(-i)^n\mathfrak{a}^kB_n^+(\varphi)=B_0^+(Q^k(P_0^*)^n\varphi)$ and $(-i)^n\mathfrak{b}^kB_n^-(\varphi)=B_0^-(Q^k(P_0^*)^n\varphi)$, it follows from (3), (4) that $\mathfrak{a}^kB_n^+(\varphi)\in l_2(\mathfrak{J}^+)$, $\mathfrak{b}^kB_n^-(\varphi)\in l_2(\mathfrak{J}^-)$, i.e., $B(\varphi)\in\mathfrak{L}_\infty$. (3) and (4) estimate $|||\mathfrak{a}^kB_n^+(\varphi)|||$ and $|||\mathfrak{b}^kB_n^-(\varphi)|||$

by a sum of seminorms of the form $\|\varphi\|_A$, $A \in \mathcal{A}_0^*$, as well. This gives the continuity of B: $\mathcal{D}(\mathcal{A}_0^*)[t_{\mathcal{A}_0^*}] \to \mathfrak{L}_\infty[\mathfrak{t}_\infty]$ and completes the proof.

Part (i) of the next lemma can be considered as a version of a classical result due to E. Borel (see, for example, [21], p. 390).

Lemma 2. (i) $\mathfrak{B}(\mathcal{D}(\mathcal{P}_0^*)) = \mathfrak{L}$, $\mathfrak{B}(\mathcal{D}(\mathcal{A}_0^*)) = \mathfrak{L}_\infty$.

(ii) *If* $\mathcal{D}$ *is a closed linear subspace of* $\mathcal{D}(\mathcal{A}_0^*)[t_{\mathcal{A}_0^*}]$ *such that* $\mathcal{D} \supseteq \mathcal{D}_0$, *then* $\mathfrak{B}(\mathcal{D})$ *is a closed linear subspace of* $\mathfrak{L}_\infty[\mathfrak{t}_\infty]$.

Remark. A similar assertion as (ii) is true for $\mathcal{P}_0^*$ as well.

Proof. The topologies $t_{\mathcal{A}_0^*}$ and $\mathfrak{t}_\infty$ are generated by the directed systems of seminorms $n_{k,m}(\varphi) := \sup\{\|t^l \varphi^{(j)}(t)\|;\ l=0, \cdots, k, j=0, \cdots, m\}$ resp. $s_{k,m}(\mathfrak{x}) := \sup\{|||(|\mathfrak{a}|+1)^k \mathfrak{x}_j^\pm|||;\ j=0, \cdots, m\}$, $k, m \in N_0$. Throughout this proof, $\mathfrak{x}$ will be of the form $\mathfrak{x} = (\mathfrak{x}^+, \mathfrak{x}^-)$, $\mathfrak{x}^\pm = (\mathfrak{x}_0^\pm, \mathfrak{x}_1^\pm, \cdots)$, $\mathfrak{x}_j^\pm = \{x_{jn}^\pm, n \in \mathfrak{J}^\pm\}$. $(|\mathfrak{a}|+1)^k \mathfrak{x}_j^\pm$ means the sequence $\{(|a_n|+1)^k x_{jn}^\pm, n \in \mathfrak{J}^\pm\}$. Now let $\varepsilon > 0$ and $k, m \in N_0$. We show that there is a $\delta = \delta(\varepsilon, k, m) > 0$ such that for any $\mathfrak{x} \in \mathfrak{L}_\infty$ with $s_{k,m}(\mathfrak{x}) \leq \delta$ there exists a $\xi \in \mathcal{D}(\mathcal{A}_0^*)$ such that $B(\xi) = \mathfrak{x}$ and $n_{k,m+1}(\varphi) \leq 4\varepsilon$.

Let $\omega(t)$ be a fixed C^∞-function on R_1 such that

$$\omega(t) \equiv 1 \text{ for } t \leq 1/2 \quad \text{and} \quad \omega(t) \equiv 0 \text{ for } t \geq 1. \tag{5}$$

Let $M_j := \sup\{|\omega^{(j)}(t)|;\ t \in R_1\}$ and $\omega_{rn}(t) := \dfrac{1}{r!}(t-a_n)^r \omega((t-a_n)/\rho_r)$ for $j, r \in N_0$ and $n \in \mathfrak{J}^+$. By substituting $t' = t - a_n$ we see that the numbers $C_{r,j} := \|\omega_{rn}^{(j)}(t)\|^2_{L_2(a_n, b_n)}$ do not depend on n. Let us take a $\delta > 0$ such that

$$\delta^2 \sum_{r=0}^{m} 2^{r+1} C_{r,j} \leq \varepsilon^2 \quad \text{for } j = 0, \cdots, m+1. \tag{6}$$

The numbers ρ_r, $r \in N_0$, will be chosen so that

(a) $0 < \rho_{r+1} < \rho_r/2 < c/2$ for $r \in N_0$,

(b) $\rho_r \left(\sum_{l=j+1}^{r-1} |||\mathfrak{x}_l^+||| + |||\mathfrak{x}_r^+||| \sum_{l=0}^{j} \binom{j}{l} M_l \right) \leq 2^{-r}$

for $r, j \in N_0$ with $r \geq j+2$ and $r \geq m+1$,

(c) $\rho_r^{1/2} \||(|\mathfrak{a}|+1)^k \mathfrak{x}_r^+\|| \sum_{l=0}^{j} \binom{j}{l} M_l \leq 2^{-r}\varepsilon$

for $r, j \in N_0$ with $r \geq j$ and $r \geq m+1$.

Note that δ does not depend on $\mathfrak{x}$ because $\rho_0, \cdots, \rho_m$ have this property. Let $\varphi = (\varphi_n)$ be the vector which is defined by $\varphi_n(t) \equiv 0$ on (a_n, b_n) for $a_n = -\infty$ and

$$\varphi_n(t) = \sum_{r=0}^{\infty} x_{rn}^+ \omega_{rn}(t) \qquad \text{on } (a_n, b_n) \text{ for } n \in \mathfrak{J}^+. \tag{7}$$

From (5) and (a) it follows that (7) is a finite sum on each closed interval contained in (a_n, b_n). In particular, $\varphi_n \in C^\infty(a_n, b_n)$ for $n \in \mathfrak{J}^+$.

Statement I. $\varphi_n^{(j)}(a_n+) = x_{jn}^+$ for $j \in N_0$ and $n \in \mathfrak{J}^+$.

Proof. First we note that for $r \geq j$ and $t \in (a_n, b_n)$

$$|\omega_{rn}^{(j)}(t)| = \left| \sum_{l=0}^{j} \binom{j}{l} \frac{1}{(r-j+l)!} (t-a_n)^{r-j+l} \rho_r^{-l} \omega^{(l)}((t-a_n)/\rho_r) \right| \tag{8}$$

$$\leq \rho_r^{r-j} \sum_{l=0}^{j} \binom{j}{l} M_l$$

and thus

$$C_{r,j} \leq \rho_r \left(\sum_{l=0}^{j} \binom{j}{l} M_l \right)^2 \quad \text{for } r \geq j. \tag{9}$$

If $t \in (a_n + \rho_{r+1}, a_n + \rho_r)$ and $r-1 \geq j+1$, then (5), (a) and (8), (b) imply

$$|\varphi_n^{(j)}(t) - x_{jn}^+| = \left| \sum_{l=j+1}^{r-1} x_{ln}^+ (t-a_n)^{l-j}/(l-j)! + x_{rn}^+ \omega_{rn}^{(j)}(t) \right|$$

$$\leq \rho_r \left(\sum_{l=j+1}^{r-1} \||\mathfrak{x}_l^+\|| + \||\mathfrak{x}_r^+\|| \sum_{l=0}^{j} \binom{j}{l} M_l \right) \leq 2^{-r}.$$

Here we also used that $\rho_r < c \leq 1$. Letting $r \to +\infty$, we obtain the assertion.

Statement II. $\varphi \in \mathcal{D}(\mathcal{A}_0^*)$ and $\nu_{k,m+1}(\varphi) \leq 2\varepsilon$.

Proof. We show that $t^l\varphi^{(j)}(t)\in\mathcal{H}$ for $l, j\in N_0$. Set $k'=\max(l,k)$ and $m'=\max(j,m+1)$. Using the Cauchy-Schwarz inequality, $\rho_r<1$ for $r\in N_0$ and finally (9) and (c), we obtain

$$\|t^l\varphi^{(j)}\|^2\leq\sum_{n\in\mathfrak{Z}^+}\Big(\sum_{r=0}^{\infty}\|t^l x_{rn}^+\omega_{rn}^{(j)}(t)\|_{L_2(a_n,b_n)}\Big)^2$$

$$\leq\sum_{n\in\mathfrak{Z}^+}\Big(\sum_{r=0}^{\infty}2^r(|a_n|+1)^{2k'}|x_{rn}^+|^2C_{r,j}\Big)\Big(\sum_{r=0}^{\infty}2^{-r}\Big)$$

$$=\sum_{r=0}^{\infty}|||(|\mathfrak{a}|+1)^{k'}\mathfrak{x}_r^+|||^2 2^{r+1}C_{r,j}\leq\mathfrak{a}_{k'm'}(\mathfrak{x})^2\sum_{r=0}^{m'}2^{r+1}C_{r,j}$$

$$+\sum_{r=m'+1}^{\infty}2^{-r+1}\varepsilon^2.$$

Therefore, $t^l\varphi^{(j)}\in\mathcal{H}$. By (2), we have $\varphi\in\mathcal{D}(\mathcal{A}_0^*)$. Now suppose that $l\leq k$ and $j\leq m+1$. Then $k'=k$ and $m'=m+1$. Combined with (6) the above estimation shows that $\|t^l\varphi^{(j)}\|^2\leq 3\varepsilon^2$ because we assumed that $\mathfrak{a}_{k,m}(\mathfrak{x})\leq\delta$. Consequently, $\mathfrak{r}_{k,m+1}(\varphi)\leq 2\varepsilon$ and the proof of statement II is complete.

Statement I means that $B^+(\varphi)=\mathfrak{x}^+$. (5) and (a) yield $B^-(\varphi)=0$. Similarly, we can find a vector $\psi\in\mathcal{D}(\mathcal{A}_0^*)$ with $B^+(\psi)=0$, $B^-(\psi)=\mathfrak{x}^-$ and $\mathfrak{r}_{k,m+1}(\psi)\leq 2\varepsilon$. Then $\xi=\varphi+\psi$ has the desired properties. Multiplying $\mathfrak{x}\in\mathfrak{L}_\infty$ by a suitable constant, it follows that $\mathfrak{B}(\mathcal{D}(\mathcal{A}_0^*))=\mathfrak{L}_\infty$. This ends the proof of part (i).

To prove (ii), let $\mathcal{D}$ be a $t_{\mathcal{A}_0^*}$-closed submanifold of $\mathcal{D}(\mathcal{A}_0^*)$. Suppose that $\mathfrak{z}\in\mathfrak{L}_\infty$ is in the $\mathfrak{t}_\infty$-closure of $\mathfrak{B}(\mathcal{D})$. According to (i), there is a $\xi\in\mathcal{D}(\mathcal{A}_0^*)$ with $B(\xi)=\mathfrak{z}$. It suffices to show that ξ is in the $t_{\mathcal{A}_0^*}$-closure of $\mathcal{D}$. For let $\varepsilon>0$ and $k, m\in N_0$. There is a vector $\psi\in\mathcal{D}$ so that $\mathfrak{a}_{k,m}(\mathfrak{z}-B(\psi))=\mathfrak{a}_{k,m}(B(\xi-\psi))\leq\delta(\varepsilon,k,m)$. Applying the preceding proof in the case $\mathfrak{x}:=B(\xi-\psi)$, we get a $\zeta\in\mathcal{D}(\mathcal{A}_0^*)$ with $\mathfrak{r}_{k,m+1}(\zeta)\leq 4\varepsilon$ and $B(\zeta)=\mathfrak{x}=B(\xi-\psi)$. Since $B(\varphi)=0$ for $\varphi:=\zeta-\xi+\psi$, $\varphi\in\mathcal{D}_0$. We have $\mathfrak{r}_{k,m+1}(\zeta)=\mathfrak{r}_{k,m+1}(\xi-(\psi-\varphi))\leq 4\varepsilon$. On the other hand, $\mathcal{D}_0\subseteq\mathcal{D}$ implies $\psi-\varphi\in\mathcal{D}$. Therefore we conclude that ξ is in the $t_{\mathcal{A}_0^*}$-closure of $\mathcal{D}$. Thus $\xi\in\mathcal{D}$ because $\mathcal{D}$ is $t_{\mathcal{A}_0^*}$-closed. This completes the proof of (ii).

We mention an easy by-product of the preceding proof. Of course, in this special case the main part of the proof of Lemma 2 is not needed.

Lemma 3. *Let A be a closed symmetric operator such that $P_0 \subseteq A \subseteq P_0^*$. Let $\mathcal{D}$ be a linear subspace of $\mathcal{D}(A^n)$, $n \in N$. Let $T := \overline{A^n \upharpoonright \mathcal{D}}$. Then, $\mathfrak{B}_n^{\pm}(\mathcal{D}(T)) \subseteq l_2^n(\mathfrak{J}^{\pm})$ and $\mathfrak{B}_n^{\pm}(\mathcal{D}(T)) = \overline{\mathfrak{B}_n^{\pm}(\mathcal{D})}$ where the closure is taken in the Hilbert space norm of $l_2^n(\mathfrak{J}^{\pm})$.*

Proof. The proof of Lemma 1 shows that $B_j^{\pm}(\varphi) \in l_2(\mathfrak{J}^{\pm})$ if $\varphi \in \mathcal{D}((P_0^*)^n)$ and $j = 0, \cdots, n-1$. Hence $\mathfrak{B}_n^{\pm}(\mathcal{D}(T)) \subseteq l_2^n(\mathfrak{J}^{\pm})$.

Now let $\varphi \in \mathcal{D}(T)$. Then there are vectors $\varphi_m \in \mathcal{D}$, $m \in N$, so that $\varphi_m \to \varphi$ and $A^n \varphi_m \to T\varphi = A^n \varphi$ in $\mathcal{H}$. Because A is a symmetric linear operator, this implies that $A^j \varphi_m \to A^j \varphi$ in $\mathcal{H}$ for all $j = 1, \cdots, n$. Since $A^j \varphi = (P_0^*)^j \varphi$ and $A^j \varphi_m = (P_0^*)^j \varphi_m$, (3) and (4) yield $B_j^{\pm}(\varphi_m) \to B_j^{\pm}(\varphi)$ in $l_2(\mathfrak{J}^{\pm})$ for $j = 0, \cdots, n-1$. Therefore, $\mathfrak{B}_n^{\pm}(\mathcal{D}(T)) \subseteq \overline{\mathfrak{B}_n^{\pm}(\mathcal{D})}$.

In order to verify the converse inclusion, we first observe that $\mathcal{D}((\overline{P_0})^n) = \{\varphi \in \mathcal{D}((P_0^*)^n) : B_j^{\pm}(\varphi) = 0 \text{ for } j = 0, \cdots, n-1\}$. Hence the argument used in the proof of part (ii) of Lemma 2 applies and gives $\overline{\mathfrak{B}_n^{\pm}(\mathcal{D})} \subseteq \mathfrak{B}_n^{\pm}(\mathcal{D}(T))$.

5.2. Now we describe the closed symmetric extensions and the self-adjoint extensions of $\overline{P_0}$ in terms of the boundary values.

Suppose that W is a partial isometry of $l_2(\mathfrak{J}^+)$ into $l_2(\mathfrak{J}^-)$ with initial space $\mathcal{W}$. This means that W is a bounded linear operator of $l_2(\mathfrak{J}^+)$ into $l_2(\mathfrak{J}^-)$ which is isometric on the closed linear subspace $\mathcal{W}$ of $l_2(\mathfrak{J}^+)$ and zero on the orthogonal complement of $\mathcal{W}$. Let $\mathcal{D}(P_W) := \{\varphi \in \mathcal{D}(P_0^*) : B_0^+(\varphi) \in \mathcal{W},\ B_0^-(\varphi) \in W\mathcal{W} \text{ and } WB_0^+(\varphi) = B_0^-(\varphi)\}$ and $P_W := P_0^* \upharpoonright \mathcal{D}(P_W)$.

Lemma 4. (i) *P_W is a closed symmetric operator with $\overline{P_0} \subseteq P_W$. Conversely, for each closed symmetric operator P with $P \supseteq P_0$ there exists a unique partial isometry W of $l_2(\mathfrak{J}^+)$ into $l_2(\mathfrak{J}^-)$ such that $P = P_W$.*

(ii) *P_W is self-adjoint if and only if W is an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$.*

(iii) *Suppose that P_W is self-adjoint. Let $\mathcal{D}$ be a linear sub-*

space of $\mathcal{D}(P_W^n)$, $n\in N$. *Then,* $P_W^n\restriction\mathcal{D}$ *is e.s.a. if and only if* $\mathfrak{B}_n^+(\mathcal{D})$ *is dense in* $l_2^n(\mathfrak{J}^+)$.

Proof. (i) First let $\varphi,\psi\in\mathcal{D}(P_0^*)$. Since $\varphi(a_n+)=0$ for $a_n=-\infty$ and $\varphi(b_n-)=0$ for $b_n=+\infty$, partial integration yields

$$\langle P_0^*\varphi,\psi\rangle-\langle\varphi,P_0^*\psi\rangle=\sum_{n\in\mathfrak{J}^-}\varphi(b_n-)\overline{\psi(b_n-)}-\sum_{n\in\mathfrak{J}^+}\varphi(a_n+)\overline{\psi(a_n+)} \tag{10}$$

$$=(B_0^-(\varphi),B_0^-(\psi))-(B_0^+(\varphi),B_0^+(\psi)).$$

For $\varphi,\psi\in\mathcal{D}(P_W)$, the boundary terms in (10) are of course vanishing. Hence P_W is a symmetric linear operator. By (3) and (4), the mappings $\mathcal{D}(P_0^*)\ni\varphi\to B_0^\pm(\varphi)\in l_2(\mathfrak{J}^\pm)$ are continuous, relative to the graph norm of $\mathcal{D}(P_0^*)$. Since the initial space $\mathcal{W}$ is closed, this implies that P_W is a closed linear operator.

Now suppose that P is a closed symmetric extension of $\overline{P_0}$. From Lemma 3 (applied in the case $P=A$, $\mathcal{D}=\mathcal{D}(A)$, $n=1$) we know that $\mathcal{W}:=\mathfrak{B}_1^+(\mathcal{D}(P))$ and $\mathfrak{B}_1^-(\mathcal{D}(P))$ are closed linear subspaces of $l_2(\mathfrak{J}^+)$ resp. $l_2(\mathfrak{J}^-)$. Since P is symmetric, the right-hand side of (10) is vanishing for all $\varphi,\psi\in\mathcal{D}(P)$. Therefore, there exists a unique isometry, say W, from $\mathcal{W}$ onto $\mathfrak{B}_1^-(\mathcal{D}(P))$ such that $WB_0^+(\varphi)=B_0^-(\varphi)$ for $\varphi\in\mathcal{D}(P)$. Setting $W=0$ on $\mathcal{W}^\perp$, W becomes a partial isometry of $l_2(\mathfrak{J}^+)$ into $l_2(\mathfrak{J}^-)$ with initial space $\mathcal{W}$. Clearly, $P\subseteq P_W$. Because $\mathfrak{B}_1^+(\mathcal{D}(P))=\mathfrak{B}_1^+(\mathcal{D}(P_W))=\mathcal{W}$ and $\mathfrak{B}_1^-(\mathcal{D}(P))=\mathfrak{B}_1^-(\mathcal{D}(P_W))=W\mathcal{W}$ and $P\supseteq P_0$, it follows that $P=P_W$.

(ii) Suppose first that W is an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$ Suppose that $\psi\in\mathcal{D}((P_W))$. For $\varphi\in\mathcal{D}(P_W)$, (10) reads $\langle P_W\varphi,\psi\rangle-\langle\varphi,P_W\psi\rangle=0=(WB_0^+(\varphi),B_0^-(\psi))-(B_0^+(\varphi),B_0^+(\psi))$, i.e., $(B_0^+(\varphi),W^*B_0^-(\psi))=(B_0^+(\varphi),B_0^+(\psi))$. Since $\mathfrak{B}_1^+(\mathcal{D}(P_W))=l_2(\mathfrak{J}^+)$, we obtain $W^*B_0^-(\psi)=B_0^+(\psi)$. Therefore, $B_0^-(\psi)=WB_0^+(\psi)$ which means that $\psi\in\mathcal{D}(P_W)$. Hence $\mathcal{D}(P_W)=\mathcal{D}(P_W^*)$ and P_W is self-adjoint.

Conversely, assume that W is not an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$. Let $\mathcal{W}$ be the initial space of the partial isometry W. Then $\mathcal{W}\neq l_2(\mathfrak{J}^+)$ or $W\mathcal{W}\neq l_2(\mathfrak{J}^-)$. Without loss of generality we assume that $\mathcal{W}=l_2(\mathfrak{J}^+)$ (otherwise we replace W by W^*). Let $\mathfrak{x}$ be a non-zero vector in $l_2(\mathfrak{J}^+)$ which is orthogonal to $\mathcal{W}$. We choose a $\psi\in\mathcal{D}(P_0^*)$ with $B_0^-(\psi)=0$

and $B_0^+(\psi)=\mathfrak{x}$. Again by (10), $\langle P_W\varphi,\psi\rangle-\langle\varphi,P_0^*\psi\rangle=-(B_0^+(\varphi),\mathfrak{x})=0$ for $\varphi\in\mathcal{D}(P_W)$. Thus $\psi\in\mathcal{D}(P_W^*)$. On the other side, $\mathfrak{x}\notin\mathcal{W}$ gives $\psi\notin\mathcal{D}(P_W)$. Therefore, $\mathcal{D}(P_W)\neq\mathcal{D}(P_W^*)$, which completes the proof.

(iii) By Lemma 3, $\mathfrak{B}_n^+(\mathcal{D}(P_W^n))\subseteq l_2^n(\mathfrak{J}^+)$. Lemma 2 (applied in the trivial case where $\mathfrak{x}_j=0$ for $j\geq n$) yields $\mathfrak{B}_n^+(\mathcal{D}(P_W^n))=l_2^n(\mathfrak{J}^+)$.

If $P_W^n\upharpoonright\mathcal{D}$ is e.s.a., then $\overline{\mathfrak{B}_n^+(\mathcal{D})}=\mathfrak{B}_n^+(\mathcal{D}(P_W^n))$ by Lemma 3 and hence $\overline{\mathfrak{B}_n^+(\mathcal{D})}=l_2^n(\mathfrak{J}^+)$. To prove the opposite inclusion, we assume that $\overline{\mathfrak{B}_n^+(\mathcal{D})}=l_2^n(\mathfrak{J}^+)$. Let $T:=\overline{P_W^n\upharpoonright\mathcal{D}}$. Using again Lemma 3, we obtain $\overline{\mathfrak{B}_n^+(\mathcal{D})}=\mathfrak{B}_n^+(\mathcal{D}(T))$. It suffices to show that $\mathcal{D}(P_W^n)\subseteq\mathcal{D}(T)$. Let $\varphi\in\mathcal{D}(P_W^n)$. Since $\mathfrak{B}_n^+(\mathcal{D}(T))=\mathfrak{B}_n^+(\mathcal{D}(P_W^n))$, there is a $\psi\in\mathcal{D}(T)$ such that $B_j^+(\varphi)=B_j^+(\psi)$ for $j=0,\cdots,n-1$. Because $T\subseteq P_W^n$, this implies $B_j^-(\varphi)=B_j^-(\psi)$ for $j=0,\cdots,n-1$. Therefore, $\varphi-\psi\in\mathcal{D}((\overline{P_0})^n)\subseteq\mathcal{D}(T)$ and $\varphi=\psi+(\varphi-\psi)\in\mathcal{D}(T)$, which completes the proof.

5.3. In this subsection, we take up the classification of all representations $\pi\in\mathscr{C}$ which extend π_0. First we establish some terminology.

Suppose W is a partial isometry of $l_2(\mathfrak{J}^+)$ in $l_2(\mathfrak{J}^-)$ with initial space $\mathcal{W}$. Let us define

$$(11)\qquad \mathfrak{S}(W):=\{\mathfrak{x}\in\mathcal{D}_\infty(\mathfrak{a})\cap\mathcal{W}:W\mathfrak{x}\in\mathcal{D}_\infty(\mathfrak{b}) \text{ and } \mathfrak{b}^rW\mathfrak{x}=W\mathfrak{a}^r\mathfrak{x} \text{ for } r\in N\}.$$

Sometimes it will be convenient to use the following definition. $\mathfrak{S}(W)$ is the set of all $\mathfrak{x}\in\mathcal{D}_\infty(\mathfrak{a})\cap\mathcal{W}$ for which there exists a $\mathfrak{y}\in\mathcal{D}_\infty(\mathfrak{b})$ such that $\mathfrak{b}^r\mathfrak{y}=W\mathfrak{a}^r\mathfrak{x}$ for all $r\in N_0$. The equivalence of both definitions is obvious because $r=0$ yields $\mathfrak{y}=W\mathfrak{x}$. Further, let $\mathfrak{L}_\infty^+(W):=\{\mathfrak{x}=(\mathfrak{x}_0,\mathfrak{x}_1,\cdots)\in\mathfrak{L}_\infty^+:\mathfrak{x}_j\in\mathfrak{S}(W) \text{ for } j\in N_0\}$. Obviously, $\mathfrak{S}(W)$ is complete in the graph topology $\mathfrak{t}_\mathfrak{a}$. Hence $\mathfrak{L}_\infty^+(W)$ is closed linear subspace of $\mathfrak{L}_\infty^+[\mathfrak{t}_\infty]$.

Definition 1. *An isometry W of $l_2(\mathfrak{J}^+)$ in $l_2(\mathfrak{J}^-)$ is called a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$ if $\mathfrak{S}(W)$ is dense in the Hilbert space $l_2(\mathfrak{J}^+)$.*

In other words, an isometry of $l_2(\mathfrak{J}^+)$ on $l_2(\mathfrak{J}^-)$ is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$ iff there is a dense linear subspace $\mathfrak{D}$ of $l_2(\mathfrak{J}^+)$ such that $\mathfrak{D}\subseteq\mathcal{D}_\infty(\mathfrak{a})$, $\mathfrak{a}\mathfrak{D}\subseteq\mathfrak{D}$ and $\mathfrak{b}W\mathfrak{x}=W\mathfrak{a}\mathfrak{x}$ for all $\mathfrak{x}\in\mathfrak{D}$.

$\mathfrak{S}(W)$ is the largest linear subspace of $l_2(\mathfrak{F}^+)$ having this property. The existence of such operators for given unbounded sequences $\mathfrak{a}$ and $\mathfrak{b}$ has been investigated in [18]. In Section 7 we will come back to this point.

Define linear operators $\mathfrak{P}, \mathfrak{Q}: \mathfrak{L}_\infty^+ \to \mathfrak{L}_\infty^+$ by $\mathfrak{P}(\mathfrak{x}_0, \mathfrak{x}_1, \cdots) := (-i\mathfrak{x}_1, -i\mathfrak{x}_2, \cdots)$ and $\mathfrak{Q}(\mathfrak{x}_0, \mathfrak{x}_1, \cdots) := (\mathfrak{a}\mathfrak{x}_0, \mathfrak{a}\mathfrak{x}_1 + \mathfrak{x}_0, \cdots, \mathfrak{a}\mathfrak{x}_j + j\mathfrak{x}_{j-1}, \cdots)$. Obviously, $\mathfrak{P}\mathfrak{Q}\mathfrak{x} - \mathfrak{Q}\mathfrak{P}\mathfrak{x} = -i\mathfrak{x}$ for $\mathfrak{x} \in \mathfrak{L}^+$.

