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L_p-Spaces for von Neumann Algebra with Reference to a Faithful Normal Semifinite Weight

By

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Abstract

The L_p-space $L_p(M, \phi_0)$ for a von Neumann algebra M and its faithful normal semifinite weight ϕ_0 is constructed as a linear space of closed linear operators acting on the standard representation Hilbert space H_{ϕ_0}.

Any L_p-element has the polar and the Jordan decompositions relative to a positive part $L_p^+(M, \phi_0)$. Any positive element in L_p-space has an interpretation as the $(1/p)^{\text{th}}$ power $\phi^{1/p}$ of a $\phi \in M_*^+$ with its L_p-norm given by $\phi(1)^{1/p}$.

The product of an L_p-element and an L_q-element is explicitly defined as an L_r-element with $r^{-1}=p^{-1}+q^{-1}$ provided $1 \leqq r$ and the Hölder inequality is proved. Also $L_p(M, \phi_0)$ are shown to be isomorphic for the same p and different ϕ_0.

There exists a vector subspace $D_{\phi_0}^\infty$ of the Tomita algebra associated with ϕ_0 and a p-dependent injection $T_p: D_{\phi_0}^\infty \to L_p(M, \phi_0)$ with dense range. The sesquilinear form on $L_p(M, \phi_0) \times L_{p'}(M, \phi_0), p^{-1}+(p')^{-1}=1$ is a natural extension of the inner product of H_{ϕ_0} through the mapping $T_p \times T_{p'}$.

The present work is an extension of our joint work with Araki to weights, and our L_p-spaces are isomorphic to those defined by Haagerup, Hilsum, Kosaki, and Terp.

§ 1. Introduction

The L_p-space $L_p(M, \tau)$ of a semifinite von Neumann algebra M with respect to a faithful normal semifinite trace τ is defined as the linear space of all τ-measurable operators T satisfying $\|T\|_p \equiv \tau(|T|^p)^{1/p} < \infty$ (see [22, 26]). Extension to non semifinite cases have been worked out by Haagerup [14] (see also Terp [30]), Hilsum [16], Kosaki [19, 20, 21], Terp [29] and our previous paper [8]. In the present paper, we shall define L_p-spaces with methods based on the technique of our previous L_p-theory [8] and different from the other authors.

In our previous L_p-theory, we assumed the existence of a faithful normal state, or equivalently the σ-finiteness of M. In this paper, we

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define L_p-spaces with reference to a faithful normal semifinite weight.

Our construction has an advantage in defining L_p-spaces on the standard representation Hilbert space rather than going over to the crossed product of M with the modular action. Another salient feature of our construction is the use of a functional defined on the L_1-space which behaves like trace and allows us trace-like calculation for products of L_p-elements.

The rest of this paper is organized as follows. In Section 2, we state the main results of this paper. In Section 3, we introduce a multilinear form which plays an important role throughout the theory. Also a Hölder type inequality is proved. The result presented in Section 3 is a straightforward generalization of Lemma A in our previous paper [8] and includes multiple KMS condition as a special case (for multiple KMS condition, see [1] or [17]). In Section 4, we introduce the concept of "measurability" in our approach and fully discuss L_p-spaces for $2 \leqq p \leqq \infty$. Section 5 is devoted to the study of linear structure of L_p-spaces for $2 \leqq p < \infty$ which is not so trivial in our situation and in Section 6, we discuss on the product between the elements of L_p-spaces for different p's. In Section 7, special cases $p=1, 2$ are discussed and the isomorphisms between the L_1-space and the predual M_* and between the L_2-space and the standard representation Hilbert space H_{ϕ_0} are constructed. In Section 8, the uniform convexity as well as the uniform smoothness of L_p-spaces, $1 < p < \infty$, are proved and used for discussing the L_p-spaces for $1 < p < 2$. In Section 9, relations between L_p-spaces for different faithful normal semifinite weights are discussed and shown to be isomorphic. In Section 10, the positive part of an L_p-space is discussed and the linear polar decomposition is shown. In the same section, bilinear form is introduced and it is shown to be symmetric. In Section 11, a product of L_p-element and L_q-element is discussed and the Hölder inequality is shown. In Section 12, the dense subspace $D^{\infty}_{\phi_0}$ and its p-dependent injection to L_p-space are discussed. In Section 13, the modular action on L_p-space is discussed. Section 14 provides a summary of proofs of Theorems in Section 2. A brief discussion on our theory and its connection with other works is presented in Section 15.

In Appendix A, for the sake of completeness, we define and show some properties of positive cones $V^{\alpha}_{\phi_0}$, $0 \leqq \alpha \leqq 1/2$, for faithful normal semifinite weights ϕ_0, which are more or less known. In Appendix B, we discuss the polar decomposition in $D(\Delta_{\phi_0}^{(1/2)-2\alpha})$ in terms of the positive cone $V_{\phi_0}^{\alpha}$, $0\leqq\alpha\leqq 1/2$.

§ 2. Main Results

In this paper, we use standard results on the Tomita-Takesaki theory (see [27]), and relative modular operators (see Appendix C of [8] or [4, 5, 6]). Let ϕ_0 be a faithful normal semifinite weight on M and $\sigma_t^{\phi_0}$ be the associated modular automorphism of M. We denote N_{ϕ_0} the set of all $y\in M$ such that $\phi_0(y^*y)<\infty$, M_0 the set of all entire analytic elements of $N_{\phi_0}^*\cap N_{\phi_0}$ with respect to $\sigma_t^{\phi_0}$. We also denote the GNS mapping by $y\in N_{\phi_0}\mapsto\eta_{\phi_0}(y)\in H_{\phi_0}$. Let J_{ϕ_0} and Δ_{ϕ_0} denote the modular conjugation operator and the modular operator on H_{ϕ_0} associated with ϕ_0, respectively.

Definition 2.1. A linear operator T acting on H_{ϕ_0} is said to be (ϕ_0, p)*-measurable* $(1\leqq p\leqq\infty)$ if it is closed as an operator and satisfies

$$TJ_{\phi_0}\sigma_{-i/p}^{\phi_0}(y)J_{\phi_0}\supset J_{\phi_0}yJ_{\phi_0}T\ , \tag{2.1}$$

for and $y\in M_0$.

The definition of L_p-norm is the following.

Definition 2.2. For a (ϕ_0, p)-measurable operator T,

$$\|T\|_{p,\phi_0}\equiv\{\sup_{x\in M_0,\|x\|\leqq 1}\||T|^{p/2}\eta_{\phi_0}(x)\|\}^{2/p}\quad(1\leqq p<\infty), \tag{2.2}$$

where $T=u|T|$ is the polar decomposition of T. For $p=\infty$, $\|T\|_{\infty,\phi_0}$ denotes the usual operator norm of T.

Definition 2.3. The L_p-space $L_p(M,\phi_0)$ is the set of all (ϕ_0, p)-measurable operators T with finite L_p-norm $\|T\|_{p,\phi_0}$.

Theorem 1. (1) *$L_p(M,\phi_0)$ is a Banach space with the L_p-norm $\|\cdot\|_{p,\phi_0}$, $1\leqq p\leqq\infty$, and the linear structure given by that of M if $p=\infty$ (see Theorem 3, (1)), strong sum (in the sense that $D=D(T_1)\cap D(T_2)$, $T_1, T_2\in L_p(M,\phi_0)$ is dense and $\overline{(T_1+T_2)|_D}\in L_p(M,\phi_0)$) if $2\leqq p<\infty$,* *that of* $L_{p'}(M,\phi_0)^*$ *(see* (2) *below) of* $1<p<2$, *and that of* M_* *if* $p=1$ *(see Theorem* 3, (2)), *where* $p^{-1}+(p')^{-1}=1$.

(2) *Sesquilinear dual pairing between* $L_p(M,\phi_0)$ *and* $L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, $1\leqq p\leqq\infty$, *is defined in terms of the inner product of* H_{ϕ_0} *by*

$$\langle T, T'\rangle_{\phi_0}\equiv\lim_{y\to1}(T\eta_{\phi_0}(y), T'\eta_{\phi_0}(y)), \tag{2.3}$$

where $T\in L_p(M,\phi_0)$ *and* $T'\in L_{p'}(M,\phi_0)$. *The formula in the parenthesis of the right hand side is well defined for the pair* $(T, T')\in L_p(M,\phi_0)\times L_{p'}(M,\phi_0)$ *(the meaning of the inner product on the right hand side is* $(T'^*T\eta_{\phi_0}(y),\eta_{\phi_0}(y))$ *defined in Notation* 3.3) *and the limit* $y\to1$ *is taken in the* $*$*-strong operator topology with the restriction* $y\in M_0$, $\|y\|\leqq1$. *Then,*

$$\|T\|_{p,\phi_0}=\sup\{|\langle T, T'\rangle_{\phi_0}|: T'\in L_{p'}(M,\phi_0),\ \|T'\|_{p',\phi_0}\leqq1\} \tag{2.4}$$

for $T\in L_p(M,\phi_0)$ *for all* p. *Through this pairing,* $L_{p'}(M,\phi_0)$ *is the dual of* $L_p(M,\phi_0)$ *if* $1\leqq p<\infty$.

(3) *For* $1<p<\infty$, *the* L_p*-norm* $\|\cdot\|_{p,\phi_0}$ *of* $L_p(M,\phi_0)$ *is uniformly convex and uniformly smooth. For* $2\leqq p<\infty$, *the following Clarkson's inequality holds:*

$$\begin{aligned}(\|T_1+T_2\|_{p,\phi_0})^p+(\|T_1-T_2\|_{p,\phi_0})^p\\ \leqq2^{p-1}\{(\|T_1\|_{p,\phi_0})^p+(\|T_2\|_{p,\phi_0})^p\},\end{aligned} \tag{2.5}$$

where $T_1, T_2\in L_p(M,\phi_0)$.

We define the positive part of the L_p-spaces as follows;

$$L_p^+(M,\phi_0)=\{T\in L_p(M,\phi_0): T^*=T\geqq0\}. \tag{2.6}$$

Theorem 2. (1) *Any* $T\in L_p(M,\phi_0)$ *has a unique polar decomposition* $T=u|T|$, *where* u *is a partial isometry in* M *and* $|T|\in L_p^+(M,\phi_0)$ *satisfying* $u^*u=s(|T|)$.

(2) $\|T\|_{p,\phi_0}=\||T|\|_{p,\phi_0}$, $T\in L_p(M,\phi_0)$.

(3) *If* $T\in L_p^+(M,\phi_0)$, $1\leqq p<\infty$, *there exists a unique* $\phi\in M_*^+$ *such that* $T=\Delta_{\phi,\phi_0}^{1/p}$. *For such a unique* ϕ, $\|T\|_{p,\phi_0}=\phi(1)^{1/p}$, *where* Δ_{ϕ,ϕ_0} *is the relative modular operator.*

(4) *For* $\phi\in M_*^+$ *and a partial isometry* $u\in M$ *such that* u^*u

$= s(\phi)$, $T = u\Delta_{\phi,\phi_0}^{1/p} \in L_p(M, \phi_0)$, $(1 \leqq p < \infty)$ *with its* L_p*-norm* $\|T\|_{p,\phi_0} = \phi(1)^{1/p}$.

The case $p = \infty$, excluded in Theorem 2 (3), and the case $p = 1, 2$ reduce to well-known objects.

Theorem 3. (1) $L_\infty(M, \phi_0) = M$.

(2) *The mapping from* $T \in L_1(M, \phi_0)$ *to* $\psi \in M_*$ *given by*

$$\psi(x) = \langle T, x^* \rangle_{\phi_0}, \quad x \in M \tag{2.7}$$

is an isometric isomorphism from $L_1(M, \phi_0)$ *onto* M_*, *where the right hand side of* (2.7) *is given in Theorem* 1 (2) *for* $p = 1$. $\psi(x) = \phi(xu)$, $x \in M$, *holds if* $T = u\Delta_{\phi,\phi_0}$ *is the polar decomposition of* T.

(3) *The mapping from* $T \in L_2(M, \phi_0)$ *to* $u\xi(\phi) \in H_{\phi_0}$ *is an isometric isomorphism from* $L_2(M, \phi_0)$ *onto* H_{ϕ_0}, *where* $T = u\Delta_{\phi,\phi_0}^{1/2}$ *is the polar decomposition of* T *given by Theorem* 2, *and* $\xi(\phi)$ *is the unique representative in a fixed natural positive cone associated with* M.

A linear polar decomposition is given by the following.

Theorem 4. *Any* $T \in L_p(M, \phi_0)$ *has a unique polar decomposition*

$$T = T_1 - T_2 + i(T_3 - T_4) \tag{2.8}$$

where $T_k \in L_p^+(M, \phi_0)$, $k = 1, \cdots, 4$, $s(T_1) \perp s(T_2)$ *and* $s(t_3) \perp s(T_4)$.

By Theorem 2 and the property of the relative modular operators, the adjoint operation on closed operators maps $L_p(M, \phi_0)$ onto $L_p(M, \phi_0)$ isometrically. We introduce a new bilinear form on $L_p(M, \phi_0) \times L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$ as folows:

$$[T, T']_{\phi_0} \equiv \langle T, T'^* \rangle_{\phi_0}, \tag{2.9}$$

where $T \in L_p(M, \phi_0)$ and $T' \in L_{p'}(M, \phi_0)$.

Theorem 5. *Let* $p^{-1} + (p')^{-1} = 1$.

(1) $[T, T']_{\phi_0} = [T', T]_{\phi_0}$,

where $T \in L_p(M, \phi_0)$ *and* $T' \in L_{p'}(M, \phi_0)$.

(2) $L_p^+(M, \phi_0) = \{T \in L_p(M, \phi_0) : [T, T']_{\phi_0} \geqq 0$
for any $T' \in L_{p'}^+(M, \phi_0)\}$.

Next, we consider the products between L_p-elements.

Theorem 6. *Let* $T_k \in L_{p_k}(M, \phi_0)$, $k=1, \cdots, n$ *and* $\sum_{k=1}^{n} p_k^{-1} = r^{-1} = 1-(r')^{-1} \leqq 1$. *Then the product* $T = T_1 \cdots T_n$ *is well defined as an element of* $L_{r'}(M, \phi_0)^* \cong L_r(M, \phi_0)$ *for* $r \neq 1$ *and as an element of* $L_\infty(M, \phi_0)_* \cong L_1(M, \phi_0)$ *for* $r=1$. *If* $2 \leqq r \leqq \infty$, *the product* $T = T_1 \cdots T_n$ *is defined as a strong product i.e.* $T_1 \cdots T_n$ *is a densely defined preclosed operator, and* T *is defined as its closure. If* $1 \leqq r < 2$, *the product is understood as the element of* $L_{r'}(M, \phi_0)^*$ *through the multilinear pairing (see* (4) *below). This product satisfies the followings.*

(1) *The product is associative and the mapping* $(T_1, \cdots, T_n) \mapsto T_1 \cdots T_n$ *is multilinear.*

(2) $\|T\|_{r,\phi_0} \leqq \|T_1\|_{p_1,\phi_0} \cdots \|T_n\|_{p_n,\phi_0}$.

(3) *Assume* $r=1$. *Then,*

$$[T, 1]_{\phi_0} = [T_1 \cdots T_k, T_{k+1} \cdots T_n]_{\phi_0}$$

for any $i \leqq k \leqq n-1$. *Hence we denote it simply as* $[T_1 \cdots T_n]_{\phi_0}$.

(4) $[T_1 \cdots T_n]_{\phi_0} = [T_{k+1} \cdots T_n T_1 \cdots T_k]_{\phi_0}$, $1 \leqq k \leqq n-1$.

(5) $|[T_1 \cdots T_n]_{\phi_0}| \leqq \|T_1\|_{p_1,\phi_0} \cdots \|T_n\|_{p_n,\phi_0}$.

As a corollary, we get the inequality which is originally obtained in [9].

Corollary. *Let* $T_1, T_2 \in L_2(M, \phi_0)$. *Then*

$$\| |T_2| - |T_1| \|_{2,\phi_0} \leqq 2^{1/2} \|T_2 - T_1\|_{2,\phi_0}.$$

The L_p-spaces for different reference weights are related as follows.

Theorem 7. *Let* ϕ_0 *and* $\tilde{\phi}_0$ *be two faithful normal semifinite weights. There exists isometric conjugate linear isomorphisms* $J_p(\tilde{\phi}_0, \phi_0)$ *and isometric linear isomorphisms* $\tau_p(\tilde{\phi}_0, \phi_0)$ *from* $L_p(M, \phi_0)$ *onto* $L_p(M, \tilde{\phi}_0)$ *having the following properties.*

(1) *For* $1\leqq p<\infty$,

$$J_p(\widetilde{\phi}_0, \phi_0) : u\Delta_{\phi,\phi_0}^{1/p}\mapsto\Delta_{\phi,\tilde{\phi}_0}^{1/p}u^* ,$$

$$\tau_p(\widetilde{\phi}_0, \phi_0) : u\Delta_{\phi,\phi_0}^{1/p}\mapsto u\Delta_{\phi_0,\tilde{\phi}_0}^{1/p} ,$$

and for $p=\infty$, J_∞ *is the usual* $*$*-operation,* τ_∞ *is the identify, where* $T=u\Delta_{\phi,\phi_0}^{1/p}\in L_p(M,\phi_0)$, $1\leqq p<\infty$ *is the polar decomposition given in Theorem* 2. *If* $\widetilde{\phi}_0=\phi_0$, *then the map* $J_p(\phi_0,\phi_0)$ *coincides with the adjoint of a closed operator.*

(2) *Let* $T\in L_p(M,\phi_0)$ *and* $T'\in L_{p'}(M,\widetilde{\phi}_0)$ *with* $p^{-1}+(p')^{-1}=1$. *Then,*

$$[J_p(\widetilde{\phi}_0, \phi_0)(T)T']_{\tilde{\phi}_0}=\overline{[TJ_{p'}(\phi_0,\widetilde{\phi}_0)(T')]}_{\phi_0} ,$$

$$[\tau_p(\widetilde{\phi}_0, \phi_0)(T)T']_{\tilde{\phi}_0}=[T\tau_{p'}(\phi_0,\widetilde{\phi}_0)(T')]_{\phi_0} .$$

(3) *Let* $T_k\in L_p(M,\phi_0)$, $k=1,\cdots,n$ *and* $\sum_{k=1}^{n}p_k^{-1}=r^{-1}\leqq 1$. *Then*

$$J_r(\widetilde{\phi}_0,\phi_0)(T_1\cdots T_n)=J_{p_n}(\widetilde{\phi}_0,\phi_0)(T_n)\cdots J_{p_1}(\widetilde{\phi}_0,\phi_0)(T_1),$$

$$\tau_r(\widetilde{\phi}_0,\phi_0)(T_1\cdots T_n)=\tau_{p_1}(\widetilde{\phi}_0,\phi_0)(T_1)\cdots \tau_{p_n}(\widetilde{\phi}_0,\phi_0)(T_n).$$

The incompatibility of the measurability condition for different p makes the intersection of the L_p-spaces for different p trivial. As a substitute for such an intersection, we have a certain vector subspace $D_{\phi_0}^\infty$ of a Tomita algebra $\mathfrak{A}_0=\eta_{\phi_0}(M_0)$ and a p-dependent injection $T_p: D_{\phi_0}^\infty\to L_p(M,\phi_0)$ with dense image. The vector space $D_{\phi_0}^\infty$ is defined as the linear span of the product of two elements in $\mathcal{A}_0=\eta_{\phi_0}(M_0)$. For $\zeta=\eta_{\phi_0}(x_1x_2)$, $x_k\in M_0$, $k=1,2$, $T_p(\zeta)$ is defined by

$$T_p(\zeta)\equiv(\Delta_{\phi_0}^{1/(2p)}x_1^*)^*\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_2))}, \tag{2.10}$$

where the bar denotes the closure. For an arbitrary $\zeta\in D_{\phi_0}^\infty$, $T_p(\zeta)$ is defined as the sum of the operators of the form (2.10) in $L_p(M,\phi_0)$.

We note that $D_{\phi_0}^\infty$ can be viewed as a Tomita algebra.

Let $V_{\phi_0}^\alpha(0\leqq\alpha\leqq 1/2)$ be the positive cones for the weight ϕ_0 which are defined by the closure of (A.1) in Appendix A just as for the positive cones for states defined by Araki [2] (see also [10]).

Theorem 8. (1) *The mapping,*

$$\zeta\in D_{\phi_0}^\infty\to T_p(\zeta)\in L_p(M,\phi_0)$$

is injective and its image is dense with respect to norm topology for $1\leqq p<\infty$ *and* $\sigma(M, M_*)$*-topology for* $p=\infty$.

(2) $T_p(D_{\phi_0}^{\infty}\cap V_{\phi_0}^{1/(2p)})\subset L_p^+(M, \phi_0)$.

(3) *Let* $\zeta, \zeta'\in D_{\phi_0}^{\infty}$, $p^{-1}+(p')^{-1}=1$. *Then,*

$$(\zeta, \zeta')=\langle T_p(\zeta), T_{p'}(\zeta')\rangle_{\phi_0}.$$

(4) *The following diagram is commutative.*

$$\begin{array}{ccc} D_{\phi_0}^{\infty} & \xrightarrow{J_p^{\phi_0}} & D_{\phi_0}^{\infty} \\ \downarrow T_p & & \downarrow T_p \\ L_p(M, \phi_0) & \xrightarrow{*} & L_p(M, \phi_0), \end{array}$$

where $J_p^{\phi_0}=J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}$ *and* $*$ *is the adjoint operation on* $L_p(M, \phi_0)$.

Remark. The case $p=1$ in the above Theorem is already known to J. Phillips [24].

Group action on L_p-spaces is defined in our setting in Section 13 and is shown to be continuous in Lemma 13.4.

§ 3. A Multilinear Form

In this section, we shall introduce some notations which will be used throughout this paper, and also show some associated results. Throughout this section, we fix a faithful normal semifinite weight ϕ_0.

Lemma 3.1. *Let* $\phi_j\in M_*^+$, $j=1, \cdots, n$, $x_k\in M$, $k=0, \cdots, n$. *Then for* $y\in N_{\phi_0}$ *with* $\|y\|\leqq 1$,

$$\zeta(z)=x_0\Delta_{\phi_1,\phi_0}^{z_1}x_1\cdots x_{n-1}\Delta_{\phi_n,\phi_0}^{z_n}x_n\eta_{\phi_0}(y) \tag{3.1}$$

is well-defined for $z=(z_1, \cdots, z_n)\in I_{1/2}^{(n)}$ *(in the sense that* $\eta_{\phi_0}(y)$ *is in the domain of the product of operators in front), holomorphic in the interior of* $I_{1/2}^{(n)}$ *and strongly continuous on* $I_{1/2}^{(n)}$ *with the bound*

$$\|\zeta(z)\|\leqq(\prod_{k=0}^{n}\|x_k\|)(\prod_{k=1}^{n}\|\phi_k\|^{\mathrm{Re}\,z_k})\phi_0(y^*y)^{\mathrm{Re}\,z_0/2}\|y\|^{1-\mathrm{Re}\,z_0} \tag{3.2}$$

where $\|\phi_k\|=\phi_k(1)$, $z_0\equiv(1/2)-\sum_{k=1}^{n}z_k$ *and* $I_a^{(n)}$ *is defined for* $a\geqq 0$ *as*

(3. 3) $$I_a^{(n)}=\{z\in \boldsymbol{C}^n : \operatorname{Re} z_k\geqq 0,\ \sum_{k=1}^{n}\operatorname{Re} z_k\leqq a\}.$$

Proof. Follows from Lemma A. 1 and the corresponding proof of [8] by replacing x_n there by $x_n y$ and replacing $y\xi_0$ by $\eta_{\phi_0}(y)$. Q.E.D.

Just as Lemma A follows from Lemma A. 1 in [8], the above lemma implies the following.

Lemma 3. 2. *Let $\phi_j\in M_*^+$, $j=1, \cdots, n$, $x_k\in M$, $k=0, \cdots, n$, complex numbers $z=(z_1, \cdots, z_n)\in I_1^{(n)}$, and $y_l\in N_{\phi_0}$ with $\|y_l\|\leqq 1$, $l=1, 2$, the expression*

(3. 4) $$F_{y_1, y_2}(z)=(\Delta_{\phi_k,\phi_0}^{z'_k}x_k\Delta_{\phi_{k+1},\phi_0}^{z_{k+1}}\cdots\Delta_{\phi_n,\phi_0}^{z_n}x_n\eta_{\phi_0}(y_1),$$
$$\Delta_{\phi_k,\phi_0}^{\bar{z}''_k}x_{k-1}^*\Delta_{\phi_{k-1},\phi_0}^{\bar{z}_{k-1}}\cdots\Delta_{\phi_1,\phi_0}^{\bar{z}_1}x_0^*\eta_{\phi_0}(y_2))$$

is well defined and independent of the splitting $z_k=z'_k+z''_k$ if

(3. 5. a) $$\operatorname{Re} z_1+\cdots+\operatorname{Re} z_{k-1}+\operatorname{Re} z''_k\leqq 1/2,\quad \operatorname{Re} z''_k\geqq 0,$$

(3. 5. b) $$\operatorname{Re} z_n+\cdots+\operatorname{Re} z_{k+1}+\operatorname{Re} z'_k\leqq 1/2,\quad \operatorname{Re} z'_k\geqq 0.$$

It defines a function of $z=(z_1, \cdots, z_n)$ which is

(1) *holomorphic in the interior of $I_1^{(n)}$*

(2) *continuous on $I_1^{(n)}$*

(3) *bounded on $I_1^{(n)}$ by*

(3. 6) $$|F_{y_1,y_2}(z)|$$
$$\leqq(\prod_{k=0}^{n}\|x_k\|)(\|\eta_{\phi_0}(y_1)\|\,\|\eta_{\phi_0}(y_2)\|)^{\operatorname{Re} z_0}(\prod_{k=1}^{n}\phi_k(1)^{\operatorname{Re} z_k})$$

where $z_0=1-\sum_{k=1}^{n}z_k$.

Notation 3. 3. We denote the function $F_{y_1,y_2}(z)$ defined by (3. 4) simply as

(3. 7) $$F_{y_1,y_2}(z)=(x_0\Delta_{\phi_1,\phi_0}^{z_1}\cdots\Delta_{\phi_n,\phi_0}^{z_n}x_n\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2)).$$

Next Lemma is used to define the sesquilinear form on $L_p(M, \phi_0)\times L_{p'}(M, \phi_0)$, $p^{-1}+(p')^{-1}=1$.

Lemma 3.4. *Let* $\phi_k \in M_*^+$, $k=1, \cdots, n$, $x_k \in M$, $k=0, \cdots, n$ *and* $\mathrm{Re}\, z_k \geqq 0$, $k=1, \cdots, n$ *such that* $\sum_{k=1}^{n} z_k = 1$. *Then the limit*

$$\lim_{\substack{y_1 \to 1 \\ y_2 \to 2}} F_{y_1, y_2}(z_1, \cdots, z_n) \equiv [x_0 \phi_1^{z_1} x_1 \phi_2^{z_2} \cdots x_n] \tag{3.8}$$

exists as $y_1, y_2 \to 1$ $*$*-strongly with the restriction* $y_1, y_2 \in N_{\phi_0}$, $\|y_1\| \leqq 1$ *and* $\|y_2\| \leqq 1$, *where* F_{y_1, y_2} *is defined by* (3.4).

Proof. Let $z \in I_1^{(n)}$ and $y_k \in N_{\phi_0}$, $\|y_k\| \leqq 1$, $k=1, \cdots, 4$. By Lemma 3.2, $F(z) \equiv F_{y_1, y_2}(z) - F_{y_3, y_4}(z)$ is holomorphic in the interior of $I_1^{(n)}$ and continuous on $I_1^{(n)}$. The tube domain $I_1^{(n)}$ has the following distinguished boundaries corresponding to extremal points of its base:

$$\partial_0 I_1^{(n)} = \{z\colon \mathrm{Re}\, z_k = 0,\ k=1, \cdots, n\}, \tag{3.9}$$

$$\partial_k I_1^{(n)} = \{z\colon \mathrm{Re}\, z_l = 0\ \ (l \neq k),\ \mathrm{Re}\, z_k = 1\},\ k=1, \cdots, n\,. \tag{3.10}$$

The estimates of $F(z)$ on the boundaries are

$$\begin{aligned} &|F_{y_1, y}(z) - F_{y_3, y}(z)| \\ &\qquad \leqq \Big(\prod_{k=0}^{n} \|x_k\|\Big) \big(\|\eta_{\phi_0}(y_1)\|\, \|\eta_{\phi_0}(y)\| + \|\eta_{\phi_0}(y_3)\|\, \|\eta_{\phi_0}(y)\|\big) \end{aligned} \tag{3.11}$$

where $z \in \partial_0 I_1^{(n)}$, and

$$\begin{aligned} &F_{y_1, y}(z) - F_{y_3, y}(z) \\ &\qquad = (J_{\phi_0} \sigma_{s_k'}^{\phi_0}(y_1^* - y_3^*) w_k'(t) \xi(\phi_k),\ J_{\phi_0} \sigma_{-s_k}^{\phi_0}(y^*) w_k(t) \xi(\phi_k)), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} w_k'(t) = {} & (D\phi_k\colon D\phi_0)_{t_k} \sigma_{t_k}^{\phi_0}(x_k (D\phi_{k+1}\colon D\phi_0)_{t_{k+1}} \cdots (D\phi_{n-1}\colon D\phi_0)_{t_{n-1}} \\ & \times \sigma_{t_{n-1}}^{\phi_0}(x_{n-1} (D\phi_n\colon D\phi_0)_{t_n} \sigma_{t_n}^{\phi_0}(x_n)) \cdots), \end{aligned} \tag{3.13}$$

$$s_k' = t_k + \cdots + t_n\,, \tag{3.14}$$

$$\begin{aligned} w_k(t) = {} & x_{k-1}^* (D\phi_{k-1}\colon D\phi_0)_{-t_{k-1}} \sigma_{-t_{k-1}}^{\phi_0}(x_{k-2}^* (D\phi_{k-2}\colon D\phi_0)_{-t_{k-2}} \cdots \\ & \times (D\phi_2\colon D\phi_0)_{-t_2} \sigma_{-t_2}^{\phi_0}(x_1^* (D\phi_1\colon D\phi_0)_{-t_1} \sigma_{-t_1}^{\phi_0}(x_0^*)) \cdots), \end{aligned} \tag{3.15}$$

$$s_k = t_1 + \cdots + t_{k-1}\,, \tag{3.16}$$

where $z \in \partial_k I_1^{(n)}$ and we have used the formulas such as $\Delta_{\phi_k, \phi_0}^{it_k} a = (D\phi_k\colon D\phi_0)_{t_k} \sigma_{t_k}^{\phi_0}(a) \Delta_{\phi_0}^{it_k}$, $\Delta_{\phi_0}^{is} \eta_{\phi_0}(y) = \eta_{\phi_0}(\sigma_s^{\phi_0}(y))$, and $\Delta_{\phi_k, \phi_0}^{1/2} w \eta_{\phi_0}(y) =$

$J_{\phi_0}y^*w^*\xi(\phi_k)$. Therefore the values of $\exp(-\sum_{k=1}^{n} z_k^2)F_{y_1,y}(z)$ at distinguished boundaries of $I_1^{(n)}$ are Cauchy nets when $y_1 \to 1$ if $y_1, y \in N_{\phi_0}$, $\|y_1\| \leqq 1$, and $\|y\| \leqq 1$ due to (3.12) and the uniform boundedness of w_k and w_k'. By making use of the generalized three line theorem, we see that $F_{y_1,y}(z_1, \cdots, z_n)$ is a Cauchy net in y_1 as $y_1 \to 1$. Similarly $F_{y,y_2}(z_1, \cdots, z_n)$ is a Cauchy net as $y_2 \to 1$ if $y, y_2 \in N_{\phi_0}$, $\|y\| \leqq 1$, $\|y_2\| \leqq 1$. Q.E.D.

Throughout this paper, we use the notation $M_0^{(1)} \equiv \{y \in M_0 \colon \|y\| \leqq 1\}$;

We shall see, the functional $(x_0, x_1, \cdots, x_n, \phi_1, \cdots, \phi_n) \mapsto [x_0\phi_1^{z_1}\cdots x_n]$ plays an important role throughout this paper for $n=2$ and in Sections 9 and 11 for general n. The conclusions in Sections 9 and 11 are then used to derive the properties of the multilinear form $(T_1, \cdots, T_n) \mapsto [T_1 \cdots T_n]_{\phi_0}$ in Theorem 6, which coincides with a value of this functional under a proper identification of T_k in terms of x's and ϕ's.

§ 4. Polar Decomposition

In this section, we shall give a characterization of elements in $L_p(M, \phi_0)$, $1 \leqq p \leqq \infty$ by making use of the relative modular operator.

Lemma 4.1. *Let T be a (ϕ_0, p)-measurable operator $(1 \leqq p < \infty)$, then there exists a unique normal semifinite weight ϕ and a partial isometry $u \in M$ satisfying $u^*u = s(\phi)$, and $T = u\Delta_{\phi,\phi_0}^{1/p}$.*

Proof. Starting with Definition 2.1 of measurability and following the proof of Lemmas 3.2 and 3.3 of [8], where $y \in M_0'$ should be replaced by $J_{\phi_0}yJ_{\phi_0}$ with $y \in M_0$, we obtain

$$(4.1) \qquad |T|^{ipt}J_{\phi_0}\sigma_{-t}^{\phi_0}(y)J_{\phi_0} = J_{\phi_0}yJ_{\phi_0}|T|^{ipt}$$

for any $y \in M$ and $t \in \mathbf{R}$ as well as $s(|T|) \in M$ and $u \in M$.

By the proof of Lemma 3.4 in [8], $|T|$ is of the form $\Delta_{\phi,\phi_0}^{1/p}$ for some normal semifinite weight ϕ on M. The uniqueness of this decomposition follows from the uniqueness of polar decomposition as closed operators and the one-to-one correspondence between the normal semifinite weight and $\sigma_t^{\phi_0}$ one cocycles with a specified support property (see

Appendix B of [8]). Q.E.D.

Lemma 4.2. *Let* $T=u\Delta^{1/p}_{\phi,\phi_0}$ *be the polar decomposition,* $1\leqq p<\infty$. *Then,* $\|T\|_{p,\phi_0}=\phi(1)^{1/p}$.

Proof. Assume $\phi(1)<\infty$. Then there exists $\xi(\phi)\in P^{\natural}_{\phi_0}$ such that $\phi=\omega_{\xi(\phi)}$. It follows that

$$(4.2)\qquad \sup_{x\in M_0^{(1)}}\|\Delta^{1/2}_{\phi,\phi_0}\eta_{\phi_0}(x)\|=\sup_{x\in M_0^{(1)}}\|x^*\xi(\phi)\|\leqq\|\xi(\phi)\|.$$

Due to the existence of a $*$-strong net $\{x_\alpha\}\subset M_0^{(1)}$ which converges to 1, inequality (4.2) is actually attained. Hence

$$(4.3)\qquad \|T\|_{p,\phi_0}=\{\sup_{x\in M_0^{(1)}}\|\Delta^{1/2}_{\phi,\phi_0}\eta_{\phi_0}(x)\|\}^{2/p}=\|\xi(\phi)\|^{2/p}=\phi(1)^{1/p}.$$

Next we assume $\phi(1)=\infty$. Because ϕ is normal and semifinite, there exists an increasing net $\{\phi_\alpha\}\subset M_*^+$ with supremum ϕ. For any $N>0$, there exists α_N such that $\phi_{\alpha_N}(1)>N^p$. Due to Lemma C.3 of [8], $D(\Delta^{1/2}_{\phi,\phi_0})\subset D(\Delta^{1/2}_{\phi_\alpha,\phi_0})$ and $\|\Delta^{1/2}_{\phi_\alpha,\phi_0}\xi\|\leqq\|\Delta^{1/2}_{\phi,\phi_0}\xi\|$ for $\xi\in D(\Delta^{1/2}_{\phi,\phi_0})$. Hence

$$(4.4)\qquad \|T\|_{p,\phi_0}\geqq\{\sup_{x\in M_0^{(1)}}\|\Delta^{1/2}_{\phi_{\alpha_N},\phi_0}\eta_{\phi_0}(x)\|\}^{2/p}.$$

By the preceding argument, the right hand side of (4.4) is $\phi_{\alpha_N}(1)^{1/p}>N$. It follows $\|T\|_{p,\phi_0}=\infty$. Q.E.D.