Definition 2. *A linear subspace $\mathfrak{M}$ of $\mathfrak{L}_\infty^+(W)$ is called an admissible boundary space (with respect to W, $\mathfrak{a}$ and $\mathfrak{b}$) if $\mathfrak{P}\mathfrak{M} \subseteq \mathfrak{M}$, $\mathfrak{Q}\mathfrak{M} \subseteq \mathfrak{M}$ and $\mathfrak{M}$ is a closed subspace of $\mathfrak{L}_\infty^+[\mathfrak{t}_\infty]$.*

We let $L_0(\mathfrak{M})$ denote the set of all first components $\mathfrak{x}_0$ for $(\mathfrak{x}_0, \mathfrak{x}_1, \cdots) \in \mathfrak{M}$.

Theorem 5. I. *Suppose W is a partial isometry of $l_2(\mathfrak{F}^+)$ in $l_2(\mathfrak{F}^-)$ and $\mathfrak{M}$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$, $\mathfrak{b}$. Let $\mathcal{D}_{W,\mathfrak{M}} := \{\varphi \in \mathcal{D}(\mathcal{A}_0^*) : B^+(\varphi) \in \mathfrak{M}$ and $WB_j^+(\varphi) = B_j^-(\varphi)$ for $j \in N_0\}$, $\pi_{W,\mathfrak{M}}(\boldsymbol{p}) := P_W \upharpoonright \mathcal{D}_{W,\mathfrak{M}}$ and $\pi_{W,\mathfrak{M}}(\boldsymbol{q}) := Q \upharpoonright \mathcal{D}_{W,\mathfrak{M}}$. Then:*

(i) *$\pi_{W,\mathfrak{M}}$ defines a closed $*$-representation of the Weyl algebra $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}_{W,\mathfrak{M}}$ which extends π_0. Also, $\mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}}) = \mathfrak{M}$.*

(ii) *$\pi_{W,\mathfrak{M}} \in \mathscr{C}$ if and only if W is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$ and $L_0(\mathfrak{M})$ is dense in $l_2(\mathfrak{F}^+)$.*

(iii) *A representation $\pi_{W,\mathfrak{M}} \in \mathscr{C}$ is self-adjoint if and only if $\mathfrak{M} = \mathfrak{L}_\infty^+(W)$.*

II. *If π is a closed $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}$ such that $\pi \supseteq \pi_0$, then there exist a partial isometry W of $l_2(\mathfrak{F}^+)$ in $l_2(\mathfrak{F}^-)$ and an admissible boundary space $\mathfrak{M}$ w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$ such that $\pi = \pi_{W,\mathfrak{M}}$, i.e., $\mathcal{D} = \mathcal{D}_{W,\mathfrak{M}}$ and $\pi(\boldsymbol{p}) = P_W \upharpoonright \mathcal{D}_{W,\mathfrak{M}}$.*

Let $\mathcal{D}_W$ and π_W denote the domain $\mathcal{D}_{W,\mathfrak{M}}$ and the representation $\pi_{W,\mathfrak{M}}$, respectively, in case $\mathfrak{M} = \mathfrak{L}_\infty^+(W)$.

First we will prove two lemmas.

Lemma 6. *Let π be a $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}$ which extends π_0. Suppose that $\overline{\pi(\boldsymbol{p})} = P_W$, where W is a partial isometry* *of* $l_2(\mathfrak{J}^+)$ *in* $l_2(\mathfrak{J}^-)$ *with initial space* $\mathcal{W}$. *Then* $\mathcal{D}(\pi^*) = \{\psi \in \mathcal{D}(\mathcal{A}_0^*) :$ $B_0^-(Q^r(P_0^*)^j\psi) - WB_0^+(Q^r(P_0^*)^j\psi) \perp W\mathcal{W}$ *in* $l_2(\mathfrak{J}^-)$ *for* $r, j \in N_0\}$.

Proof. Setting $S=\pi(\boldsymbol{p})$, $T=\pi(\boldsymbol{q})$ in Lemma 1.1, we obtain $\mathcal{D}(\pi^*) = \bigcap_{j,r=0}^{\infty} \mathcal{D}((P_W^*)^jQ^r)$. Since $B_0^+(\varphi) = W^*B_0^-(\varphi)$ for $\varphi \in \mathcal{D}(P_W)$, we see from (10) that a vector $\xi \in \mathcal{D}(P_0^*)$ is in $\mathcal{D}(P_W^*)$ if and only if $\langle P_W\varphi, \xi\rangle - \langle\varphi, P_0^*\xi\rangle = (B_0^-(\varphi), B_0^-(\xi)) - (W^*B_0^-(\varphi), B_0^+(\xi)) = (B_0^-(\varphi), B_0^-(\xi) - WB_0^+(\xi)) = 0$ for all $\varphi \in \mathcal{D}(P_W)$. By Lemma 3, $W\mathcal{W}$ is the closure of $\mathfrak{B}_1^-(\mathcal{D})$ in $l_2(\mathfrak{J}^-)$. Therefore, for $\psi \in \mathcal{D}(\mathcal{A}_0^*)$ we have $\psi \in \mathcal{D}(\mathcal{A}^*) = \bigcap_{j,r=0}^{\infty} \mathcal{D}((P_W^*)^jQ^r)$ if and only if $B_j^-(Q^r\psi) - WB_j^+(Q^r\psi) \perp W\mathcal{W}$ for $r, j \in N_0$. Since $(-i)^jB_j^+(Q^r\psi) = B_0^+((P_0^*)^jQ^r\psi)$ and $\mathrm{Lin}\{Q^r(P_0^*)^j\psi;\, r, j \in N_0\} = \mathrm{Lin}\{(P_0^*)^jQ^r\psi;\, r, j \in N_0\}$, the latter is equivalent to the above condition.

Lemma 7. *Let* W *be a partial isometry of* $l_2(\mathfrak{J}^+)$ *in* $l_2(\mathfrak{J}^-)$, *and let* $\psi \in \mathcal{D}(\mathcal{A}_0^*)$. *Then,* $\psi \in \mathcal{D}_W$ *if and only if* $B_j^+(\psi) \in \mathcal{W}$ *and* $B_0^-(Q^r(P_0^*)^j\psi) = WB_0^+(Q^r(P_0^*)^j\psi)$ *for all* $r, j \in N_0$.

Proof. Since $B_0^-(Q^r(P_0^*)^j\psi) = (-i)^j\mathfrak{b}^rB_0^-(\psi)$ and $B_0^+(Q^r(P_0^*)^j\psi) = (-i)^j\mathfrak{a}^rB_j^+(\psi)$, the above conditions are $\mathfrak{b}^rB_j^-(\psi) = W\mathfrak{a}^rB_j^+(\psi)$ and $B_j^+(\psi) \in \mathcal{W}$ for $r, j \in N_0$. By the second definition of $\mathfrak{S}(W)$, this is equivalent to $B_j^+(\psi) \in \mathfrak{S}(W)$ and $WB_j^+(\psi) = B_j^-(\psi)$ for $j \in N_0$, that is, $\psi \in \mathcal{D}_W$.

Proof of Theorem 5. We begin by proving (i).

First we show that $P_W\mathcal{D}_{W,\mathfrak{M}} \subseteq \mathcal{D}_{W,\mathfrak{M}}$ and $Q\mathcal{D}_{W,\mathfrak{M}} \subseteq \mathcal{D}_{W,\mathfrak{M}}$. From the definition it is clear that $\mathcal{D}_{W,\mathfrak{M}} \subseteq \mathcal{D}(P_W) \cap \mathcal{D}(Q)$. Since $\mathcal{D}(\mathcal{A}_0^*)$ is invariant under Q and P_0^* and $P_0^* \restriction \mathcal{D}_{W,\mathfrak{M}} = P_W \restriction \mathcal{D}_{W,\mathfrak{M}}$, it is sufficient to check that the conditions $B^+(\varphi) \in \mathfrak{M}$ and $WB_j^+(\varphi) = B_j^-(\varphi)$, $j \in N_0$, remain valid. Suppose that $\varphi \in \mathcal{D}_{W,\mathfrak{M}}$. Because $\mathfrak{P}\mathfrak{M} \subseteq \mathfrak{M}$ and $\mathfrak{Q}\mathfrak{M} \subseteq \mathfrak{M}$ by Definition 2, we have $B^+(P_W\varphi) = \mathfrak{P}B^+(\varphi) \in \mathfrak{M}$ and $B^+(Q\varphi) = \mathfrak{Q}B^+(\varphi) \in \mathfrak{M}$. Let $j \in N_0$. The definition of $\mathcal{D}_{W,\mathfrak{M}}$ shows that $B_j^+(\varphi) \in \mathfrak{S}(W)$. Letting $B_{-1}^+(\varphi) = 0$ for this proof and applying (11) with $\mathfrak{x} = B_j^+(\varphi)$, it follows that $WB_j^+(Q\varphi) = W\mathfrak{a}B_j^+(\varphi) + jWB_{j-1}^+(\varphi) = \mathfrak{b}WB_j^+(\varphi) + jWB_{j-1}^+(\varphi) = \mathfrak{b}B_j^-(\varphi) + jB_{j-1}^-(\varphi) = B_j^-(Q\varphi)$. Moreover, $WB_j^+(P_W\varphi) = -iWB_{j+1}^+(\varphi) = -iB_{j+1}^-(\varphi)$

$=WB_j^-(P_W\varphi)$. Therefore, $Q\mathcal{D}_{W,\mathfrak{M}}\subseteq\mathcal{D}_{W,\mathfrak{M}}$ and $P_W\mathcal{D}_{W,\mathfrak{M}}\subseteq\mathcal{D}_{W,\mathfrak{M}}$.

Since, of course, $P_WQ\varphi-QP_W\varphi=-i\varphi$ for $\varphi\in\mathcal{D}_{W,\mathfrak{M}}$, $\pi_{W,\mathfrak{M}}$ defines a $*$-representation of $A(\boldsymbol{p},\boldsymbol{q})$ on $\mathcal{D}_{W,\mathfrak{M}}$. $\mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}})=\mathfrak{M}$ follows immediately from Lemma 2, (i). It remains to prove that $\pi_{W,\mathfrak{M}}$ is closed. Assume that $\varphi=t_{\mathcal{A}_0^*}-\lim_k \varphi_k$ for $\varphi\in\mathcal{D}(\mathcal{A}_0^*)$ and $\varphi_k\in\mathcal{D}_{W,\mathfrak{M}}$, $k\in N$. Since B is continuous by Lemma 1, (ii), this implies $B(\varphi)=\mathrm{t}_\infty-\lim_k B(\varphi_k)$ and $WB_j^+(\varphi)=\lim_k WB_j^+(\varphi_k)=\lim_k B_j^-(\varphi_k)=B_j^-(\varphi)$. Since $B^+(\varphi_k)\in\mathfrak{M}$ and $\mathfrak{M}$ is closed by Definition 2, we obtain $B^+(\varphi)\in\mathfrak{M}$. Hence $\varphi\in\mathcal{D}_{W,\mathfrak{M}}$. This completes the proof of (i).

Next we consider (ii). Let $P:=\overline{P_W\upharpoonright\mathcal{D}_{W,\mathfrak{M}}}$. From the definition of the class $\mathscr{C}$ it is clear that $\pi_{W,\mathfrak{M}}\in\mathscr{C}$ iff P is self-adjoint. If P is self-adjoint, then there is an isometry W' of $l_2(\mathfrak{J}^+)$ on $l_2(\mathfrak{J}^-)$ such that $P=P_{W'}$. Of course, $P\equiv P_{W'}\subseteq P_W$ implies $W=W'$. Since $\mathfrak{B}_1^+(\mathcal{D}_{W,\mathfrak{M}})=L_0(\mathfrak{M})$, it follows from Lemma 3, (ii), that $L_0(\mathfrak{M})$ must be dense in $l_2(\mathfrak{J}^+)$. Because $L_0(\mathfrak{M})\subseteq\mathfrak{S}(W)$ by definition, W is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. The opposite direction follows in a similar way.

Now we prove (iii). Suppose that $\pi_{W,\mathfrak{M}}\in\mathscr{C}$. If $\mathfrak{M}\neq\mathfrak{L}_\infty^+(W)$, then π_W $(\equiv\pi_{W,\mathfrak{L}_\infty^+(W)})\neq\pi_{W,\mathfrak{M}}$ because $\mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}})=\mathfrak{M}\neq\mathfrak{L}_\infty^+(W)=\mathfrak{B}^+(\mathcal{D}_W)$. Moreover, π_W is a $*$-representation of $A(\boldsymbol{p},\boldsymbol{q})$ which extends $\pi_{W,\mathfrak{M}}$. Thus $\pi_{W,\mathfrak{M}}$ is not self-adjoint. Conversely, assume that $\mathfrak{M}=\mathfrak{L}_\infty^+(W)$, that is, $\pi_{W,\mathfrak{M}}=\pi_W$. Since $\pi_{W,\mathfrak{M}}\in\mathscr{C}$ by assumption, it follows from (ii) that W is an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$. Clearly, $\overline{\pi_W(\boldsymbol{p})}=P_W$ by (ii). Let $\psi\in\mathcal{D}(\pi_W^*)$. Since $W\mathcal{W}=l_2(\mathfrak{J}^-)$, Lemma 6 gives $B_0^-(Q^r(P_0^*)^j\psi)=WB_0^+(Q^r(P_0^*)^j\psi)$ for $r,j\in N_0$. Since $\mathcal{W}=l_2(\mathfrak{J}^+)$, Lemma 7 shows that $\psi\in\mathcal{D}_W$. Hence $\pi_W=\pi_W^*$, that is, π_W is self-adjoint.

Finally, we prove part II. Let π be a closed $*$-representation of $A(\boldsymbol{p},\boldsymbol{q})$ on $\mathcal{D}$. By Lemma 4, (i), there is a partial isometry W with initial space $\mathcal{W}$ such that $\overline{\pi(\boldsymbol{p})}=P_W$. Let $\mathfrak{M}:=\mathfrak{B}^+(\mathcal{D})$. Our aim is to show that $\mathfrak{M}$ is an admissible boundary space and $\pi=\pi_{W,\mathfrak{M}}$. Suppose that $\psi\in\mathcal{D}$. In particular, we have $\psi\in\mathcal{D}(\pi^*)$ and $P_0^*\psi=P_W\psi$. Hence Lemma 6 shows that $\mathfrak{x}:=B_0^-(Q^rP_W^j\psi)-WB_0^+(Q^rP_W^j\psi)\perp W\mathcal{W}$ in $l_2(\mathfrak{J}^-)$ for all $r,j\in N_0$. On the other side, $Q^rP_W^j\psi\in\mathcal{D}(P_W)$ yields $\mathfrak{x}\in W\mathcal{W}$. Therefore, $\mathfrak{x}=0$. Applying now Lemma 7, we obtain $\psi\in\mathcal{D}_W$ and thus $B^+(\varphi)\in\mathfrak{L}_\infty^+(W)$. Hence $\mathfrak{M}\subseteq\mathfrak{L}_\infty^+(W)$. From $B^+(P_W\psi)=\mathfrak{P}B^+(\psi)$ and $B^+(Q\psi)=\mathfrak{Q}B^+(\psi)$ we see that $\mathfrak{P}\mathfrak{M}\subseteq\mathfrak{M}$ and $\mathfrak{Q}\mathfrak{M}\subseteq\mathfrak{M}$. Since π is a closed representation, Lemma 2, (ii), shows that $\mathfrak{B}^+(\mathcal{D}) = \mathfrak{M}$ is $\mathfrak{t}_\infty$-closed in $\mathfrak{L}_\infty^+$. Therefore, $\mathfrak{M}$ is an admissible boundary space with respect to W, $\mathfrak{a}$ and $\mathfrak{b}$. We have $B^+(\psi) \in \mathfrak{M}$ by definition and $WB_j^+(\psi) = B_j^-(\psi)$ because of $\psi \in \mathcal{D}(P_W^j)$ for $\psi \in \mathcal{D}$, that is, $\mathcal{D} \subseteq \mathcal{D}_{W,\mathfrak{M}}$. Since $\mathfrak{B}^+(\mathcal{D}) = \mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}}) = \mathfrak{M}$ and also $\mathfrak{B}^-(\mathcal{D}) = \mathfrak{B}^-(\mathcal{D}_{W,\mathfrak{M}})$, we conclude that $\mathcal{D} = \mathcal{D}_{W,\mathfrak{M}}$. Now the the proof of Theorem 5 is complete.

Remarks. 1) In general, the partial isometry W in part II of Theorem 5 is not unique. One reason is that different partial isometries W, W' may have the same space $\mathfrak{S}(W) = \mathfrak{S}(W')$ and hence the same representations $\pi_W = \pi_{W'}$. However, for representations π of the class $\mathscr{C}$, $\overline{\pi(\boldsymbol{p})}$ is self-adjoint (by definition) and hence W is uniquely determined by π. In the general case we may assume without loss of generality that $\overline{\pi(\boldsymbol{p})} = P_W$. Then, by Lemma 4, (i), π defines W uniquely. From $\mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}}) = \mathfrak{M}$ we conclude that $\mathfrak{M}$ is always uniquely determined by π.

2) Theorem 5 shows that the representations $\pi \in \mathscr{C}$ which extend π_0 are uniquely characterized by a weak intertwining operator W for $\mathfrak{a}$ and $\mathfrak{b}$ and an admissible boundary space $\mathfrak{M}$ for which $L_0(\mathfrak{M})$ is dense in $l_2(\mathfrak{J}^+)$. Moreover, there is a one-to-one correspondence between self-adjoint extensions $\pi \in \mathscr{C}$ of π_0 and weak intertwining operators for $\mathfrak{a}$ and $\mathfrak{b}$.

3) It should be noted that there are self-adjoint extensions of π_0 which are not of the class $\mathscr{C}$. To construct examples of this kind in Section 7, we need the following

Corollary 8. *Let W and $\mathfrak{M}$ as in part I of Theorem* 5. *Suppose that $\overline{\pi_W(\boldsymbol{p})} = P_W$. Then, $\pi_{W,\mathfrak{M}}$ is self-adjoint if and only if the following two conditions are fulfilled*:

(a) $\mathfrak{M} = \mathfrak{L}_\infty^+(W)$.

(b) *If* $\mathfrak{x} \in \mathcal{D}_\infty(\mathfrak{a})$ *and* $\mathfrak{y} \in \mathcal{D}_\infty(\mathfrak{b})$ *satisfy* $\mathfrak{b}^r\mathfrak{y} - W\mathfrak{a}^r\mathfrak{x} \perp W\overline{\mathfrak{S}(W)}$ *for all* $r \in N_0$, *then* $\mathfrak{b}^r\mathfrak{y} = W\mathfrak{a}^r\mathfrak{x}$ *for* $r \in N_0$ *and* $\mathfrak{x} \in \mathscr{W}$ (*or equivalently,* $\mathfrak{x} \in \mathfrak{S}(W)$ *and* $\mathfrak{y} = W\mathfrak{x}$).

Proof. We have already seen that $\pi_{W,\mathfrak{M}}$ is not self-adjoint for $\mathfrak{M} \neq \mathfrak{L}_\infty^+(W)$. Suppose now that $\mathfrak{M} = \mathfrak{L}_\infty^+(W)$, i.e., $\pi_{W,\mathfrak{M}} = \pi_W$.

Assume that (b) is satisfied. Since $\overline{\pi_W(\boldsymbol{p})} = P_W$ by assumption, $\mathcal{W} = \overline{\mathfrak{B}_1^+(\mathcal{D}_W)} = \overline{\mathfrak{S}(W)}$ in $l_2(\mathfrak{J}^+)$ by Lemma 3. Let $\psi \in \mathcal{D}(\pi_W^*)$. Lemma 1 and Lemma 6 show that $\mathfrak{x} := B_j^+(\psi) \in \mathcal{D}_\infty(\mathfrak{a})$, $\mathfrak{y} := B_j^-(\psi) \in \mathcal{D}_\infty(\mathfrak{b})$ for $j \in N_0$ and $\mathfrak{b}^r\mathfrak{y} - W\mathfrak{a}^r\mathfrak{x} \perp W\mathcal{W}$ for $r \in N_0$. Combining now (b) and Lemma 7, we obtain $\psi \in \mathcal{D}_W$. Hence, π_W is self-adjoint.

Conversely, assume that (b) is not fulfilled, that is, there are $\mathfrak{x} \in \mathcal{D}_\infty(\mathfrak{a})$, $\mathfrak{y} \in \mathcal{D}_\infty(\mathfrak{b})$ and $k \in N_0$ such that $\mathfrak{b}^r\mathfrak{y} - W\mathfrak{a}^r\mathfrak{x} \perp W\mathcal{W}$ for all $r \in N_0$, but $\mathfrak{b}^k\mathfrak{y} \neq W\mathfrak{a}^k\mathfrak{x}$. We take a $\psi \in \mathcal{D}(\mathcal{A}_0^*)$ so that $B^+(\psi) = (\mathfrak{x}, 0, \cdots)$ and $B^-(\psi) = (\mathfrak{y}, 0, \cdots)$. Then, $\psi \in \mathcal{D}(\pi_W^*)$ by Lemma 6. Since $B_0^-(Q^k\psi) = \mathfrak{b}^k\mathfrak{y} \neq W\mathfrak{a}^k\mathfrak{x} = WB_0^+(Q^k\psi)$, $Q^k\psi \notin \mathcal{D}(P_W)$ and thus $\psi \notin \mathcal{D}(\pi_W)$. This shows that π_W is not self-adjoint.

5.4. We conclude this section by characterizing the representations $\pi_W \in \mathcal{K}$ and the integrable representations $\pi_{W,\mathfrak{M}}$.

Let $\mathbf{I}_r^+ := \{n \in \mathfrak{J}^+ : a_r = a_n\}$ and $\mathbf{I}_r^- := \{n \in \mathfrak{J}^- : a_r = b_n\}$ for $r \in \mathfrak{J}^+$. We denote by $G_r^\pm$ the projection in $l_2(\mathfrak{J}^\pm)$ with range $l_2(\mathbf{I}_r^\pm)$. (For $\mathfrak{J}' \subseteq \mathfrak{J}''$ we always consider $l_2(\mathfrak{J}')$ as a subspace of $l_2(\mathfrak{J}'')$ in an obvious way).

Proposition 9. (i) *Suppose that* $\overline{\pi_{W,\mathfrak{M}}(\boldsymbol{p})} = P_W$. *If* $\pi_{W,\mathfrak{M}} \in \mathcal{K}$, *then* $WG_r^+\mathfrak{x} = G_r^-W\mathfrak{x}$ *for all* $\mathfrak{x} \in \mathcal{W}$ *and* $r \in \mathfrak{J}^+$. *Conversely, if* $WG_r^+\mathfrak{x} = G_r^-W\mathfrak{x}$ *for* $\mathfrak{x} \in \mathcal{W}$, $r \in \mathfrak{J}^+$, *then* $\pi_W \in \mathcal{K}$ *and* $\mathfrak{S}(W) = \mathcal{D}_\infty(\mathfrak{a}) \cap \mathcal{W}$.

(ii) $\pi_{W,\mathfrak{M}}$ *is integrable with respect to the Weyl relation if and only if* $\mathfrak{M} = \mathfrak{L}_\infty^\pm(W)$ *and* W *is an isometry of* $l_2(\mathfrak{J}^+)$ *onto* $l_2(\mathfrak{J}^-)$ *such that* $\mathfrak{a} = W^*\mathfrak{b}W$ (*i.e.,* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are unitarily equivalent and* W *implements the unitary equivalence*).

Proof. (i) Suppose first that $\pi_{W,\mathfrak{M}} \in \mathcal{K}$. Let $r \in \mathfrak{J}^+$. By Lemma 3, $\mathcal{W} = \overline{\mathfrak{B}_1^+(\mathcal{D}_{W,\mathfrak{M}})}$. Let $f_\varepsilon \in C_0^\infty(R_1)$ be such that $f_\varepsilon(a_r) = 1$, $0 \leq f_\varepsilon(t) \leq 1$ on R_1 and $\operatorname{supp} f_\varepsilon \subseteq (a_r - \varepsilon, a_r + \varepsilon)$ for $\varepsilon > 0$. Then, for $\varphi \in \mathcal{D}_{W,\mathfrak{M}}$,

$$\|\!|(G_r^+ - I)B_0^+(f_\varepsilon(Q)\varphi)\|\!|^2 \leq \sum_{a_n \in (a_r - \varepsilon, a_r + \varepsilon)} |\varphi_n(a_n+)|^2 .$$

Since $(\varphi_n(a_n+), n \in \mathfrak{J}^+) \in l_2(\mathfrak{J}^+)$, the right-hand side tends to zero as $\varepsilon \to +0$. Because $G_r^+B_0^+(f_\varepsilon(Q)\varphi) = G_r^+B_0^+(\varphi)$, this shows that $\lim_{\varepsilon \to +0} B_0^+(f_\varepsilon\varphi)$

$=G_r^+B_0^+(\varphi)$. Similarly, $\lim_{\varepsilon\to+0} B_0^-(f_\varepsilon\varphi)=G_r^-B_0^-(\varphi)$. On the other side, $\pi_{W,\mathfrak{M}}\in\mathcal{K}$ implies that $f_\varepsilon(Q)\varphi\in\mathcal{D}_{W,\mathfrak{M}}$ by [17], Prop. 3.1. Hence $WB_0^+(f_\varepsilon\varphi)=B_0^-(f_\varepsilon\varphi)$ which gives $WG_r^+B_0^+(\varphi)=G_r^-B_0^-(\varphi)=G_r^-WB_0^+(\varphi)$. Therefore, $WG_r^+\mathfrak{x}=G_r^-W\mathfrak{x}$ for all $\mathfrak{x}\in\mathcal{W}\equiv\mathfrak{B}_1^+(\mathcal{D}_{W,\mathfrak{M}})$.

Next we prove the opposite direction. Suppose that $WG_r^+\mathfrak{x}=G_r^-W\mathfrak{x}$ for $\mathfrak{x}\in\mathcal{W}$ and $r\in\mathfrak{J}^+$. Let f be a function on R_1 and let $G_{r_j}^+$, $j=1,\cdots,l$, $l\in N\cup\{+\infty\}$, be an enumeration of all projections G_r^+, $r\in\mathfrak{J}^+$. For $\mathfrak{x}\in\mathcal{D}(f(\mathfrak{a}))\cap\mathcal{W}$, we have

$$Wf(\mathfrak{a})\mathfrak{x}=Wf(\mathfrak{a})\sum_j G_{r_j}^+\mathfrak{x}=W\sum_j f(a_{r_j})G_{r_j}^+\mathfrak{x} \tag{12}$$

$$=\sum_j f(a_{r_j})G_{r_j}^-W\mathfrak{x}=\sum_j f(\mathfrak{b})G_{r_j}^-W\mathfrak{x}$$

$$=f(\mathfrak{b})\sum_j WG_{r_j}^+\mathfrak{x}=f(\mathfrak{b})W\mathfrak{x}.$$

Setting $f(t)=t^k$, $k\in N_0$, we get $W\mathfrak{a}^k\mathfrak{x}=\mathfrak{b}^kW\mathfrak{x}$ for $\mathfrak{x}\in\mathcal{D}_\infty(\mathfrak{a})\cap\mathcal{W}$, that is, $\mathfrak{S}(W)=\mathcal{D}_\infty(\mathfrak{a})\cap\mathcal{W}$.

To prove $\pi_W\in\mathcal{K}$, let $\varphi\in\mathcal{D}_W$. It suffices to check that $e^{ist}\varphi\in\mathcal{D}_W$ for all $s\in R_1$. Applying (12) with $f(t)=e^{ist}$ and $\mathfrak{x}=B_0^+((P_W+s)^j\varphi)$ $\in\mathfrak{S}(W)$ for $j\in N_0$, it follows that $i^jWB_j^+(e^{ist}\varphi)=WB_0^+((P_0^*)^je^{ist}\varphi)$ $=WB_0^+(e^{ist}(P_W+s)^j\varphi)=We^{is\mathfrak{a}}B_0^+((P_W+s)^j\varphi)=e^{is\mathfrak{b}}WB_0^+((P_W+s)^j\varphi)$ $=i^jB_j^-(e^{ist}\varphi)$, i.e., $e^{ist}\varphi\in\mathcal{D}_W$.

(ii) Suppose that $\pi_{W,\mathfrak{M}}$ is integrable. Then, $\pi_{W,\mathfrak{M}}$ is self-adjoint ([14]) and contained in $\mathcal{K}$ and in $\mathcal{C}$. Combining Theorem 5, (ii) and (iii), and part (i), it follows that $\mathfrak{M}=\mathfrak{L}_\infty^+(W)$, W is an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$ and $\mathfrak{S}(W)=\mathcal{D}_\infty(\mathfrak{a})$. The latter means that $\mathfrak{a}\mathfrak{x}=W^*\mathfrak{b}W\mathfrak{x}$ for $\mathfrak{x}\in\mathcal{D}_\infty(\mathfrak{a})$.

Conversely, assume that the above conditions are satisfied. $\mathfrak{a}=W^*\mathfrak{b}W$ and $\mathcal{W}=l_2(\mathfrak{J}^+)$ imply that $WG_r^+\mathfrak{x}=G_r^-W\mathfrak{x}$ for all $r\in\mathfrak{J}^+$, $\mathfrak{x}\in\mathcal{W}$. Hence $\pi_{W,\mathfrak{M}}=\pi_W\in\mathcal{K}$ by (i). Moreover, $\mathfrak{S}(W)\equiv\mathcal{D}_\infty(\mathfrak{a})$ is dense in $l_2(\mathfrak{J}^+)$ and therefore W is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. Thus $\pi_{W,\mathfrak{M}}=\pi_W$ is a self-adjoint representation in $\mathcal{C}$ by Theorem 5, (ii) and (iii). We already noted in 2.2, Remark 7), that a self-adjoint representation which is contained in the intersection of $\mathcal{K}$ and $\mathcal{C}$ is integrable. Hence $\pi_{W,\mathfrak{M}}$ is integrable.

§ 6. Irreducibility and Unitary Equivalence

We continue the study of the representation $\pi_{W,\mathfrak{M}}$. Throughout this section, the set $\{(a_n, b_n); n\in\mathfrak{J}\}$ will be fixed.

6.1. We define $H_k\varphi := \varphi_k$ for $\varphi = (\varphi_n, n\in\mathfrak{J}) \in \mathcal{H}$ and $k\in\mathfrak{J}$. We say that an operator $C\in B(\mathcal{H})$ is *constant* if there are complex numbers c_{kn}, $k, n\in\mathfrak{J}$, such that $H_kCH_n\varphi = c_{kn}\chi_{kn}H_n\varphi$ for $k, n\in\mathfrak{J}$ and $\varphi\in\mathcal{H}$ where χ_{kn} is the characteristic function of $(a_n, b_n)\cap(a_k, b_k)$.

Lemma 1. $\mathcal{I}(\pi_0, \pi_0^*) = \{C\in B(\mathcal{H}) : C$ *is constant*$\}$.

Proof. Suppose first that $C\in\mathcal{I}(\pi_0, \pi_0^*)$. Let $n\in\mathfrak{J}$. Take a $\xi_n = (\delta_{nk}\xi_n, k\in\mathfrak{J}) \in \mathcal{D}_0$ such that $\xi_n(t) > 0$ on (a_n, b_n) and the set $\{f(Q)\xi_n, f\in C_0^\infty(R_1)\}$ is dense in $L_2(a_n, b_n)$. Let $C\xi_n = (\eta_{kn}, k\in\mathfrak{J})$. Since $C\in\mathcal{I}(\pi_0, \pi_0^*)$, $C\xi_n\in\mathcal{D}(\mathcal{A}_0^*)$. Since $CQ\varphi = QC\varphi$ for $\varphi\in\mathcal{D}_0$ and $Q\upharpoonright\mathcal{D}_0$ is e.s.a., C commutes with all functions $f(Q)$, $f\in L_\infty(R_1)$, i.e., $Cf(Q)\xi_n = (f(t)\eta_{kn}(t), k\in\mathfrak{J})$. In particular, this implies $\eta_{kn}(t) = 0$ on $(a_k, b_k)\setminus(a_n, b_n)$ for $k\in\mathfrak{J}$. If $f\in C_0^\infty(R_1)$ and $\operatorname{supp} f\subseteq(a_n, b_n)$, then $f(Q)\xi_n \in\mathcal{D}_0$ and $f(t)\xi_n'(t)/\xi_n(t)\in C_0^*(R_1)$. By $CP_0\varphi = P_0^*C\varphi$ for $\varphi\in\mathcal{D}_0$, we get $CP_0f(Q)\xi_n = C(-i\delta_{kn}[f'\xi_n + f\xi_n'], k\in\mathfrak{J}) = (-if'\eta_{kn} - if\xi_n'/\xi_n, k\in\mathfrak{J})$ $= P_0^*Cf(Q)\xi_n = (-if'\eta_{kn} - if\eta_{kn}', k\in\mathfrak{J})$ and therefore

$$\xi_n'(t)\eta_{kn}(t) = \xi_n(t)\eta_{kn}'(t) \quad \text{for } t\in(a_n, b_n)\cap(a_k, b_k) \text{ and } k\in\mathfrak{J}.$$

This shows that $(\eta_{kn}/\xi_n)' \equiv 0$ on $(a_n, b_n)\cap(a_k, b_k)$. Thus there is a constant c_{kn} such that $\eta_{kn}(t) = c_{kn}\xi_n(t)$ for $t\in(a_n, b_n)\cap(a_k, b_k)$. Consequently, $Cf(Q)\xi_n = (c_{kn}\chi_{kn}f(t)\xi_n(t), k\in\mathfrak{J})$ and $H_kCH_n\varphi = c_{kn}\chi_{kn}H_n\varphi$ for $\varphi\in\mathcal{H}$, i.e., C is constant.

Conversely, suppose that $C\in B(\mathcal{H})$ is constant. Let $k, r\in N_0$ and let $\varphi = (\varphi_n)\in\mathcal{D}_0$. Since $\varphi_n^{(j)}(a_n+) = \varphi_n^{(j)}(b_n-) = 0$ for all $j\in N_0$ and $\varphi_n\in C^\infty[a_n, b_n]$, we have $\chi_{mn}(t)\varphi_n(t)\in C^\infty[a_m, b_m]$ for $m\in\mathfrak{J}$ and $(\delta_{mn}\chi_{mn}\varphi_n, m\in\mathfrak{J})\in\mathcal{D}(\mathcal{A}_0^*)$. Clearly, for $m, n\in\mathfrak{J}$, $H_mCH_nP_0^rQ^k\varphi = c_{mn}\chi_{mn}H_nP_0^rQ^k\varphi = c_{mn}(P_0^*)^rQ^k\chi_{mn}\varphi_n = (P_0^*)^rQ^kH_mCH_n\varphi$. Since $(P_0^*)^rQ^k$ is closable, this implies $C\varphi\in\mathcal{D}(\overline{(P_0^*)^rQ^k})$ and $CP_0^rQ^k\varphi = \overline{(P_0^*)^rQ^k}C\varphi$. Hence

$C\varphi\in\bigcap_{k,r=0}^{\infty}\mathcal{D}(\overline{(P_0^*)^r Q^k})$. Because π_0^* is closed, $C\varphi\in\mathcal{D}(\pi_0^*)=\mathcal{D}(\mathcal{A}_0^*)$ and $C\varphi\in\mathcal{J}(\pi_0,\pi_0^*)$. Now the proof is complete.

6.2. To describe the intertwining space of two representations $\pi_{W,\mathfrak{M}}$ and $\pi_{W',\mathfrak{M}'}$, we have to study constant operators C which map $\mathcal{D}_{W,\mathfrak{M}}$ into $\mathcal{D}_{W',\mathfrak{M}'}$. To avoid several difficulties (for example, if the set of all a_n, $n\in\mathfrak{J}$, is dense in R_1) we assume that:

(+) *There is a $c>0$ such that $b_n-a_n\geq c$ for all $n\in\mathfrak{J}$.*

(++) *If $k,n\in\mathfrak{J}$, then either $a_k<a_n$, $b_k<b_n$ or $a_k=a_n$, $b_k=b_n$ or $a_k>a_n$, $b_k>b_n$.*

If for intervals (a_n,b_n), $n\in\mathfrak{J}$, the distance between two different point of the set $\{a_n,b_n;\ n\in\mathfrak{J}\}$ is always greater than $c>0$, then (+) and (++) can be satisfied by dividing the intervals and adding "trivial" boundary conditions. For example, if $a_n<a_k<b_n<b_k$, then we replace (a_n,b_n), (a_k,b_k) by (a_n,a_k), (a_k,b_n), (a_k,b_n), (b_n,b_k) and we put the "trivial" boundary condition $\varphi_n(a_k+)=\varphi_n(a_k-)$, $\varphi_k(b_n+)=\varphi_k(b_n-)$ into the operator W.

For $r\in\mathfrak{J}$, let $\mathfrak{J}_r:=\{n\in\mathfrak{J}\colon a_r=a_n \text{ and } b_r=b_n\}$ and let E_r be the orthogonal projection of $l_2(\mathfrak{J})$ onto $l_2(\mathfrak{J}_r)$.

Suppose that $C\in B(\mathcal{H})$ is constant. Since $\chi_{kn}(t)\equiv 0$ or $\chi_{kn}(t)\equiv 1$ on (a_k,b_k) by assumption (++), we can always assume that $c_{kn}=0$ if $n\notin\mathfrak{J}_k$. Let $\widetilde{C}$ be the operator on $l_2(\mathfrak{J})$ given by the infinite matrix $(c_{kn})_{k,n\in\mathfrak{J}}$ relative to the basis $\mathfrak{e}_n=\{\delta_{kn},k\in\mathfrak{J}\}$, $n\in\mathfrak{J}$. Then, $\widetilde{C}\in B(l_2(\mathfrak{J}))$ and $\widetilde{C}E_r=E_r\widetilde{C}$ for all $r\in\mathfrak{J}$. To prove these, let $\mathfrak{x}=(x_n)\in l_2(\mathfrak{J})$. For each set $\mathfrak{J}_r$, take a continuous function, say φ_r, in $L_2(a_r,b_r)$ such that $\|\varphi_r\|_{L_2(a_r,b_r)}=1$. Then, $\psi:=(x_n\varphi_n,n\in\mathfrak{J})\in\mathcal{H}$ and $\|\psi\|_{\mathcal{H}}=\|\mathfrak{x}\|_{l_2(\mathfrak{J})}$. Hence the series $\sum_{n\in\mathfrak{J}}c_{kn}x_n\varphi_n(t)=\varphi_k(t)\sum_{n\in\mathfrak{J}_k}c_{kn}x_n=\varphi_k(t)\sum_{n\in\mathfrak{J}}c_{kn}x_n$ is converging on (a_k,b_k) for all $k\in\mathfrak{J}$ and $\|C\psi\|^2=\sum_{k\in\mathfrak{J}}\int_{a_k}^{b_k}|\sum_{n\in\mathfrak{J}_k}c_{kn}x_n\varphi_k(t)|^2dt=\sum_{k\in\mathfrak{J}}|\sum_{n\in\mathfrak{J}}c_{kn}x_n|^2$ $\leq\|C\|^2\|\mathfrak{x}\|^2=\|C\|^2\|\psi\|^2$. This shows that $\widetilde{C}\in B(l_2(\mathfrak{J}))$. Clearly, $\widetilde{C}E_r=E_r\widetilde{C}$, $r\in\mathfrak{J}$, because $c_{kn}=0$ for $n\notin\mathfrak{J}_k$.

Conversely, for each operator $D\in B(l_2(\mathfrak{J}))$ commuting with all E_r, $r\in\mathfrak{J}$, there is a unique constant operator $C\in B(\mathcal{H})$ such that $D=\widetilde{C}$.

Indeed, let (c_{kn}) be the matrix of D relative to the basis $\mathfrak{e}_n, n\in\mathfrak{J}$. Suppose that $\varphi=(\varphi_n)\in\mathcal{D}(\mathcal{P}_0^*)$. The proof of Lemma 5.1, (i), shows that the sequence $(\varphi_n(t), n\in\mathfrak{J}_k)$ is in $l_2(\mathfrak{J}_k)$ for all $t\in[a_k, b_k]$ and $k\in\mathfrak{J}$. Therefore, the series $\sum_{n\in\mathfrak{J}_k} c_{kn}\varphi_n(t)$ is converging on R_1 for $k\in\mathfrak{J}$ and

$$\sum_{k\in\mathfrak{J}}\int_{a_k}^{b_k}|\sum_{n\in\mathfrak{J}_k} c_{kn}\varphi_n(t)|^2 dt=\int\sum_{k\in\mathfrak{J}}|\sum_{n\in\mathfrak{J}_k} c_{kn}\varphi_n(t)|^2 dt$$
$$\leq\int|||D|||^2\sum_{n\in\mathfrak{J}}|\varphi_n(t)|^2 dt=|||D|||^2\|\varphi\|^2.$$

Hence $C\varphi:=(\sum_{n\in\mathfrak{J}_k} c_{kn}\varphi_n, k\in\mathfrak{J})$ is in $\mathcal{H}$ and $\|C\varphi\|\leq|||D|||\,\|\varphi\|$ for φ in the dense subset $\mathcal{D}(\mathcal{P}_0^*)$ of $\mathcal{H}$. By continuity, C extends to an operator in $B(\mathcal{H})$. Since $DE_r=E_rD$ for $r\in\mathfrak{J}$, $c_{kn}=0$ if $n\notin\mathfrak{J}_k$. Hence $\widetilde{C}=D$. The uniqueness of C is obvious.

Suppose now that $C\in B(\mathcal{H})$ is constant. Since $\widetilde{C}E_r=E_r\widetilde{C}$ for $r\in\mathfrak{J}$, $\widetilde{C}l_r(\mathfrak{J}^{\pm})\subseteq l_2(\mathfrak{J}^{\pm})$. We also denote by $\widetilde{C}$ the linear mapping of $\mathfrak{L}^+$ defined by $\widetilde{C}(\mathfrak{x}_0, \mathfrak{x}_1, \cdots):=(\widetilde{C}\mathfrak{x}_0, \widetilde{C}\mathfrak{x}_1, \cdots)$. From the definition of $\widetilde{C}$ it is clear that

(1) $$B_j^{\pm}(C\varphi)=\widetilde{C}B_j^{\pm}(\varphi)\quad\text{for } j\in N_0 \text{ and } \varphi\in\mathcal{D}(\mathcal{P}_0^*).$$

For $\mathfrak{M}\subseteq\mathfrak{L}^+$, let $L_{\mathfrak{M}}$ be the projection on the smallest closed subspace of $l_2(\mathfrak{J}^+)$ which contains all $\mathfrak{x}_j$ for $\mathfrak{x}=(\mathfrak{x}_0, \mathfrak{x}_1, \cdots)\in\mathfrak{M}$ and $j\in N_0$.

We now characterize the intertwining space of two representations $\pi_{W,\mathfrak{M}}$ and $\pi_{W',\mathfrak{M}'}$ relative to the same set $\{(a_n, b_n), n\in\mathfrak{J}\}$.

Proposition 2. *$\mathcal{I}(\pi_{W,\mathfrak{M}}, \pi_{W',\mathfrak{M}'})=\{C\in B(\mathcal{H}): C$ is constant, $\widetilde{C}\mathfrak{M}\subseteq\mathfrak{M}'$ and $W'\widetilde{C}L_{\mathfrak{M}}=\widetilde{C}WL_{\mathfrak{M}}\}$.*

Proof. Suppose that $C\in\mathcal{I}(\pi_{W,\mathfrak{M}}, \pi_{W',\mathfrak{M}'})$. Then, of course, $C\in\mathcal{I}(\pi_0, \pi_0^*)$ and C is constant by Lemma 1. By definition, $C\mathcal{D}_{W,\mathfrak{M}}\subseteq\mathcal{D}_{W',\mathfrak{M}'}$. For $\varphi\in\mathcal{D}_{W,\mathfrak{M}}$, we have $B^+(\varphi)\in\mathfrak{M}$ and $B^+(C\varphi)=CB^+(\varphi)\in\mathfrak{M}'$. Since $\mathfrak{B}^+(\mathcal{D}_{W,\mathfrak{M}})=\mathfrak{M}$, this gives $\widetilde{C}\mathfrak{M}\subseteq\mathfrak{M}'$. Again by (1), $W'B_j^+(C\varphi)=W'\widetilde{C}B_j^+(\varphi)=B_j^-(C\varphi)=\widetilde{C}B_j^-(\varphi)=\widetilde{C}WB_j^+(\varphi)$ for $\varphi\in\mathcal{D}_{W,\mathfrak{M}}$ and $j\in N_0$. Hence $W'\widetilde{C}L_{\mathfrak{M}}=\widetilde{C}WL_{\mathfrak{M}}$.

Suppose now that C is in the set as defined on the right-hand side Reasoning as in the proof of Lemma 1, we see that $CP_W^rQ^k\varphi=(P_0^*)^rQ^kC\varphi$

for $\varphi\in\mathcal{D}_{W,\mathfrak{M}}$. Similarly as above, $\tilde{C}\mathfrak{M}\subseteq\mathfrak{M}'$ and $W'\tilde{C}L_{\mathfrak{M}}=W\tilde{C}L_{\mathfrak{M}}$ imply $C\mathcal{D}_{W,\mathfrak{M}}\subseteq\mathcal{D}_{W',\mathfrak{M}'}$. Since $P_{W'}^{r}\restriction\mathcal{D}_{W',\mathfrak{M}'}=(P_0^*)^r\restriction\mathcal{D}_{W',\mathfrak{M}'}$, $C\in\mathcal{J}(\pi_{W,\mathfrak{M}},\pi_{W',\mathfrak{M}'})$.

Theorem 3. *Suppose that $\pi_{W,\mathfrak{M}}\in\mathscr{C}$ and $\pi_{W',\mathfrak{M}'}\in\mathscr{C}$ relative to the same set of intervals (a_n,b_n), $n\in\mathfrak{J}$.*

(i) *$\pi_{W,\mathfrak{M}}$ is irreducible if and only if there is no projection $E\neq 0$, I in $l_2(\mathfrak{J})$ such that $EE_r=E_rE$ for all $r\in\mathfrak{J}$, $EW\mathfrak{x}=WE\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)$ and $E\mathfrak{M}\subseteq\mathfrak{M}$. In case $\mathfrak{M}=\mathfrak{L}_\infty^{\pm}(W)$, i.e., $\pi_{W,\mathfrak{M}}=\pi_W$ the latter condition can be omitted.*

(ii) *$\pi_{W,\mathfrak{M}}$ is unitarily equivalent to $\pi_{W',\mathfrak{M}'}$ if and only if there is a unitary operator $U\in B(l_2(\mathfrak{J}))$ such that $UE_r=E_rU$ for $r\in\mathfrak{J}$, $UW\mathfrak{x}=W'U\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)$ and $U\mathfrak{M}=\mathfrak{M}'$. For $\mathfrak{M}=\mathfrak{L}_\infty^{\pm}(W)$ and $\mathfrak{M}'=\mathfrak{L}_\infty^{\pm}(W')$, the condition $U\mathfrak{M}=\mathfrak{M}'$ can be omitted.*

Proof. We prove part (i). As we have noted in Section 1, $\pi_{W,\mathfrak{M}}$ is irreducible if and only if there is no projection $C\neq 0, I$ in $\mathcal{J}(\pi_{W,\mathfrak{M}},\pi_{W,\mathfrak{M}})$. Since $\pi_{W,\mathfrak{M}}\in\mathscr{C}$, range $L_{\mathfrak{M}}=l_2(\mathfrak{J}^+)$, Since the mapping $C\to\tilde{C}$ is a $*$-isomorphism, the above criterion is simply a reformulation of Proposition 2 in this case. We have to verify that $EE_r=E_rE$ for $r\in\mathfrak{J}$ and $EW\mathfrak{x}=WE\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)$ imply that $E\mathfrak{M}\subseteq\mathfrak{M}$ for $\mathfrak{M}=\mathfrak{L}_\infty^{\pm}(W)$. Indeed, since $EE_r=E_rE$ for $r\in\mathfrak{J}$, $El_2(\mathfrak{J}^{\pm})\subseteq l_2(\mathfrak{J}^{\pm})$ and E commutes with the diagonal operators $\mathfrak{a}$ and $\mathfrak{b}$ in $l_2(\mathfrak{J}^+)$ resp. $l_2(\mathfrak{J}^-)$. Together with $EW\mathfrak{x}=WE\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)$, this yields $E\mathfrak{S}(W)\subseteq\mathfrak{S}(W)$. And thus $E\mathfrak{L}_\infty^{\pm}(W)\subseteq\mathfrak{L}_\infty^{\pm}(W)$.

Part (ii) follows similarly.

For later use we state some facts of the preceding discussion separately as

Corollary 4. *Suppose that $\pi_{W,\mathfrak{M}}\in\mathscr{C}$. The mapping $(\pi_W)_s'\ni C\to\tilde{C}$ is a $*$-isomorphism of $(\pi_W)_s'$ on the W^*-algebra $\mathfrak{A}=\{D\in B(l_2(\mathfrak{J})):DE_r=E_rD$ for $r\in\mathfrak{J}$ and $DW\mathfrak{x}=WD\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)\}$.*

If $D\in\mathfrak{A}$, then $D\mathfrak{x}\in\mathfrak{S}(W)$ and $D\mathfrak{a}\mathfrak{x}=\mathfrak{a}D\mathfrak{x}$ for $\mathfrak{x}\in\mathfrak{S}(W)$.

Proof. The only thing we have to check is that $\mathfrak{A}$ is a W^*-algebra. Let E_+ be the projection of $l_2(\mathfrak{J})$ on $l_2(\mathfrak{J}^+)$. If $DE_r = E_r D$ for $r \in \mathfrak{J}$, then $DE_+ = E_+ D$. Therefore, $\mathfrak{A}$ is the commutant of $\{WE_+, E_r; r \in \mathfrak{J}\}$ in the Hilbert space $l_2(\mathfrak{J})$.

Remarks. We briefly discuss the case where the the operator Q has a simple spectrum. Obviously, this is equivalent to the requirement $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for all $n, m \in \mathfrak{J}$, $n \neq m$.

1) Except from the (uninteresting) case where the set $\{(a_n, b_n); n \in \mathfrak{J}\}$ reduces to the single interval $(-\infty, +\infty)$, we then have $\mathfrak{J} = \mathfrak{J}^+$ or $\mathfrak{J} = \mathfrak{J}^-$. Let us assume that $\mathfrak{J} = \mathfrak{J}^+$. Then, π_W is irreducible if and only if the automorphism $A \to WAW^*$ of the W^*-algebra $l_\infty(\mathfrak{J})$ is ergodic (or equivalently, the fix point algebra $\{A \in l_\infty(\mathfrak{J}) : A = WAW^*\}$ contains only the scalar multiples of the identity). π_W and $\pi_{W'}$ are unitarily equivalent if and only if W and W' are conjugated by an inner automorphism of $l_\infty(\mathfrak{J})$, i.e. $W = \mathfrak{u} W' \mathfrak{u}^*$ in $l_2(\mathfrak{J})$ for some unitary diagonal operator $\mathfrak{u} \in l_\infty(\mathfrak{J})$. Both statements follow immediately from Theorem 6.3 and the fact that (under the assumption that Q has a simple spectrum) $l_\infty(\mathfrak{J})$ is the W^*-algebra generated by $\{E_r; r \in \mathfrak{J}\}$.

2) Suppose that $\pi_{W,\mathfrak{M}} \in \mathscr{C}$. Let $\{\Delta_r, r \in N\}$ be the supporting sequence of the largest pair $(P_1, Q; \mathscr{D}_1) \in \mathscr{K}$ which is a restriction of $\pi_{W,\mathfrak{M}}$ (see Prop. 4.1). Since Q has a simple spectrum, $\Delta_r = \emptyset$ for $r \geq 2$. Put $\Delta = \bigcup_{n \in \mathfrak{J}} (a_n, b_n)$. Clearly, $\Delta \subseteq \Delta_1$. We now show that there is no loss of generality if we assume that $\Delta = \Delta_1$. Suppose that $t_0 \in \Delta_1 \backslash \Delta$. Then, $t_0 = a_n = b_m$ for some $n \in \mathfrak{J}^+$ and $m \in \mathfrak{J}^-$. There is a $\xi \in \mathscr{D}_1$ so that $\xi(a_n+) \neq 0$. Replacing ξ by $f(Q)\xi$ for some $f \in C_0^\infty(R_1)$ if necessary, we can assume that $\xi(a_{n'}+) = \xi(b_{m'}-) = 0$ for all $n' \in \mathfrak{J}^+$, $m' \in \mathfrak{J}^-$, $n' \neq n$, $m' \neq m$. Then, $WB_0^+(\xi) = B_0^-(\xi)$ and $\xi(a_n+) \neq 0$ imply that $W\mathfrak{e}_n = z\mathfrak{e}_m$, where $z \in C_1$, $|z| = 1$. We may assume, by the unitary transformation $(V\varphi)(t) = z\varphi(t)$ if $t > a_n$ and $(V\varphi)(t) = \varphi(t)$ if $t < a_n$, that $z = 1$. Finally, we replace the intervals (a_m, b_m) and (a_n, b_n) by the single interval (a_m, b_n) and we modify W by omitting the trivial boundary condition $\varphi(a_n+) = \varphi(b_m-)$. Using the above arguments and proceeding by induction (note that $\Delta_1 \backslash \Delta$ is either empty or countable), we can "remove" all points in $\Delta_1 \backslash \Delta$.

3) Theorem 6.3, (ii), characterizes the unitary equivalence for representations $\pi_{W,\mathfrak{M}}\in\mathscr{C}$ and $\pi_{W',\mathfrak{M}'}\in\mathscr{C}$ with respect to the same set of intervals (a_n, b_n), $n\in\mathfrak{J}$. The general case can be easily reduced to this case if Q has a simple spectrum (without this assumption further manipulations are needed). Let $\pi_{W,\mathfrak{M}}$ and $\pi_{W',\mathfrak{M}'}$ be representations in $\mathscr{C}$ relative to the sets $\{(a_n, b_n); n\in\mathfrak{J}\}$ resp. $\{(a_n', b_n'), n\in\mathfrak{J}'\}$ of intervals. Let $(P_1, Q; \mathscr{D}_1)$, Δ_1 resp. $(P_1', Q'; \mathscr{D}_1')$, Δ_1' be as in Remark 2). As we have seen in Remark 2), we can assume without loss of generality that $\Delta_1=\bigcup_{n\in\mathfrak{J}}(a_n, b_n)$ and $\Delta_1'=\bigcup_{n\in\mathfrak{J}'}(a_n', b_n')$. Assume now that $\pi_{W,\mathfrak{M}}\cong\pi_{W',\mathfrak{M}'}$. Then $(P_1, Q; \mathscr{D}_1)$ and $(P_1', Q; \mathscr{D}_1')$ are unitarily equivalent. By Corollary 4.3 in [17], $\Delta_1=\Delta_1'$, that is, $\bigcup_{n\in\mathfrak{J}}(a_n, b_n)=\bigcup_{n\in\mathfrak{J}'}(a_n', b_n')$. Therefore, except from the enumeration, both sets of intervals coincide.