Remark. Conversely, if $T=u\Delta^{1/p}_{\phi,\phi_0}$ with a partial isometry $u\in M$ and $\phi\in M_*^+$ such that $u^*u=s(\phi)$, T is (ϕ_0,p)-measurable and $\|T\|_{p,\phi_0}=\phi(1)^{1/p}$. It follows that if $T=u\Delta^{1/p}_{\phi,\phi_0}\in L_p(M,\phi_0)$, $1\leqq p<\infty$, then $T^*=\Delta^{1/p}_{\phi,\phi_0}u^*=u^*\Delta^{1/p}_{\phi_u,\phi_0}\in L_p(M,\phi_0)$ $(\phi_u(x)=\phi(u^*xu),x\in M)$. It follows that the adjoint operation is a bijective isometry of $L_p(M,\phi_0)$.

The case $p=\infty$ excluded from the above is given by the following lemma. It is an easy consequence of the definition.

Lemma 4.3. *If* T *is* (ϕ_0,∞)*-measurable, then* T *is a closed operator affiliated with* M. $L_\infty(M,\phi_0)=M$.

§ 5. Linear Structure of $L_p(M, \phi_0)$ for $2 \leqq p < \infty$

In this section, we examine the linear structure of $L_p(M, \phi_0)$, $2 \leqq p < \infty$.

Lemma 5.1. *$L_p(M, \phi_0)$, $2 \leqq p < \infty$, is a linear space under strong sum.*

Proof. Let $T_k \in L_p(M, \phi_0)$, $k=1,2$. According to the polar decomposition, T_k is of the form $T_k = u_k \Delta_{\phi_k, \phi_0}^{1/p}$, $u_k \in M$, $\phi_k \in M_*^+$, $k=1,2$. By the property of relative modular operators, $\eta_{\phi_0}(N_{\phi_0})$ is a subset of $D(T)$ for any $T \in L_p(M, \phi_0)$ $(p \geqq 2)$ and actually it is a core of T due to the fact that $\eta_{\phi_0}(N_{\phi_0})$ is a core of $\Delta_{\phi, \phi_0}^{1/2}$ for any $\phi \in M_*^+$. It follows that $T_1 + T_2$ *and* $T_1^* + T_2^*$ are densely defined operators. By $(T_1+T_2)^* \supset T_1^* + T_2^*$, $T_1 + T_2$ is preclosed. For any $\xi \in D(T_1+T_2)$ and $y \in M_0$,

$$
\begin{aligned}
(5.1) \qquad J_{\phi_0} y J_{\phi_0} (T_1+T_2)\xi &= J_{\phi_0} y J_{\phi_0} T_1 \xi + J_{\phi_0} y J_{\phi_0} T_2 \xi \\
&= T_1 J_{\phi_0} \sigma_{-i/p}^{\phi_0}(y) J_{\phi_0} \xi + T_2 J_{\phi_0} \sigma_{-i/p}^{\phi_0}(y) J_{\phi_0} \xi \\
&= (T_1+T_2) J_{\phi_0} \sigma_{-i/p}^{\phi_0}(y) J_{\phi_0} \xi \,.
\end{aligned}
$$

Passing to the closure $\overline{T_1+T_2}$ of T_1+T_2 in the formula (5.1), we have the (ϕ_0, p)-measurability condition of $\overline{T_1+T_2}$, the closure of T_1+T_2. Next we show $\|\overline{T_1+T_2}\|_{p,\phi_0} < \infty$. We put $T = \overline{T_1+T_2} \supset T_1+T_2$. Let $T = u\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition and let $S = u^* \Delta_{\psi,\phi_0}^{1/q}$, where $1/p + 1/q = 1/2$, and $\psi \in M_*^+$, $\|\psi\| \leqq 1$. Let $y \in M_0^{(1)}$, then by Lemma 3.1, $S\eta_{\phi_0}(y) \in D(T_1) \cap D(T_2) \subset D(T)$ and hence,

$$
(5.2) \qquad \|TS\eta_{\phi_0}(y)\| \leqq (\|\phi_1\|^{1/p} + \|\phi_2\|^{1/p}).
$$

Assume $\phi(1) = +\infty$. Because ϕ is normal and semifinite, there exists an increasing net $\{\phi_\alpha\} \subset M_*^+$ with supremum $\phi_u = \phi(u^* \cdot u)$. For an arbitrary $N > 0$, there exists ϕ_{α_N} such that $\phi_{\alpha_N}(1) \geqq N^p$. Then by choosing $\psi = \phi_{\alpha_N} / \|\phi_{\alpha_N}\|$,

$$
\begin{aligned}
(5.3) \qquad \|TS\eta_{\phi_0}(y)\| &= \|\Delta_{\phi_u,\phi_0}^{1/p} \Delta_{\psi,\phi_0}^{1/q} \eta_{\phi_0}(y)\| \\
&\geqq \|\Delta_{\phi_{\alpha_N},\phi_0}^{1/p} \Delta_{\psi,\phi_0}^{1/q} \eta_{\phi_0}(y)\| \\
&= (1/\|\phi_{\alpha_N}\|)^{1/q} \phi_{\alpha_N}(yy^*)^{1/2} \,.
\end{aligned}
$$

By taking the $*$-strong limit $y^*\to 1$ in (5.3), we get

$$\|TS\eta_{\phi_0}(y)\|\geqq\phi_{\alpha_N}(1)^{1/p}\geqq N\,. \tag{5.4}$$

This contradicts (5.2). Therefore $\phi\in M_*^+$ and $T\in L_p(M,\phi_0)$. Associativity of the sum follows from the fact that $\eta_{\phi_0}(N_{\phi_0})$ is a core for any $T\in L_p(M,\phi_0)$. Q.E.D.

Lemma 5.2. *Let $2<p<\infty$ and $p^{-1}+q^{-1}=1/2$.*

(1) *Let $T=u\Delta_{\phi,\phi_0}^{1/p}\in L_p(M,\phi_0)$, $S=v\Delta_{\psi,\phi_0}^{1/q}\in L_q(M,\phi_0)$. Then ST is preclosed and the closure $\overline{ST}\in L_2(M,\phi_0)$. Furthermore,*

$$\|\overline{ST}\|_{2,\phi_0}\leqq\|S\|_{q,\phi_0}\|T\|_{p,\phi_0}\,. \tag{5.5}$$

(2) *Let $T\in L_p(M,\phi_0)$. Then*

$$\|T\eta_{\phi_0}(y)\|\leqq\|T\|_{p,\phi_0}\|y\|^{2/p}\|\eta_{\phi_0}(y)\|^{1-(2/p)}\,, \tag{5.6}$$

where $y\in N_{\phi_0}$.

Proof. (1) By Lemma 3.1, $\eta_{\phi_0}(N_{\phi_0})\subset D(ST)$ and $\eta_{\phi_0}(N_{\phi_0})\subset D(T^*S^*)$. Hence ST is preclosed. For $\xi\in\eta_{\phi_0}(N_{\phi_0})$ and $y\in M_0$,

$$\begin{aligned} J_{\phi_0}yJ_{\phi_0}ST\xi &= SJ_{\phi_0}\sigma_{-i/q}^{\phi_0}(y)J_{\phi_0}T\xi \\ &= STJ_{\phi_0}\sigma_{-i/2}^{\phi_0}(y)J_{\phi_0}\xi\,. \end{aligned} \tag{5.7}$$

By taking the closure of ST in (5.7), we get the $(\phi_0,2)$-measurability condition of $\overline{ST}$, the closure of ST. By Lemma 3.1, $\|(\overline{ST})\eta_{\phi_0}(y)\|\leqq\|S\|_{q,\phi_0}\|T\|_{p,\phi_0}$, where $y\in M_0^{(1)}$. Hence $\overline{ST}\in L_2(M,\phi_0)$ and (5.5) holds by Definition 2.2.

(2) By Lemma 3.1. Q.E.D.

Lemma 5.3. *Let $T=u\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition of a (ϕ_0,p)-measurable operator, $2\leqq p<\infty$. Then*

$$\begin{aligned} \sup\{\|TS\eta_{\phi_0}(y)\|: y\in M_0^{(1)},\ S\in L_q(M,\phi_0),\ \|S\|_{q,\phi_0}\leqq 1\} \\ =\phi(1)^{1/p} \end{aligned} \tag{5.8}$$

where $p^{-1}+q^{-1}=1/2$ for $p>2$ and $q=\infty$ for $p=2$.

Proof. First assume $\phi(1)<\infty$. Then by Lemma 3.1, inequality

$\leqq$ holds. By substituting $S=\Delta_{\phi,\phi_0}^{1/q}$ we obtain equality. If $\phi(1)=\infty$, the same proof as (5.4) implies that the left hand side is ∞. Q.E.D.

Lemma 5.4. *Let* $T_1, T_2\in L_p(M,\phi_0)$, $2\leqq p<\infty$. *Then*

$$\|T_1^*+T_2^*\|_{p,\phi_0}=\|T_1+T_2\|_{p,\phi_0}. \tag{5.9}$$

Proof. Let T and $\tilde{T}$ be the closures of T_1+T_2 and $T_1^*+T_2^*$ respectively. Then $T^*=(T_1+T_2)^*\supset\tilde{T}$. Because $\eta_{\phi_0}(M_0)$ is the core of $\Delta_{\phi,\phi_0}^{1/p}$ for any $\phi\in M_*^+$ and $2\leqq p<\infty$, the polar decomposition of Lemmas 4.1 and 4.2 and remark after Lemma 4.2 imply that it is in the domains of T_k^* and hence of T^*. Furthermore, Lemma 5.1 and Remark after Lemma 4.2 imply that $T^*\in L_p(M,\phi_0)$ and hence the same reasoning implies that $\eta_{\phi_0}(M_0)$ is the core of T^*.

$$\|T^*\|_{p,\phi_0}=\|T\|_{p,\phi_0},\quad T\in L_p(M,\phi_0),\quad 2\leqq p\leqq\infty \tag{5.10}$$

would then imply the assertion. Equality (5.10) is an easy consequence of Lemma 4.2 and its Remark. Q.E.D.

Lemma 5.5. $L_p(M,\phi_0)$, $2\leqq p<\infty$ *with norm* $\|\cdot\|_{p,\phi_0}$ *is a Banach space under strong sum.*

Proof. By Lemma 5.1 as well as (5.8), $L_p(M,\phi_0)$ is a normed space with $\|\cdot\|_{p,\phi_0}$. The rest is to show the completeness. Let $\{T_n\}\subset L_p(M,\phi_0)$ be a Cauchy sequence and $K=\sup_n\|T_n\|_{p,\phi_0}$.

Due to Lemma 5.4, $\|T_m^*-T_n^*\|_{p,\phi_0}=\|T_m-T_n\|_{p,\phi_0}$ and hence $\{T_n^*\}$ is also a Cauchy sequence. By Lemma 5.2 (2), $\{T_n\eta_{\phi_0}(y)\}$ and $\{T_n^*\eta_{\phi_0}(y)\}$ are Cauchy sequences in H_{ϕ_0}. Hence there exist densely defined operators T_∞ and T'_∞ defined on $\eta_{\phi_0}(N_{\phi_0})$ by,

$$T_\infty\eta_{\phi_0}(y)=\lim_{n\to\infty}T_n\eta_{\phi_0}(y),\quad T'_\infty\eta_{\phi_0}(y)=\lim_{n\to\infty}T_n^*\eta_{\phi_0}(y). \tag{5.11}$$

Since $T'_\infty\subset(T_\infty)^*$, T_∞ and T'_∞ are preclosed. Their closures, $\overline{T}_\infty$ and $\overline{T}'_\infty$, are easily seen to be (ϕ_0, p)-measurable. The rest is to show the following two facts.

(1) $\overline{T}_\infty\in L_p(M,\phi_0)$.

(2) $\|T_n-\overline{T}_\infty\|_{p,\phi_0}\to 0$ as $n\to\infty$.

First we assume $p=2$. Then by (5.11), for $y\in M_0^{(1)}$,

$$\|T_\infty\eta_{\phi_0}(y)\| = \lim_{n\to\infty}\|T_n\eta_{\phi_0}(y)\| \leqq K\,. \tag{5.12}$$

By Definition 2.2, $\|T_\infty\|_{2,\phi_0}\leqq K$. For $y\in N_{\phi_0}^{(1)}$,

$$\|(T_n-T_m)\eta_{\phi_0}(y)\| \leqq \|T_n-T_m\|_{2,\phi_0}\,. \tag{5.13}$$

For any $\varepsilon>0$, there exists $N(\varepsilon)>0$ such that $n, m\geqq N(\varepsilon)$ implies $\|T_n-T_m\|_{2,\phi_0}<\varepsilon$. By taking the limit $m\to\infty$ in (5.13), we get $\|(T_n-T_\infty)\eta_{\phi_0}(y)\|\leqq\varepsilon$ if $n\geqq N(\varepsilon)$ and $y\in N_{\phi_0}^{(1)}$. Hence $\|T_n-\overline{T}_\infty\|_{2,\phi_0}\leqq\varepsilon$ if $n\geqq N(\varepsilon)$ and we have (2).

Next we assume $2<p<\infty$. Let $S\in L_q(M,\phi_0)$ with $p^{-1}+q^{-1}=1/2$. Then, for $y_1, y_2\in N_{\phi_0}$, $(T_n\eta_{\phi_0}(y_1), S\eta_{\phi_0}(y_2))$ converges to $(T_\infty\eta_{\phi_0}(y_1), S\eta_{\phi_0}(y_2))$. Since $\overline{T_n^*S}$ is Cauchy in $L_2(M,\phi_0)$ due to $\|\overline{(T_n^*-T_m^*)S}\|_{2,\phi_0}\leqq\|T_n-T_m\|_{p,\phi_0}\|S\|_{q,\phi_0}$ by Lemma 5.2, $T_n^*S\eta_{\phi_0}(y_2)$ converges to some ζ as $n\to\infty$. Furthermore ζ has a bound $\|\zeta\|\leqq K\|S\|_{q,\phi_0}\|y_2\|$ due to (5.5). By the fact that $\eta_{\phi_0}(N_{\phi_0})$ is a core of T_∞, we have $S\eta_{\phi_0}(y_2)\in D((T_\infty)^*)$ and,

$$\|(T_\infty)^*S\eta_{\phi_0}(y_2)\| = \|\zeta\| \leqq K\|S\|_{q,\phi_0}\,, \tag{5.14}$$

if $y_2\in N_{\phi_0}^{(1)}$. Hence by Lemma 5.3, we have $T_\infty^*\in L_p(M,\phi_0)$ and we have (1). For $y\in N_{\phi_0}^{(1)}$,

$$\|(T_n^*-T_m^*)S\eta_{\phi_0}(y)\| \leqq \|T_n^*-T_m^*\|_{p,\phi_0}\|S\|_{q,\phi_0}\,. \tag{5.15}$$

For any $\varepsilon>0$, there exists $N(\varepsilon)>0$ such that $n, m\geqq N(\varepsilon)$ implies $\|T_n^*-T_m^*\|_{p,\phi_0}<\varepsilon$. By taking the limit $m\to\infty$ in (5.15), we get $\|(T_n^*-(T_\infty)^*)S\eta_{\phi_0}(y)\|\leqq\varepsilon\|S\|_{q,\phi_0}$ if $n\geqq N(\varepsilon)$ and $y\in N_{\phi_0}^{(1)}$. By Lemmas 5.3 and 5.4, $\|T_n-T_\infty\|_{p,\phi_0}=\|T_n^*-(T_\infty)^*\|_{p,\phi_0}<\varepsilon$ if $n\geqq N(\varepsilon)$ and we have (2).

Q.E.D.

§ 6. Product

Notation 6.1. Let $L_p^*(M,\phi_0)$ be the set of all formal expressions

$$T = x_0\Delta_{\phi_1,\phi_0}^{1/p_1}x_1\cdots\Delta_{\phi_n,\phi_0}^{1/p_n}x_n \tag{6.1}$$

with $x_k\in M$, $j=0,\cdots,n$, $\phi_k\in M_*^+$, $p_k>0$, $k=1,\cdots,n$ such that $\sum_{k=1}^n p_k^{-1}=1-p^{-1}$. The adjoint T^* of T in $L_p^*(M,\phi_0)$ is defined as

(6. 2) $$T^* = x_n^* \Delta_{\phi_n,\phi_0}^{1/p_n} \cdots x_1^* \Delta_{\phi_1,\phi_0}^{1/p_1} x_0^* .$$

The product $TS \in L_r^*(M, \phi_0)$ of $T \in L_p^*(M, \phi_0)$ and $S \in L_q^*(M, \phi_0)$ is defined if $r^{-1} = p^{-1} + q^{-1} - 1$ and $1 \leqq r, p, q \leqq \infty$ as the expression obtained by writing expressions for T and S together in that order and combine the last x in T and the top x in S according to the product operation in M. By Lemmas 4. 1 and 4. 2, $L_p(M, \phi_0) \subset L_p^*(M, \phi_0)$ with $p^{-1} + (p')^{-1} = 1$.

Lemma 6. 2. *$L_p^*(M, \phi_0)$ is embedded in the dual space $L_p(M, \phi_0)^*$ of $L_p(M, \phi_0)$, $2 \leqq p < \infty$, through the mapping*

(6. 3) $$\begin{array}{ccc} L_p^*(M, \phi_0) & \rightarrow & L_p(M, \phi_0)^* , \\ \Downarrow & & \Downarrow \\ T & \longmapsto & \langle \cdot, T^* \rangle_{\phi_0} \end{array}$$

where the form $\langle \cdot, \cdot \rangle_{\phi_0}$ is defined by

(6. 4) $$\langle S, T^* \rangle_{\phi_0} = \lim_{y \to 1} (TS\eta_{\phi_0}(y), \eta_{\phi_0}(y)), \quad S \in L_p(M, \phi_0).$$

The limit in (6. 4) is the special case of Lemma 3. 4, $y_1 = y_2$. In particular, $L_{p'}(M, \phi_0) \subset L_p(M, \phi_0)^*$, $p^{-1} + (p')^{-1} = 1$, through the form (6. 4) and

(6. 5) $$\|T\|_{p',\phi_0} = \|T\|_{p',\phi_0}^*, \quad T \in L_{p'}(M, \phi_0),$$

where $\|\cdot\|_{p',\phi_0}^*$ is the dual norm in $L_p(M, \phi_0)^*$ through the dual pairing (6. 4).