§ 7. Construction of Canonical Pairs I: Weak Intertwining Operators

In the preceding sections all $\pi\in\mathscr{C}$ which extend π_0 have been classified in terms of weak intertwining operators W and admissible boundary spaces $\mathfrak{M}$. In this section we are dealing with the construction of such operators for a given set $\{(a_n, b_n), n\in\mathfrak{J}\}$.

7.1. Let (a_n, b_n), $n\in\mathfrak{J}$, be a set of intervals satisfying $(+)$, that is, $b_n-a_n\geq c>0$ for all $n\in\mathfrak{J}$.

Suppose that there is a $\pi\in\mathscr{C}$ such that $\pi\supseteq\pi_0$. By Theorem 5.5, π is of the form $\pi_{W,\mathfrak{M}}$ where W is a weak intertwining operator of $l_2(\mathfrak{J}^+)$ on $l_2(\mathfrak{J}^-)$ for $\mathfrak{a}$ and $\mathfrak{b}$. Hence

(1) $\mathfrak{a}\mathfrak{x}=W^*\mathfrak{b}W\mathfrak{x}$ for $\mathfrak{x}$ in the dense domain $\mathfrak{S}(W)$ of $l_2(\mathfrak{J}^+)$.

Because W is an isometry, $\dim l_2(\mathfrak{J}^+)=\dim l_2(\mathfrak{J}^-)$. If $\mathfrak{a}$ [resp. $\mathfrak{b}$] is bounded, then $\mathfrak{b}$ [resp. $\mathfrak{a}$] is bounded and therefore $\mathfrak{S}(W)=\mathscr{D}_\infty(\mathfrak{a})=l_2(\mathfrak{J}^+)$. But, then, by Proposition 5.9, π_W is integrable. If both operators $\mathfrak{a}$ and $\mathfrak{b}$ are unbounded, then (1) immediately implies (see [18], p. 246) that $\sup_{n\in\mathfrak{J}^+}a_n=\sup_{n\in\mathfrak{J}^-}b_n=+\infty$ or $\inf_{n\in\mathfrak{J}^+}a_n=\inf_{n\in\mathfrak{J}^-}b_n=-\infty$.

Conversely, if $\sup_{n\in\mathfrak{J}^+}a_n=\sup_{n\in\mathfrak{J}^-}b_n=+\infty$ or $\inf_{n\in\mathfrak{J}^+}a_n=\inf_{n\in\mathfrak{J}^-}b_n=-\infty$, then there exist a weak intertwining operator W for $\mathfrak{a}$ and $\mathfrak{b}$ ([18], Theorem 4.5) and thus a $*$-representation $\pi\in\mathscr{C}$ which extends π_0. We now prove a little more.

Theorem 1. *Let (a_n, b_n), $n\in\mathfrak{J}$, be intervals which satisfy the assumptions $(+)$, $(++)$ from 6.2. Suppose $\mathfrak{J}=\mathfrak{J}^+\cup\mathfrak{J}^-$. Suppose that $\sup_{n\in\mathfrak{J}^+} a_n=\sup_{n\in\mathfrak{J}^-} b_n=+\infty$ or $\inf_{n\in\mathfrak{J}^+} a_n=\inf_{n\in\mathfrak{J}^-} b_n=-\infty$.*

Then there exists an uncountable set of pairwise inequivalent irreducible self-adjoint $*$-representations of $A(\boldsymbol{p},\boldsymbol{q})$ in the class $\mathscr{C}$ which extend π_0.

First we recall a result from [18] stated in a convenient form. We use the notion of a weak intertwining operator as introduced in 5.3, but for arbitrary index sets instead of $\mathfrak{J}^+$ and $\mathfrak{J}^-$.

Proposition 2. *Let $\mathfrak{c}=\{c_r, r\in N\}$, $\mathfrak{d}=\{d_s, s\in N\}$ be real sequences so that $\sup_r c_r=\sup_s d_s=+\infty$. Let $\gamma=\{\gamma_j, j\in N\}$ be a monotone positive sequence. Suppose that $\gamma_1+c_{r_1}<d_{s_1}$, $\gamma_1+d_{s_1}<c_{r_2}$, $\gamma_2+c_{r_2}<d_{s_2}$ for natural numbers $r_1<r_2$, $s_1<s_2$. Then there exist (unbounded) subsequences $\mathfrak{c}'=\{c_{r_j}, j\in N\}$, $\mathfrak{d}'=\{d_{s_j}, j\in N\}$ and a weak intertwining operator W of $l_2(N)$ for $\mathfrak{c}'$ and $\mathfrak{d}'$ such that*

$$\gamma_k+c_{r_k}<d_{s_k},\ \gamma_k+d_{s_k}<c_{r_{k+1}}\ \textit{for}\ k\in N \tag{2}$$

and

$$(W\mathfrak{e}_j, \mathfrak{e}_l)\neq 0\ \textit{for all}\ j, l\in N\ \textit{where}\ \mathfrak{e}_j:=\{\delta_{jk}, k\in N\}. \tag{3}$$

By passing to a subsequence if necessary, we may assume that

$$\gamma_n+c_n<d_n,\ \gamma_n+d_n<c_{n+1}\ \text{for}\ n\in N. \tag{4}$$

Corollary 4.4 in [18][2] gives the existence of subsequences $\mathfrak{c}'$, $\mathfrak{d}'$ such that $\mathfrak{c}'$ and $\mathfrak{d}'$ are 1-related in the terminology of [18] and $c_{r_k}<d_{s_k}<c_{r_{k+1}}$, $k\in N$. Because γ is monotone, the latter and (4) imply (2). Since $\mathfrak{c}'$ and $\mathfrak{d}'$ satisfy the assumptions of Theorem 4.1, Corollary 4.2 in [18] (or formula (15) on p. 244) yields (3).

[2] See the appendix to Section 7.

Proof of Theorem 1. Without loss of generality assume that $\sup_{n\in\mathfrak{J}^+} a_n = \sup_{n\in\mathfrak{J}^-} b_n = +\infty$. There are infinitely many different sets $\mathfrak{J}_r$, $r\in\mathfrak{J}$. Let $\mathfrak{J}_{r_j}$, $j\in N$, be an enumeration of these sets. Clearly, $N_+ := \{j\in N: a_{r_j}>0\}$ is infinite. By $(++)$, $b_{r_j}<+\infty$ for $j\in N_+$. We identify $\mathfrak{J}$ with N. For subsets $\mathfrak{J}', \mathfrak{J}''\subseteq\mathfrak{J}$, let $d(\mathfrak{J}', \mathfrak{J}'')$ be the number of $j\in N$ with $\mathfrak{J}_{r_j}\cap\mathfrak{J}'\neq\emptyset$ and $\mathfrak{J}_{r_j}\cap\mathfrak{J}''\neq\emptyset$.

Let $\gamma=\{\gamma_j, j\in N\}$ be a given monotone positive sequence. We show that there are mutually disjoint infinite sets $N_l\equiv\{n_{lj}, j\in N\}\subseteq\mathfrak{J}^+$ resp. $M_l\equiv\{m_{lj}, j\in N\}\subseteq\mathfrak{J}^-$, $l\in N$, such that $\mathfrak{J}^+=\bigcup_{l=1}^{\infty} N_l$, $\mathfrak{J}^-=\bigcup_{l=1}^{\infty} M_l$ and the following properties hold:

(a) $d(N_j, N_l)+d(M_j, M_l)\leq 1$ and $d(N_j, M_s)<+\infty$ for all $j, l, s\in N$, $j\neq l$.

(b) $M_l\cap N_{l-1}\neq\emptyset$ and $N_l\cap M_{l-1}\neq\emptyset$ for $l\in N$, $l\geq 2$.

(c) The sequences $\mathfrak{c}'=\mathfrak{a}_{2l}$, $\mathfrak{d}'=\mathfrak{b}_{2l}$ resp. $\mathfrak{c}'=\mathfrak{a}_{2l+1}$, $\mathfrak{d}'=\mathfrak{b}_{2l+1}$, $l\in N_0$, satisfy the assertion of Proposition 2.

(d) $A_l := \{j\in N_+ : \mathfrak{J}_{r_j}\cap(N_s\cup M_s)=\emptyset$ for $s=1, \cdots, l-1\}$ is infinite for $l\in N$.

Here we set $\mathfrak{a}_l := \{a_{n_{lj}}, j\in N\}$ and $\mathfrak{b}_l := \{b_{m_{lj}}, j\in N\}$ for $l\in N$.

To prove the existence of such sets, we proceed by induction. Let $k\in N$, $k\geq 2$. Suppose that $N_1, \cdots, N_{k-1}, M_1, \cdots, M_{k-1}$ are constructed such that (a)-(d) are true for these sets. First assume that k is even. Let n_{k1} be the smallest number in $\mathfrak{J}^+\setminus\bigcup_{j=1}^{k-1} N_j$. Because of (d), we can find an $n\in A_k$ and an $m_{k1}\in\mathfrak{J}_{r_n}$ such that $b_{m_{k1}}-a_{n_{k1}}>\gamma_1$. By (a), there are numbers $j, l\in N$ and $n_{k2}\in M_{k-1}\cap\mathfrak{J}_{r_j}$, $m_{k2}\in N_{k-1}\cap\mathfrak{J}_{r_l}$ such that $n_{k2}>n_{k1}$, $m_{k2}>m_{k1}$, $b_{m_{k2}}-a_{n_{k2}}>\gamma_2$, $a_{n_{k2}}-b_{m_{k1}}>\gamma_1$ and $\mathfrak{J}_{r_j}\cap N_s=\emptyset$, $\mathfrak{J}_{r_l}\cap M_s=\emptyset$ for $s=1, \cdots, k-1$. Now we decompose $A_k\setminus\{j, l, n\}$ as a union of mutually disjoint infinite subsets U_k, V_k, W_k. We apply Proposition 2 with $\mathfrak{c}=\{a_{n_{k1}}, a_{n_{k2}}, a_n; n\in\bigcup_{j\in U_k}\mathfrak{J}_{r_j}\}$ and $\mathfrak{d}=\{b_{m_{k1}}, b_{m_{k2}}, b_n; n\in\bigcup_{j\in V_k}\mathfrak{J}_{r_j}\}$ (written as sequences in an obvious way) and we obtain subsequences $\mathfrak{c}'=\mathfrak{a}_k\equiv\{a_{n_{kj}}, j\in N\}$, $\mathfrak{d}'=\mathfrak{b}_k=\{b_{m_{kj}}, j\in N\}$. If k is odd, we only change the role of n_{kj} and m_{kj}. The same construction, with $\bigcup_{j=1}^{k-1} M_j$ replaced by $\emptyset$, works for $k=1$.

It is almost trivial to check that (a)-(d) are true for $N_1, \cdots, N_k$,

$M_1, \cdots, M_k$. (d) follows from $W_k \subseteq A_{k+1}$.

Clearly, if W is a weak intertwining operator for $\mathfrak{c}'$ and $\mathfrak{d}'$, then W^* is a weak intertwining operator for $\mathfrak{d}'$ and $\mathfrak{c}'$. Therefore, by (c) and Proposition 2, there is a weak intertwining operator W_l, $l \in N$, of $l_2(N)$ onto $l_2(N)$ for $\mathfrak{a}_l$ and $\mathfrak{b}_l$. It induces a weak intertwining operator (denoted again by W_l) of $l_2(N_l)$ onto $l_2(M_l)$ for the sequences $\{a_n, n \in N_l\}$ and $\{b_n, n \in M_l\}$. The isometry $W_r := \sum_{l=1}^{\infty}{}^{\oplus} W_l$ of $l_2(\mathfrak{J}^+) = \sum_{l=1}^{\infty}{}^{\oplus} l_2(N_l)$ onto $l_2(\mathfrak{J}^-) = \sum_{l=1}^{\infty}{}^{\oplus} l_2(M_l)$ is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. By Theorem 5. 5, π_{W_r} is a self-adjoint $*$-representation of the class $\mathscr{C}$.

To prove that π_{W_r} is irreducible, we apply Theorem 6. 3. For let $E \neq 0$ be a projection in $l_2(\mathfrak{J})$ so that $EE_r = E_r E$ for $r \in \mathfrak{J}$ and $EW_r \mathfrak{x}_+ = W_r E \mathfrak{x}_+$ for $\mathfrak{x}_+ \in l_2(\mathfrak{J}^+)$. Our aim is to show that $E = I$.

Let $\mathfrak{x}$ be a non-zero vector in $\mathcal{E} :=$ range E. By $(++)$ and $\sup_{n \in \mathfrak{J}^+} a_n = +\infty$, we have $\mathfrak{J}^- = \mathfrak{J}$. Hence $\mathfrak{x} \in l_2(\mathfrak{J}^-)$ and there is a $\mathfrak{y} \equiv \{y_n, n \in \mathfrak{J}^+\} \in l_2(\mathfrak{J}^+)$ so that $\mathfrak{x} = W_r \mathfrak{y}$. Since W_r is an isometry, it follows from $\mathfrak{x} = W_r \mathfrak{y} = EW_r \mathfrak{y} = W_r E \mathfrak{y}$ that $\mathfrak{y} \in \mathcal{E}$ and $\mathfrak{y} \neq 0$. Then $y_{n_0} \neq 0$ for some $n_0 \in \mathfrak{J}^+$. There are numbers $s, l, j \in N$ such that $n_0 = n_{sl} \in \mathfrak{J}_{r_j}$. Let $m_0 := m_{s, l+1}$. Because $EE_r = E_r E$ for $r \in \mathfrak{J}$ and $EW_r \mathfrak{x}_+ = W_r E \mathfrak{x}_+$ for $\mathfrak{x}_+ \in l_2(\mathfrak{J}^+)$, we conclude that $E_{m_0} W_r E_{r_j} \mathfrak{y} = \sum_{n \in \mathfrak{J}_{r_j}} y_n E_{m_0} W_r \mathfrak{e}_n \in \mathcal{E}$. Let $n \in N_r \cap \mathfrak{J}_{r_j}$ so that $n \neq n_0$. By the monotonocity of $\mathfrak{a}_r$, $r \neq s$. Since $\mathfrak{J}_{r_j} \cap N_r \neq \emptyset$ and $\mathfrak{J}_{r_j} \cap N_s \neq \emptyset$, we have $d(N_r, N_s) \geq 1$. By (a) and $r \neq s$, this implies $d(M_r, M_s) = 0$. Because $m_0 \in \mathfrak{J}_{m_0} \cap M_s$, the latter shows that $\mathfrak{J}_{m_0} \cap M_r = \emptyset$. Since $W_r \mathfrak{e}_n \in l_2(M_r)$ by construction, it follows that. $E_{m_0} W_r \mathfrak{e}_n = 0$. Hence $E_{m_0} W_r E_{r_j} \mathfrak{y} = y_{n_0} (W_r \mathfrak{e}_{n_0}, \mathfrak{e}_{m_0}) \mathfrak{e}_{m_0}$. Since $(W_r \mathfrak{e}_{n_0}, \mathfrak{e}_{m_0}) \neq 0$ by (3), we obtain $\mathfrak{e}_{m_0} \in \mathcal{E}$.

From $(++)$ and (c) it follows that $b_{m_0} = b_{m_{s, l+1}} > a_{m_0} \geq b_{m_{sl}} > -\infty$, i.e., $m_0 \in \mathfrak{J}^+$. Let $m_0 \in N_k$, $k \in N$. Since the sequence $\mathfrak{a}_k$ is monotone, we have $E_m W_r \mathfrak{e}_{m_0} = (W_r \mathfrak{e}_{m_0}, \mathfrak{e}_m) \mathfrak{e}_m$ for any $m \in M_k$. From (3) we again deduce that $\mathfrak{e}_m \in \mathcal{E}$ for $m \in M_k$. By (b), there is a number $m_1 \in M_k \cap N_{k+1}$. Repeating the last argument, with m_0 replaced by m_1, we get $\mathfrak{e}_m \in \mathcal{E}$ for $m \in M_{k+1}$. Using (b), induction shows that $\mathfrak{e}_m \in \mathcal{E}$ for all $m \in \bigcup_{j=1}^{\infty} M_j = \mathfrak{J}^-$. Hence $l_2(\mathfrak{J}^-) = l_2(\mathfrak{J}) \subseteq \mathcal{E}$ and $E = I$. This proves that π_{W_r} is irreducible.

Depending on γ, we define a sequence $\delta_j(\gamma):=\inf\{b_s-a_{r_j}: s\in\mathfrak{J}^-, b_s\geq a_{r_j}$ and $E_sW_\gamma E_{r_j}\neq 0\}$, $j\in N$. First we note that $\delta_j(\gamma)\geq\gamma_j$ for any $j\in N$. For suppose that $b_s\geq a_{r_j}$ and $E_sW_\gamma E_{r_j}\neq 0$. Then $E_sW_\gamma\mathfrak{e}_n\neq 0$ for some $n\in\mathfrak{J}_{r_j}$. Let $n\in N_l$. Because $W_\gamma\mathfrak{e}_n\in l_2(M_l)$, there is a $m\in\mathfrak{J}_s\cap M_l$. Clearly, $b_m=b_s$. Since $b_s\geq a_{r_j}$, (c) and (2) imply that $b_s-a_{r_j}>\gamma_j$. Hence $\delta_j(\gamma)\geq\gamma_j$.

Assume that π_{W_γ} and $\pi_{W_{\gamma'}}$ are unitarily equivalent by a unitary operator U. We have seen in Section 6 that $\tilde{U}E_r=E_r\tilde{U}$ for $r\in\mathfrak{J}$. Therefore, $\delta_j(\gamma)=\delta_j(\gamma')$ for $j\in N$. Now it is clear that the set of equivalence classes of all representations π_{W_γ} is not denumerable. Otherwise the set of all positive sequences would have a countable cofinal subset which is, of course, not true.

This completes the proof of Theorem 1.

Remark. In the above formulation Theorem 1 is still valid if the assumption $(++)$ is omitted.

7.2. In 5.3 we already stated (without proof) that there are self-adjoint representations $\pi_{W,\mathfrak{M}}$ which are not in $\mathscr{C}$. We now prove

Theorem 3. *Suppose* $\mathfrak{a}=\{a_n, n\in N\}$ *is a real sequence so that* $\sup_n a_n=+\infty$ *and* $\inf_n a_n=-\infty$. *Let* $\mathfrak{J}:=N$. *Then there exist a real sequence* $\mathfrak{b}=\{b_n, n\in N\}$ *and a partial isometry* W *of* $l_2(N)$ *in* $l_2(N)$ *such that* $\inf_n b_n-a_n>0$ *(i.e., the intervals* (a_n, b_n), $n\in N$, *satisfy* $(+)$*) and the corresponding* $*$*-representation* π_W *of* $A(\boldsymbol{p},\boldsymbol{q})$ *is self-adjoint, but not in* $\mathscr{C}$ *(i.e.,* $\pi_W(\boldsymbol{p})$ *is not e.s.a.).*

Proof. In the case described in Theorem 3 we have $\mathfrak{J}^+=\mathfrak{J}^-=\mathfrak{J}=N$. Let us abbreviate $l_2:=l_2(N)$. We shall prove the existence of a partial isometry W of l_2 on l_2 with initial space $\mathscr{W}$ such that:

(5) $\qquad \mathscr{W}=\overline{\mathfrak{S}(W)}\neq l_2,$

(6) $\qquad W\mathscr{W}=l_2,$

(7) $\qquad \mathfrak{y}=W\mathfrak{x},\ \mathfrak{x}\in\mathscr{D}(\mathfrak{a})$ and $\mathfrak{y}\in\mathscr{D}(\mathfrak{b})$ imply that $\mathfrak{x}\in\mathscr{W}$.

We then have $\pi_W(\boldsymbol{p})=P_W$ by Lemma 5.3, since $\mathscr{W}=\overline{\mathfrak{S}(W)}$ and $\mathfrak{S}(W)$

$=\mathfrak{B}_0^+(\mathcal{D}_W)$. Hence Corollary 5.8 applies. From (6) and (7) we see that condition (b) in Corollary 5.8 is satisfied. Therefore, π_W is self-adjoint. Since $\mathfrak{S}(W)$ is not dense in l_2 by (5), $\pi_W(\boldsymbol{p})$ is not e.s.a. by Lemma 5.4 and hence $\pi_W \notin \mathscr{C}$.

Before going to construct a sequence $\mathfrak{b}$ and an operator W satisfying (5)-(7), let us assume without loss of generality that in addition $\lim_n |a_n| = +\infty$. In the general case we take a partition of $\mathfrak{a}$ into subsequences $\mathfrak{a}_k = \{a_{r_{kn}}, n \in N\}$, $k \in N$, such that $\sup_n a_{r_{kn}} = +\infty$, $\inf_n a_{r_{kn}} = -\infty$ and $\lim_n |a_{r_{kn}}| = +\infty$ for $k \in N$. The proof given below shows that for each $k \in N$ there is a sequence $\mathfrak{b}_k = \{b_{kn}, n \in N\}$ such that $b_{kn} - a_{r_{kn}} \geq 1$ for $n \in N$ and a partial isometry W_k satisfying (5)-(7) for $\mathfrak{a}_k$ and $\mathfrak{b}_k$. Now it suffices to take the direct sums $W = \sum^{\oplus} W_k$ and $\mathfrak{b} = \sum^{\oplus} \mathfrak{b}_k$ in $l_2 = \sum^{\oplus} l_2(N_k)$ where $N_k := \{r_{kn}, n \in N\}$.

Let $V = (\mathfrak{a} - i)(\mathfrak{a} + i)^{-1}$ be the Cayley transform of the diagonal operator $\mathfrak{a}$ in l_2. Since $\mathfrak{a}$ is unbounded, we can find a $\xi \in l_2$, $\|\xi\| = 1$, so that $\xi \notin \mathcal{D}(\mathfrak{a})$. Put $C := (V\xi, \xi)$, $\psi_+ := \overline{C}\xi - V^*\xi$ and $\psi_- := C\xi - V\xi$. We shall use the notation $[\varphi_1, \cdots, \varphi_r] := \mathrm{Lin}\{\varphi_1, \cdots, \varphi_r\}$. Let $\mathcal{H}_1 := l_2 \ominus [\xi]$, $\mathcal{D}_+ := [\xi, V^*\xi]$ and $\mathcal{D}_- := [V\xi, \xi]$. It is easy to see $\mathcal{D}_+ \cap \mathcal{H}_1 = [\psi_+]$ and $\mathcal{D}_- \cap \mathcal{H}_1 = [\psi_-]$. Let E_+ be the projection of l_2 on $[\psi_+]$. Put $V_1 = V(I - E_+) \upharpoonright \mathcal{H}_1$. Let $\varphi \in \mathcal{H}_1$. Since $(I - E_+)\varphi \in \mathcal{H}_1$, we have $(I - E_+)\varphi \perp \xi$ and $(I - E_+)\psi \perp \psi_+$. Hence $(I - E_+)\varphi \perp V^*\xi$ and $(V_1\varphi, \xi) = ((I - E_+)\varphi, V^*\xi) = 0$. This proves that $V_1\mathcal{H}_1 \subseteq \mathcal{H}_1$. Let T be the closed symmetric operator in $\mathcal{H}_1$ which has the Cayley transform V_1.

We now show that $\mathcal{D}(T)$ is dense in $\mathcal{H}_1$. Clearly, $\mathcal{D}(T) = (I - V_1)(I - E_+)\mathcal{H}_1 = (I - V)(I - E_+)\mathcal{H}_1$. Suppose that $\eta \perp \mathcal{D}(T)$ for $\eta \in \mathcal{H}_1$. Then $(\eta, (I - V)(I - E_+)\varphi) = ((I - E_+)(I - V^*)\eta, \varphi) = 0$ for each $\varphi \in \mathcal{H}_1$. Hence $(I - V^*)\eta = E_+(I - V^*)\eta + \mu\xi$ for some $\mu \in C_1$. Since $(I - V^*)\eta \in \mathcal{D}(\mathfrak{a})$, $E_+(I - V^*)\eta + \mu\xi \in \mathcal{D}_+$ and $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}_+ = (I - V^*)[\xi]$, the injectivity of $I - V^*$ yields $\eta \in [\xi]$. Because $\eta \in \mathcal{H}_1$, we obtain $\eta = 0$.

Let $\mathcal{H}_\pm$ be the deficiency spaces for $\pm i$ of the operator T in $\mathcal{H}_1$. Since $\mathcal{H}_+$ and $\mathcal{H}_-$ are the orthogonal complements of the initial space resp. the range of V_1 in $\mathcal{H}_1$, we have $\mathcal{H}_+ = [\psi_+]$ and $\mathcal{H}_- = [\psi_-]$. Clearly, $\psi_+ \neq 0$ and $\psi_- \neq 0$. Otherwise ξ would be an eigenvector of V^* or V which contradicts $\xi \notin \mathcal{D}(\mathfrak{a})$. Thus T has deficiency indices $(1, 1)$.

Moreover, $\|\psi_+\|^2=\|\psi_-\|^2=1-|C|^2$. Hence $C\neq 1$. Let $z:=(\overline{C}-1)/(1-C)$. Because $|z|=1$, there is a self-adjoint extension B of T in $\mathcal{H}_1$ with domain $\mathcal{D}(B)=\{\varphi+\mu\psi_++\mu z\psi_-;\ \varphi\in\mathcal{D}(T),\mu\in C_1\}$. From the special form of z it follows that $\psi_++z\psi_-=(I-V^*)\xi+z(I-V)\xi\in\mathcal{D}(\mathfrak{a})$. Since $\mathcal{D}(T)\subseteq\mathcal{D}(\mathfrak{a})$, this yields $\mathcal{D}(B)\subseteq\mathcal{D}(\mathfrak{a})$. [Note that B is not a restriction of $\mathfrak{a}$!]. Let V_2 be the Cayley transform of B. $V_3:=V_2\oplus I$ defines a unitary operator in $l_2=\mathcal{H}_1\oplus[\xi]$. If $\varphi\in l_2\ominus\mathcal{D}_+$, then $\varphi\in\mathcal{H}_1$, $(I-E_+)\varphi=\varphi$ and hence $V_3\varphi=V_1\varphi=V\varphi$. Therefore range $V-V_3\subseteq(V-V_3)\mathcal{D}_+$ is finite dimensional. Since we assumed that $\lim_n|a_n|=+\infty$, the essential spectrum of V contains only the number 1. According to a classical theorem of H. Weyl ([24]), V_3 has the same essential spectrum. Since $\sigma(V_3)=\sigma(V_2)\cup\{1\}$, V_2 and B have a complete system of eigenvectors, say $\{\mathfrak{f}_n, n\in N\}$, in $\mathcal{H}_1$. Let $\mathfrak{b}=\{b_n, n\in N\}$ be the corresponding eigenvalues of B. By construction, we have dim $\mathcal{D}(T)\ (\mathrm{mod}\,\mathcal{D}(\mathfrak{a}))=2$. Arguing now as in [1], No. 106, $\sup_n a_n=+\infty$ implies that $\sup\{(T\varphi,\varphi);\ \varphi\in\mathcal{D}(T), \|\varphi\|=1\}=+\infty$. Hence $\sup_n b_n=+\infty$. Because $\inf_n a_n=-\infty$ by assumption, we can assume (after new enumeration if necessary) that $b_n-a_n\geq 1$ for all $n\in N$, i.e., $(+)$ is satisfied.

Recall that $\mathfrak{a}$ and $\mathfrak{b}$ are diagonal operators in l_2 relative to the orthobase $\mathfrak{e}_k=\{\delta_{kn}, n\in N\}$, $k\in N$. Let W be the partial isometry of l_2 with initial space $\mathcal{H}_1$ which is defined by $W\mathfrak{f}_n=\mathfrak{e}_n$, $n\in N$. We then have $W\mathcal{D}(B)=\mathcal{D}(\mathfrak{b})$ and $B=W^*\mathfrak{b}W\restriction\mathcal{H}_1$. Since $T=\mathfrak{a}\restriction\mathcal{D}(T)=B\restriction\mathcal{D}(T)$, $\mathfrak{a}\varphi=W^*\mathfrak{b}W\varphi$ for $\varphi\in\mathcal{D}(T)$. Since $WW^*=I_{\mathcal{H}}$, we obtain $W\mathfrak{a}\varphi=\mathfrak{b}W\varphi$ for $\varphi\in\mathcal{D}(T)$ and thus $\mathcal{D}_\infty(T)\subseteq\mathfrak{S}(W)\subseteq\mathcal{H}_1=\mathcal{W}$. Because T has deficiency indices $(1,1)$ in $\mathcal{H}_1$, $\mathcal{D}_\infty(T)$ is dense in $\mathcal{H}_1$ ([18], Prop. 2.1). Therefore, $\overline{\mathfrak{S}(W)}=\mathcal{W}\neq l_2$ which proves (5). (6) is obvious, since $W\mathcal{H}_1=l_2$. To prove (7), suppose that $\mathfrak{x}\in\mathcal{D}(\mathfrak{a})$ and $\mathfrak{y}=W\mathfrak{x}\in\mathcal{D}(\mathfrak{b})$. Since $\mathfrak{z}:=\mathfrak{x}-(\mathfrak{x},\xi)\xi\in\mathcal{H}_1$ and $W\mathfrak{x}=W\mathfrak{z}$ by definition, we get $W\mathfrak{z}\in\mathcal{D}(\mathfrak{b})=W\mathcal{D}(B)$ and hence $\mathfrak{z}\in\mathcal{D}(B)$. Because $\mathcal{D}(B)\subseteq\mathcal{D}(\mathfrak{a})$ and $\xi\notin\mathcal{D}(\mathfrak{a})$, this implies that $(\mathfrak{x},\xi)=0$, i.e., $\mathfrak{x}\in\mathcal{H}_1=\mathcal{W}$. This completes the proof of Theorem 3.

Appendix to Section 7

The proof of Corollary 4.4 in [18] is not correct (condition ii) cannot be satisfied in general). Because it is essentially used in the proof of Theorem 1 in this section and in the proof of Theorem 4.5 in [18], we now give a complete proof of this corollary. We retain the notation used in [18].

We choose subsequences $(c_n = a_{k_n})$ and $(d_n = b_{m_n})$ of (a_n) resp. (b_n) such that $c_{n+1} > 2d_n > 4c_n > 0$ for $n \in N$, $n \geq r+1$. For simplicity assume that $c_n \neq 0$ and $d_n \neq 0$ for all $n \in N$ (otherwise the infinite products must be slightly modified). Let $f(z) := \prod_n (1 - zc_{n+1}^{-1})(1 - zd_n^{-1})^{-1}$ and $g(z) := \prod_n (1 - zd_n^{-1})(1 - zc_n^{-1})^{-1}$, $z \in C_1$. Our aim is to prove that $(c_n) \underset{1}{\sim} (d_n)$. Since $((d_n - c_n)/c_n) \notin l_1$ and $c_{n+1} > d_n > c_n$ for $n \in N$, it follows from Theorem 4.1 in [18] that it suffices to show that $\lim_{|y| \to \infty} |yg(iy)| = +\infty$. (The condition in Theorem 4.1 is only a reformulation of the latter). Since $\lim_n c_n = +\infty$, we have $g(z)h(z)(c_1 - z) = c_1$ for $z \in C_1$. Applying Lemma 2, i.) in [18] to $h(z)$ (in case $a_n = d_n$, $b_n = c_{n+1}$), we obtain $\lim_{|y| \to \infty} h(iy) = \prod_n d_n c_{n+1}^{-1} = 0$, since $2d_n < c_{n+1}$ for $n \geq r+1$. Combined with $\lim_{|y| \to \infty} (c_1 - iy) y^{-1} = -i$ and $c_1 \neq 0$, this implies $\lim_{|y| \to \infty} |yg(iy)| = +\infty$ thus completing the proof of Corollary 4.4 in [18].

§ 8. Restrictions of Unbounded Symmetric Operators

The results obtained in this section are preparatory for the construction of admissible boundary spaces, but they are also of some interest in its own right. (In fact, we do not need the full strength of Theorem 1.)

Let $\mathcal{G}$ be a (separable, complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Throughout this section we assume that T is an unbounded symmetric linear operator defined on a dense domain $\mathcal{F} = \mathcal{D}(T)$ of $\mathcal{G}$ such that $T\mathcal{F} \subseteq \mathcal{F}$ and $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{D}(\overline{T^n})$. Obviously, the latter means that $\mathcal{P}(T)$ is a closed $Op*$-algebra on $\mathcal{F}$.

8.1. Theorem 1. *Let $(c_{jk})_{j,k \in N_0}$ be a given matrix of positive numbers. Let $\varepsilon \in R_1$, $0 < \varepsilon < 1/4$. Then there exist sequences $\{\varphi_j^1, j \in N_0\}$ and $\{\varphi_j^2, j \in N_0\}$ of non-zero vectors in $\mathcal{F}$ such that:*

(i) $\|T^{k+1}\varphi_j^m\| \geq c_{jk} \|T^k \varphi_j^m\|$ *for* $j, k \in N_0$ *and* $m = 1, 2$,

(ii) $\mathcal{F}_m := \mathrm{Lin}\{T^k \varphi_j^m; j, k \in N_0\}$ *is dense in* $\mathcal{G}$ *for* $m = 1, 2$,

(iii) $(1-\varepsilon)\sum_{j,k=0}^{\infty}|\rho_{jk}|^2\|T^k\varphi_j^m\|^2\leq\|\sum_{j,k=0}^{\infty}\rho_{jk}T^k\varphi_j^m\|^2$

$\leq(1+\varepsilon)\sum_{j,k=0}^{\infty}|\rho_{jk}|^2\|T^k\varphi_j^m\|^2$

(iv) $(1-2\varepsilon)\sum_{j,k=0}^{\infty}(|\lambda_{jk}|^2\|T^{k+1}\varphi_j^1\|^2+|\mu_{jk}|^2\|T^{k+1}\varphi_j^2\|^2)$

$\leq\|\sum_{j,k=0}^{\infty}(\lambda_{jk}T^{k+1}\varphi_j^1+\mu_{jk}\cdot T^{k+1}\varphi_j^2)\|^2$

$\leq(1+2\varepsilon)\sum_{j,k=0}^{\infty}(|\lambda_{jk}|^2\|T^{k+1}\varphi_j^1\|^2+|\mu_{jk}|^2\|T^{k+1}\varphi_j^2\|^2)$

for all finite matrices $(\rho_{jk})_{j,k\in N_0}$, $(\lambda_{jk})_{j,k\in N_0}$ *and* $(\mu_{jk})_{j,k\in N_0}$ *of complex numbers.*

Here an infinite matrix is called *finite* if it has only finitely many non-zero entries.

Proof. By assumption, $\mathcal{G}$ is separable. Hence we can find an ortho-normal base $\{\xi_r, r\in N_0\}$ of vectors $\xi_r\in\mathfrak{F}$. We enumerate the set $N_0\times N_0$ by a diagonal procedure, that is, we define $d(0,0)=1$, $d(0,1)=2$, $d(1,0)=3$, $d(0,2)=4$, $d(1,1)=5$, $d(2,0)=6$ etc. Clearly, if $j+r>l+s$, then $d(j,r)>d(l,s)$. We first show that there are sequences $\{\varphi_{jr}^m, r\in N_0\}$, $m=1,2$ and $j\in N_0$, of vectors $\varphi_{jr}^m\in\mathfrak{F}$ satisfying the following conditions. For $j,k,l,n,r,s\in N_0$ and $m,m'=1,2$, we have

(a) $\|T^n\varphi_{jr}^m\|\leq\varepsilon 2^{-3(n+j+r)-5}\|T^n\varphi_{jn}^m\|$ if $r<n$,

(b) $\|T^r\varphi_{jn}^m\|\leq\varepsilon 2^{-3(n+j+r)-5}$ if $r<n$,

(c) $(1+\varepsilon)c_{jn}\|T^n\varphi_{jn}^m\|\leq(1-\varepsilon)\|T^{n+1}\varphi_{j,n+1}^m\|$ and $1\leq\|T^n\varphi_{jn}^m\|$,

(d) $\langle\varphi_{jr}^m, T^k\varphi_{ls}^m\rangle=0$ if $k\leq 2(j+r)$, $r\neq 0$ and $d(l,s)<d(j,r)$,

(e) $\langle\varphi_{jr}^m, T^k\varphi_{ls}^{m'}\rangle=0$ if $k\leq 2(j+r)$, $r\neq 0$, $m\neq m'$ and $d(l,s)<d(j,r)$,

(f) $\varphi_{j0}^m\perp\mathfrak{F}^{mj}:=\mathrm{Lin}\{T^s\varphi_{ls}^m;\ d(l,s)<d(j,0)\}$ and $\xi_j\in\mathfrak{F}^{m,j+1}$.

Set $\varphi_{00}^1=\varphi_{00}^2=\xi_0$. Let $d\in N$. Suppose that we have found vectors φ_{ls}^m, $d(l,s)\leq d$ and $m=1,2$, so that conditions (a)-(f) hold for these vectors. Let $d(j,n)=d+1$. To construct φ_{jn}^m, we first let $n=0$. Let k_{jm} be the smallest integer for which $(I-E_{jm})\xi_{k_{jm}}\neq 0$, where E_{jm} is the orthogonal projection on $\mathfrak{F}^{mj}$. We set $\varphi_{j0}^m=(I-E_{jm})\xi_{k_{jm}}/\|(I-E_{jm})\xi_{k_{jm}}\|$. Then (c) and (f) are satisfied. Suppose now that $n\neq 0$. Let F_{d1} and F_{d2} be the orthogonal projections on $\mathfrak{F}_d^1:=\mathrm{Lin}\{T^k\varphi_{ls}^m;\ k\leq 2(j+n),\ d(l,s)\leq d$ and $m=1,2\}$ resp. $\mathfrak{F}_d^2:=\mathfrak{F}_d^1+\mathrm{Lin}\{T^k\varphi_{jn}^1;\ k\leq 2(j+n)\}$. Given $\alpha>0$ and $\beta>0$, we can find a vector $\varphi\in(I-F_{d1})\mathfrak{F}$ so that $\|T^n\varphi\|\geq\alpha$ and

$\|T^r\varphi\|\leq\beta$ for $r=0,\cdots,n-1$. [Indeed, otherwise there would exist a constant ρ such that $\|T^n\varphi\|\leq\rho\sum_{r=0}^{n-1}\|T^r\varphi\|$ for all $\varphi\in(I-F_{d1})\mathcal{F}$. Since $\mathcal{F}_d^1$ is finite dimensional and hence T^kF_{d1} is bounded for all $k\in N_0$, it follows that T is bounded. This is the desired contradiction.] Applying this fact, we can choose a vector $\varphi_{jn}^1\in(I-F_{d1})\mathcal{F}$ such that $\|T^n\varphi_{jn}^1\|\geq c_{j,n-1}(1+\varepsilon)(1-\varepsilon)^{-1}\|T^{n-1}\varphi_{j,n-1}^1\|$, $\|T^n\varphi_{jn}^1\|\geq 1$, $\|T^n\varphi_{jn}^1\|\geq\frac{1}{\varepsilon}2^{3(n+j+r)+5}\times\|T^n\varphi_{jr}^1\|$ and $\|T^r\varphi_{jn}^1\|\leq\varepsilon 2^{-3(n+j+r)-5}$ for $r=0,\cdots,n-1$. Then, (a)-(d) are true for $m=1$. $\varphi_{jn}^1\in(I-F_{d1})\mathcal{F}$ ensures (f) in case $m=1$. Replacing $\mathcal{F}_d^1$ by $\mathcal{F}_d^2$, φ_{jn}^2 will be constructed similarly. Condition (e) follows from the symmetry of T. By induction, this proves the existence of sequences $\{\varphi_{jr}^m, r\in N_0\}$ satisfying (a)-(f).

The next step is to show that

$$(1)\qquad |\langle T^k\varphi_{jr}^m, T^n\varphi_{ls}^{m'}\rangle|\leq\varepsilon 2^{-(k+j+r+n+l+s+5)}\|T^k\varphi_{jk}^m\|\,\|T^n\varphi_{ln}^{m'}\|$$

for all $k, j, r, n, l, s\in N_0$ and $m, m'=1,2$ for which

$$(2)\quad \text{either } m=m',\ (j,k)\neq(l,n),\ (r,k)\neq(0,0) \text{ and } (s,n)\neq(0,0)$$

$$(3)\quad \text{or } m\neq m',\ k\geq 1 \text{ and } n\geq 1.$$

We divide the argument into several cases. For simplicity we shall denote by α the left-hand side of (1). First, however, we note that (a)-(c) imply that

$$(4)\quad \|T^n\varphi_{jr}^m\|\leq\varepsilon 2^{-3(n+j+r)-5}\|T^n\varphi_{jn}^m\| \text{ if } r\neq n,\ n,j,r\in N_0 \text{ and } m=1,2.$$

Case 1. $r\neq k$ and $s\neq n$

By the Cauchy-Schwarz inequality and $\varepsilon^2<\varepsilon$, (1) follows at once from (4). Thus it remains to treat the cases $r=k$ and $s=n$. Since T is a symmetric operator, there is no loss of generality if we restrict ourselves to the case $r=k$. By (2) and (3), this implies $r=k\neq 0$.

Case 2. $r=k$ and $s=n=0$

Then we are in case (2) and $\alpha=|\langle T^k\varphi_{jk}^m, \varphi_{l0}^m\rangle|$. Since $r\neq 0$, we have $\alpha=0$ for $d(l,0)<d(j,r)$ by (d). Since $(j,k)\neq(l,n)$, we have $(j,r)\neq(l,0)$ and hence $d(j,r)\neq d(l,0)$. If $d(l,0)>d(j,r)$, then $\alpha=0$ by (f).

Case 3. $r=k$ and $s=n\neq 0$

First let $m=m'$. Then $\alpha=|\langle T^k\varphi_{jk}^m, T^n\varphi_{ln}^m\rangle|=0$ by (d) if either

$d(j,k)>d(l,n)$ or $d(j,k)<d(l,n)$ because of the symmetry of T. The case $(j,k)=(l,n)$ is excluded for $m=m'$ by (2). For $m\neq m'$, $\alpha=0$ by (e).

Case 4. $r=k$, $s\neq n$ and $n+l+s\geq k+j$

By (4), $\alpha=|\langle T^k\varphi^m_{jk}, T^n\varphi^{m'}_{ls}\rangle|\leq\varepsilon 2^{-3(n+l+s)-5}\|T^k\varphi^m_{jk}\|\,\|T^n\varphi^{m'}_{ln}\|$ $\leq\varepsilon 2^{-(k+j+k+n+l+s+5)}\|T^k\varphi^m_{jk}\|\,\|T^n\varphi^{m'}_{ln}\|$.

Case 5. $r=k$, $s\neq n$ and $k+j>n+l+s$

Since $k+j>l+s$, $d(j,k)>d(l,s)$. Moreover, $k+n<2(k+j)$. Therefore, $\alpha=0$ by (d) resp. (e). This completes the proof of (1).

Now we define $\varphi^m_j:=\sum_{r=0}^{\infty}\varphi^m_{jr}$ for $j\in N_0$ and $m=1,2$. Since $\|T^n\varphi^m_{jr}\|\leq 2^{-r}$ for $r>$n by (b), the infinite series is converging in the locally convex space $\mathcal{F}[t_{\mathcal{P}(T)}]$. Recall that $\mathcal{F}[t_{\mathcal{P}(T)}]$ is complete, since we assumed that $\mathcal{F}=\bigcap_{n=1}^{\infty}\mathcal{D}(\overline{T^n})$. Therefore, $\varphi^m_j\in\mathcal{F}$. By (4), we have $\|T^k\sum_{\substack{r=0\\ r\neq k}}^{\infty}\varphi^m_{jr}\|\leq\sum_{r=0}^{\infty}\varepsilon 2^{-(r+1)}\|T^k\varphi^m_{jk}\|\leq\varepsilon\|T^k\varphi^m_{jk}\|$ and hence

$$(5)\qquad \frac{3}{4}\|T^k\varphi^m_{jk}\|\leq(1-\varepsilon)\,\|T^k\varphi^m_{jk}\|\leq\|T^k\varphi^m_j\|\leq(1+\varepsilon)\,\|T^k\varphi^m_{jk}\|$$

for all $j,k\in N_0$ and $m=1,2$. From (c) we obtain

$$\|T^{k+1}\varphi^m_j\|\geq(1-\varepsilon)\,\|T^{k+1}\varphi^m_{j,k+1}\|\geq(1+\varepsilon)\,c_{jk}\|T^k\varphi^m_{jk}\|\geq c_{jk}\|T^k\varphi^m_j\|,$$

thus proving (i).

Let $(\rho_{jk})_{j,k\in N_0}$ be a given finite matrix. We now prove (iii). For this it suffices to check that

$$\varepsilon\sum_{j,k=0}^{\infty}|\rho_{jk}|^2\|T^k\varphi^m_j\|^2-|\sum_{\substack{j,k,l,n=0\\ (j,k)\neq(l,n)}}^{\infty}\rho_{jk}\overline{\rho_{ln}}\langle T^k\varphi^m_j, T^n\varphi^m_l\rangle|\geq 0.$$

Using (5) and the definition of φ^m_j, it is enough to show that

$$(6)\qquad \frac{3}{4}\ \varepsilon\sum_{j,k=0}^{\infty}|\rho_{jk}|^2\|T^k\varphi^m_{jk}\|^2\ -2|\sum_{\substack{j,l,n,s=0\\ (j,0)\neq(l,n)}}^{\infty}\rho_{j0}\overline{\rho_{ln}}\langle\varphi^m_{j0}, T^n\varphi^m_{ls}\rangle|$$
$$-|\sum_{\substack{j,k,r,l,n,s=0\\ (j,k)\neq(l,n),\,(r,k)\neq(0,0),\,(s,n)\neq(0,0)}}^{\infty}\rho_{jk}\overline{\rho_{ln}}\langle T^k\varphi^m_{jr}, T^n\varphi^m_{ls}\rangle|\geq 0\,.$$

Let II and III denote the second resp. third sum above. Applying (1) in case (2), we obtain

$$(7)\qquad |\mathrm{III}|\leq \sum_{j,k,r,l,n,s=0}^{\infty}|\rho_{jk}|\ |\rho_{ln}|\varepsilon 2^{-(k+j+r+n+l+s+5)}\|T^k\varphi_{jk}^m\|\,\|T^n\varphi_{ln}^m\|$$

$$\leq\frac{\varepsilon}{8}\Big(\sum_{j,k=0}^{\infty}|\rho_{jk}|\,\|T^k\varphi_{jk}^m\|2^{-(k+j)}\Big)^2\leq\frac{\varepsilon}{4}\sum_{j,k=0}^{\infty}|\rho_{jk}|^2\|T^k\varphi_{jk}^m\|^2.$$

Now we turn to the second sum. First we check that $\langle\varphi_{j0}^m, T^n\varphi_{ln}^m\rangle=0$ for $(j,0)\neq(l,n)$. Indeed, if $d(j,0)>d(l,n)$, then (f) applies. For $d(j,0)<d(l,n)$, this follows from (d). The case $d(j,0)=d(l,n)$ is not possible. Moreover, we have $\langle\varphi_{j0}^m, \varphi_{l0}^m\rangle=0$ for $j\neq l$. Together with (4), we thus obtain

$$(8)\qquad 2|\mathrm{II}|\leq 2\sum_{\substack{l,n,s=0\\ s\neq n}}^{\infty}|\rho_{ln}|\,|\langle(\sum_{j=0}^{\infty}\rho_{j0}\varphi_{j0}^m),\ T^n\varphi_{ls}^m\rangle|$$

$$\leq 2\|\sum_{j=0}^{\infty}\rho_{j0}\varphi_{j0}^m\|\sum_{\substack{l,n,s=0\\ s\neq n}}^{\infty}|\rho_{ln}|\varepsilon 2^{-(n+l+s+5)}\|T^n\varphi_{ln}^m\|$$

$$\leq\frac{\varepsilon}{8}\sum_{j=0}^{\infty}|\rho_{j0}|^2\|\varphi_{j0}^m\|^2+\frac{\varepsilon}{8}\sum_{l,n=0}^{\infty}|\rho_{ln}|^2\|T^n\varphi_{ln}^m\|^2.$$

Now (6) follows from (7) and (8). This proves (iii).

The proof of (iv) is similar. We abbreviate

$$S_1=\sum_{j,k=0}^{\infty}|\lambda_{jk}|^2\|T^{k+1}\varphi_j^1\|^2,\quad S_2=\sum_{j,k=0}^{\infty}|\mu_{jk}|^2\|T^{k+1}\varphi_j^2\|^2,$$

$$S_3=\|\sum_{j,k=0}^{\infty}\lambda_{jk}T^{k+1}\varphi_j^1\|^2\quad\text{and}\quad S_4=\|\sum_{j,k=0}^{\infty}\mu_{jk}T^{k+1}\varphi_j^2\|^2.$$

Then, (iv) is equivalent to

$$(1-2\varepsilon)\ (S_1+S_2)\leq S_3+S_4+2\mathrm{Re}\sum_{j,k,l,n=0}^{\infty}\lambda_{jk}\overline{\mu_{ln}}\langle T^{k+1}\varphi_j^1,\ T^{n+1}\varphi_l^2\rangle$$

$$\leq(1+2\varepsilon)\ (S_1+S_2).$$

From (iii) we know that $(1-\varepsilon)S_1\leq S_3\leq(1+\varepsilon)S_1$ and $(1-\varepsilon)S_2\leq S_4\leq(1+\varepsilon)S_2$. Therefore it is sufficient to show that

$$(9)\qquad \varepsilon(S_1+S_2)-2|\sum_{j,k,l,n=0}^{\infty}\lambda_{jk}\overline{\mu_{ln}}\langle T^{k+1}\varphi_j^1,\ T^{n+1}\varphi_l^2\rangle|\geq 0.$$

(1) in case (3) (that is, $m=1$, $m'=2$) and (5) imply $|\langle T^{k+1}\varphi_j^1, T^{n+1}\varphi_l^2\rangle|\leq\varepsilon 2^{-(k+j+n+l+4)}\|T^{k+1}\varphi_j^1\|\,\|T^{n+1}\varphi_l^2\|$ for all $j,k,l,n\in N_0$. Applying the Cauchy-Schwarz inequality in (9), the assertion follows.

Finally, we prove (ii). Let $\mathfrak{F}^m=\mathrm{Lin}\{T^n\varphi_{ln}^m;\ l,n\in N_0\}$ for $m=1,2$.

Since $\xi_j \in \mathfrak{F}^{m,j+1} \subseteq \mathfrak{F}^m$ for all $j \in N_0$ by (f), $\mathfrak{F}^m$ is dense in $\mathcal{G}$. Recall that $\| T^n\varphi_l^m - T^n\varphi_{ln}^m \| \leq \sum_{\substack{r=0 \\ r \neq n}}^{\infty} \| T^n\varphi_{lr}^m \| \leq \sum_{r=0}^{\infty} \varepsilon 2^{-(n+l+r+2)} \| T^n\varphi_{ln}^m \| \leq \varepsilon 2^{-(n+l)} \| T^n\varphi_l^m \|$ for $l, n \in N_0$ and $m=1, 2$ by (4) and (5). Using (iii) and $\varepsilon \leq 1/4$, we have, for any finite matrix $(\rho_{ln})_{l,n \in N_0}$ of complex numbers,

$$\| \sum_{l,n=0}^{\infty} \rho_{ln} (T^n\varphi_l^m - T^n\varphi_{ln}^m) \|^2 \leq (\sum_{l,n=0}^{\infty} |\rho_{ln}| \varepsilon 2^{-(n+l)} \| T^n\varphi_l^m \|)^2$$

$$\leq 2\varepsilon^2 \sum_{l,n=0}^{\infty} |\rho_{ln}|^2 \| T^n\varphi_l^m \|^2 \leq \frac{1}{4} \| \sum_{l,n=0}^{\infty} \rho_{ln} T^n\varphi_l^m \|^2 .$$

This shows that the systems $\{T^n\varphi_{ln}^m\}$ and $\{T^n\varphi_l^m\}$ satisfy the assumptions of the Paley-Wiener stability theorem for sequences in Banach spaces, i.e., formula (9.1) in [20], p. 84, is true. Because $\mathfrak{F}^m = \mathrm{Lin}\{T^n\varphi_{ln}^m\}$ is dense, we therefore conclude from Theorem 9.2 (applied in case b.), γ)) in [20], p. 87, that $\mathfrak{F}_m = \mathrm{Lin}\{T^n\varphi_l^m\}$ is dense in $\mathcal{G}$.

This completes the proof of Theorem 1.

8.2. We retain the assumptions and notations of Theorem 1. Moreover, let $\mathfrak{F}_{1,2} = \mathfrak{F}_1 + \mathfrak{F}_2$ and let t denote the graph topology of the *Op**-algebra $\mathcal{P}(T)$ on $\mathfrak{F}$. As usual, we denote by $\underline{\mathfrak{F}}_1$, $\underline{\mathfrak{F}}_2$ and $\underline{\mathfrak{F}}_{1,2}$ the t-closures of $\mathfrak{F}_1$, $\mathfrak{F}_2$ resp. $\mathfrak{F}_{1,2}$.

Corollary 2. *Assume that* $c_{j+1,n} \geq c_{jn}$ *and* $c_{j,n+1} \geq c_{jn}$ *for all* $j, n \in N_0$ *Then:*

(i) $2\| T^{n+1}\psi \| \geq c_{0n} \| T^n\psi \|$
if either $\psi \in \underline{\mathfrak{F}}_{1,2}$ *and* $n \in N$ *or* $\psi \in \underline{\mathfrak{F}}_1 \cup \underline{\mathfrak{F}}_2$ *and* $n \in N_0$.

(ii) *If* $T^n\psi = 0$ *for some* $\psi \in \underline{\mathfrak{F}}_1 \cup \underline{\mathfrak{F}}_2$ *and* $n \in N$, *then* $\psi = 0$.

(iii) $(1-3\varepsilon)(\| T^n\psi_1 \|^2 + \| T^n\psi_2 \|^2) \leq \| T^n(\psi_1 + \psi_2) \|^2$
$\leq (1+4\varepsilon)(\| T^n\psi_1 \|^2 + \| T^n\psi_2 \|^2)$ *for* $\psi_1 \in \underline{\mathfrak{F}}_1$, $\psi_2 \in \underline{\mathfrak{F}}_2$, $n \in N$.

(iv) $\underline{\mathfrak{F}}_1 \cap \underline{\mathfrak{F}}_2 = \{0\}$.