Proof. Let T be of the form (6. 1). By (3. 6) and Lemmas 4. 1, 4. 2 as well as Remark after Lemma 4. 2,

(6. 6) $$|(TS\eta_{\phi_0}(y), \eta_{\phi_0}(y))| \leqq (\prod_{k=0}^{n} \|x_k\|) (\prod_{k=1}^{n} \phi_k(1)^{1/p_k}) \|S\|_{p,\phi_0} .$$

By taking the limit in (6. 6) in the sense of Lemma 3. 4, we get the first half of the assertion i.e. $L_p^*(M, \phi_0)$ is embedded in $L_p(M, \phi_0)^*$. Next, assume $T = u\Delta_{\phi,\phi_0}^{1/p'} \neq 0$, $\phi \in M_*^+$, $u^*u = s(\phi)$ (u is a partial isometry in M). By Lemmas 4. 1, 4. 2 as well as (6. 6),

(6. 7) $$|\langle S, T^* \rangle_{\phi_0}| \leqq \|T\|_{p',\phi_0} \|S\|_{p,\phi_0} .$$

Hence we have $\|T\|_{p',\phi_0}^* \leqq \|T\|_{p',\phi_0}$. Inequality in (6. 7) is actually attained by taking $S^*=u\Delta_{\phi,\phi_0}^{1/p}$. So we have (6.5). Q.E.D.

Remark. Let $T\in L_p(M,\phi_0)$ $1<p<\infty$. Then there exists $T'\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$ such that $\langle T,T'\rangle_{\phi_0}=\|T\|_{p,\phi_0}\|T'\|_{p',\phi_0}$, where $\langle\cdot,\cdot\rangle_{\phi_0}$ is defined by (6.4).

Proof. Let $T=u\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition. Then $T'=u\Delta_{\phi,\phi_0}^{1/p'}$ satisfies the equality in question by Lemma 4.2. Q.E.D.

Lemma 6.3. *Let $T\in L_p^*(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, $2\leqq p'<\infty$. Then T is preclosed and its closure belongs to $L_{p'}(M,\phi_0)$.*

Proof. By Lemma 3.1, $\eta_{\phi_0}(N_{\phi_0})\subset D(T)\cap D(T^*)$ for $T\in L_p^*(M,\phi_0)$. Then for $y_1,y_2\in N_{\phi_0}$, $(T\eta_{\phi_0}(y_1),\eta_{\phi_0}(y_2))=(\eta_{\phi_0}(y_1),T^*\eta_{\phi_0}(y_2))$ and hence T is preclosed. It is also easily seen that T satisfies (2.1) for p' and hence the closure $\overline{T}$ is (ϕ_0,p')-measurable. Let T be of the form (6.1). If $2<p'<\infty$,

$$\|TS\eta_{\phi_0}(y)\|\leqq\Big(\prod_{k=0}^{n}\|x_k\|\Big)\Big(\prod_{k=1}^{n}\|\phi_k\|^{1/p_k}\|S\|_{q,\phi_0}\tag{6.8}$$

for $y\in N_{\phi_0}^{(1)}$ and $S\in L_q(M,\phi_0)$, $\sum_{k=1}^{n}p_k^{-1}+q^{-1}=1/2$, due to (3.2). Hence by Lemma 5.3, the closure of T belongs to $L_{p'}(M,\phi_0)$. Q.E.D.

Notation 6.4. For $1\leqq p\leqq 2$, T_1 and T_2 in $L_p^*(M,\phi_0)$ is said to be equivalent if $T_1\eta_{\phi_0}(y)=T_2\eta_{\phi_0}(y)$ for any $y\in N_{\phi_0}$ (by Lemma 3.1, $\eta_{\phi_0}(N_{\phi_0})\subset D(T)$ for any $T\in L_p^*(M,\phi_0)$). If $p=1$, this equivalence is the same as that of M. For $2\leqq p\leqq\infty$, T_1 and T_2 in $L_p^*(M,\phi_0)$ is said to be equivalent if $\langle T_1,S\rangle_{\phi_0}=\langle T_2,S\rangle_{\phi_0}$ for all $S\in L_p(M,\phi_0)$, where $\langle\cdot,\cdot\rangle_{\phi_0}$ for $2\leqq p<\infty$ is defined by (6.4) and $\langle T,x\rangle_{\phi_0}=\lim_{y\to 1}(x^*T\eta_{\phi_0}(y),\eta_{\phi_0}(y))$ for $p=\infty$ (in the sense of Lemma 3.4). Note that for $1\leqq p\leqq 2$, T_1 and $T_2\in L_p^*(M,\phi_0)$ are equivalent iff $\langle T_1,S\rangle_{\phi_0}=\langle T_2,S\rangle_{\phi_0}$ for all $S\in L_p(M,\phi_0)$.

Lemma 6.5. *If $T_k\in L_p^*(M,\phi_0)$ $k=1,\cdots,n$ and $\sum_{k=1}^{n}T_k=0$ in $L_{p'}(M,\phi_0)$ for $1\leqq p\leqq 2$ and $p^{-1}+(p')^{-1}=1$ (Lemma 6.3), then $\sum_{k=1}^{n}T_k^*=0$ in the same sense and $\sum_{k=1}^{n}T_kS=\sum_{k=1}^{n}ST_k=0$ in $L_{r'}(M,\phi_0)$ for $1\leqq r\leqq 2$*

and $r^{-1}+(r')^{-1}=1$ *or in* $L_r(M,\phi_0)^*$ *for* $2\leqq r\leqq\infty$ (*Lemma* 6.2 *and Notation* 6.4) *where* $S\in L_q^*(M,\phi_0)$, $r^{-1}=p^{-1}+q^{-1}-1$ *and* $1\leqq q,\ r\leqq\infty$.

Proof. From the assumption,

$$(6.9)\qquad (\sum_{k=1}^{n} T_k^*\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=(\eta_{\phi_0}(y_1),\ \sum_{k=1}^{n} T_k\eta_{\phi_0}(y_2))=0$$

for any $y_1, y_2\in N_{\phi_0}$. Hence $\sum_{k=1}^{n} T_k^*=0$ in $L_{p'}(M,\phi_0)$ due to the fact that $\eta_{\phi_0}(N_{\phi_0})$ is a core for any $T\in L_{p'}(M,\phi_0)$. If $1\leqq r\leqq 2$,

$$(6.10)\qquad \sum_{k=1}^{n}(T_kS\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=\overline{(\sum_{k=1}^{n} T_k^*\eta_{\phi_0}(y_2),\ S\eta_{\phi_0}(y_1))}=0$$

$$(6.11)\qquad \sum_{k=1}^{n}(ST_k\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=(S\sum_{k=1}^{n} T_k\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=0$$

for any $y_1, y_2\in N_{\phi_0}$. Hence $\sum_{k=1}^{n} T_kS=\sum_{k=1}^{n} ST_k=0$ in $L_{r'}(M,\phi_0)$. If $2\leqq r\leqq\infty$, let

$$(6.12)\qquad S=x_0\Delta_{\phi_1,\phi_0}^{1/q_1}\cdots\Delta_{\phi_m,\phi_0}^{1/q_m}x_m$$

with $\sum_{k=1}^{m} q_k^{-1}+q^{-1}=1$, $x_k\in M$, $k=1,\cdots,m$ and $R=v\Delta_{\psi,\phi_0}^{1/r}\in L_r(M,\phi_0)$. Then,

$$(6.13)\qquad \sum_{k=1}^{n}(\Delta_{\psi,\phi_0}^{z_0}v^*x_0\Delta_{\phi_1,\phi_0}^{z_1}\cdots\Delta_{\phi_m,\phi_0}^{z_m}x_mT_k\eta_{\phi_0}(y_1),\ n_{\phi_0}(y_2))$$

$$=(\sum_{k=1}^{n} T_k\eta_{\phi_0}(y_1),\ x_m^*\Delta_{\phi_m,\phi_0}^{\bar z_m}\cdots\Delta_{\phi_1,\phi_0}^{\bar z_1}x_0^*v\Delta_{\psi,\phi_0}^{\bar z_0}\eta_{\phi_0}(y_2))=0\ ,$$

$$(6.14)\qquad \sum_{k=1}^{n}(\Delta_{\psi,\phi_0}^{z_0}v^*T_kx_0\Delta_{\phi_1,\phi_0}^{z_1}\cdots\Delta_{\phi_m,\phi_0}^{z_m}x_m\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))$$

$$=(\sum_{k=1}^{n} T_kx_0\Delta_{\phi_1,\phi_0}^{z_1}\cdots\Delta_{\phi_m,\phi_0}^{z_m}x_m\eta_{\phi_0}(y_1),\ v\Delta_{\psi,\phi_0}^{\bar z_0}\eta_{\phi_0}(y_2))=0$$

for $z=(z_0,\cdots,z_m)\in i\mathbf{R}^{m+1}$ and $y_1, y_2\in N_{\phi_0}$ because $\sum_{k=1}^{n} T_k=0$ as an operator. Hence by analytic continuation,

$$(6.15)\qquad \sum_{k=1}^{n}(R^*ST_k\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=0\ ,$$

$$(6.16)\qquad \sum_{k=1}^{n}(R^*T_kS\eta_{\phi_0}(y_1),\ \eta_{\phi_0}(y_2))=0\ .$$

By taking the limit $y_1\to 1$ and $y_2\to 1$, we have

$$(6.17)\qquad \sum_{k=1}^{n}\langle ST_k,\ R\rangle_{\phi_0}=0\ ,$$

(6.18) $$\sum_{k=1}^{n} \langle T_k S, R\rangle_{\phi_0}=0 .$$ Q.E.D.

Lemma 6.6. *Let* $2\leqq p<\infty$, $u_k\in M$, $\phi_k\in M_*^+$ *and* $T_k=u_k\Delta_{\phi_k,\phi_0}^{1/p}\in L_p(M,\phi_0)$, $k=1,\cdots,n$. *Then*

(6.19) $$\{\|\sum_{k=1}^{n} T_k\|_{p,\phi_0}\}^2=\|\sum_{k,l=1}^{n} T_k^*T_l\|_{p/2,\phi_0}^* .$$

Proof. We put $T=\sum_{k=1}^{n} T_k$ in $L_p(M,\phi_0)$. By Lemma 6.5, $T^*=\sum_{k=1}^{n} T_k^*$ and $\sum_{k,l=1}^{n} T_k^*T_l=\sum_{k=1}^{n} T^*T_k=T^*T$ in $L_q(M,\phi_0)^*$, $(2/p)+q^{-1}=1$. Let $T=u\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition given by Lemmas 4.1 and 4.2. Then $\sum_{k,l=1}^{n} T_k^*T_l=\Delta_{\phi,\phi_0}^{2/p}$ and hence the assertion follows by Lemma 6.2.

Q.E.D.

Lemma 6.7. *Let* $T\in L_p^*(M,\phi_0)$ *be of the form* (6.1). Then

(6.20) $$\|T\|_{p',\phi_0}^*\leqq(\prod_{k=0}^{n}\|x_k\|)(\prod_{k=1}^{n}\phi_k(1)^{1/p_k}).$$

Proof. By the proof of Lemma 6.2 and inequality (3.6).

Q.E.D.

§7. Special Cases $p=1,2$

By Lemma 4.3, $L_\infty(M,\phi_0)$ is identical with M. In this section, we shall give canonical isomorphisms of $L_1(M,\phi_0)$ to M_* and of $L_2(M,\phi_0)$ to H_{ϕ_0}.

Lemma 7.1. *Let* $T=u\Delta_{\phi,\phi_0}\in L_1(M,\phi_0)$. *Then,*

(7.1) $$\psi_T(x)=[xu\phi],\quad x\in M$$

in the sense of Lemma 3.4 *defines the bijection between* $L_1(M,\phi_0)$ *and* M_* *satisfying* $\psi_T(x)=\phi(xu)$ *and*

(7.2) $$\|\psi_T\|=\|T\|_{1,\phi_0} .$$

Through the mapping $T\mapsto\psi_T$, $L_1(M,\phi_0)$ *is isomorphic to* M_* *as a Banach space.*

Proof. Any $T \in L_1(M, \phi_0)$ is of the form $T = u\Delta_{\phi,\phi_0}$ by Lemma 4.1. (7.1) is computed as

$$(7.3) \qquad \psi_T(x) = \lim_{y \to 1} (xu\Delta_{\phi,\phi_0}\eta_{\phi_0}(y), \eta_{\phi_0}(y))$$
$$= \lim_{y \to 1} (\Delta^{1/2}_{\phi,\phi_0}\eta_{\phi_0}(y), \Delta^{1/2}_{\phi,\phi_0}u^*x^*\eta_{\phi_0}(y))$$
$$= \lim_{y \to 1} (y^*xu\xi(\phi), y^*\xi(\phi))$$
$$= \phi(xu)$$

where $\xi(\phi) \in \mathcal{P}^{\natural}_{\phi_0}$ is the representative vector of $\phi \in M^+_*$ ($\phi \in M^+_*$ is due to Lemma 4.2). Hence $\psi_T \in M_*$ and $\|\psi_T\| = \|\phi\| = \|T\|_{1,\phi_0}$. Surjectivity of this mapping follows from the usual polar decomposition of $\psi \in M_*$ (Theorem 1.14.4 of [25]). Q.E.D.

Lemma 7.2. *Let $T \in L_2(M, \phi_0)$. Then the limit*

$$(7.4) \qquad \zeta_T = \lim_{y \to 1} T\eta_{\phi_0}(y)$$

exists and the mapping $T \mapsto \zeta_T$ is an isomorphism of, $L_2(M, \phi_0)$ onto H_{ϕ_0}.

Proof. Let $T \in L_2(M, \phi_0)$ and $T = u\Delta^{1/2}_{\phi,\phi_0}$ be the polar decomposition given by Lemma 4.1. Then (7.4) is computed as

$$(7.5) \qquad \zeta_T = \lim_{y \to 1} u\Delta^{1/2}_{\phi,\phi_0}\eta_{\phi_0}(y)$$
$$= \lim_{y \to 1} uJ_{\phi_0}y^*\xi(\phi)$$
$$= uJ_{\phi_0}\xi(\phi)$$
$$= u\xi(\phi)$$

where $\xi(\phi) \in \mathcal{P}^{\natural}_{\phi_0}$ is the representative vector of $\phi \in M^+_*$ ($\phi \in M^+_*$ is due to Lemma 4.2), and we have used the J_{ϕ_0}-invariance of $\xi(\phi) \in \mathcal{P}^{\natural}_{\phi_0}$ (see Theorem C.1 of [8]). The mapping $T \mapsto \zeta_T$ is linear by (7.4). By (7.5), $\zeta_T = u\xi(\phi) \in H_{\phi_0}$ and $\|\zeta_T\| = \|\xi(\phi)\| = \phi(1)^{1/2} = \|T\|_{2,\phi_0}$. Hence $T \mapsto \zeta_T$ is an isomorphism of Banach spaces. Since its image contains $\mathcal{P}^{\natural}_{\phi_0}$ which is total in H_{ϕ_0} (see [15] or Appendix A), we have the assertion. Q.E.D.

§ 8. Dual Pair

In this section, we first show the uniform convexity of the norm of $L_p(M, \phi_0)$. By making use of this fact, we show the polar decomposition of the element of $L_p(M, \phi_0)^*$ for $2 \leqq p < \infty$.

Lemma 8.1. *For $2 \leqq p < \infty$ and $T_1, T_2 \in L_p(M, \phi_0)$, the following Clarkson's inequality holds,*

$$(\|T_1+T_2\|_{p,\phi_0})^p + (\|T_1-T_2\|_{p,\phi_0})^p \leqq 2^{p-1}\{(\|T_1\|_{p,\phi_0})^p + (\|T_2\|_{p,\phi_0})^p\}. \tag{8.1}$$

Proof. The same as Lemma 8.1 of [8] by replacing η by $\eta_{\phi_0}(y)$, $y \in M_0^{(1)}$ and taking the limit $y \to 1$, $*$-strongly. Q.E.D.

For the uniform convexity, the uniform smootheness as well as the uniform strong differentiability of the norm of a Banach space, see [18]. We follow the line of proof in Section 9, [8].

Lemma 8.2. *The norm $\|\cdot\|_{p,\phi_0}$, $2 \leqq p < \infty$ is uniformly strongly differentiable.*

Proof. Follow from the proof of Lemma 9.1, [8]. Q.E.D.

Lemma 8.3. *Let $1 < p \leqq 2$ and $p^{-1} + (p')^{-1} = 1$.*

(1) *For $T_1, T_2 \in L_p(M, \phi_0)$, $T_1 = T_2$ in $L_{p'}(M, \phi_0)^*$ iff $T_1 = T_2$ as operator.*

(2) *For any $\Phi \in L_{p'}(M, \phi_0)^*$, there exists $T \in L_p(M, \phi_0)$ such that $\Phi(S) = \langle S, T\rangle_{\phi_0}$ for $S \in L_{p'}(M, \phi_0)$, where the right hand side is the sequilinear pairing defined by (4.24). Through this pairing, $L_{p'}(M, \phi_0) \cong L_p(M, \phi_0)^*$.*

Proof. (1) For the polar decomposition $T_k = u_k \Delta_{\phi_k, \phi_0}^{1/p}$, $k = 1, 2$, $S_k = u_k \Delta_{\phi_k, \phi_0}^{1/p'} \in L_{p'}(M, \phi_0)$ satisfies

$$\langle T_k, S_k\rangle_{\phi_0} = \|T_k\|_{p,\phi_0} \|S_k\|_{p',\phi_0}. \tag{8.2}$$

(See Remark after Lemma 6.2.) By the assumtion, $T_1=T_2$ in $L_{p'}(M, \phi_0)^*$ i.e. $\langle T_1, S_k\rangle_{\phi_0}=\langle T_2, S_k\rangle_{\phi_0}$ and by Lemma 6.2, $\|T_1\|_{p,\phi_0}=\|T_2\|_{p,\phi_0}$. It follows that from (6.5) and the Clarkson's inequality (8.1) (see [8], the proof of Lemma 10.1) $S_1=S_2$ and hence $u_1=u_2$, $\phi_1=\phi_2$. Therefore we obtain $T_1=T_2$.

(2) By Lemma 6.2, $L_p(M, \phi_0)\subset L_{p'}(M, \phi_0)^*$. Let $\Phi\in L_{p'}(M, \phi_0)^*$, $\Phi\neq 0$. There exists an $S\in L_{p'}(M, \phi_0)$, $\|S\|_{p',\phi_0}=1$, such that $\Phi(S)=\|\Phi\|$. Let $S=u\Delta_{\phi,\phi_0}^{1/p'}$ be the polar decomposition and put $T=u\Delta_{\|\Phi\|^p\phi,\phi_0}^{1/p}$. Then $\langle S, T\rangle_{\phi_0}=\|T\|_{p,\phi_0}=\|\Phi\|$. By the uniform convexity of $L_{p'}(M, \phi_0)^*$ which is an immediate consequence of Lemma 8.2, $\Phi(\cdot)=\langle\cdot, T\rangle_{\phi_0}$ and we get the assertion. Q.E.D.

Remark. By Lemma 8.3, the dual space $L_p(M, \phi_0)^*$, $2\leqq p<\infty$, has a unique polar decomposition and it is realized by the set $L_{p'}(M, \phi_0)$, $p^{-1}+(p')^{-1}=1$. Through this correspondence, $L_{p'}(M, \phi_0)$ is a Banach space. Hence by Lemma 6.2, we have the following.

Lemma 8.4. *$L_p(M, \phi_0)$, $1\leqq p\leqq\infty$, is a Banach space.*

§ 9. Change of Reference Weight

In this section, we discuss the change of reference weight ϕ_0 and the associated isomorphism $\tau_p(\tilde{\phi}_0, \phi_0)$ from $L_p(M, \phi_0)$ to $L_p(M, \tilde{\phi}_0)$. Let ϕ_0 and $\tilde{\phi}_0$ be two faithful normal semifinite weights.

Lemma 9.1. *Let $T\in L_p(M, \phi_0)$, $1\leqq p<\infty$. Then $x^*T\eta_{\phi_0}(y)\in D(\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-(1/p)})$ and*

$$J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-(1/p)}x^*T\eta_{\phi_0}(y)=y^*J_p(\tilde{\phi}_0, \phi_0)(T)\eta_{\tilde{\phi}_0}(x) \tag{9.1}$$

where we assume $y\in N_{\phi_0}$, $\eta_{\phi_0}(y)\in D(T)$, $x\in N_{\tilde{\phi}_0}$, $\eta_{\tilde{\phi}_0}(x)\in D(J_p(\tilde{\phi}_0, \phi_0)(T))$, and we have defined in Theorem 7 as follows

$$J_p(\tilde{\phi}_0, \phi_0)(T)=\Delta_{\phi,\tilde{\phi}_0}^{1/p}u^*=u^*\Delta_{\phi_u,\tilde{\phi}_0}^{1/p}\in L_p(M, \tilde{\phi}_0) \tag{9.2}$$

for the polar decomposition $T=u\Delta_{\phi,\phi_0}^{1/p}$.

Proof. If we have the following formula,

$$(9.3)\qquad J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-it}x^*u\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y)=y^*\Delta_{\phi,\tilde{\phi}_0}^{-it}u_*\eta_{\tilde{\phi}_0}(x)$$

for $t\in\boldsymbol{R}$, the assertion follows by analytic continuation.

Formula (9.3) is shown as follows.

$$\begin{aligned}(9.4)\qquad & J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-it}x^*u\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y)\\ &=J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{1/2}\Delta_{\tilde{\phi}_0,\phi_0}^{-it}x^*u\Delta_{\tilde{\phi}_0,\phi_0}^{it}\Delta_{\tilde{\phi}_0,\phi_0}^{-it}\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y)\\ &=J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{1/2}\sigma_{-t}^{\tilde{\phi}_0}(x^*u)\,(D\phi:D\tilde{\phi}_0)^*_{-t}\eta_{\phi_0}(y)\\ &=\eta_{\tilde{\phi}_0}(y^*(D\phi:D\tilde{\phi}_0)_{-t}\sigma_{-t}^{\tilde{\phi}_0}(u^*x))\\ &=y^*\Delta_{\phi,\tilde{\phi}_0}^{-it}\Delta_{\tilde{\phi}_0}^{it}\eta_{\tilde{\phi}_0}(\sigma_{-t}^{\tilde{\phi}_0}(u^*x))\\ &=y^*\Delta_{\phi,\tilde{\phi}_0}^{-it}u^*\eta_{\tilde{\phi}_0}(x).\end{aligned}$$

Q.E.D.

Next we discuss the property of $J_p(\tilde{\phi}_0,\phi_0)$, which is defined (in Theorem 7) by (9.3) for $1\leqq p<\infty$ and by the following for $p=\infty$:

$$(9.5)\qquad \begin{array}{rccc} J_\infty(\tilde{\phi}_0,\phi_0): & L_\infty(M,\phi_0) & \to & L_\infty(M,\tilde{\phi}_0)\\ & \Cup & & \Cup\\ & x & \mapsto & x^* \end{array}.$$

Lemma 9.2. *For $1\leqq p\leqq\infty$, $J_p(\tilde{\phi}_0,\phi_0)$ is a conjugate linear isometry from $L_p(M,\phi_0)$ onto $L_p(M,\tilde{\phi}_0)$ and satisfies*

$$(9.6)\qquad \langle T,T'\rangle_{\phi_0}=\langle J_{p'}(\tilde{\phi}_0,\phi_0)(T'),J_p(\tilde{\phi}_0,\phi_0)(T)\rangle_{\tilde{\phi}_0}$$

where $T\in L_p(M,\phi_0)$, $T'\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$.

Proof. The isometric and surjective property is immediate from the definition. In addition, the conjugate linear property will follow from (9.6). To prove (9.6), we first consider the case $1<p<\infty$. Let $T=u\Delta_{\phi,\phi_0}^{1/p}$, $T'=v\Delta_{\psi,\phi_0}^{1/p'}$ be the polar decomposition. By taking analytic elements $\int_{\boldsymbol{R}}\alpha_t(x)f(t)\,dt$ of the one parameter group of mappings $\alpha_t(x)=(D\psi:D\phi_0)_t\sigma_t^{\phi_0}(x)$ with $x\in N_{\phi_0}$ and the Fourier transform of f having a compact support, we can choose a σ-weakly dense (and hence $*$-strongly dense) convex subset E of the unit ball of N_{ϕ_0} such that $\eta_{\phi_0}(E)\subset D(T')$. If $2\leqq p<\infty$, $\eta_{\phi_0}(N_{\phi_0})\subset D(T)$ and hence $\eta_{\phi_0}(E)\subset D(T)\cap D(T')$. For the case $1<p\leqq 2$, the same reasoning is applicable if we interchange the role of T and T'. Hence we can choose convex subsets E and $\tilde{E}$ of the unit balls of N_{ϕ_0} and $N_{\tilde{\phi}_0}$, respectively, such that $\eta_{\phi_0}(E)\subset D(T)\cap$

$D(T')$ and $\eta_{\tilde{\phi}_0}(\widetilde{E}) \subset D(J_p(\widetilde{\phi}_0, \phi_0)(T)) \cap D(J_{p'}(\widetilde{\phi}_0, \phi_0)(T'))$. By the three line method, we obtain for $y \in E$ and $\tilde{y} \in \widetilde{E}$

$$(9.7) \quad |\tilde{y}^* T \eta_{\phi_0}(y), \tilde{y}^* T' \eta_{\phi_0}(y))|$$
$$\leqq |\phi(yy^*v^*\tilde{y}\tilde{y}^*u)|^{1/p} |\psi(v^*\tilde{y}\tilde{y}^*uyy^*)|^{1/p'}$$
$$\leqq \|yy^*\xi(\phi)\|^{1/p} \|\tilde{y}\tilde{y}^*u\xi(\phi)\|^{1/p} \|\tilde{y}\tilde{y}^*v\xi(\psi)\|^{1/p'} \|yy^*\xi(\psi)\|^{1/p'}.$$

Furthermore, by (9.1) and the conjugate isometric property of $J_{\tilde{\phi}_0, \phi_0}$ we obtain

$$(9.8) \quad (y^* J_{p'}(\widetilde{\phi}_0, \phi_0)(T') \eta_{\tilde{\phi}_0}(\tilde{y}), y^* J_p(\widetilde{\phi}_0, \phi_0)(T) \eta_{\tilde{\phi}_0}(\tilde{y}))$$
$$= (\tilde{y}^* T \eta_{\phi_0}(y), \tilde{y}^* T' \eta_{\phi_0}(y)).$$

Hence the limits

$$(9.9) \quad \lim_{y \to 1} (\tilde{y}^* T \eta_{\phi_0}(y), \tilde{y}^* T' \eta_{\phi_0}(y))$$

and

$$(9.10) \quad \lim_{\tilde{y} \to 1} (\tilde{y}^* T \eta_{\phi_0}(y), \tilde{y}^* T' \eta_{\phi_0}(y))$$

are uniform with respect to $\tilde{y}$ and y respectively for our choice of $y \in E$ and $\tilde{y} \in \widetilde{E}$. (Note that $\|y\| \leqq 1$ and $\|\tilde{y}\| \leqq 1$). Then we have

$$(9.11) \quad \langle T, T' \rangle_{\phi_0} = \lim_{y \to 1} (T \eta_{\phi_0}(y), T' \eta_{\phi_0}(y))$$
$$= \lim_{y \to 1} \lim_{\tilde{y} \to 1} (\tilde{y}^* T \eta_{\phi_0}(y), \tilde{y}^* T' \eta_{\phi_0}(y))$$
$$= \lim_{\tilde{y} \to 1} \lim_{y \to 1} (y^* J_{p'}(\widetilde{\phi}_0, \phi_0)(T') \eta_{\tilde{\phi}_0}(\tilde{y}),$$
$$y^* J_p(\widetilde{\phi}_0, \phi_0)(T) \eta_{\tilde{\phi}_0}(\tilde{y}))$$
$$= \lim_{\tilde{y} \to 1} (J_{p'}(\widetilde{\phi}_0, \phi_0)(T') \eta_{\tilde{\phi}_0}(\tilde{y}), J_p(\widetilde{\phi}_0, \phi_0)(T) \eta_{\tilde{\phi}_0}(\tilde{y}))$$
$$= \langle J_{p'}(\widetilde{\phi}_0, \phi_0)(T'), J_p(\widetilde{\phi}_0, \phi_0)(T) \rangle_{\tilde{\phi}_0},$$

where the first and the last equalities are due to the definition (6.4), Notation 3.3, (3.4) and proper redistribution of Δ's between two members of the inner product and the third equality is due to (9.9) and the uniformity of limits. Next we assume $p=1$. Let $T = u\Delta_{\phi, \phi_0}$ be the polar decomposition and $x \in M$. Then,

$$(9.12)\qquad \langle T, x\rangle_{\phi_0}=\lim_{y\to 1}(T\eta_{\phi_0}(y), x\eta_{\phi_0}(y))$$
$$=\lim_{y\to 1}\phi(yy^*x^*u)=\phi(x^*u).$$

By the same computation,

$$(9.13)\qquad \langle J_\infty(\tilde{\phi}_0, \phi_0)(x), J_1(\tilde{\phi}_0, \phi_0)(T)\rangle_{\tilde{\phi}_0}=\phi(x^*u).$$

The case $p=\infty$ is proved by exactly the same computation. This shows formula (9.6) for $1\leqq p\leqq\infty$. To show the conjugate linearity of $J_p(\tilde{\phi}_0, \phi_0)$, let $T_1+T_2=T$ in $L_p(M, \phi_0)$. Then for any $S\in L_{p'}(M, \tilde{\phi}_0)$,

$$(9.14)\qquad \langle J_p(\tilde{\phi}_0, \phi_0)(T_1), S\rangle_{\tilde{\phi}_0}+\langle J_p(\tilde{\phi}_0, \phi_0)(T_2), S\rangle_{\tilde{\phi}_0}$$
$$=\langle J_{p'}(\phi_0, \tilde{\phi}_0)(S), T_1\rangle_{\phi_0}+\langle J_{p'}(\phi_0, \tilde{\phi}_0)(S), T_2\rangle_{\phi_0}$$
$$=\langle J_{p'}(\phi_0, \tilde{\phi}_0)(S), T\rangle_{\phi_0}$$
$$=\langle J_p(\tilde{\phi}_0, \phi_0)(T), S\rangle_{\tilde{\phi}_0}.$$

Hence $J_p(\tilde{\phi}_0, \phi_0)(T_1)+J_p(\tilde{\phi}_0, \phi_0)(T_2)=J_p(\tilde{\phi}_0, \phi_0)(T)$ and we have the conjugate linearity of $J_p(\tilde{\phi}_0, \phi_0)$. Q.E.D.

Remark. If $\tilde{\phi}_0=\phi_0$, then the mapping $J_p(\phi_0, \phi_0)$ is actually the adjoint operation for closed operators. (Hence it must be conjugate linear.) Equality (9.6) then reduces to

$$(9.15)\qquad \langle T, T'\rangle_{\phi_0}=\langle T'^*, T^*\rangle_{\phi_0}$$

for $T\in L_p(M, \phi_0)$, $T'\in L_{p'}(M, \phi_0)$, $p^{-1}+(p')^{-1}=1$.

From Lemma 9.2, we immediately obtain the following lemma.

Lemma 9.3. *Let ϕ_0, ϕ_0', ϕ_0'' and $\tilde{\phi}_0$ be faithful normal semifinite weights and set $\tau_p(\phi_0', \phi_0)=J_p(\phi_0', \tilde{\phi}_0)J_p(\tilde{\phi}_0, \phi_0)$. Then $\tau_p(\phi_0', \phi_0)$ is a linear isometry from $L_p(M, \phi_0)$ onto $L_p(M, \phi_0')$ and explicitly given by*

$$(9.16)\qquad \begin{array}{ccc} \tau_p(\phi_0', \phi_0): L_p(M, \phi_0) & \to & L_p(M, \phi_0') \\ \Cup & & \Cup \\ u\Delta_{\phi,\phi_0}^{1/p} & \mapsto & u\Delta_{\phi,\phi_0'}^{1/p} \end{array} \qquad (1\leqq p<\infty),$$

$$(9.17)\qquad \begin{array}{ccc} \tau_\infty(\phi_0', \phi_0): L_\infty(M, \phi_0) & \to & L_\infty(M, \phi_0') \\ \Cup & & \Cup \\ x & \mapsto & x \end{array}\ .$$

The map $\tau_p(\phi_0', \phi_0)$ *is independent of the choice of* $\tilde{\phi}_0$ *and satisfies*

$$\langle \tau_p(\phi_0', \phi_0)(T), \tau_{p'}(\phi_0', \phi_0)(S)\rangle_{\phi_0'} = \langle T, S\rangle_{\phi_0}, \tag{9.18}$$

$$\tau_p(\phi_0'', \phi_0')\tau_p(\phi_0', \phi_0) = \tau_p(\phi_0'', \phi_0), \tag{9.19}$$

where $T \in L_p(M, \phi_0)$, $S \in L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$.

§ 10. Positivity

In this section, the positive cone $L_p^+(M, \phi_0)$ in L_p-space is discussed and the linear polar decomposition is shown. Also, a bilinear dual pairing between $L_p(M, \phi_0)$ and $L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$, is introduced in the present section.

Lemma 10.1. *Let* $T \in L_p(M, \phi_0)$ *such that* $T^* = T$. *Then there exist positive selfadjoint operators* T_+ *and* T_- *belonging to* $L_p(M, \phi_0)$ *such that*

$$T = T_+ - T_-. \tag{10.1}$$

This decomposition is unique under the condition,

$$s(T_+) \perp s(T_-). \tag{10.2}$$

Proof. Same as the proof of Lemma 13.1 in [9]. Q.E.D.

Corollary. *Any* $T \in L_p(M, \phi_0)$, $1 \leqq p \leqq \infty$, *has a unique decomposition* $T = (T_{r+} - T_{r-}) + i(T_{i+} - T_{i-})$ *such that* $T_{\tau\sigma} \in L_p^+(M, \phi_0)$ *and* $s(T_{\tau+}) \perp s(T_{\tau-})$ *where* $\tau = r, i$ *and* $\sigma = +, -$.

Proof. Same as the proof of Corollary 13.2, [8]. Q.E.D.

Lemma 10.2. *Let* $T \in L_p^+(M, \phi_0)$, $1 \leqq p \leqq \infty$. *If* $y \in N_{\phi_0}$ *satisfies* $\eta_{\phi_0}(y) \in D(T)$, *then* $y^* T \eta_{\phi_0}(y) \in V_{\phi_0}^{1/(2p)}$.

Proof. The case $p = \infty$ is clear. We assume $1 \leqq p < \infty$ and $T = \Delta_{\phi, \phi_0}^{1/p}$. Due to the polar relation of positive cones and the density of the positive cone (see Appendix A), it is enough to show the following:

$$(10.3)\qquad (y^{*}\Delta_{\phi,\phi_0}^{1/p}\eta_{\phi_0}(y),\ \Delta_{\phi_0}^{(1/2)-(1/2p)}x^{*}\eta_{\phi_0}(x))\geqq 0$$

for any $x\in M_0$. We have

$$(10.4)\qquad \Delta_{\phi_0}^{(1/2)-(1/2p)}x^{*}\eta_{\phi_0}(x)=\Delta_{\phi_0}^{1/2}\eta_{\phi_0}(\sigma_{i/(2p)}^{\phi_0}(x^{*}x))$$
$$=J_{\phi_0}\eta_{\phi_0}(\sigma_{-i/(2p)}^{\phi_0}(x^{*}x))$$
$$=x'^{*}J_{\phi_0}\eta_{\phi_0}(\sigma_{-i/(2p)}^{\phi_0}(x))$$

where $x'=j(\sigma_{i/(2p)}^{\phi_0}(x))\in M_0'$. Since

$$(10.5)\qquad x'\Delta_{\phi,\phi_0}^{1/p}\eta_{\phi_0}(y)=\Delta_{\phi,\phi_0}^{1/p}(\Delta_{\phi_0}^{-1/p}x'\Delta_{\phi_0}^{1/p})\eta_{\phi_0}(y)$$
$$=\Delta_{\phi,\phi_0}^{1/p}j(\sigma_{-i/(2p)}^{\phi_0}(x))\eta_{\phi_0}(y)$$
$$=\Delta_{\phi,\phi_0}^{1/p}yJ_{\phi_0}\eta_{\phi_0}(\sigma_{-i/(2p)}^{\phi_0}(x)),$$

we obtain

$$(10.6)\qquad (y^{*}\Delta_{\phi,\phi_0}^{1/p}\eta_{\phi_0}(y),\ \Delta_{\phi_0}^{(1/2)-(1/2p)}x^{*}\eta_{\phi_0}(x))$$
$$=\|\Delta_{\phi,\phi_0}^{1/(2p)}yJ_{\phi_0}\eta_{\phi_0}(\sigma_{-i/(2p)}^{\phi_0}(x))\|^2\geqq 0.$$

Q.E.D.