Proof. (i) We only carry out the proof in case $\psi \in \underline{\mathfrak{F}}_{1,2}$, $n \in N$. Replacing part (iv) of Theorem 1 by part (iii), the case $\psi \in \underline{\mathfrak{F}}_1 \cup \underline{\mathfrak{F}}_2$, $n \in N_0$, can be treated similarly. By Theorem 1, (i), we have $\| T^{n+1+k}\varphi_j^m \| \geq c_{j,n+k} \| T^{n+k}\varphi_j^m \| \geq c_{0n} \| T^{n+k}\varphi_j^m \|$ for $j, k \in N_0$, $n \in N$ and $m=1, 2$. Let $\psi = \psi_1 + \psi_2 = \sum_{j,k} \lambda_{jk} T^k\varphi_j^1 + \sum_{j,k} \mu_{jk} T^k\varphi_j^2$, where (λ_{jk}) and (μ_{jk}) are finite matrices of complex numbers. Letting $\varepsilon=\frac{1}{4}$ and using Theorem 1, (iv), we obtain for $n\in N$

$$\|T^{n+1}\psi\|^2\geq\frac{1}{2}\sum_{j,k=0}(|\lambda_{jk}|^2\|T^{n+1+k}\varphi_j^1\|^2+|\mu_{jk}|^2\|T^{n+1+k}\varphi_j^2\|^2)$$

$$\geq\frac{1}{2}c_{0n}^2\sum_{j,k=0}^{\infty}(|\lambda_{jk}|^2\|T^{n+k}\varphi_j^1\|^2+|\mu_{jk}|^2\|T^{n+k}\varphi_j^2\|^2)$$

$$\geq\frac{1}{2}c_{0n}^2\frac{2}{3}\|T^n\psi\|^2\geq\frac{1}{4}c_{0n}^2\|T^n\psi\|^2.$$

Here we essentially used that $n\geq 1$. This proves the assertion for $\psi\in\mathcal{F}_{1,2}$. Since T is continuous in $\mathcal{F}[t]$, the above inequality is still true for $\psi\in\underline{\mathcal{F}}_{1,2}$.

(ii) follows immediately from (i) because we assumed that $c_{0n}\neq 0$ for all $n\in N_0$.

(iii) We retain the notation from the proof of (i). Let $n\in N$. Applying Theorem 1, (iv) and (iii), we get

$$\|T^n(\psi_1+\psi_2)\|^2\geq(1-2\varepsilon)\sum_{j,k=0}^{\infty}(|\lambda_{jk}|^2\|T^{n+k}\varphi_j^1\|^2+|\mu_{jk}|^2\|T^{n+k}\varphi_j^2\|^2)$$

$$\geq(1-2\varepsilon)(1+\varepsilon)^{-1}(\|T^n\psi_1\|^2+\|T^n\psi_2\|^2)\geq(1-3\varepsilon)(\|T^n\psi_1\|^2+\|T^n\psi_2\|^2).$$

Again, by the continuity of T in $\mathcal{F}[t]$, this inequality remains valid for $\psi_1\in\underline{\mathcal{F}}_1$ and $\psi_2\in\underline{\mathcal{F}}_2$. The other inequality follows similarly. Instead of $(1-2\varepsilon)(1+\varepsilon)^{-1}$ we obtain the constant $(1+2\varepsilon)(1-\varepsilon)^{-1}\leq 1+4\varepsilon$, since $\varepsilon\leq 1/4$.

(iv) Let $\psi\in\underline{\mathcal{F}}_1\cap\underline{\mathcal{F}}_2$. Then we can find sequences $\{\psi_r^m, r\in N\}$ of vectors $\psi_r^m\in\underline{\mathcal{F}}_m$ such that $\psi=t\text{-}\lim_r\psi_r^m$, for $m=1,2$. Hence we have $T(\psi_r^1-\psi_r^2)\to 0$ and $T\psi_r^1\to T\psi$ in $\mathcal{G}$ as $r\to+\infty$. By (iii), $T\psi_r^1\to 0$ in $\mathcal{G}$. Therefore, $T\psi=0$ which gives $\psi=0$ by (ii). This completes the proof of Corollary 2.

Corollary 3. *There exists an uncountable set $\{\mathcal{F}_i, i\in\mathcal{J}\}$ of t-closed linear subspaces $\mathcal{F}_i$ of $\mathcal{F}$ such that $T\mathcal{F}_i\subseteq\mathcal{F}_i$ and $\mathcal{F}_i$ is dense in $\mathcal{G}$ for each $i\in\mathcal{J}$ and that the operators $T\upharpoonright\mathcal{F}_i$ and $T\upharpoonright\mathcal{F}_j$ are not unitarily equivalent for all $i,j\in\mathcal{J}$, $i\neq j$.*

Proof. Let $\gamma=\{\gamma_n, n\in N_0\}$ be a positive monotone sequence. We set $c_{jn}=2\gamma_n$ for all $j, n\in N_0$ and apply Theorem 1. Rename the space $\underline{\mathcal{D}}_1$ in this case by $\mathcal{D}_\gamma$. Put $\delta_n(\gamma):=\sup\{\rho\in R_1: \|T^{n+1}\psi\|\geq\rho\|T^n\psi\|$ for all $\psi\in\mathcal{D}_\gamma\}$ for $n\in N_0$. Corollary 2, (i), shows that $\delta_n(\gamma)\geq\gamma_n$ for all $n\in N_0$. Obviously, if $T\upharpoonright\mathcal{D}_\gamma$ and $T\upharpoonright\mathcal{D}_{\gamma'}$ are unitarily equivalent for positive monotone sequences γ and γ', then $\delta_n(\gamma)=\delta_n(\gamma')$ for all $n\in N$. Since the set of all positive sequences has no countable cofinal subset (with respect to the coordinatewise order), we conclude from the preceding that the family $\{T\upharpoonright\mathcal{D}_\gamma\}$ contains an uncountable subset of mutually inequivalent (that is, not unitarily equivalent) operators. This ends the proof.

Remarks. 1) Some arguments of the proof of Theorem 1 are taken from [19]. They have been used to show that (under the above assumtions) the strong operator topology $\sigma^{\mathcal{D}}$ and the strongest locally convex topology (denoted by τ_{st}) on $\mathcal{P}(T)$ are identical. For an $Op*$-algebra $\mathcal{A}$ on $\mathcal{D}$, $\sigma^{\mathcal{D}}$ is the locally convex topology on $\mathcal{A}$ defined by the family of seminorms $\|A\|_\varphi:=\|A\varphi\|, \varphi\in\mathcal{D}$. On the other hand, the equality $\sigma^{\mathcal{D}}=\tau_{st}$ follows easily from Theorem 1, (i) and (iii). Given a seminorm $\mathfrak{p}$ on $\mathcal{P}(T)$, we choose the matrix (c_{jk}) so that $c_{0,n-1}c_{0,n-2}\cdots c_{00}\geq 2^{n/2+1}\mathfrak{p}(T^n)$ for all $n\in N$. Norming φ_0^1 by $\|\varphi_0^1\|=\max(1, 2\mathfrak{p}(I))$, we then have $\frac{3}{4}\|T^n\varphi_0^1\|^2\geq 2^{n+1}\mathfrak{p}(T^n)^2$ for $n\in N$ and therefore $\mathfrak{p}(\sum\rho_nT^n)^2\leq\sum|\rho_n|^2 2^{n+1}\mathfrak{p}(T^n)^2\leq\|\sum\rho_nT^n\varphi_0^1\|^2$ for each polynomial $\sum\rho_nT^n$.

2) We briefly indicate some reformulations and easy consequences of the preceding results. Corollary 2, (iii) and (iv), show that $\underline{\mathcal{D}}_{1,2}[\mathfrak{t}]$ is the topological direct sum of $\underline{\mathcal{D}}_1[\mathfrak{t}]$ and $\underline{\mathcal{D}}_2[\mathfrak{t}]$. From Theorem 1, (iii), it follows that $\underline{\mathcal{D}}_1$ and $\underline{\mathcal{D}}_2$ can be described as sequence spaces in an obvious way (i.e., intersections of certain weighted l_2-spaces). Applying Theorem 1 again to $T\upharpoonright\underline{\mathcal{D}}_2$ and continuing this procedure we obtain a sequence of $\mathfrak{t}$-closed dense subspaces $\underline{\mathcal{D}}_j$, $j\in N$, of $\mathcal{D}$ such that $\underline{\mathcal{D}}_j\cap\underline{\mathcal{D}}_l=\{0\}$ for all $j, l\in N$, $j\neq l$. Moreover, given a positive number δ, the sequence $\{\underline{\mathcal{D}}_j, j\in N\}$ can be chosen so that $(1-\delta)\sum_{l=1}^{j}\|T^n\psi_l\|^2\leq\|T^n(\sum_{l=1}^{j}\psi_l)\|^2\leq(1+\delta)\sum_{l=1}^{j}\|T^n\psi_l\|^2$ for all $\psi_1\in\underline{\mathcal{D}}_1, \cdots, \psi_j\in\underline{\mathcal{D}}_j$ and all $j, n\in N$. For this, it suffices to choose the numbers ε_j, $0<\varepsilon_j<1/4$, in the j-step of the construction such that $\prod_{j=1}^{\infty}(1-3\varepsilon_j)\geq 1-\delta$ and $\prod_{j=1}^{\infty}(1+4\varepsilon_j)\leq 1+\delta$.

3) We now state an additional fact concerning Theorem 1. It will be used in the proof of Theorem 9.2.

Let $(d_{jn})_{j\in N_0, n\in N}$ be a given matrix of positive entries. The sequences $\{\varphi_j^m, j\in N_0\}$, $m=1,2$, in Theorem 1 can be chosen such that in addition $\|\varphi_j^m\|=1$ and $\|T^n\varphi_{j+1}^m\|\geq d_{jn}\|T^n\varphi_j^m\|$ for $j\in N_0$, $n\in N$ and $m=1,2$. (In particular, we can assume that $\|T^n\varphi_{j+1}^m\|\geq\|T^n\varphi_j^m\|$ for $j, n\in N_0$.)

Indeed, since $d(j+1, n)>d(j, n)$, the preceding proof of Theorem 1 shows that the vectors φ_{jn}^m can be chosen such that $\|T^n\varphi_{j+1,n}^m\|\geq 4d_{jn}\|T^n\varphi_{jn}^m\|$ for $j\in N_0$ and $n\in N$ as well. By (5), we have $\frac{3}{4}\|T^n\varphi_{jn}^m\|\leq\|T^n\varphi_j^m\|\leq\frac{5}{4}\|T^n\varphi_{jn}^m\|$ for $j, n\in N_0$. Therefore, $\|T^n\varphi_{j+1}^m\|\geq 2d_{jn}\|T^n\varphi_j^m\|$ for $j\in N_0$, $n\in$ N, $m=1,2$. Further, we have $\|\varphi_{j0}^m\|=1$ by construction and $3/4\leq\|\varphi_j^m\|\leq 5/4$ by (5). Replacing φ_j^m by $\varphi_j^m/\|\varphi_j^m\|$, we obtain $\|T^n\varphi_{j+1}^m\|\geq d_{jn}\|T^n\varphi_j^m\|$ and the other assertions (i)-(iv) in Theorem 1 remain unaffected.

§ 9. Construction of Canonical Pairs II: Admissible Boundary Spaces

For a given set of intervals (a_n, b_n), $n\in\mathfrak{J}$, we know already from Section 7 when a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$ (or equivalently, a representation $\pi\in\mathscr{C}$ such that $\pi\supseteq\pi_0$) exists. The main purpose of this section is to construct admissible boundary spaces $\mathfrak{M}$ relative to given W, $\mathfrak{a}$ and $\mathfrak{b}$ such that the corresponding representations $\pi_{W,\mathfrak{M}}$ have some special properties.

9.1. As usual, let (a_n, b_n), $n\in\mathfrak{J}$, be a fixed set of intervals satisfying (+). Condition (++) is not needed in 9.1. We assume in 9.1 and 9.2 that the sequence $\mathfrak{a}=\{a_n, n\in\mathfrak{J}^+\}$ is unbounded and that W is a fixed weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$.

We first prove the existence of admissible boundary spaces satisfying some growth conditions.

Theorem 1. *For any $k\in N$ and $\varepsilon>0$, there exists an admissible boundary space $\mathfrak{M}$ with respect to W, $\mathfrak{a}$ and $\mathfrak{b}$ such that:*

(i) $|||\mathfrak{z}_{k+j}||| \leq \varepsilon \sum_{l=0}^{k-1} |||\mathfrak{z}_l|||$ *for all* $(\mathfrak{z}_0, \mathfrak{z}_1, \cdots) \in \mathfrak{M}$ *and* $j \in N_0$.

(ii) $\mathfrak{B}_k^+(\mathfrak{M})$ *is dense in* $l_2^k(\mathfrak{F}^+)$.

(iii) $\pi_{W,\mathfrak{M}}(p)^n = P_W^n \upharpoonright \mathcal{D}_{W,\mathfrak{M}}$ *is e.s.a. if and only if* $n \leq k$. *In particular,* $\pi_{W,\mathfrak{M}} \in \mathscr{C}$.

Proof. Clearly, (iii) follows immediately from (i) and (ii). Indeed, (i) shows that $(0, \cdots, 0, \mathfrak{z}_k, \mathfrak{z}_{k+1}, \cdots) \in \mathfrak{M}$ implies that $\mathfrak{z}_{k+j} = 0$ for $j \in N_0$. That is, $\mathfrak{B}_n^+(\mathfrak{M})$ is not dense in $l_2^n(\mathfrak{F}^+)$ for $n \geq k+1$. From Lemma 5.4, (iii), it follows that $\pi_{W,\mathfrak{M}}(p)^n$ is e.s.a. if and only if $n \leq k$. Since $\pi_{W,\mathfrak{M}}(p)$ is e.s.a., $\pi_{W,\mathfrak{M}} \in \mathscr{C}$.

In order to construct $\mathfrak{M}$, we make use of Theorem 8.1. First we introduce some notation. Let $j \in N_0$. We set $(\mathfrak{x})^j := (\delta_{jn}\mathfrak{x}, n \in N_0)$ and $L_j(\mathfrak{x}_0, \mathfrak{x}_1, \cdots) := \mathfrak{x}_j$ for $\mathfrak{x} \in l_2(\mathfrak{F}^+)$ and $(\mathfrak{x}_0, \mathfrak{x}_1, \cdots) \in \mathfrak{L}^+$. By assumption, $T := \mathfrak{a} \upharpoonright \mathfrak{S}(W)$ is an unbounded symmetric operator on the dense invariant domain $\mathcal{F} := \mathfrak{S}(W)$ of the Hilbert space $\mathcal{G} := l_2(\mathfrak{F}^+)$. As we have noted already, $\mathfrak{S}(W)$ is $\mathfrak{t}_{\mathfrak{a}}$-complete, that is, $\mathcal{F} = \mathfrak{S}(W) = \bigcap_{n=1}^{\infty} \mathcal{D}(\overline{T^n})$. We set $c_{jn} = \varepsilon^{-1} \cdot 2 \cdot k^2(n+1)$ for $j, n \in N_0$ and apply Theorem 8.1. Let $\mathfrak{N}_s = \mathrm{Lin}\{\mathfrak{Q}^n(\varphi_l^1)^s; l, n \in N_0\}$ for $s = 0, \cdots, k-1$, where $\{\varphi_l^1\}$ is the sequence occuring in Theorem 8.1. We claim that for each $\mathfrak{y}^s \in \mathfrak{N}_s$, $s = 0, \cdots, k-1$, and $r = s, s+1, \cdots$

$$|||L_{r+1}\mathfrak{y}^s||| \leq \frac{\varepsilon}{k} |||L_r \mathfrak{y}^s|||. \tag{1}$$

Let $n, r, s \in N_0$ and $\mathfrak{x} \in \mathfrak{S}(W)$. Then

$$\begin{cases} L_r\mathfrak{Q}^n(\mathfrak{x})^s = \binom{n}{r-s} r(r-1)\cdots(s+1)\, T^{n+s-r}\mathfrak{x} \text{ if } s+1 \leq r \leq n+s, \\ L_r\mathfrak{Q}^n(\mathfrak{x})^s = T^n\mathfrak{x} \text{ if } r = s \text{ and } L_r\mathfrak{Q}^n(\mathfrak{x})^s = 0 \text{ if } r < s. \end{cases} \tag{2}$$

This can be shown by induction on n. We omit the details. Therefore, by (2), if $p(\mathfrak{Q}) \in \mathcal{P}(\mathfrak{Q})$ and $r \geq s$, then there is a polynomial $\sum_{n \geq 0} \rho_n T^n$ so that

$$L_r p(\mathfrak{Q})(\cdot)^s \equiv \sum_{n \geq 0} \rho_n T^n \text{ and } L_{r+1} p(\mathfrak{Q})(\cdot)^s \equiv \sum_{n \geq 1} \rho_n \frac{r+1}{r+1-s} n T^{n-1}. \tag{3}$$

Fix $\mathfrak{y}^s \in \mathfrak{N}_s$. By (3), there is a finite matrix of complex numbers, say

(ρ_{ln}), such that

$$L_r\mathfrak{y}^s=\sum_{l,n=0}^{\infty}\rho_{ln}T^n\varphi_l^1 \text{ and } L_{r+1}\mathfrak{y}^s=\sum_{l=0,n=1}^{\infty}\rho_{ln}\frac{r+1}{r+1-s}nT^{n-1}\varphi_l^1 .$$

Since $s\in\{0,\cdots,k-1\}$ and $r\geq s$, $\dfrac{r+1}{r+1-s}=1+\dfrac{s}{r+1-s}\leq k$. Using Theorem 8.1, (i) and (iii), we thus obtain

$$\begin{aligned}|||L_{r+1}\mathfrak{y}^s|||^2&\leq\frac{5}{4}\sum_{l=0,n=1}^{\infty}|\rho_{ln}|^2\left(\frac{r+1}{r+1-s}\right)^2n^2|||T^{n-1}\varphi_l^1|||^2\\&\leq\varepsilon^2k^{-2}\frac{1}{2}\sum_{l=0,n=1}^{\infty}|\rho_{ln}|^2|||T^n\varphi_l^1|||^2\\&\leq\varepsilon^2k^{-2}\frac{1}{2}\,\frac{4}{3}\,|||\sum_{l,n=0}^{\infty}\rho_{ln}T^n\varphi_l^1|||^2\leq\varepsilon^2k^{-2}|||L_r\mathfrak{y}^s|||^2 ,\end{aligned}$$

thus proving (1).

Let $\mathfrak{M}_0=\mathfrak{N}_0+\cdots+\mathfrak{N}_{k-1}$. Obviously, $\mathfrak{Q}\mathfrak{N}_s\subseteq\mathfrak{N}_s$ for $s=0,\cdots,k-1$. Hence $\mathfrak{Q}\mathfrak{M}_0\subseteq\mathfrak{M}_0$. From $\mathfrak{P}\mathfrak{Q}^n=\mathfrak{Q}^n\mathfrak{P}-in\mathfrak{Q}^{n-1}$ for $n\in N$, $\mathfrak{P}(\mathfrak{x})^s=-i(\mathfrak{x})^{s-1}$ for $s\in N$ and $\mathfrak{P}(\mathfrak{x})^0=0$ we conclude that $\mathfrak{P}\mathfrak{M}_0\subseteq\mathfrak{M}_0$. Let $\mathfrak{M}$ be the $\mathfrak{t}_\infty$-closure of $\mathfrak{M}_0$ in $\mathfrak{L}_\infty^+(W)$. Obviously, $\mathfrak{P}$ and $\mathfrak{Q}$ are $\mathfrak{t}_\infty$-continuous. Hence $\mathfrak{P}\mathfrak{M}\subseteq\mathfrak{M}$ and $\mathfrak{Q}\mathfrak{M}\subseteq\mathfrak{M}$. That is, $\mathfrak{M}$ is an admissible boundary space.

From the definition of $\mathfrak{M}_0$ it is easy to check that $(\mathfrak{x}_0,\cdots,\mathfrak{x}_{k-1})\in\mathfrak{B}_k^+(\mathfrak{M}_0)\subseteq\mathfrak{B}_k^+(\mathfrak{M})$ for all $\mathfrak{x}_0,\cdots,\mathfrak{x}_{k-1}\in\mathcal{F}_1=\mathrm{Lin}\{T^n\varphi_l^1;\,l,n\in N_0\}$. Hence $\mathfrak{B}_k^+(\mathfrak{M})$ is dense in $l_2^k(\mathfrak{H}^+)$, because $\mathcal{F}_1$ is dense in $l_2(\mathfrak{H}^+)$ by Theorem 8.1, (ii). This proves (ii).

Finally, we prove (i). Obviously, we can assume that $\varepsilon\leq1$. Since $\mathfrak{M}$ is the closure of $\mathfrak{M}_0$ relative to the product topology $\mathfrak{t}_\infty$, it suffices to prove the assertion for $(\mathfrak{z}_0,\mathfrak{z}_1,\cdots)\in\mathfrak{M}_0$. Then $(\mathfrak{z}_0,\mathfrak{z}_1,\cdots)=\mathfrak{y}^0+\mathfrak{y}^1+\cdots+\mathfrak{y}^{k-1}$, where $\mathfrak{y}^l\in\mathfrak{N}_l$. For $r\in N_0$, put $S_r:=|||\mathfrak{z}_0|||+\cdots+|||\mathfrak{z}_r|||$. We first show that

(4) $$|||L_r\mathfrak{y}^r|||\leq S_r \text{ for } r=0,\cdots,k-1 .$$

(In fact, finer estimations would be possible.) We reason by induction. In case $r=0$ we have $|||L_0\mathfrak{y}^0|||=|||\mathfrak{z}_0|||=S_0$. Let $n\in N_0$, $n+1\leq k-1$. Assume that (4) is valid for $r=0,\cdots,n$. Using (1) and $\varepsilon\leq1$, we get

$$|||L_{n+1}\mathfrak{y}^{n+1}||| = |||\mathfrak{z}_{n+1} - L_{n+1}(\mathfrak{y}^0 + \cdots + \mathfrak{y}^n)||| \leq |||\mathfrak{z}_{n+1}||| + \sum_{r=0}^{n} \frac{\varepsilon}{k} |||L_r \mathfrak{y}^r|||$$

$$\leq |||\mathfrak{z}_{n+1}||| + \frac{\varepsilon}{k} \sum_{r=0}^{n} S_r \leq |||\mathfrak{z}_{n+1}||| + S_n = S_{n+1}\,.$$

Let $j \in N_0$. Combining (1), (4) and $\varepsilon \leq 1$, we obtain

$$|||\mathfrak{z}_{k+j}||| = |||L_{k+j}(\mathfrak{y}^0 + \cdots + \mathfrak{y}^{k-1})||| \leq \sum_{r=0}^{k-1} \frac{\varepsilon}{k} |||L_r \mathfrak{y}^r||| \leq \frac{\varepsilon}{k} \sum_{r=0}^{k-1} S_r \leq \varepsilon S_n\,.$$

This ends the proof of Theorem 1.

Remark. Theorem 1 is still valid if (i) is replaced by the (quite stronger) inequality $|||\mathfrak{z}_{k+j}||| \leq \varepsilon \sum_{l=0}^{k-1} |||\mathfrak{z}_{l+j}|||$ for $(\mathfrak{z}_0, \mathfrak{z}_1, \cdots) \in \mathfrak{M}$ and $j \in N_0$. The proof of this assertion requires a more careful estimation in the last part of the preceding proof by using (3) and again Theorem 8.1, (iii). Moreover, c_{jn} must be modified.

Theorem 2. *For any positive sequence $\{\alpha_r, r \in N_0\}$, there is an admissible boundary space $\mathfrak{M}$ with respect to W, $\mathfrak{a}$ and $\mathfrak{b}$ such that $\pi_{W,\mathfrak{M}} \in \mathscr{C}$ and*

(5) $$\alpha_r |||\mathfrak{z}_r||| \leq |||\mathfrak{z}_{r+1}||| \textit{ for all } (\mathfrak{z}_0, \mathfrak{z}_1, \cdots) \in \mathfrak{M} \textit{ and } r \in N_0.$$

Proof. Let T, $\mathcal{F}$ and $\mathcal{G}$ be as in the preceding proof. Moreover, we retain the notations L_j and $(\mathfrak{x})^j$ introduced there. By taking a larger sequence if necessary, we can assume without loss of generality that

(6) $$\alpha_{r+1+l} \geq 8\alpha_l \alpha_{r+l} \text{ and } \alpha_r \geq 2^{(r-1)/2+4} r! \text{ for all } l, r \in N_0.$$

Set $c_{jn} = 2^{n+1}\alpha_j$ for $j, n \in N_0$. We now apply Theorem 8.1 and obtain a sequence $\{\varphi_l^1, l \in N_0\}$. Rename φ_l^1 by φ_l. As we have noted in Remark 3) in 8.2, we can assume in addition that

(7) $$|||T^n \varphi_{j+1}||| \geq |||T^n \varphi_j||| \quad \text{for} \quad j, n \in N_0\,.$$

Let $\mathfrak{x}^l$, $l \in N_0$, denote the vector $(\alpha_l \varphi_l, \alpha_{l+1}\varphi_{l+1}, \cdots)$ in $\mathfrak{L}_\infty^+(W)$. Put $\mathfrak{M}^0 = \mathrm{Lin}\{\mathfrak{Q}^n \mathfrak{x}^l; l, n \in N_0\}$. Let $\mathfrak{M}$ be the $\mathfrak{t}_\infty$-closure of $\mathfrak{M}^0$ in $\mathfrak{L}_\infty^+(W)$. Since $\mathfrak{P}\mathfrak{x}^l = \mathfrak{x}^{l+1}$ for $l \in N_0$, we can argue as in the proof of Theorem 1 to show that $\mathfrak{M}$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$. Because

$\mathfrak{B}_1^+(\mathfrak{M}) \supseteq \mathcal{F}_1$ is dense in $\mathcal{G}=l_2(\mathfrak{I}^+)$, $\pi_{W,\mathfrak{M}}(p)$ is e.s.a. and $\pi_{W,\mathfrak{M}}\in\mathscr{C}$.