Lemma 10.3. *Let* $p^{-1}+(p')^{-1}=1$. *Then*

$L_p^{+}(M,\phi_0)=\{T\in L_p(M,\phi_0):\langle T,S\rangle_{\phi_0}\geqq 0$ *for any* $S\in L_{p'}^{+}(M,\phi_0)\}$.

Proof. Let $T\in L_p^{+}(M,\phi_0)$ and $S\in L_{p'}^{+}(M,\phi_0)$. By taking suitable analytic elements as in the proof of Lemma 9.2, we can choose σ-weakly dense (and hence $*$-strongly dense) convex subset E of the unit ball of N_{ϕ_0} such that $\eta_{\phi_0}(y)\in D(S)\cap D(T)$ for any $y\in E$. Again by the same uniformity argument as in the proof of Lemma 9.2,

$$(10.7)\qquad \langle T,S\rangle_{\phi_0}=\lim_{y\to 1}(T\eta_{\phi_0}(y),S\eta_{\phi_0}(y))$$
$$=\lim_{y\to 1}(y^{*}T\eta_{\phi_0}(y),\ y^{*}S\eta_{\phi_0}(y)).$$

By Lemma 10.2 and the polar relation of the positive cone, $(y^{*}T\eta_{\phi_0}(y),$ $y^{*}S\eta_{\phi_0}(y))\geqq 0$. Hence $\langle T,S\rangle_{\phi_0}\geqq 0$.

Conversely, assume $T\in L_p(M,\phi_0)$ satisfies $\langle T,S\rangle_{\phi_0}\geqq 0$ for any $S\in L_{p'}^{+}(M,\phi_0)$. By the remark after Lemma 9.2,

$$(10.8)\qquad \langle T^{*},S\rangle_{\phi_0}=\langle S^{*},T\rangle_{\phi_0}=\langle S,T\rangle_{\phi_0}=\overline{\langle T,S\rangle_{\phi_0}}=\langle T,S\rangle_{\phi_0}.$$

Due to the corollary after Lemma 10.1, $L_{p'}^{+}(M,\phi_0)$ is total in $L_{p'}(M,\phi_0)$.

Together with (10.8), T is self-adjoint. Again by Lemma 10.1, T has a polar decomposition $T=T_+-T_-$ with orthogonal supports. We assume $T_-\neq 0$. Let $T_-=\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition. If we set $S=\Delta_{\phi,\phi_0}^{1/p'}$. Then $\langle T,S\rangle_{\phi_0}=-\langle T_-,S\rangle_{\phi_0}=-\phi(1)<0$ and this contradicts to the assumption. Hence we have $T\geqq 0$ and complete the proof. Q.E.D.

Next we introduce a bilinear dual pairing between $L_p(M,\phi_0)$ and $L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$.

Notation 10.4. For $T\in L_p(M,\phi_0)$ and $S\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, we define

(10.9) $$[T,S]_{\phi_0}\equiv\langle T,S^*\rangle_{\phi_0}.$$

By the remark after Lemma 9.2,

(10.10) $$[S,T]_{\phi_0}=\langle S,T^*\rangle_{\phi_0}=\langle T,S^*\rangle_{\phi_0}=[T,S]_{\phi_0}.$$

This shows that this bilinear dual pairing is symmetric.

§ 11. Product and Hölder Inequality

Let us recall Notation 6.1 for $L_p^*(M,\phi_0)$, $1\leqq p\leqq\infty$ as well as for the adjoint and the product of its elements. We may identify $L_p^*(M,\phi_0)$ (modulo induced equivalence) with a subset of $L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$ (by Lemma 3.1 as operators after taking closure for $2\leqq p'\leqq\infty$, and through Lemma 6.2 and duality $L_{p'}(M,\phi_0)\cong L_p(M,\phi_0)^*$ (Lemma 8.3 (2)) for $1<p'\leqq 2$ and through $L_1(M,\phi_0)=L_\infty(M,\phi_0)_*$).

Lemma 11.1. *Let ϕ_0 and $\tilde{\phi}_0$ be faithful normal semifinite weights and let $T\in L_p^*(M,\phi_0)$, $S\in L_p(M,\phi_0)$. Then $\langle J_{p'}(\tilde{\phi}_0,\phi_0)(T), J_p(\tilde{\phi}_0,\phi_0)(S)\rangle_{\tilde{\phi}_0}=\langle S,T\rangle_{\phi_0}$, where $J_p(\tilde{\phi}_0,\phi_0)$ for $L_p(M,\phi_0)$ is defined by (9.2) and $J_{p'}(\tilde{\phi}_0,\phi_0)$ is defined for $L_{p'}^*(M,\phi_0)$ by*

(11.1) $$J_{p'}(\tilde{\phi}_0,\phi_0)(T)=x_n^*\Delta_{\phi_n,\tilde{\phi}_0}^{1/p_n}\cdots x_1^*\Delta_{\phi_1,\tilde{\phi}_0}^{1/p_1}x_0^*$$

if T is given by (6.1). When $L_{p'}^*(M,\phi_0)$ is identified with a subset of $L_p(M,\phi_0)$, the two definitions of $J_p(\tilde{\phi}_0,\phi_0)$ coincide.

Proof. Let $y\in N_{\phi_0}$ and $\tilde{y}\in N_{\tilde{\phi}_0}$. Then for $t=\sum_{k=1}^{n} t_k$,

$$J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-it}\tilde{y}^* x_0 \Delta_{\phi_1,\phi_0}^{it_1}\cdots\Delta_{\phi_n,\phi_0}^{it_n} x_n \eta_{\phi_0}(y) \tag{11.2}$$
$$=J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{1/2}\sigma_{-t_n}^{\tilde{\phi}_0}(\cdots\sigma_{-t_2}^{\tilde{\phi}_0}(\sigma_{-t_1}^{\tilde{\phi}_0}(\tilde{y}^* x_0)(D\phi_1: D\tilde{\phi}_0)_{-t_1}^* x_1)\cdots)$$
$$\times (D\phi_n: D\tilde{\phi}_0)_{-t_n}^* x_n \eta_{\phi_0}(y)$$
$$=\eta_{\tilde{\phi}_0}(y^* x_n^* (D\phi_n: D\tilde{\phi}_0)_{-t_n}\sigma_{-t_n}^{\tilde{\phi}_0}(x_{n-1}^*(D\phi_{n-1}: D\tilde{\phi}_0)_{-t_{n-1}}$$
$$\times \sigma_{-t_{n-1}}^{\tilde{\phi}_0}(\cdots\sigma_{-t_1}^{\tilde{\phi}_0}(x_0^*\tilde{y})\cdots))$$
$$=y^* x_n^* \Delta_{\phi_n,\tilde{\phi}_0}^{-it_n}\cdots\Delta_{\phi_1,\tilde{\phi}_0}^{-it_1} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y}).$$

Hence, applying the same argument as in Section 3,

$$F(z)=J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-\bar{z}_0}\tilde{y}^* x_0 \Delta_{\phi_1,\phi_0}^{\bar{z}_1}\cdots\Delta_{\phi_n,\phi_0}^{\bar{z}_n} x_n \eta_{\phi_0}(y) \tag{11.3}$$
$$=y^* x_n^* \Delta_{\phi_n,\tilde{\phi}_0}^{z_n}\cdots\Delta_{\phi_1,\tilde{\phi}_0}^{z_1} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y})$$

is continuous for $z=(z_1,\cdots,z_n)\in I_{1/2}^{(n)}$ and holomorphic in the interior, where $z_0=\sum_{k=1}^{n} z_k$. Let $1<p<\infty$ and $S=u\Delta_{\phi,\phi_0}^{1/p}$ be the polar decomposition. If $z\in\partial_k I_{1/2}^{(n)}$, $k=1,\cdots,n$ then

$$(y^* x_n^* \Delta_{\phi_n,\tilde{\phi}_0}^{z_n}\cdots\Delta_{\phi_1,\tilde{\phi}_0}^{z_1} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y}),\ y^*\Delta_{\phi,\tilde{\phi}_0}^{1-\bar{z}_0} u^* \eta_{\tilde{\phi}_0}(\tilde{y})) \tag{11.4}$$
$$=(J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{(1/2)-z_0}\tilde{y}^* x_0 \Delta_{\phi_1,\phi_0}^{\bar{z}_1}\cdots\Delta_{\phi_n,\phi_0}^{\bar{z}_n} x_n \eta_{\phi_0}(y),$$
$$J_{\tilde{\phi}_0,\phi_0}\Delta_{\tilde{\phi}_0,\phi_0}^{z_0-(1/2)}\tilde{y}^* u \Delta_{\phi,\phi_0}^{1-z_0}\eta_{\phi_0}(y))$$
$$=(\tilde{y}^* u \Delta_{\phi_1,\phi_0}^{1-z_0}\eta_{\phi_0}(y),\ \tilde{y}^* x_0 \Delta_{\phi_1,\phi_0}^{\bar{z}_1}\cdots\Delta_{\phi_n,\phi_0}^{\bar{z}_n} x_n \eta_{\phi_0}(y))$$

by (11.3) for general n and for $n=1$. By the analytic continuation of (11.4) for $z_k\to p_k^{-1}$, $k=1,\cdots,n$ we have

$$(u\Delta_{\phi,\tilde{\phi}_0}^{1/p} y y^* x_n^* \Delta_{\phi_n,\tilde{\phi}_0}^{1/p_n}\cdots\Delta_{\phi_1,\tilde{\phi}_0}^{1/p_1} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y}),\ \eta_{\tilde{\phi}_0}(\tilde{y})) \tag{11.5}$$
$$=(x_n^* \Delta_{\phi_n,\phi_0}^{1/p_n}\cdots\Delta_{\phi_1,\phi_0}^{1/p_1} x_0^* \tilde{y}\tilde{y}^* u \Delta_{\phi,\phi_0}^{1/p}\eta_{\phi_0}(y),\ \eta_{\phi_0}(y)),$$

where $p^{-1}=1-\sum_{k=1}^{n} p_k^{-1}$. By taking the limit $y\to 1$ and $\tilde{y}\to 1$ and by using the uniformity argument as in the proof of Lemma 9.2, we get the assertion for $1<p<\infty$.

Proof for remaining cases are as follows. The case $p=1$ is already shown by Lemma 9.2. Let $p=\infty$. Then for $z\in\partial_k I_{1/2}^{(n)}$, $k=1,\cdots,n$, $x\in M$ and $y\in N_{\phi_0}$,

$$(11.6)\qquad (\tilde{y}^* x\eta_{\phi_0}(y), \tilde{y}^* x_0 \Delta^{\bar{z}_1}_{\phi_1,\phi_0}\cdots \Delta^{\bar{z}_n}_{\phi_n,\phi_0} x_n \eta_{\phi_0}(y))$$
$$= (y^* x_n^* \Delta^{z_n}_{\phi_n,\tilde{\phi}_0}\cdots \Delta^{z_1}_{\phi_1,\tilde{\phi}_0} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y}), J_{\tilde{\phi}_0,\phi_0}\Delta^{z_0-(1/2)}_{\tilde{\phi}_0,\phi_0}\tilde{y}^* x\eta_{\phi_0}(y))$$

where $z_0-(1/2)\in i\boldsymbol{R}$.
By the analytic continuation of (11.6) for $z_k\to p_k^{-1}$, $k=1,\cdots,n$, we have

$$(11.7)\qquad (x_n^* \Delta^{1/p_n}_{\phi_n,\phi_0}\cdots \Delta^{1/p_1}_{\phi_1,\phi_0} x_0^* \tilde{y}\tilde{y}^* x\eta_{\phi_0}(y), \eta_{\phi_0}(y))$$
$$= (xyy^* x_n^* \Delta^{1/p_n}_{\phi_n,\tilde{\phi}_0}\cdots \Delta^{1/p_1}_{\phi_1,\tilde{\phi}_0} x_0^* \eta_{\tilde{\phi}_0}(\tilde{y}), \eta_{\tilde{\phi}_0}(\tilde{y})).$$

By taking the limit $y\to 1$ and by the uniformity argument, we get $\langle J_1(\tilde{\phi}_0,\phi_0)(T), x^*\rangle_{\tilde{\phi}_0}=\langle x, T\rangle_{\phi_0}$.

Lastly we show that the two definitions of $J_p(\tilde{\phi}_0,\phi_0)$ coincide when $L^*_{p'}(M,\phi_0)$ is identified with a subset of $L_p(M,\phi_0)$. The case $p=1$ (equivalently $p'=\infty$) is immediate. Let T be of the form of (6.1), $\tilde{T}=v\Delta^{1/p'}_{\psi,\phi_0}$ and assume $T=\tilde{T}$ in $L_p(M,\phi_0)^*$. By replacing S by $J_p(\phi_0,\tilde{\phi}_0)(\tilde{S})$, $\tilde{S}\in L_p(M,\tilde{\phi}_0)$, in the formula of Lemma 11.1 which we have just proved, we obtain

$$\langle J_{p'}(\tilde{\phi}_0,\phi_0)(T), \tilde{S}\rangle_{\tilde{\phi}_0}=\langle J_p(\phi,\tilde{\phi}_0)(\tilde{S}), T\rangle_{\phi_0}$$
$$=\langle J_p(\phi,\tilde{\phi}_0)(\tilde{S}), \tilde{T}\rangle_{\phi_0}$$
$$=\langle J_{p'}(\tilde{\phi}_0,\phi_0)(\tilde{T}), \tilde{S}\rangle_{\tilde{\phi}_0}.$$

So the two definitions coincide. Q.E.D.

Lemma 11.2. *If $T_k\in L^*_p(M,\phi_0)$, $k=1,\cdots,n$ and $\sum_{k=1}^n T_k=0$ in $L_p(M,\phi_0)^*$ for $2\leqq p\leqq\infty$, then $\sum_{k=1}^n J_{p'}(\tilde{\phi}_0,\phi_0)(T_k)=0$ in $L_p(M,\tilde{\phi}_0)^*$ and $\sum_{k=1}^n T_kS=\sum_{k=1}^n ST_k=0$ in $L_r(M,\phi_0)^*$ where $S\in L^*_q(M,\phi_0)$, $r^{-1}=p^{-1}+q^{-1}-1$ and $1\leqq q$, $r\leqq\infty$.*

Proof. If $\sum_{k=1}^n T_k=0$. Then $\sum_{k=1}^n J_{p'}(\tilde{\phi}_0,\phi_0)(T_k)=0$ by Lemma 11.1. Let $R\in L_r(M,\phi_0)$. Again by Lemma 11.1 for $\tilde{\phi}_0=\phi_0$, and by definition (6,2) of the sesquilinear form $\langle\cdot,\cdot\rangle$,

$$(11.8)\qquad \sum_{k=1}^n \langle T_kS, R\rangle_{\phi_0}=\sum_{k=1}^n\langle R^*, S^*T_k^*\rangle_{\phi_0}=\sum_{k=1}^n\langle SR^*, T_k^*\rangle_{\phi_0}=0\,,$$

$$(11.9)\qquad \sum_{k=1}^n\langle ST_k, R\rangle_{\phi_0}=\sum_{k=1}^n\langle T_k, S^*R\rangle_{\phi_0}=0\,.$$

R is arbitrary, we get the assertion. Q.E.D.

Lemma 11.3. *Let* $1 \leqq p, q, r \leqq \infty$, $p^{-1}+(p')^{-1}=q+(q')^{-1}=r+(r')^{-1}=1$, $p^{-1}+q^{-1}=r^{-1}$.

(1) *If* T_1 *and* T_2 *in* $L^*_{p'}(M, \phi_0)$ *are equal as elements of* $L_p(M, \phi_0)$, *then* $T_1^* = T_2^*$ *in* $L_p(M, \phi_0)$, $T_1S = T_2S$ *and* $ST_1 = ST_2$ *in* $L_r(M, \phi_0)$ *where* $S \in L^*_{q'}(M, \phi_0)$.

(2) T^* *is conjugate linear in* T *and* TS *is bilinear in* T *and* S.

(3) *The product is associative and* $(TS)^* = S^*T^*$.

(4) *For* $T \in L^*_{p'}(M, \phi_0)$ *and* $S \in L^*_{q'}(M, \phi_0)$,

$\|TS\|_{r,\phi_0} \leqq \|T\|_{p,\phi_0} \|S\|_{q,\phi_0}$.

Proof. Viewing $\beta \in \mathbf{C}$ as an element of $L_1^*(M, \phi_0)$, it is easy to check $(\beta T)^* = \bar{\beta} T^*$, $(\beta T)S = T(\beta S) = \beta TS$ and the equivalence of T_1 and T_2 with $T_1 + (-1)T_2 = 0$ in $L_p(M, \phi_0)$. Therefore Lemmas 6.5 and 11.2 imply (1), (2) and (3). Assertion (4) follows from (3.6).

Q.E.D.

§ 12. Dense Subspaces of L_p-spaces

In this section, we discuss the p-dependent injection T_p of $D^\infty_{\phi_0}$ (defined after Theorem 7) into $L_p(M, \phi_0)$. We start with some preliminary lemmas.

Lemma 12.1. *Let* $x \in N^*_{\phi_0} \cap N_{\phi_0}$, $y \in N_{\phi_0}$ *and* $\|y\| = 1$. *Then for partial isometry* $u \in M$ *and* $\phi \in M^+_*$ *satisfying* $u^*u = s(\phi)$,

$$\|\Delta_{\phi_0}^{(1/2)-\lambda} x u \Delta^{\lambda}_{\phi,\phi_0} \eta_{\phi_0}(y)\| \leqq \|\eta_{\phi_0}(x^*)\|^{1-2\lambda} \|x\|^{2\lambda} \phi(1)^{\lambda} \tag{12.1}$$

for $0 \leqq \lambda \leqq 1/2$.

Proof. Follows from

$$\|\Delta_{\phi_0}^{(1/2)-\lambda} x u \Delta^{\lambda}_{\phi,\phi_0} \eta_{\phi_0}(y)\| = \|y^* \Delta^{\lambda}_{\phi,\phi_0} u^* \eta_{\phi_0}(x^*/\|x\|)\| \, \|x\|$$

and Lemma 3.2. Q.E.D.

Lemma 12.2. *Let* $x \in N^*_{\phi_0} \cap N_{\phi_0}$ *and* $2 \leqq p \leqq \infty$. *Then* $\Delta^{1/p}_{\phi_0} x$ *is* *preclosed and the closure $S_p(x)$ belongs to $L_p(M, \phi_0)$, and*

$$(12.2) \qquad \|S_p(x)\|_{p,\phi_0} \leqq \|\eta_{\phi_0}(x^*)\|^{2/p}\|x\|^{1-(2/p)}.$$

Proof. The case $p=\infty$ is immediate. We assume $2\leqq p<\infty$. By the assumption, $\eta_{\phi_0}(N_{\phi_0}^*\cap N_{\phi_0})\subset D(\Delta_{\phi_0}^{1/p}x)\cap D(\Delta_{\phi_0}^{1/p})$. Hence the operators $\Delta_{\phi_0}^{1/p}x$ and $x^*\Delta_{\phi_0}^{1/p}$ are densely defined. Because $\Delta_{\phi_0}^{1/p}x$ is a product of closed operator and bounded operator, it is preclosed. For $y\in M_0$,

$$(12.3) \qquad \Delta_{\phi_0}^{1/p}xJ_{\phi_0}\sigma_{-i/p}^{\phi_0}(y)J_{\phi_0}=\Delta_{\phi_0}^{1/p}J_{\phi_0}\sigma_{-i/p}^{\phi_0}(y)J_{\phi_0}x\supset J_{\phi_0}yJ_{\phi_0}\Delta_{\phi_0}^{1/p}x.$$

By taking the closure of $\Delta_{\phi_0}^{1/p}x$ in (12.3), we have the (ϕ_0, p)-measurability of the closure $S_p(x)$ of $\Delta_{\phi_0}^{1/p}x$. If $2<p<\infty$, the assertion of Lemma follows from Lemmas 5.3 and 12.1 for $(1/2)-(1/p)=\lambda$. If $p=2$, then for $y\in M_0^{(1)}$,

$$(12.4) \qquad \||S_2(x)|\eta_{\phi_0}(y)\|=\|\Delta_{\phi_0}^{1/2}x\eta_{\phi_0}(y)\|=\|y^*\eta_{\phi_0}(x^*)\|\leqq\|\eta_{\phi_0}(x^*)\|.$$

In view of (2.2), $\|S_2(x)\|_{2,\phi_0}\leqq\|\eta_{\phi_0}(x^*)\|$ and the assertion follows.

Q.E.D.

Now, we recall the definition of the p-dependent injection T_p of $D_{\phi_0}^\infty$ into $L_p(M, \phi_0)$ defined by the linear combination of (2.10).

Lemma 12.3. *Let $\zeta\in D_{\phi_0}^\infty$. Then $T_p(\zeta)\in L_p(M, \phi_0)$ and is independent of the expression of ζ by the linear combination of the products of two elements of the Tomita algebra $\eta_{\phi_0}(M_0)$. Moreover for $\zeta=\eta_{\phi_0}(x_1x_2)$, $x_k\in M_0$, $k=1, 2$, $1\leqq p\leqq\infty$,*

$$(12.5) \qquad \|T_p(\zeta)\|_{p,\phi_0}\leqq\{\|\eta_{\phi_0}(x_1)\|\ \|\Delta_{\phi_0}^{(1/2)-(1/p)}\eta_{\phi_0}(x_2)\|\}^{1/p} \times\{\|x_1\|\ \|\sigma_{i/p}^{\phi_0}(x_2)\|\}^{1-(1/p)}.$$

Proof. The case $p=\infty$ is immediate. Let $1<p<\infty$, and assume $\zeta=\eta_{\phi_0}(x_1x_2)$. By Lemma 12.2, $\overline{\Delta_{\phi_0}^{1/(2p)}x_1^*}$ and $\overline{\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_2)}$ belong to $L_{2p}(M, \phi_0)$. Hence by taking adjoint and product, $T_p(\zeta)=(\Delta_{\phi_0}^{1/(2p)}x_1)^*\times\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_1))}\in L_p(M, \phi_0)$. The assertion for general ζ follows by linear combination. Inequality (12.5) follows from (12.2) and Lemma 11.3 (4). Next we show that $T_p(\zeta)$ is independent of the expression of ζ. Let

$\zeta_k \in D^{\infty}_{\phi_0}$, $k=1,2,3$, and assume $\zeta_1+\zeta_2=\zeta_3$. If $\zeta_k=\eta_{\phi_0}(x_k)$, $k=1,2,3$, then $x_1+x_2=x_3$. Therefore by the explicit form of $T_p(\zeta_k)$, $k=1,2,3$, $T_p(\zeta_1)\eta_{\phi_0}(y)+T_p(\zeta_2)\eta_{\phi_0}(y)=T_p(\zeta_3)\eta_{\phi_0}(y)$, $y\in M_0$. If $2\leqq p<\infty$, $\eta_{\phi_0}(M_0)$ is a core for any $T\in L_p(M,\phi_0)$, $T_p(\zeta_1)+T_p(\zeta_2)=T_p(\zeta_3)$ as elements of $L_p(M,\phi_0)$. If $1\leqq p<2$, $(T_p(\zeta_1)\eta_{\phi_0}(y), S\eta_{\phi_0}(y))+(T_p(\zeta_2)\eta_{\phi_0}(y), S\eta_{\phi_0}(y))$ $=(T_p(\zeta_3)\eta_{\phi_0}(y), S\eta_{\phi_0}(y))$, $S=u\Delta^{1/p'}_{\phi,\phi_0}\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, and a proper distribution of the power of Δ_{ϕ,ϕ_0} between the two members of each inner products. By taking limit $y\to1$, we obtain $\langle T_p(\zeta_1), S\rangle_{\phi_0}+\langle T_p(\zeta_2), S\rangle_{\phi_0}=\langle T_p(\zeta_3), S\rangle_{\phi_0}$, and hence $T_p(\zeta_1)+T_p(\zeta_2)=T_p(\zeta_3)$. Q.E.D.

Next we consider the special case $p=1,2,\infty$.

Lemma 12.4. *Let $\zeta\in D^{\infty}_{\phi_0}$ be of the form,*

$$\zeta=\eta_{\phi_0}\Big(\sum_{k=1}^{n} a_k x_k^{(1)} x_k^{(2)}\Big) \tag{12.6}$$

where $a_k\in\mathbf{C}$, $x_k^{(l)}\in M_0$, $k=1,\cdots,n$, $l=1,2$.

(1) *$T_\infty(\zeta)=\sum_{k=1}^{n} a_k x_k^{(1)} x_k^{(2)}$. Hence $T_\infty(D^{\infty}_{\phi_0})$ is σ-weakly dense in $L_\infty(M,\phi_0)=M$.*

(2) *Let $T=T_2(\zeta)$. Then $\zeta_T=\zeta$ where $T\mapsto\zeta_T$ is the isomorphic mapping $L_2(M,\phi_0)\to H_{\phi_0}$ defined by (7.4). Hence $T_2(D^{\infty}_{\phi_0})$ is norm dense in $L_2(M,\phi_0)\cong H_{\phi_0}$.*

(3) *Let $T=T_1(\zeta)$. Then for $x\in M$,*

$$\psi_T(x)=\sum_{k=1}^{n} a_k\big(x\eta_{\phi_0}(x_k^{(1)}), J_{\phi_0}\Delta_{\phi_0}^{-1/2}\eta_{\phi_0}(x_k^{(2)})\big) \tag{12.7}$$

where $T\mapsto\psi_T$ is the isomorphic mapping $L_1(M,\phi_0)\to M_*$ defined by (7.1). Hence $T_1(D^{\infty}_{\phi_0})$ is norm dense in $L_1(M,\phi_0)\cong M_*$.

Proof. (1) Clear from definition (2.10).

(2) By the density argument and Lemma 7.2,

$$\zeta_T=\lim_{y\to1} T\eta_{\phi_0}(y) \tag{12.8}$$

where the limit means $y\in M_0^{(1)}$ and y converges to 1 $*$-strongly. Hence for $T=T_2(\zeta)$,

$$\zeta_T=\sum_{k=1}^{n} a_k \lim_{y\to1}(\Delta_{\phi_0}^{1/4}x_k^{(1)*})^*\,\overline{(\Delta_{\phi_0}^{1/4}\sigma_{i/2}^{\phi_0}(x_k^{(2)}))}\,\eta_{\phi_0}(y) \tag{12.9}$$

$$= \sum_{k=1}^{n} a_k \lim_{y\to 1} x_k^{(1)} \Delta_{\phi_0}^{1/2} \sigma_{i/2}^{\phi_0}(x_k^{(2)}) \eta_{\phi_0}(y)$$

$$= \sum_{k=1}^{n} a_k \lim_{y\to 1} \Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_{i/2}^{\phi_0}(x_k^{(1)} x_k^{(2)})\, y)$$

$$= \sum_{k=1}^{n} a_k \lim_{y\to 1} J_{\phi_0} y^* \eta_{\phi_0}(\sigma_{i/2}^{\phi_0}(x_k^{(1)} x_k^{(2)})^*)$$

$$= \sum_{k=1}^{n} a_k J_{\phi_0} \eta_{\phi_0}(\sigma_{i/2}^{\phi_0}(x_k^{(1)} x_k^{(2)})^*)$$

$$= \zeta .$$

(3) By the density argument and Lemma 7.1,

$$\psi_T(x) = \lim_{y\to 1} (xT\eta_{\phi_0}(y), \eta_{\phi_0}(y)), \quad x \in M \tag{12.10}$$

where the limit means $u \in M_0^{(1)}$ and yy^* converges to 1 $*$-strongly. Hence for $T = T_1(\zeta)$,

$$\begin{aligned}\psi_T(x) &= \sum_{k=1}^{n} a_k \lim_{y\to 1} ((\overline{\Delta_{\phi_0}^{1/2} \sigma_i^{\phi_0}(x_k^{(2)})})\eta_{\phi_0}(y), (\overline{\Delta_{\phi_0}^{1/2} x_k^{(1)*}}) x^* \eta_{\phi_0}(y)) \\ &= \sum_{k=1}^{n} a_k \lim_{y\to 1} (\Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_i^{\phi_0}(x_k^{(2)})\, y), \Delta_{\phi_0}^{1/2} \eta_{\phi_0}(x_k^{(1)*} x^* y)) \\ &= \sum_{k=1}^{n} a_k \lim_{y\to 1} (\eta_{\phi_0}(y^* x x_k^{(1)}), \eta_{\phi_0}(y^* \sigma_i^{\phi_0}(x_k^{(2)})^*)) \\ &= \sum_{k=1}^{n} a_k (x\eta_{\phi_0}(x_k^{(1)}), J_{\phi_0} \Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_i^{\phi_0}(x_k^{(2)}))) \\ &= \sum_{k=1}^{n} a_k (x\eta_{\phi_0}(x_k^{(1)}), J_{\phi_0} \Delta_{\phi_0}^{-1/2} \eta_{\phi_0}(x_k^{(2)})) .\end{aligned} \tag{12.11}$$

Q.E.D.

Next we show that the image of $D_{\phi_0}^{\infty}$ by the mapping T_p is norm dense in $L_p(M, \phi_0)$, $1 < p < \infty$. The case $p = 2$ is already shown in Lemma 12.4 (2).

Lemma 12.5. *Let $x = x_0 x_1 x_2$ with $x_0 \in M$, x_1, $x_2 \in M_0$, a partial isometry $u \in M$ and $\phi \in M_*^+$. Then the limit*

$$F_x(z) = \lim_{y\to 1} (x\Delta_{\phi_0}^{z} \eta_{\phi_0}(y), u\Delta_{\phi,\phi_0}^{1-\bar{z}} \eta_{\phi_0}(y)) \tag{12.12}$$

exists for $0 \leqq \operatorname{Re} z \leqq 1$ and,

(12. 13) $$F_x(z)=(\Delta_{\phi,\phi_0}^{1-z}u^*x_0\eta_{\phi_0}(x_1),\ \Delta_{\phi_0}^{\bar z}\eta_{\phi_0}(x_2^*))$$

for $0\leqq\mathrm{Re}\,z\leqq 1$. *Moreover for a fixed* z *such that* $0<\mathrm{Re}\,z<1$, $F_x(z)$ *is* $*$*-strongly continuous in* x_0

Proof. Let $y\in M_0^{(1)}$ and consider the function

(12. 14) $$F_{x,y}(z)=(x\Delta_{\phi_0}^{z}\eta_{\phi_0}(y),\ u\Delta_{\phi,\phi_0}^{1-\bar z}\eta_{\phi_0}(y)).$$

Then $F_{x,y}(z)$ is continuous on $0\leqq\mathrm{Re}\,z\leqq 1$ and holomorphic in the interior of this strip region with the following estimates on the boundary. For $t\in\boldsymbol{R}$,

(12. 15)
$$\begin{aligned}F_{x,y}(it)&=(x\Delta_{\phi_0}^{it}\eta_{\phi_0}(y),\ u\Delta_{\phi,\phi_0}^{1+it}\eta_{\phi_0}(y))\\&=(\Delta_{\phi,\phi_0}^{(1/2)-it}u^*x\Delta_{\phi_0}^{it}\eta_{\phi_0}(y),\ \Delta_{\phi,\phi_0}^{1/2}\eta_{\phi_0}(y))\\&=(\Delta_{\phi,\phi_0}^{1/2}(D\phi:D\phi_0)_{-t}\sigma_{-t}^{\phi_0}(u^*x)\eta_{\phi_0}(y),\ \Delta_{\phi,\phi_0}^{1/2}\eta_{\phi_0}(y))\\&=(y^*\xi(\phi),\ y^*\sigma_{-t}^{\phi_0}(x^*u)(D\phi:D\phi_0)_{-t}^*\xi(\phi))\\&=(yy^*\xi(\phi),\ \Delta_{\phi_0}^{-it}x^*u\xi(\phi)),\end{aligned}$$

(12. 16)
$$\begin{aligned}F_{x,y}(1+it)&=(x_0x_1x_2\Delta_{\phi_0}^{1+it}\eta_{\phi_0}(y),\ u\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y))\\&=(x_0x_1\Delta_{\phi_0}^{1+it}\sigma_{i-t}^{\phi_0}(x_2)\eta_{\phi_0}(y),\ u\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y))\\&=(\Delta_{\phi_0}^{1/2}\eta_{\phi_0}(\sigma_{i-t}^{\phi_0}(x_2)y),\ \Delta_{\phi_0}^{(1/2)-it}x_1^*x_0^*u\Delta_{\phi,\phi_0}^{it}\eta_{\phi_0}(y))\\&=(\Delta_{\phi_0}^{1/2}\eta_{\phi_0}(\sigma_{i-t}^{\phi_0}(x_2)y),\\&\qquad \Delta_{\phi_0}^{1/2}\sigma_{-t}^{\phi_0}(x_1^*x_0^*u)(D\phi:D\phi_0)_{-t}^*\eta_{\phi_0}(y))\\&=(y^*(D\phi:D\phi_0)_{-t}\eta_{\phi_0}(\sigma_{-t}^{\phi_0}(u^*x_0x_1)),\ y^*\eta_{\phi_0}(\sigma_{-i-t}^{\phi_0}(x_2^*)))\\&=(\Delta_{\phi,\phi_0}^{-it}u^*x_0\eta_{\phi_0}(x_1),\ yy^*\eta_{\phi_0}(\sigma_{-i-t}^{\phi_0}(x_2^*))),\end{aligned}$$

where $\xi(\phi)$ is the unique vector representative of ϕ in $\mathcal{P}_{\phi_0}^{\natural}$. Hence for $y, y_1, y_2\in M_0^{(1)}$, $x'=x_0'x_1x_2$,

(12. 17) $$|F_{x,y_1}(it)-F_{x,y_2}(it)|\leqq\|(y_1y_1^*-y_2y_2^*)\xi(\phi)\|\,\|x^*u\xi(\phi)\|,$$

(12. 18) $$|F_{x,y_1}(1+it)-F_{x,y_2}(1+it)|\leqq 2\|x_0\eta_{\phi_0}(x_1)\|\,\|\eta_{\phi_0}(\sigma_{-i}^{\phi_0}(x_2^*))\|,$$

(12. 19) $$|F_{x',y}(it)-F_{x,y}(it)|\leqq\|\xi(\phi)\|\,\|(x'-x)^*u\xi(\phi)\|,$$

(12. 20) $$|F_{x',y}(1+it)-F_{x,y}(1+it)|\leqq\|(x_0'-x_0)\eta_{\phi_0}(x_1)\|\,\|\Delta_{\phi_0}^{-1/2}\eta_{\phi_0}(x_2)\|.$$

For any compact set K contained in the half open strip $0\leqq\mathrm{Re}\,z<1$, there exists a real number β satisfying,

(12.21) $$0 \leqq \sup_{z \in K} \operatorname{Re} z \leqq \beta < 1 .$$

Hence due to (12.17) and (12.18),

(12.22) $$|F_{x,y_1}(z) - F_{x,y_2}(z)| \leqq \{\|(y_1 y_1^* y_2 y_2^*)\xi(\phi)\| \, \|x\| \, \|\xi(\phi)\|\}^{1-\gamma} \times \{2\|x_0 \eta_{\phi_0}(x_1)\| \, \|\eta_{\phi_0}(\sigma_{-i}^{\phi_0}(x_2^*))\|\}^{\gamma}$$

for $0 \leqq \operatorname{Re} z = \gamma \leqq \beta$. It follows that the convergence of limit in (12.12) is uniform on K. The compact convergence theorem implies that the limit is holomorphic in z for $0 < \operatorname{Re} z < 1$ and continuous in z for $0 \leqq \operatorname{Re} z \leqq 1$. For the proof of equality (12.13), we denote the right hand side of (12.13) by $G(z)$. Then $G(z)$ is continuous on $0 \leqq \operatorname{Re} z \leqq 1$ and holomorphic in the interior. By (12.15), the boundary values coinside:

(12.23) $$\begin{aligned} G(it) &= (\Delta_{\phi,\phi_0}^{1/2} \eta_{\phi_0}(u^* x_0 x_1), \Delta_{\phi,\phi_0}^{1/2} \eta_{\phi_0}((D\phi : D\phi_0)_t x_2^*)) \\ &= (x_2 (D\phi : D\phi_0)_t^* \xi(\phi), x_1^* x_0^* u \xi(\phi)) \\ &= (\Delta_{\phi_0}^{it} \xi(\phi), x^* u \xi(\phi)) = F_x(it). \end{aligned}$$

Hence $G(z) = F_x(z)$ for $0 \leqq \operatorname{Re} z < 1$. For $\operatorname{Re} z = 1$, the following direct calculation by using (12.16) proves the convergence (12.12) and equality (12.13):

(12.24) $$\begin{aligned} F_x(1+it) &= (\Delta_{\phi,\phi_0}^{-it} u^* x_0 \eta_{\phi_0}(x_1), \Delta_{\phi_0}^{1-it} \eta_{\phi_0}(x_2^*)) \\ &= G(1+it). \end{aligned}$$

Due to (12.19) and (12.20),

(12.25) $$|F_{x',y}(z) - F_{x,y}(z)| \leqq \{\|\xi(\phi)\| \, \|(x'-x)^* u \xi(\phi)\|\}^{1-\operatorname{Re} z} \times \{\|(x_0' - x_0)\eta_{\phi_0}(x_1)\| \, \|\Delta_{\phi_0}^{-1/2} \eta_{\phi_0}(x_2)\|\}^{\operatorname{Re} z} .$$

By taking the limit $y \to 1$ in (12.25), $|F_{x'}(z) - F_x(z)|$ is dominated by the right hand side of (12.25). Hence we have the second assertion.