It remains to prove that (5) is fulfilled. Again it suffices to prove (5) for $\mathfrak{z}=(\mathfrak{z}_0, \mathfrak{z}_1, \cdots)\in\mathfrak{M}^0$. By definition, $\mathfrak{z}$ is of the form $\sum_{l,n=0}^{\infty}\rho_{ln}\mathfrak{Q}^n\mathfrak{x}^l$, where $(\rho_{ln})_{l,n\in N_0}$ is a certain finite matrix of complex numbers. Using the formulae (2), we obtain

$$\begin{aligned}\mathfrak{z}_r &= \sum_{l,n=0}^{\infty}\rho_{ln}L_r\mathfrak{Q}^n\mathfrak{x}^l=\sum_{l,n=0}^{\infty}\rho_{ln}\sum_{s=1}^{\infty}\alpha_s L_r\mathfrak{Q}^n(\varphi_s)^{s-1}\\ &=\sum_{l,n=0}^{\infty}\rho_{ln}\alpha_{r+l}T^n\varphi_{r+l}+\sum_{j=1}^{r}\sum_{l=0,n=j}^{\infty}\rho_{ln}\alpha_{r+l-j}\binom{n}{j}r(r-1)\\ &\quad\cdots(r-j+1)T^{n-j}\varphi_{r+l-j}\\ &=:\mathfrak{z}_r'+\mathfrak{z}_r''\quad\text{for}\quad r\in N_0\,.\end{aligned}$$

We first estimate the vector $\mathfrak{z}_r''$ defined by the second sum above. Clearly, $\binom{n}{j}r(r-1)\cdots(r-j+1)\le 2^n r!$ for $r\in N_0$, $j, n\in N$. Using Theorem 8.1, (iii) and (i), the definition of c_{jn} and finally (7), we get the estimate

$$\begin{aligned}(8)\quad \|\!|\mathfrak{z}_r''|\!\|^2 &\le\sum_{j=1}^{r}2^j\,\Big|\!\Big|\!\Big|\sum_{l=0,n=j}^{\infty}\cdots\Big|\!\Big|\!\Big|^2\\ &\le\sum_{j=1}^{r}2^j\frac{5}{4}\sum_{l=0,n=j}^{\infty}|\rho_{ln}|^2 2^{2n}r!^2\alpha_{r+l-j}\|\!|T^{n-j}\varphi_{r+l-j}|\!\|^2\\ &\le\sum_{j=1}^{r}2^{j+1}r!^2\sum_{l=0,n=j}^{\infty}|\rho_{ln}|^2c^2_{r+l-j,n-1}c^2_{r+l-j,n-2}\cdots c^2_{r+l-j,n-j}\|\!|T^{n-j}\varphi_{r+l-j}|\!\|^2\\ &\le\sum_{j=1}^{r}2^{j+1}r!^2\sum_{l=0,n=j}^{\infty}|\rho_{ln}|^2\|\!|T^n\varphi_{r+l-j}|\!\|^2\\ &\le 2^{r+2}r!^2\sum_{l,u=0}^{\infty}|\rho_{ln}|^2\|\!|T^n\varphi_{r+l}|\!\|^2\,.\end{aligned}$$

To estimate $\|\!|\mathfrak{z}_{r+1}'|\!\|^2$, we first apply Theorem 8.1, (iii), (6), (7) and then once more Theorem 8.1, (iii), and (8). Therefore,

$$\begin{aligned}\|\!|\mathfrak{z}_{r+1}'|\!\|^2 &\ge\frac{3}{4}\sum_{l,n=0}^{\infty}|\rho_{ln}|^2\alpha^2_{r+1+l}\|\!|T^n\varphi_{r+1+l}|\!\|^2\\ &\ge\frac{3}{4}\,\frac{1}{2}\sum_{l,n=0}^{\infty}|\rho_{ln}|^2(8\alpha_r\alpha_{r+l})^2\|\!|T^n\varphi_{r+1+l}|\!\|^2\\ &\quad+\frac{3}{4}\,\frac{1}{2}\sum_{l,n=0}^{\infty}|\rho_{ln}|^2(2^{r/2+4}(r+1)\,!)^2\|\!|T^n\varphi_{r+1+l}|\!\|^2\end{aligned}$$

$$\geq 24\alpha_r^2 \sum_{l,n=0}^{\infty} |\rho_{ln}|^2 \alpha_{r+l}^2 |||T^n \varphi_{r+l}|||^2 + 8\cdot 2^{r+3}(r+1)!$$

$$\times \sum_{l,n=0}^{\infty} |\rho_{ln}|^2 |||T^n \varphi_{r+1+l}|||^2$$

$$\geq 24\alpha_r^2 \frac{5}{4} |||\mathfrak{z}_r'|||^2 + 8|||\mathfrak{z}_{r+1}''|||^2.$$

Hence we have

(9) $$|||\mathfrak{z}_{r+1}'||| \geq 3\alpha_r |||\mathfrak{z}_r'||| + 2|||\mathfrak{z}_{r+1}''||| \quad \text{for all } r\in N_0.$$

For $r\in N_0$, (9) gives

$$|||\mathfrak{z}_{r+1}||| \geq |||\mathfrak{z}_{r+1}'||| - |||\mathfrak{z}_{r+1}''||| \geq \frac{1}{2}|||\mathfrak{z}_{r+1}'||| \geq \frac{1}{2}\cdot 3\alpha_r |||\mathfrak{z}_r'|||$$

$$\geq \alpha_r (|||\mathfrak{z}_r'||| + |||\mathfrak{z}_r''|||) \geq \alpha_r |||\mathfrak{z}_r||| .$$

This ends the proof of Theorem 2.

Let π be a representation of $A(\boldsymbol{p}, \boldsymbol{q})$ on $\mathcal{D}$. For $A\in\pi(A(\boldsymbol{p}, \boldsymbol{q}))$, $\mathcal{D}_a(\bar{A})$ is defined as the set of all $\varphi\in\mathcal{D}$ which are analytic vectors for $\bar{A}$.

Corollary 3. *Let* $\{\alpha_r, r\in N_0\}$ *and* $\mathfrak{M}$ *be as in Theorem* 2. *As usual* (*see Remark* 3) *in* 2.2) *we set* $P=\overline{\pi_{W,\mathfrak{M}}(\boldsymbol{p})}$. *Assume that* $\mathfrak{J}=\mathfrak{J}^+\cup\mathfrak{J}^-$. *If* $\alpha_r\geq r^2$ *for* $r\in N$, *then* $\mathcal{D}_a(P)=\{0\}$.

Proof. Let $\varphi=(\varphi_n, n\in\mathfrak{J})$ be a fixed vector in $\mathcal{D}_a(P)$. We first prove that

(10) $\varphi_n^{(j)}(a_n+)=\varphi_m^{(j)}(b_m-)=0$ for all $j\in N_0$, $n\in\mathfrak{J}^+$ and $m\in\mathfrak{J}^-$.

By definition of an analytic vector, there is a constant $M>0$ so that $\|P^n\varphi\|\leq M^n n!$ for $n\in N$. Put $\mathfrak{z}_r=B_r^+(\varphi)$ for $r\in N_0$. Fix $j\in N_0$. Since $\alpha_r\geq r^2$, (5) gives

$$|||\mathfrak{z}_{n+j}||| \geq |||\mathfrak{z}_{n+j-1}|||(n+j-1)^2 \geq \cdots \geq |||\mathfrak{z}_j|||(n+j-1)!/j! \quad \text{for all } n\in N.$$

On other hand, since $P^r\varphi=(P_0^*)^r\varphi$ for $r\in N$, formula (3) in 5.1 reads to $|||\mathfrak{z}_{n+j}|||^2\leq 2c^{-1}\|P^{n+j}\varphi\|^2+2c\|P^{n+j+1}\varphi\|^2$. Putting these inequalities together, we get

$$|||\mathfrak{z}_j|||^2 (n+j-1)!^4 \leq 2j!^2 n!^2 (c^{-1}M^{2(n+j)} + cM^{2(n+j+1)}).$$

Because $n\in N$ is arbitrary, this implies $\mathfrak{z}_j=0$, that is, $\varphi_n^{(j)}(a_n+)=0$ for all $n\in\mathfrak{J}^+$. From $B_j^-(\varphi)=WB_j^+(\varphi)$ we conclude that $\varphi_m^{(j)}(b_m-)=0$ for $m\in\mathfrak{J}^-$.

To show that $\varphi=0$, let $m\in\mathfrak{J}$. Since we assumed that $\mathfrak{J}=\mathfrak{J}^+\cup\mathfrak{J}^-$, at least one of the numbers a_m, b_m, say a_m, is finite. Let S denote the (self-adjoint) ordinary differential operator $-i\dfrac{d}{dx}$ in the Hilbert space $L_2(R_1)$. Define $\psi_m(x)=\varphi_m(x)$ for $x\in(a_m, b_m)$ and $\psi_m(x)=0$ for $x\in R_1\setminus(a_m, b_m)$. By (10) and $\varphi\in\mathcal{D}_\infty(P)$, we have $\psi_m\in C^\infty(R_1)$ and $\psi_m\in\mathcal{D}_\infty(S)$. Because $\|S^n\psi_m\|_{L_2(R_1)}=\|\varphi_m^{(n)}\|_{L_2(a_m,b_m)}\leq\|P^n\varphi\|$ for $n\in N$, ψ_m is an analytic vector for S. If ξ_m denotes the Fourier transform of ψ_m; then $\|x^n\xi_m\|=\|S^n\psi_m\|\leq M^n n!$, $n\in N$, implies that $e^{\beta x}\xi_m(x)\in L_2(R_1)$ for $0<\beta<M^{-1}$. By the classical Paley-Wiener theorem, ψ_m is the restriction to R_1 of a function which is holomorphic in some strip $|\mathrm{Im}\, z|<\gamma$, $\gamma>0$. Since $\psi_m(x)\equiv 0$ on $(-\infty, a_m)$, this yields $\psi_m=0$ and hence $\varphi_m=0$. Thus $\varphi=0$ and the proof is complete.

For the representation $\pi_{W,\mathfrak{M}}$ of Theorem 1, the space $\mathcal{D}_a(P)$ is not dense in $\mathcal{H}$. This is an immediate consequence of statement (iii) in Theorem 1 and the following lemma.

Lemma 4. *Let π be a closed $*$-representation of $A(\boldsymbol{p},\boldsymbol{q})$ on $\mathcal{D}$. Suppose that $\mathcal{D}_a(\overline{\pi(\boldsymbol{p})})$ is dense in $\mathcal{H}$. Then $\pi(\boldsymbol{p})^n$ is e.s.a. on $\mathcal{D}$ for all $n\in N$.*

Proof. Set $P=\overline{\pi(\boldsymbol{p})}$. Since $\mathcal{D}_a(P)$ is dense in $\mathcal{H}$, P is e.s.a. on $\mathcal{D}$ ([11], Lemma 5.1). Let $V(s)=e^{isP}$, $s\in R_1$. Arguing as at the beginning of the proof of Proposition. 4.1, we see that $V(s)\mathcal{D}_a(P)\subseteq\mathcal{D}_a(P)$ for all $s\in R_1$. It is well-known (see, for instance, [12], Cor. 1.3) that the latter implies that $\mathcal{D}_a(P)$ is a core for P^n, $n\in N$, i.e., $P^n\restriction\mathcal{D}_a(P)$ is e.s.a. Since $\mathcal{D}_a(P)\subseteq\mathcal{D}$, P^n is e.s.a. on $\mathcal{D}$ for each $n\in N$.

Remark. As noted in the first remark of 2.2, condition (2.5) in Definition 2.2 cannot be replaced by the (weaker) condition (2.6) in general. Now it is easy to construct examples. For instance, take the representation $\pi_{W,\mathfrak{M}}$ of Theorem 2 and define $P=\overline{\pi_{W,\mathfrak{M}}(q)}$, $Q=\overline{\pi_{W,\mathfrak{M}}(p)}$ and $\mathscr{D}=\mathscr{D}_{W,\mathfrak{M}}$. Then, (2.1)-(2.4) and (2.6) are fulfilled because of $\pi_{W,\mathfrak{M}}\in\mathscr{C}$. But, by Corollary 3, there is no non-zero analytic vector for Q in $\mathscr{D}$.

9.2. We now show that (for given W, $\mathfrak{a}$ and $\mathfrak{b}$) there are "sufficiently many" inequivalent representations $\pi_{W,\mathfrak{M}}\in\mathscr{C}$. Throughout this subsection, we assume in addition that $(++)$ is satisfied. (The reason is that we shall use Theorem 6.3.)

Theorem 5. *There is an uncountable family* $\{\mathfrak{M}_i, i\in\mathscr{I}\}$ *of admissible boundary spaces with respect to* W, $\mathfrak{a}$ *and* $\mathfrak{b}$ *such that the representations* $\pi_{W,\mathfrak{M}_i}$, $i\in\mathscr{I}$, *of* $A(p,q)$ *belong to the class* $\mathscr{C}$ *and are pairwise inequivalent.*

Proof. Let T, $\mathscr{F}$ and $\mathscr{G}$ be as in the proof of Theorem 1. We take the family $\{\mathscr{F}_i, i\in\mathscr{I}\}$ of subspaces of $\mathscr{F}$ occuring in Corollary 8.3 and define $\mathfrak{M}_i=\{(\mathfrak{x}_0,\mathfrak{x}_1,\cdots);\ \mathfrak{x}_j\in\mathscr{F}_i$ for all $j\in N_0\}$, $i\in\mathscr{I}$. Since $\mathscr{F}_i$ is $\mathfrak{t}_{\mathfrak{a}}$-closed and $T\mathscr{F}_i=\mathfrak{a}\mathscr{F}_i\subseteq\mathscr{F}_i$, $\mathfrak{M}_i$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathscr{F}_i$ is dense in $\mathscr{G}=l_2(\mathfrak{J}^+)$ it follows from Lemma 5.4 that $\pi_{W,\mathfrak{M}_i}(p)^n$ is e.s.a. for all $n\in N$. Hence $\pi_{W,\mathfrak{M}_i}\in\mathscr{C}$ for $i\in\mathscr{I}$.

Assume now that $\pi_{W,\mathfrak{M}_i}$ and $\pi_{W,\mathfrak{M}_j}$ are unitarily equivalent for some $i\in\mathscr{I}$ and $j\in\mathscr{I}$. By Theorem 6.3, there is a unitary operator $U\in B(l_2(\mathfrak{J}))$ such that $UE_r=E_rU$ for $r\in\mathfrak{J}$, $UW\mathfrak{x}=WU\mathfrak{x}$ for $\mathfrak{x}\in l_2(\mathfrak{J}^+)$ and $U\mathfrak{M}_i=\mathfrak{M}_j$. The latter gives $U\mathscr{F}_i=\mathscr{F}_j$. By Corollary 6.4, U commutes with $T=\mathfrak{a}\restriction\mathfrak{S}(W)$. Therefore, $T\restriction\mathscr{F}_i$ and $T\restriction\mathscr{F}_j$ are unitarily equivalent. By Corollary 8.3, this yields $i=j$ and completes the proof.

Theorem 6.3 shows that if a representation $\pi_W\in\mathscr{C}$ is irreducible, then $\pi_{W,\mathfrak{M}}\in\mathscr{C}$ is irreducible for any admissible boundary space $\mathfrak{M}$. The converse is, however, not true. As a sample, we prove

Theorem 6. *Suppose in addition that* π_W *is an* (*orthogonal*) *direct sum of countably many irreducible* $*$*-representations of* $A(p,q)$.

There exists an uncountable set $\{\mathfrak{M}_\iota, \iota \in \mathcal{J}\}$ *of admissible boundary spaces with respect to* W, $\mathfrak{a}$ *and* $\mathfrak{b}$ *such that the representations* $\pi_{W,\mathfrak{M}_\iota}$, $\iota \in \mathcal{J}$, *are pairwise inequivalent irreducible representations of the class* $\mathscr{C}$.

Proof. For simplicity we assume that π_W is a direct sum of infinitely many irreducible $*$-representations $\pi_j, j \in N_0$, of $A(\boldsymbol{p}, \boldsymbol{q})$, i.e., $\mathcal{H} = \sum_{j=0}^{\infty} {}^{\oplus} \mathcal{H}_j$ and $\pi_W = \sum_{j=0}^{\infty} {}^{\oplus} \pi_j$, where $\mathcal{H}_j \neq \{0\}$ for all $j \in N_0$. Let C_j be the projection of $\mathcal{H}$ on $\mathcal{H}_j$. Then $C_j \in (\pi_W)'_s$. By Corollary 6.4, the mapping $(\pi_W)'_s \ni C \to \widetilde{C}$ is a $*$-isomorphism of $(\pi_W)'_s$ on $\mathfrak{A} = \{A \in B(l_2(\mathfrak{J})): AE_r = E_r A$ for $r \in \mathfrak{J}$ and $AW\mathfrak{x} = WA\mathfrak{x}$ for $\mathfrak{x} \in l_2(\mathfrak{J}^+)\}$. $\mathfrak{A}$ is a W^*-algebra. Let $D_j := \widetilde{C}_j$, $j \in N_0$. Since π_j is assumed to be irreducible, C_j is a minimal (non-zero) projection in $(\pi_W)'_s$. Moreover, $C_j C_l = 0$ for $j \neq l$ and $\sum_{j=0}^{\infty} {}^{\oplus} C_j = I$. Because these properties are preserved under $*$-isomorphism, they are also true for the projections $D_j, j \in N_0$.

Now let A be a fixed operator in $\mathfrak{A}$ and let $j, l \in N_0$. It is easy to see that either

$$D_j A D_l = 0 \quad \text{or} \quad D_j A D_l = \mu_{jl} U_{jl}, \tag{11}$$

where μ_{jl} is a non-zero complex number, U_{jl} is a partial isometry with initial space $D_l l_2(\mathfrak{J})$ and range $D_j l_2(\mathfrak{J})$ and $U_{jl} = D_j$ in case $j = l$. We denote by N_l the set of all $j \in N_0$ for which $D_j A D_l \neq 0$. The proof of (11) repeats some standard arguments from the theory of von Neumann algebras. $A_{jl} := (D_j A D_l)^* (D_j A D_l) = D_l A^* D_j A D_l$ is a positive self-adjoint operator in $\mathfrak{A}$ and commutes with D_l. Let $E(\lambda)$ be a spectral projection of A_{jl}. Since $\mathfrak{A}$ is a W^*-algebra, $E(\lambda) \in \mathfrak{A}$. Moreover, $E(\lambda) D_l = D_l E(\lambda)$. Because D_l is a minimal projection in $\mathfrak{A}$ we have either $E(\lambda) D_l = 0$ or $E(\lambda) D_l = D_l$. Hence there is a non-negative number α_{jl} so that $A_{jl} = \alpha_{jl} D_l$. Similarly, $B_{jl} := (D_j A D_l)(D_j A D_l)^* = \beta_{jl} D_j$ for some $\beta_{jl} \in R_1, \beta_{jl} \geq 0$. From $D_j A D_l (D_j A D_l)^* D_j A D_l = \alpha_{jl} D_j A D_l = \beta_{jl} D_j A D_l$ we see that either $D_j A D_l = 0$ or $\alpha_{jl} = \beta_{jl} > 0$. Setting $U'_{jl} = \alpha_{jl}^{-1/2} D_j A D_l$ and $\mu'_{jl} = \alpha_{jl}^{1/2}$ in the latter case, we have $U'_{jl} U'^*_{jl} = D_j$ and $U'^*_{jl} U'_{jl} = D_l$, that is, U'_{jl} is an isometry of $D_l l_2(\mathfrak{J})$ onto $D_j l_2(\mathfrak{J})$. For $U_{jl} := U'_{jl}$ and $\mu_{jl} := \mu'_{jl}$ for $j \neq l$, (11) is proven in this case. Now suppose that $j = l$. $U'_{jj} \upharpoonright D_j l_2(\mathfrak{J})$ is a unitary operator in the

Hilbert space $D_j l_2(\mathfrak{J})$. The spectral projections of this operator are in $\mathfrak{A}$, because $U'_{jj}\in\mathfrak{A}$. Using again the minimality of D_j in $\mathfrak{A}$, we obtain $U'_{jj}=\rho_j D_j$, where $|\rho_j|=1$. We set $U_{jj}=D_j$, $\mu_{jj}=\mu'_{jj}\rho_j$ and the proof of (11) is complete.

Put $T:=\mathfrak{a}\upharpoonright\mathfrak{S}(W)$. Let $l\in N_0$. Since $D_l\in\mathfrak{A}$, we know from Section 6 that $D_l l_2(\mathfrak{J}^+)\subseteq l_2(\mathfrak{J}^+)$, $D_l\mathfrak{S}(W)\subseteq\mathfrak{S}(W)$ and $D_l T\subseteq T D_l$. Define $\mathcal{G}^l:=D_l l_2(\mathfrak{J}^+)$, $\mathcal{F}^l:=D_l\mathfrak{S}(W)$ and $T_l:=T\upharpoonright\mathcal{F}^l$. We now check that the triple $(T_l, \mathcal{F}^l, \mathcal{G}^l)$ satisfies the assumptions of Theorem 8.1. Since $\mathfrak{S}(W)$ is $\mathfrak{t}_{\mathfrak{a}}$-complete, $\mathcal{F}^l=D_l\mathfrak{S}(W)$ is $\mathfrak{t}_{\mathfrak{a}}$-complete as well. Hence $\mathcal{F}^l=\bigcap_{n=1}^{\infty}\mathcal{D}(\overline{T_l^n})$. We show that T_l is unbounded. Assume the contrary, that is, T_l is bounded. Recall that $\mathfrak{S}(W)$ is dense in $l_2(\mathfrak{J}^+)$ because W is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. Hence $\mathcal{F}^l=D_l\mathfrak{S}(W)$ is dense in $\mathcal{G}^l=D_l l_2(\mathfrak{J}^+)$. Since $T_l=\mathfrak{a}\upharpoonright\mathcal{F}^l$ is bounded and $\mathfrak{a}\mathcal{F}^l\subseteq\mathcal{F}^l$, the latter implies $\mathcal{G}^l\subseteq\mathcal{D}(\mathfrak{a})$ and $\mathfrak{a}\mathcal{G}^l\subseteq\mathcal{G}^l$. We have $W\mathfrak{a}\mathfrak{x}=\mathfrak{b}W\mathfrak{x}$ for all $\mathfrak{x}\in\mathcal{F}^l\subseteq\mathfrak{S}(W)$. Because $\mathfrak{a}\upharpoonright\mathcal{F}^l$ is bounded and $\mathfrak{b}$ is closed, this is also true for $\mathfrak{x}\in\mathcal{G}^l=\overline{\mathcal{F}^l}$. $\mathfrak{a}\mathcal{G}^l\subseteq\mathcal{G}^l$ implies $W\mathcal{G}^l\subseteq\mathcal{D}(\mathfrak{b})$ and $\mathfrak{b}W\mathcal{G}^l\subseteq W\mathcal{G}^l$. That is, $\mathfrak{a}_l:=\mathfrak{a}\upharpoonright\mathcal{G}^l$ and $\mathfrak{b}_l:=\mathfrak{b}\upharpoonright W\mathcal{G}^l$ are unitarily equivalent. Combined with $(+)$, this gives $\inf\{a_r: r\in\mathfrak{J}^+$ and $D_l E_r\neq 0\}=\inf\{\lambda:\lambda\in\sigma(\mathfrak{a}_l)\}$ $=\inf\{\lambda:\ \lambda\in\sigma(\mathfrak{b}_l)\}=\inf\{b_s: s\in\mathfrak{J}^-$ and $D_l E_r\neq 0\}=:\alpha$. Because of $(+)$, there are numbers $r\in\mathfrak{J}^+$ and $s\in\mathfrak{J}^-$ so that $\alpha=a_r=b_s$. Since $a_s<b_s$, this implies that $a_s=-\infty$. From $(++)$ it follows that $a_n\geq\alpha$ and $b_m\geq\alpha$ for $n\in\mathfrak{J}^+$ and $m\in\mathfrak{J}^-$. Replacing inf by sup, we see that there is a $\beta\in R_1$ such that $a_n\leq\beta$ for $n\in\mathfrak{J}^+$. Thus $\mathfrak{a}$ would be bounded which is the desired contradiction.

Let $\gamma=\{\gamma_n, n\in N\}$ be a given positive sequence. We now shall apply Theorem 8.1 to each operator T_l, $l\in N_0$. We denote by φ_j^{ml}, $\underline{\mathcal{F}}_m^l$, $\underline{\mathcal{F}}_{1,2}^l$ and c_{ljn}, where $j, l, n\in N_0$ and $m=1, 2$, the corresponding quantities occuring in Theorem 8.1 resp. Corollary 8.2. For $l=0$, we let $c_{0jn}=1$ for all $j, n\in N_0$ and apply Theorem 8.1. Let $k\in N$. Suppose that the vectors φ_j^{ml} are constructed according to Theorem 8.1 for $j, n\in N_0$, $m=1, 2$ and $l\leq k-1$. Take an increasing positive sequence $\{\alpha_n, n\in N_0\}$ so that $\alpha_n\cdot\alpha_{n-1}\cdots\alpha_1\geq(n+1)\|\!|T_l^{n+1}\varphi_0^{1l}\|\!|$, $l\in N_0$, $n\in N$, $l<k$, and

$$\alpha_n\geq 2\gamma_n \quad \text{for} \quad n\in N. \tag{12}$$

Letting $c_{kjn}=\alpha_n$ for $j, n\in N_0$, we apply Theorem 8.1 to T_k. Since the matrix $(c_{kjn})_{j,n\in N_0}$ satisfies the assumptions of Corollary 8.2, we thus obtain

$$(13) \qquad |||T_k^n\psi||| \geq \alpha_{n-1}\cdot\alpha_{n-2}\cdots\alpha_1 |||T_k\psi||| \geq n|||T_l^n\varphi_0^{1l}|||\,|||T_k\psi|||$$

for all $\psi\in\underline{\mathfrak{F}}_{1,2}^k$, $l\in N_0$, $n\in N$, $n\geq 2$ and $k>l$.

Next we choose positive numbers β_l, $l\in N_0$, such that the series $\sum_{l=0}^{\infty}\beta_l\varphi_0^{2l}$ converges in $\mathfrak{S}(W)[\mathfrak{t}_\mathfrak{a}]$ and defines a vector $\xi\in\mathfrak{S}(W)$. [It suffices to take $\beta_0=1$ and $\beta_l=\max\{2^l|||T^j\varphi_0^{2l}|||;\ j=0,\cdots,l-1\}$ for $l\in N$.] Let $\mathfrak{F}_\xi$ be the $\mathfrak{t}_\mathfrak{a}$-closure of $\mathcal{P}(T)\xi$ and let $\mathfrak{F}_r$ be the $\mathfrak{t}_\mathfrak{a}$-closure of $\mathrm{Lin}\{\mathfrak{F}_\xi, \underline{\mathfrak{F}}_1^l;\ l\in N_0\}$ in $\mathfrak{S}(W)$. Define $\mathfrak{M}_r:=\{(\mathfrak{x}_0,\mathfrak{x}_1,\cdots);\ \mathfrak{x}_j\in\mathfrak{F}_r$ for $j\in N_0\}$. As in the proof of Theorem 5, $\mathfrak{M}_r$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$ and $\pi_{W,\mathfrak{M}_r}\in\mathscr{C}$. [Moreover, $\pi_{W,\mathfrak{M}_r}(\boldsymbol{p})^n$ is e.s.a. for each $n\in N$.]

In order to show that $\pi_{W,\mathfrak{M}_r}$ is irreducible, we take a non-zero projection $A\in\mathfrak{A}$ such that $A\ \mathfrak{M}_r\subseteq\mathfrak{M}_r$. By Theorem 6.3, we are done if we have shown that $A=I$. First note that $A\mathfrak{M}_r\subseteq\mathfrak{M}_r$ implies $A\mathfrak{F}_r\subseteq\mathfrak{F}_r$. Since $A\in\mathfrak{A}$, (11) applies to A. Let $j,l\in N$ such that $l<j$ and $j\in N_l$. The definition of $\mathfrak{F}_r$ yields $D_jA\varphi_0^{1l}\in\underline{\mathfrak{F}}_{1,2}^j$. Using $AT\subseteq TA$ and (13), we get

$$|||AT^n\varphi_0^{1l}|||^2=\sum_{k=0}^{\infty}|||D_kAD_lT_l^n\varphi_0^{1l}|||^2=\sum_{k\in N_l}|\mu_{kl}|^2|||U_{kl}T_l^n\varphi_0^{1l}|||^2$$
$$=|||T_l^n\varphi_0^{1l}|||^2\sum_{k\in N_l}|\mu_{kl}|^2=|||T^nA\varphi_0^{1l}|||^2\geq|||T_j^nD_jA\varphi_0^{1l}|||^2$$
$$\geq n^2|||T_l^n\varphi_0^{1l}|||^2|||T_jD_jA\varphi_0^{1l}|||^2 \quad \text{for} \quad n\in N,\ n\geq 2.$$

Corollary 8.2, (ii), and $\varphi_0^{1l}\neq 0$ imply $T_l^n\varphi_0^{1l}\neq 0$. Therefore, the above inequality is true for all $n\in N$, $n\geq 2$, only if $T_jD_jA\varphi_0^{1l}=0$. Again by Corollary 8.2, (ii), this gives $D_jA\varphi_0^{1l}=\mu_{jl}U_{jl}\varphi_0^{1l}=0$. Hence $\mu_{jl}=0$, since $\varphi_0^{1l}\neq 0$. Because A is self-adjoint and $D_jAD_l=0$ if $j\notin N_l$, we thus obtain $D_jAD_l=0$ for all $j,l\in N_0$, $j\neq l$. We now come to the case $j=l$. By (11), we have either $D_jAD_j=0$ or $D_jAD_j=\mu_{jj}D_j$, where $\mu_{jj}\neq 0$. Because A is a projection, $\mu_{jj}=1$ in the latter case. We claim that $D_jAD_j=D_j$ for all $j\in N_0$. Suppose, to the contrary, that $D_jAD_j=0$ for some $j\in N_0$. Since $A\neq 0$ by assumption, there is a $r\in N_0$ such that $D_rAD_r\neq 0$, that is, $D_rAD_r=D_r$. From $A\xi=\sum_k D_kAD_k\beta_k\varphi_0^{2k}$ we conclude that $D_j A\xi = 0$, $D_r A\xi = \beta_r \varphi_0^{2r}$ and $D_s A\xi \in \underline{\mathcal{F}}_2^s$ for all $s \in N_0$. By Corollary 8.2, (iii), the latter implies that $A\xi \in \mathcal{F}_\xi$. That is, there are polynomials $p_n(T) \equiv \sum_k \rho_{nk} T^k$, $n \in N$, such that $A\xi = \mathfrak{t}_{\mathfrak{a}}\text{-}\lim_n p_n(T)\xi$. Hence $D_r A\xi = \mathfrak{t}_{\mathfrak{a}}\text{-}\lim p_n(T_r)\beta_r \varphi_0^{2r} = \beta_r \varphi_0^{2r}$. Since $\beta_r \neq 0$, Theorem 8.1, (iii), gives $|||(p_n(T_r) - I)\varphi_0^{2r}|||^2 \geq \frac{3}{4}|\rho_{n0} - 1|^2 |||\varphi_0^{2r}|||^2$ for $n \in N$. Therefore, $\rho_{n0} \to 1$ as $n \to \infty$. Similarly, $D_j A\xi = \mathfrak{t}_{\mathfrak{a}}\text{-}\lim p_n(T_j)\beta_j \varphi_0^{2j} = 0$ yields $\rho_{n0} \to 0$ as $n \to \infty$. This contradiction proves that $A = I$.

We now turn to the unitary equivalence of the representations $\pi_{W, \mathfrak{M}_\gamma}$. Assume that $\pi_{W, \mathfrak{M}_\gamma} \cong \pi_{W, \mathfrak{M}_{\gamma'}}$. As in the proof of Theorem 5, this implies that $T \upharpoonright \mathcal{F}_\gamma$ and $T \upharpoonright \mathcal{F}_{\gamma'}$ are unitarily equivalent. Arguing as in the proof of Corollary 8.3, we conclude that the set $\{\pi_{W, \mathfrak{M}_\gamma}\}$ contains an uncountable subset of pairwise inequivalent representations. This completes our proof of Theorem 6.

Remarks. 1) We sketch two variations of the preceding results. We retain the assumptions of 9.1.

I. Given $m \in N$, $\alpha \in C_1$ and $\varepsilon > 0$, there is an admissible boundary space $\mathfrak{M}$ w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$ such that $\pi_{W, \mathfrak{M}} \in \mathscr{C}$ and $|||\mathfrak{z}_j - \alpha^j \mathfrak{z}_0||| \leq \varepsilon |||\mathfrak{z}_0|||$ for all $(\mathfrak{z}_0, \mathfrak{z}_1, \cdots) \in \mathfrak{M}$ and $j = 1, \cdots, m$. In proving this assertion, we let $\mathfrak{M}$ be the $\mathfrak{t}_\infty$-closure of $\text{Lin}\{\mathfrak{Q}^n(\varphi_l^1, \alpha\varphi_l^1, \alpha^2\varphi_l^1, \cdots); l, n \in N_0\}$ and argue as in the proof of Theorem 1.

II. Let us add the assumption $(++)$ in the hypothesis of Theorem 1. Then there are uncountably many admissible boundary spaces $\mathfrak{M}_\iota, \iota \in \mathcal{J}$, w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$ having the properties stated in Theorem 1 such that the representations $\pi_{W, \mathfrak{M}_\iota}$, $\iota \in \mathcal{J}$, are mutually inequivalent. Indeed, it suffices to combine the proof of Theorem 1 with the argument used to prove Corollary 8.3. Of course, the same modification works for Theorem 2 and for the result states in I as well.

2) Set $\mathfrak{J} = Z$, $a_n = n$ and $b_n = n + 1$ for $n \in Z$. Let W be the bilateral shift in $l_2(Z)$. Obviously, $\mathfrak{a} = W^* \mathfrak{b} W$ and W is a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. π_W is of course unitarily equivalent to the Schrödinger representation on the Schwartz space. Hence π_W is integrable. This is also clear by Proposition 5.9. Choosing an admissible boundary space $\mathfrak{M}$ as in Theorem 1 (or in Theorem 2), $\mathcal{D}_{W, \mathfrak{M}} \neq \mathcal{D}_W$

and hence $\pi_{W,\mathfrak{M}}$ is not integrable.

3) The main technical tool on the preceding constructions and in Section 8 are growth conditions for the unbounded operator T. This technique has the advantage that it works for arbitrary W, $\mathfrak{a}$ and $\mathfrak{b}$ (provided only that $\mathfrak{a}$ is unbounded). For special W, $\mathfrak{a}$ and $\mathfrak{b}$ some results could be derived easier. For instance, if $\mathfrak{a}\upharpoonright\mathfrak{S}(W)$ has finite deficiency indices, then Proposition 2.1 in [18] can be used to obtain "sufficiently many" $\mathfrak{t}_{\mathfrak{a}}$-closed dense $\mathfrak{a}$-invariant subspaces $\mathcal{F}\subsetneqq\mathfrak{S}(W)$. This leads to a result like Theorem 5.

9.3. In this subsection we briefly explain another method of construction of admissible boundary spaces. Again let W be a fixed weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that there is a dense linear subspace $\mathcal{F}\subseteq\mathfrak{S}(W)$ of $l_2(\mathfrak{J}^+)$ and a symmetric linear operator S defined on $\mathcal{F}$ such that

$$(14) \qquad \mathfrak{a}\mathcal{F}\subseteq\mathcal{F},\ S\mathcal{F}\subseteq\mathcal{F} \text{ and } \mathfrak{a}S\mathfrak{x}-S\mathfrak{a}\mathfrak{x}=-i\mathfrak{x} \text{ for all } \mathfrak{x}\in\mathcal{F}.$$

Set $T=\mathfrak{a}\upharpoonright\mathcal{F}$. Let $\mathfrak{N}_0$ be the set of all vectors

$$\mathcal{S}_0(\mathfrak{x}):=(\mathfrak{x},\ (iS)\mathfrak{x},\ (iS)^2\mathfrak{x},\ \cdots).$$

Proposition 7. *If the Op*-algebra $\mathcal{A}(S,T)$ is closed on $\mathcal{F}$, then $\mathfrak{N}_0$ is an admissible boundary space with respect to W, $\mathfrak{a}$ and $\mathfrak{b}$ and $\pi_{W,\mathfrak{N}_0}\in\mathscr{C}$.*

Proof. Let $\mathfrak{x}\in\mathcal{F}$. Since $\mathfrak{a}(iS)^n\mathfrak{x}=(iS)^n\mathfrak{a}\mathfrak{x}-n(iS)^{n-1}\mathfrak{x}$ for $n\in N$ by (14), we have $\mathfrak{Q}\mathcal{S}_0(\mathfrak{x})=\mathcal{S}_0(\mathfrak{a}\mathfrak{x})$. Moreover, $\mathfrak{P}\mathcal{S}_0(\mathfrak{x})=\mathcal{S}_0(S\mathfrak{x})$. Because $\mathfrak{a}\mathcal{F}\subseteq\mathcal{F}$ and $S\mathcal{F}\subseteq\mathcal{F}$, this gives $\mathfrak{Q}\mathfrak{N}_0\subseteq\mathfrak{N}_0$ and $\mathfrak{P}\mathfrak{N}_0\subseteq\mathfrak{N}_0$.

We now show that $\mathfrak{N}_0$ is $\mathfrak{t}_\infty$-closed. Assume that $\mathfrak{z}\equiv(\mathfrak{z}_0,\mathfrak{z}_1,\cdots)=\mathfrak{t}_\infty\text{-}\lim_m\mathcal{S}_0(\mathfrak{x}^m)$, where $\mathfrak{z}\in\mathfrak{L}_\infty^+(W)$ and $\mathfrak{x}^m\in\mathcal{F}$ for $m\in N$. Considering the first component, this means $\mathfrak{z}_0=\mathfrak{t}_{\mathfrak{a}}\text{-}\lim_m\mathfrak{x}^m$. Moreover, $\mathfrak{z}_r=\lim_m(iS)^r\mathfrak{x}^m$ in the Hilbert space norm of $l_2(\mathfrak{J}^+)$. Since S is a symmetric operator and (14) holds, Lemma 1.1 in [17] applies and shows that the graph topology $t_{\mathcal{A}}$ of the *Op**-algebra $\mathcal{A}(S,T)$ is generated by the seminorms $|||\mathfrak{x}|||_{S^n}:=|||S^n\mathfrak{x}|||$ and $|||\mathfrak{x}|||_{T^n}:=|||T^n\mathfrak{x}|||$ for $n\in N_0$. Therefore, $\{\mathfrak{x}^m\}$ is a $t_{\mathcal{A}}$-Cauchy sequence. By assumption, $\mathcal{A}(S,T)$ is closed on $\mathcal{F}$. Hence

$\{\mathfrak{x}^m\}$ converges in $\mathscr{F}[t_{\mathscr{A}}]$. Thus $\mathfrak{z}_0 = t_{\mathscr{A}}\text{-}\lim_m \mathfrak{x}^m$, $\mathfrak{z}_0 \in \mathscr{F}$ and $\mathfrak{z}_r = \lim_m (iS)^r \mathfrak{x}^m = (iS)^r \mathfrak{z}_0$ for $r \in N$. That is, $\mathfrak{z} = \mathscr{S}_0(\mathfrak{z}_0)$. Therefore, $\mathfrak{M}_0$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$.

Since $\mathscr{F}$ is dense in $l_2(\mathfrak{J}^+)$, $\pi_{W,\mathfrak{N}_0}(\boldsymbol{p})$ is e.s.a. and thus $\pi_{W,\mathfrak{N}_0} \in \mathscr{C}$. This completes the proof.

It follows easily that $\mathscr{A}(S, T)$ is closed on $\mathscr{F}$ if $\mathfrak{N}_0$ is t_∞-closed in $\mathfrak{L}_\infty^+(W)$. Since obviously $\mathfrak{B}_2^+(\mathfrak{N}_0) = \{(\mathfrak{x}, S\mathfrak{x});\ \mathfrak{x} \in \mathscr{F}\}$ is not dense in $l_2^2(\mathfrak{J}^+)$, $\pi_{W,\mathfrak{N}_0}(\boldsymbol{p})^n$ is not e.s.a. for $n \in N, n \geq 2$.

Let S and $\mathscr{F}$ as above. Assume in addition that $\mathscr{F}$ is $t_{\mathfrak{a}}$-closed and that S is bounded. [In fact, it suffices to assume that $|||S\mathfrak{x}||| \leq \rho(|||\mathfrak{a}^k\mathfrak{x}||| + |||\mathfrak{x}|||)$ on $\mathscr{F}$ for some $k \in N$ and some constant ρ. Moreover, we do not need that S is symmetric on $\mathscr{F}$.] For $j \in N$ and $\mathfrak{x} \in \mathscr{F}$, let $\mathscr{S}_j(\mathfrak{x})$ denote the vector in $\mathfrak{L}_\infty^+(W)$ which is defined by $L_r\mathscr{S}_j(\mathfrak{x}) = 0$ for $0 \leq r \leq j-1$ and $L_r\mathscr{S}_j(\mathfrak{x}) = \binom{r}{r-j}(iS)^{r-j}\mathfrak{x}$ for $r \geq j$. Set $\mathfrak{N}_j := \{\mathscr{S}_j(\mathfrak{x});\ \mathfrak{x} \in \mathscr{F}\}$ and $\mathfrak{M}_k := \mathfrak{N}_0 + \cdots + \mathfrak{N}_{k-1}$ for $k \in N$. We then have

Proposition 8. *For each $k \in N$, $\mathfrak{M}_k$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$ and $\pi_{W,\mathfrak{M}_k} \in \mathscr{C}$. For $n \in N$, $\pi_{W,\mathfrak{M}_k}(\boldsymbol{p})^n$ is e.s.a. if and only if $n \leq k$.*

Proof. Let $j \in N$. Again by the commutation rule we obtain $\mathfrak{Q}\mathscr{S}_j(\mathfrak{x}) = \mathscr{S}_j(\mathfrak{a}\mathfrak{x})$. From $L_r\mathfrak{P}\mathscr{S}_j(i\mathfrak{x}) = L_{r+1}\mathscr{S}_j(\mathfrak{x}) = \binom{r+1}{r+1-j}(iS)^{r+1-j}\mathfrak{x} = \binom{r}{r-j}(iS)^{r-j+1}\mathfrak{x} + \binom{r}{r-(j-1)}(iS)^{r-(j-1)}\mathfrak{x} = L_r\mathscr{S}_j(iS\mathfrak{x}) + L_r\mathscr{S}_{j-1}(\mathfrak{x})$ for $r+1 \geq j$ and $L_r\mathfrak{P}\mathscr{S}_j(i\mathfrak{x}) = L_r\mathscr{S}_j(iS\mathfrak{x}) + L_r\mathscr{S}_{j-1}(\mathfrak{x}) = 0$ for $r+1 < j$ we get $\mathfrak{P}\mathscr{S}_j(i\mathfrak{x}) = \mathscr{S}_j(iS\mathfrak{x}) + \mathscr{S}_{j-1}(\mathfrak{x})$ for all $\mathfrak{x} \in \mathscr{F}$. This implies $\mathfrak{Q}\mathfrak{N}_j \subseteq \mathfrak{N}_j$ and $\mathfrak{P}\mathfrak{N}_j \subseteq \mathfrak{N}_j$. Therefore, $\mathfrak{Q}\mathfrak{M}_k \subseteq \mathfrak{M}_k$ and $\mathfrak{P}\mathfrak{M}_k \subseteq \mathfrak{M}_k$.

Now let $\mathfrak{z} = (\mathfrak{z}_0, \mathfrak{z}_1, \cdots) \in \mathfrak{L}_\infty^+(W)$ be in the t_∞-closure of $\mathfrak{M}_k$. Then there are sequences $\{\mathfrak{x}^{m,0}, m \in N\}, \cdots, \{\mathfrak{x}^{m,k-1}, m \in N\}$ of vectors in $\mathscr{F}$ so that $\mathfrak{z} = t_\infty\text{-}\lim_m \mathfrak{z}^m$, where $\mathfrak{z}^m := \mathscr{S}_0(\mathfrak{x}^{m,0}) + \cdots + \mathscr{S}_{k-1}(\mathfrak{x}^{m,k-1})$ for $m \in N$. In particular, $\mathfrak{z}_0 = t_{\mathfrak{a}}\text{-}\lim_m L_0\mathfrak{z}^m = t_{\mathfrak{a}}\text{-}\lim_m \mathfrak{x}^{m,0}$. Since $\mathscr{F}$ is $t_{\mathfrak{a}}$-closed, $\mathfrak{z}_0 \in \mathscr{F}$. Since S is bounded and $\mathfrak{a}S - S\mathfrak{a} = -i$ on $\mathscr{F}$, it follows easily that S is $t_{\mathfrak{a}}$-continuous. Hence $S^r\mathfrak{z}_0 = t_{\mathfrak{a}}\text{-}\lim_m S^r\mathfrak{x}^{m,0}$, i.e., $\mathscr{S}_0(\mathfrak{z}_0) = t_\infty\text{-}\lim_m \mathscr{S}_0(\mathfrak{x}^{m,0})$.

Replacing $\mathfrak{z}^m, \mathfrak{z}$ and L_0 by $\mathfrak{z}'^m := \mathfrak{z}^m - \mathscr{S}_0(\mathfrak{x}^{m,0})$, $\mathfrak{z}' := \mathfrak{z} - \mathscr{S}_0(\mathfrak{z}_0)$ and L_1, respectively, the same argument yields $\mathscr{S}_1(\mathfrak{z}_1') = \mathfrak{t}_\infty\text{-}\lim_m \mathscr{S}_1(\mathfrak{x}^{m,1})$ and $\mathfrak{z}_1' \in \mathscr{F}$, where $\mathfrak{z}' = (0, \mathfrak{z}_1', \cdots)$. Proceding by induction, we see that there are vectors $\mathfrak{x}^j \in \mathscr{F}, j = 0, \cdots, k-1$, such that $\mathscr{S}_j(\mathfrak{x}^j) = \mathfrak{t}_\infty\text{-}\lim_m \mathscr{S}_j(\mathfrak{x}^{m,j})$. Hence $\mathfrak{z} = \mathscr{S}_0(\mathfrak{x}^0) + \cdots + \mathscr{S}_{k-1}(\mathfrak{x}^{k-1})$ which means that $\mathfrak{z} \in \mathfrak{M}_k$. This shows that $\mathfrak{M}_k$ is an admissible boundary space w.r.t. W, $\mathfrak{a}$ and $\mathfrak{b}$.

Clearly, $\pi_{W,\mathfrak{M}_k}(\boldsymbol{p})^n$ is e.s.a. for $n \in N$ if and only if $\mathfrak{B}_n^+(\mathfrak{M}_k)$ is dense in $l_2^n(\mathfrak{J}^+)$, that is, $n \leq k$. Thus $\pi_{W,\mathfrak{M}_k} \in \mathscr{C}$, which completes the proof.

We close this subsection by showing how the above conditions can be fulfilled. A more detailed study of this point will be given elsewhere.

Let π be a closed $*$-representation of $\boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on the dense domain $\mathscr{F}$ on the Hilbert space $\mathcal{G}$. Assume that $\pi(\boldsymbol{p})$ is unbounded and has self-adjoint extensions R and R' which both admit complete systems (denoted by $\{\xi_n, n \in N\}$ resp. $\{\xi_n', n \in N\}$) of eigenvectors. Let $\{a_n, n \in N\}$ and $\{a_n', n \in N\}$ be the corresponding sequences of eigenvalues. Without loss of generality we assume that $\sup_n a_n' = +\infty$. Then we can choose a subsequence $\{a_{k_n}'\}$ of $\{a_n'\}$ such that $a_{k_n}' - a_n \geq 1$ for $n \in N$. We now take the intervals (a_n, a_{k_n}') for $n \in N$ and $(-\infty, a_n')$ for $n \in N \setminus \{k_n; n \in N\}$. Then $(+)$ is satisfied. For short, we identify $\xi_n = \mathfrak{e}_n := \{\delta_{nk}, k \in \mathfrak{J}^+\}$, $\mathcal{G} = l_2(\mathfrak{J}^+)$ and $R = \mathfrak{a}$. $W\mathfrak{e}_n := \xi_n'$ defines a weak intertwining operator for $\mathfrak{a}$ and $\mathfrak{b}$. Obviously, $\mathscr{F} \subseteq \mathfrak{S}(W)$. Then, $S := \pi(\boldsymbol{q})$, $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the above conditions. It should be noted that π_W (but not $\pi_{W,\mathfrak{M}_k}$!) can be integrable in this case, we have not excluded that $a_n' = a_{\tau(n)}$ for some permutation τ of N (see Proposition 5. 9).

The simplest example of this kind is the representation π on $\mathscr{F} = \mathring{C}^\infty[a, b]$ in $\mathcal{G} = L_2(a, b)$, $a, b \in R_1, a < b$, which is defined by $\pi(\boldsymbol{p}) = -i \dfrac{d}{dx}$ and $\pi(\boldsymbol{q}) = x$. Given $\alpha, \alpha' \in R_1$, there are extensions R and R' such that $a_n = \alpha + 2\pi n/(b-a)$ and $a_n' = \alpha' + 2\pi n/(b-a)$ for $n \in Z$.

9. 4. Let π be a $*$-representation of $\boldsymbol{A} := \boldsymbol{A}(\boldsymbol{p}, \boldsymbol{q})$ on a domain $\mathscr{D}$. Put $P' = \pi(\boldsymbol{p})$ and $Q' = \pi(\boldsymbol{q})$. (Recall from Section 2 that if $\pi \in \mathscr{C}$, then we usually denote by P and Q the operators $\overline{\pi(\boldsymbol{p})}$ resp. $\overline{\pi(\boldsymbol{q})}$.) We then have

$$\underline{\mathcal{D}}(\pi(\boldsymbol{A})) = \bigcap_{A\in\boldsymbol{A}} \mathcal{D}(\overline{\pi(A)}) \text{ and } \mathcal{D}_*(\pi(\boldsymbol{A})) = \bigcap_{k,r=0}^{\infty} \mathcal{D}((P'^*)^k (Q'^*)^r).$$

The latter follows from Lemma 1.1. We consider the domains

$$\mathcal{D}_1 := \bigcap_{k,r=0}^{\infty} \mathcal{D}(\overline{P'^k Q'^r}) \text{ and } \mathcal{D}_2 := \bigcap_{k,r=0}^{\infty} \mathcal{D}((\overline{P'})^k (\overline{Q'})^r) \text{ as well.}$$

Lemma 9. $\mathcal{D} \subseteq \underline{\mathcal{D}}(\pi(\boldsymbol{A})) \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_*(\pi(\boldsymbol{A}))$.

Proof. $\mathcal{D} \subseteq \mathcal{D}(\pi(\boldsymbol{A}))$ is trivial. Since P' and Q' are symmetric linear opearators, we have $\overline{P'^k} \subseteq (\overline{P'})^k \subseteq (P'^*)^k$ and $\overline{Q'^r} \subseteq (\overline{Q'})^r \subseteq (Q'^*)^r$ for $k, r \in N_0$. Therefore, $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_*(\pi(\boldsymbol{A}))$. The argument used in the proof of Lemma 1.2 shows that $\underline{\mathcal{D}}(\pi(\boldsymbol{A})) \subseteq \mathcal{D}_1$.

If π is an integrable representation, then π is self-adjoint and hence $\mathcal{D} = \underline{\mathcal{D}}(\pi(\boldsymbol{A})) = \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_*(\pi(\boldsymbol{A}))$. Suppose now that $\pi \in \mathscr{C}$. We then have $\mathcal{D} = \underline{\mathcal{D}}(\pi(\boldsymbol{A}))$ by definition and $\mathcal{D}_2 = \mathcal{D}_*(\pi(\boldsymbol{A}))$ by Lemma 1.1, because $\overline{P'} = P'^*$ and $\overline{Q'} = Q'^*$ in that case. But, in general, $\underline{\mathcal{D}}(\pi(\boldsymbol{A}))$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are different from each other as we shall see now. Retaining the notation of Section 5, we first state

Lemma 10. *If* $\pi_{W,\mathfrak{M}} \in \mathscr{C}$, *then* $\pi^*_{W,\mathfrak{M}} = \pi_W$. *Moreover,*

$$\mathcal{D}_W = \bigcap_{k,r=0}^{\infty} \mathcal{D}(P_W^k Q^r) = \bigcap_{k,r=0}^{\infty} (Q^r P_W^k).$$

Proof. Since $\pi_{W,\mathfrak{M}} \in \mathscr{C}$, $\pi_{W,\mathfrak{M}}(\boldsymbol{p})$ is e.s.a. Therefore, W is an isometry of $l_2(\mathfrak{J}^+)$ onto $l_2(\mathfrak{J}^-)$ and $\overline{\pi_{W,\mathfrak{M}}(\boldsymbol{p})} = P_W$. Combining Lemmas 5.6 and 5.7, we thus obtain $\mathcal{D}(\pi^*_{W,\mathfrak{M}}) = \mathcal{D}_W$, i.e., $\pi^*_{W,\mathfrak{M}} = \pi_W$. The second part follows (for instance) from Lemma 1.1 and the fact that π_W is self-adjoint.

Now let W, $\mathfrak{a}$ and $\mathfrak{b}$ be as in the subsection 9.1.

Example 11. $\mathcal{D} \neq \mathcal{D}_1$.

We take an admissible boundary space $\mathfrak{M}_i \neq \mathfrak{L}^{\pm}_{\infty}(W)$ as defined in the proof of Theorem 5. Then, $P'^n = \pi_{W,\mathfrak{M}_i}(\boldsymbol{p})^n$ is e.s.a. for each $n \in N$. Since Q'^n, $n \in N$, is e.s.a., $\overline{P'^k Q'^r} = (\overline{P'})^k (\overline{Q'})^r$ for $k, r \in N_0$. Hence $\mathcal{D}_1 = \mathcal{D}_2$. By Lemma 10, $\mathcal{D}_2 = \mathcal{D}_*(\pi_{W,\mathfrak{M}_i}(\boldsymbol{A})) = \mathcal{D}_W$. Since $\mathfrak{M}_i \neq \mathfrak{L}^{\pm}_{\infty}(W)$, we have $\mathcal{D} \equiv \mathcal{D}_{W,\mathfrak{M}_i} \neq \mathcal{D}_W$ and thus $\mathcal{D} \neq \mathcal{D}_1$.

Example 12. $\mathcal{D}_1 \neq \mathcal{D}_2$.

By Theorem 1, there is an admissible boundary space $\mathfrak{M}$ such that $\pi_{W,\mathfrak{M}} \in \mathscr{C}$ and $P'^2 = \pi_{W,\mathfrak{M}}(\boldsymbol{p})^2$ is not e.s.a. Therefore, $P'^2 \restriction \mathcal{D}_1$ is not e.s.a. On the other hand, $\mathcal{D}_2 = \mathcal{D}_W$ by Lemma 10 and $P^2_W \restriction \mathcal{D}_W$ is, of course, e.s.a. Since $P'^2 \restriction \mathcal{D}_1 = P^2_W \restriction \mathcal{D}_1$, we conclude that $\mathcal{D}_1 \neq \mathcal{D}_2$.

Closing Remarks. Throughout the whole discussion in Sections 5–9 we assumed that condition $(+)$ is fulfilled, that is, $\inf_{n \in \mathfrak{J}} b_n - a_n > 0$. It is easy to check that $(+)$ is satisfied if and only if the vector space $\mathcal{H}_+ + \mathcal{H}_-$ is closed in $\mathcal{H}$. Here $\mathcal{H}_+$ and $\mathcal{H}_-$ are the deficiency spaces of the closed symmetric operator $\overline{P_0}$ for $+i$ resp. $-i$. Another equivalent condition is that the sequences $B_0^+(\varphi) = \{\varphi_n(a_n+), n \in \mathfrak{J}^+\}$ and $B_0^-(\varphi) = \{\varphi_n(b_n-), n \in \mathfrak{J}^-\}$ are in $l_2(\mathfrak{J}^+)$ resp. $l_2(\mathfrak{J}^-)$ for any $\varphi = (\varphi_n) \in \mathcal{D}(P_0^*)$. [The necessity is shown in the proof of Lemma 5.1.] The latter has been the basic ingredient in the construction of weak intertwining operators in Section 7 and of admissible boundary spaces in Section 9 as well. Both methods use restrictions of the diagonal operator $\mathfrak{a}$ in $l_2(\mathfrak{J}^+)$ as a common technical tool. That is, they strongly depend on the Hilbert space $l_2(\mathfrak{J}^+)$. The case in which condition $(+)$ is not fulfilled (i.e., $\inf_{n \in \mathfrak{J}} b_n - a_n = 0$) will be treated elsewhere.

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Added in proof.

Correction to: "On the Heisenberg commutation relation. I", *J. Funct. Analysis* **50** (1983), 8-49.

On p. 10, l. -5, the formula "$\cap_{n,k=0}^{\infty}\mathcal{D}(\overline{Q^nP^k})=\cap_{n,k=0}^{\infty}\mathcal{D}(\overline{P^kQ^n})$" should be replaced by "$\bigcap_{A\in\boldsymbol{A}}\mathcal{D}(\bar{A})$". (The proof of Lemma 1.1 does not give more). In a similar way, l. 4 on p. 12, the assertion of Lemma 2.2 and the proof of Proposition 3.1 should be modified. There are no consequences for other parts of the paper.