Q.E.D.

Lemma 12.6. *The image of $D_{\phi_0}^{\infty}$ by the mapping T_p is norm dense in $L_p(M, \phi_0)$ for $1 < p < \infty$.*

Proof. By making use of the Hahn-Banach theorem, it is enough to show that for $S\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, $\langle T_p(\zeta),S\rangle_{\phi_0}=0$ for any $\zeta\in D_{\phi_0}^{\infty}$ implies $S=0$.

Let $S=u\Delta_{\phi,\phi_0}^{1/p'}$ be the polar decomposition. We assume

$$(12.26)\qquad \langle T_p(\eta_{\phi_0}(x)),S\rangle_{\phi_0} = \lim_{y\to 1}(x\Delta_{\phi_0}^{1/p}\eta_{\phi_0}(y),\, u\Delta_{\phi,\phi_0}^{1/p'}\eta_{\phi_0}(y))=0\,,\quad x=x_1x_2$$

for any $x_1, x_2\in M_0$, where $y\in M_0^{(1)}$ and the limit $y\to 1$ is in the sense of Lemma 3.4. By Lemma 12.4 and the strong density of M_0 in M, (12.26) also holds for $x=x_0x_1x_2$ with $x_0\in M$, $x_1, x_2\in M_0$. We put $x_0=u$. By (12.13), $(\Delta_{\phi,\phi_0}^{1/p'}\eta_{\phi_0}(x_1),\, \Delta_{\phi_0}^{1/p}\eta_{\phi_0}(x_2^*))=0$ for any $x_1, x_2\in M_0$. By putting $x_2=\sigma_{-i/p}^{\phi_0}(x_1^*)$, $\|\Delta_{\phi,\phi_0}^{1/(2p')}\eta_{\phi_0}(x_1)\|=0$ for any $x_1\in M_0$. Since $\eta_{\phi_0}(M_0)$ is a core for $\Delta_{\phi,\phi_0}^{\alpha}$, $0<\alpha\leqq 1/2$, we conclude $\phi=0$ and hence, $S=0$.

Q.E.D.

Lemma 12.7.

(1) *Let* $\zeta,\zeta'\in D_{\phi_0}^{\infty}$, $p^{-1}+(p')^{-1}=1$. *Then*

$$(12.27)\qquad \langle T_p(\zeta),T_{p'}(\zeta')\rangle_{\phi_0}=(\zeta,\zeta').$$

(2) *The following diagram is commutative.*

$$(12.28)\qquad \begin{array}{ccc} D_{\phi_0}^{\infty} & \xrightarrow{J_p^{\phi_0}} & D_{\phi_0}^{\infty} \\ \Big\downarrow T_p & & \Big\downarrow T_p \\ L_p(M,\phi_0) & \xrightarrow{*} & L_p(M,\phi_0), \end{array}\quad 1\leqq p\leqq\infty$$

where $J_p^{\phi_0}=J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}$ *and* $*$ *is the adjoint operation in* $L_p(M,\phi_0)$.

Proof. (1) Let

$$(12.29)\qquad \zeta=\sum_{j=1}^{n}a_j\eta_{\phi_0}(x_j^{(1)}x_j^{(2)}),\quad \zeta'=\sum_{k=1}^{m}b_k\eta_{\phi_0}(y_k^{(1)}y_k^{(2)})$$

where $a_j, b_k\in \mathbf{C}$, $x_j^{(l)}, y_k^{(l)}\in M_0$, $j=1,\cdots,n$, $k=1,\cdots,m$, $l=1,2$. By (2.10) and (2.3), we obtain (with the convention $y\in M_0^{(1)}$)

$$(12.30)\qquad \langle T_p(\zeta),T_{p'}(\zeta')\rangle_{\phi_0} = \sum_{j=1}^{n}\sum_{k=1}^{m}a_j\bar{b}_k\lim_{y\to 1}((\Delta_{\phi_0}^{1/(2p)}x_j^{(1)*})^*\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_j^{(2)}))}\eta_{\phi_0}(y), (\Delta_{\phi_0}^{1/(2p')}y_k^{(1)*})^*\overline{(\Delta_{\phi_0}^{1/(2p')}\sigma_{i/p'}^{\phi_0}(y_k^{(2)})}\eta_{\phi_0}(y))$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \bar{b}_k \lim_{y \to 1} (\Delta_{\phi_0}^{1/2} \sigma_{i/p}^{\phi_0}(x_j^{(1)} x_j^{(2)}) \eta_{\phi_0}(y),$$

$$\Delta_{\phi_0}^{1/2} \sigma_{i/p'}^{\phi_0}(y_k^{(1)} y_k^{(2)}) \eta_{\phi_0}(y))$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \bar{b}_k \lim_{y \to 1} (y^* \eta_{\phi_0}(\sigma_{i/p'}^{\phi_0}(y_k^{(1)} y_k^{(2)})^*),$$

$$y^* \eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(x_j^{(1)} x_j^{(2)})^*))$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \bar{b}_k (\eta_{\phi_0}(\sigma_{i/p'}^{\phi_0}(y_k^{(1)} y_k^{(2)})^*), \eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(x_j^{(1)} x_j^{(2)})^*))$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \bar{b}_k (\Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(x_j^{(1)} x_j^{(2)})),$$

$$\Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_{i/p'}^{\phi_0}(y_k^{(1)} y_k^{(2)})))$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \bar{b}_k (\eta_{\phi_0}(x_j^{(1)} x_j^{(2)}), \eta_{\phi_0}(y_k^{(1)} y_k^{(2)}))$$

$$= (\zeta, \zeta').$$

(2) The case $p=\infty$ is clear from Lemma 12.4 (1). So we assume $1 \leqq p < \infty$. Let ζ be of the form (12.29). Then

$$T_p(\zeta) = \sum_{k=1}^{n} a_k (\Delta_{\phi_0}^{1/(2p)} x_k^{(1)*})^* \overline{(\Delta_{\phi_0}^{1/(2p)} \sigma_{i/p}^{\phi_0}(x_k^{(2)}))}, \tag{12.31}$$

$$J_p^{\phi_0} \zeta = \sum_{k=1}^{n} \bar{a}_k J_{\phi_0} \Delta_{\phi_0}^{1/2} \eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(x_k^{(1)}) \sigma_{i/p}^{\phi_0}(x_k^{(2)})) \tag{12.32}$$

$$= \sum_{k=1}^{n} \bar{a}_k \eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(x_k^{(2)})^* \sigma_{i/p}^{\phi_0}(x_k^{(1)})^*),$$

$$T_p(J_p^{\phi_0} \zeta) = \sum_{k=1}^{n} \bar{a}_k (\Delta_{\phi_0}^{1/(2p)} \sigma_{i/p}^{\phi_0}(x_k^{(2)}))^* \overline{(\Delta_{\phi_0}^{1/(2p)} x_k^{(1)*})}. \tag{12.33}$$

It follows $T_p(\zeta)^* \eta_{\phi_0}(y) = T_p(J_p^{\phi_0} \zeta) \eta_{\phi_0}(y)$ for any $y \in M_0$ and hence, $\langle T_p(\zeta)^*, S \rangle_{\phi_0} = \langle T_p(J_p^{\phi_0} \zeta), S \rangle_{\phi_0}$ for any $S \in L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$. This shows (12.28). Q.E.D.

Next, we discuss the relation between $D_{\phi_0}^\infty \cap V_{\phi_0}^{1/(2p)}$ and $L_p^+(M, \phi_0)$ via mapping T_p.

Lemma 12.8. $T_p(D_{\phi_0}^\infty \cap V_{\phi_0}^{1/(2p)}) \subset L_p^+(M, \phi_0)$, $1 \leqq p \leqq \infty$.

Proof. We start with

$$\zeta=\sum_{k=1}^{n} a_k\eta_{\phi_0}(x_k^{(1)}x_k^{(2)})\in D_{\phi_0}^{\infty}\cap V_{\phi_0}^{1/(2p)}. \tag{12.34}$$

If,

$$y^*T_p(\zeta)\eta_{\phi_0}(y)\in V_{\phi_0}^{1/(2p)}, \quad y\in M_0 \tag{12.35}$$

hold, then by Lemma 10.2 and the polar relation of the positive cones (see Appendix A), $(y^*T_p(\zeta)\eta_{\phi_0}(y), \tilde{y}^*S\eta_{\phi_0}(\tilde{y}))\geqq 0$ for $S\in L_{p'}^{+}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$, and $\tilde{y}\in N_{\phi_0}$ such that $\eta_{\phi_0}(\tilde{y})\in D(S)$. Actually we can choose a $*$-strongly dense convex subset E of the unit ball of N_{ϕ_0} such that $\eta_{\phi_0}(E)\subset D(S)$ (see the proof of Lemma 9.2). By taking limits $y\to 1$ and $\tilde{y}\to 1$, we obtain $\langle T_p(\zeta), S\rangle_{\phi_0}\geqq 0$, $S\in L_{p'}^{+}(M,\phi_0)$ and by Lemma 10.3, we obtain the assertion. So we have only to prove (12.35). By the polar relation of positive cones and the density of the set of vectors $\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y})$, $\tilde{y}\in M_0$ in $V_{\phi_0}^{(1/2)-(1/2p)}$, we compute as follows.

$$\begin{aligned}
&(y^*T_p(\zeta)\eta_{\phi_0}(y), \Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y}))\\
&=(\sum_{k=1}^{n} a_k(\Delta_{\phi_0}^{1/(2p)}x_k^{(1)*})^*\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_k^{(2)}))}\,\eta_{\phi_0}(y),\\
&\qquad y\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y}))\\
&=(\sum_{k=1}^{n} a_k x_k^{(1)}x_k^{(2)}\Delta_{\phi_0}^{1/p}\eta_{\phi_0}(y), y\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y}))\\
&=(J_{\phi_0}\Delta_{\phi_0}^{(1/p)-(1/2)}y\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y}),\\
&\qquad J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}\sum_{k=1}^{n} a_k x_k^{(1)}x_k^{(2)}\Delta_{\phi_0}^{1/p}\eta_{\phi_0}(y)),
\end{aligned} \tag{12.36}$$

$$\begin{aligned}
J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}\tilde{x}\Delta_{\phi_0}^{1/p}\eta_{\phi_0}(y)&=S_{\phi_0}\sigma_{i/p}^{\phi_0}(\tilde{x})\eta_{\phi_0}(y)\\
&=y^*\eta_{\phi_0}(\sigma_{i/p}^{\phi_0}(\tilde{x})^*)\\
&=y^*J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}\eta_{\phi_0}(\tilde{x})\\
&=y^*J_{\phi_0}\Delta_{\phi_0}^{(1/2)-(1/p)}\zeta \quad (\tilde{x}=\sum_{k=1}^{n} a_k x_k^{(1)}x_k^{(2)})\\
&=y^*\zeta,
\end{aligned} \tag{12.37}$$

$$\begin{aligned}
&yJ_{\phi_0}\Delta_{\phi_0}^{(1/p)-(1/2)}y\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\eta_{\phi_0}(\tilde{y})\\
&=yJ_{\phi_0}\Delta_{\phi_0}^{1/(2p)}\sigma_{-i[(1/2p)-(1/2)]}^{\phi_0}(y)\,\tilde{y}^*\eta_{\phi_0}(\tilde{y})
\end{aligned} \tag{12.38}$$

$$= y\Delta_{\phi_0}^{(1/2)-(1/2p)}\tilde{y}^*\tilde{y}\eta_{\phi_0}(\sigma^{\phi_0}_{-i[(1/2p)-(1/2)]}(y)^*)$$
$$= \Delta_{\phi_0}^{(1/2)-(1/2p)}\sigma^{\phi_0}_{-i[(1/2p)-(1/2)]}(y)\tilde{y}^*$$
$$\times\eta_{\phi_0}(\{\sigma^{\phi_0}_{-i[(1/2p)-(1/2)]}(y)\tilde{y}^*\}^*)\in V_{\phi_0}^{(1/2)-(1/2p)}$$

where we used the $J_p^{\phi_0}$-invariance of ζ in the last equality of (12.37). By applying (12.37) and (12.38) to (12.36), we get (12.35).

Q.E.D.

§ 13. Group Action on L_p-Spaces

On the standard representation of a given von Neumann algebra M by a faithful normal semifinite weight ϕ_0, the associated modular action $\sigma_t^{\phi_0}$ on M plays an important role in the structure theory of the von Neumann algebra M ([7], [11], [28]). This one parameter action can be extended to L_p-spaces. In this section, we discuss the group action on L_p-spaces. Throughout this section, we fix a W^*-dynamical system (M, G, α). The modular action on L_p-spaces can be considered as its special case of this discussion.

Lemma 13.1. *Let $T\in L_p(M, \phi_0)$. Then*

(13.1.a) $$\alpha_g^{(p)}(T)=\alpha_g(u)\Delta_{\phi_g,\phi_0}^{1/p}, \quad 1\leqq p<\infty,$$

(13.1.b) $$\alpha_g^{(\infty)}(x)=\alpha_g(x), \quad x\in L_\infty(M, \phi_0)=M,$$

defines an isometric G-action on $L_p(M, \phi_0)$, where $T=u\Delta_{\phi,\phi_0}^{1/p}$ is the polar decomposition and $\phi_g(x)=\phi\circ\alpha_{g^{-1}}(x)$, $x\in M$. furthermore, this action $\alpha_g^{(p)}$ ($1\leqq p\leqq\infty$) preserves the adjoint operation and the positive part $L_p^+(M, \phi_0)$.

Proof. The case $p=\infty$ is the α-action on M. For the case $1\leqq p<\infty$, the group property is checked by direct computation. It is also immediately seen that this action preserves the L_p-norm, the adjoint operation and the positive part $L_p^+(M, \phi_0)$. Q.E.D.

Now, recall that the continuous group action is implemented by the strongly continuous unitary representation on the standard form of von

Neumann algebra (see [15]). Furthermore, this unitary representation preserves the natural positive cone. The unitary representation is actually defined by the following family of unitaries and the identification of two GNS representations H_{ϕ_0} and $H_{\phi_{0,g}}$ ($\phi_{0,g}=\phi_0\circ\alpha_{g^{-1}}$) through $\xi(\phi)=\xi_g(\phi)$

$$U(g): H_{\phi_0}\to H_{\phi_0} \tag{13.2}$$

$$U(g)\xi(\phi)=\xi(\phi_g),\quad \phi_g=\phi\circ\alpha_{g^{-1}}, \tag{13.3}$$

where $\xi(\phi)$, $\xi(\phi_g)\in\mathcal{P}^{\natural}_{\phi_0}$ are the (unique) representative vectors of ϕ and ϕ_g, respectively.

Lemma 13.2. $U(g)\Delta^z_{\phi,\phi_0}U(g)^*=\Delta^z_{\phi_g,\phi_{0,g}}$, $z\in\boldsymbol{C}$, $g\in G$.

Proof. Let $y\in N_{\phi_{0,g}}$. Then

$$\begin{aligned} U(g)\Delta^{1/2}_{\phi,\phi_0}U(g)^*\eta_{\phi_{0,g}}(y) &= U(g)\Delta^{1/2}_{\phi,\phi_0}\eta_{\phi_0}(\alpha_{g^{-1}}(y)) \\ &= U(g)J_{\phi_0}\alpha_{g^{-1}}(y^*)\xi(\phi) \\ &= J_{\phi_0}U(g)\alpha_{g^{-1}}(y^*)\xi(\phi) \\ &= J_{\phi_0}y^*\xi(\phi_g) \\ &= \Delta^{1/2}_{\phi_g,\phi_{0,g}}\eta_{\phi_{0,g}}(y) \end{aligned} \tag{13.4}$$

where we used $U(g)J_{\phi_0}=J_{\phi_0}U(g)$ for the third equality and (13.3) for the fourth equality. By the fact that the set $\eta_{\phi_{0,g}}(N_{\phi_{0,g}})=U(g)\eta_{\phi_0}(N_{\phi_0})$ is a core for the both of operators, we obtain the assertion for $z=1/2$. The general case follows from functional calculus of self-adjoint operators.

Q.E.D.

Lemma 13.3. (1) $\alpha_g^{(p)}(T)=\tau_p(\phi_0,\phi_{0,g})(U(g)TU(g)^*)$, $g\in G$, $T\in L_p(M,\phi_0)$.

(2) *Let* $T_k\in L_{p_k}(M,\phi_0)$, $k=1,\cdots,n$ *and* $\sum_{k=1}^n p_k^{-1}=p^{-1}\leqq 1$. *Then* $\alpha_g^{(p)}(T)=\alpha_g^{(p_1)}(T_1)\cdots\alpha_g^{(p_n)}(T_n)$ *if* $T=T_1\cdots T_n$.

(3) *The mapping* $T\in L_1(M,\phi_0)\to[T]_{\phi_0}$ *is G-invariant.*

Proof. (1) The case $p=\infty$ follows from the definition. The case $1\leqq p<\infty$ follows from Lemma 13.2 for $z=1/p$ and Lemma 9.3:

(2) By the definition of the product, $U(g)TU(g)^* = U(g)T_1U(g)^* \cdots U(g)T_nU(g)^*$ in $L_p(M, \phi_{0,g})$. Hence by (1) and Lemmas 9.3 and 11.1, we obtain,

$$
\begin{aligned}
(13.5) \qquad \alpha_g^{(p)}(T_1\cdots T_n) &= \tau_p(\phi_0, \phi_{0,g})(U(g)T_1\cdots T_nU(g)^*) \\
&= \tau_{p_1}(\phi_0, \phi_{0,g})(U(g)T_1U(g)^*)\cdots \\
&\qquad \times \tau_{p_n}(\phi_0, \phi_{0,g})(U(g)T_nU(g)^*) \\
&= \alpha_g^{(p_1)}(T_1)\cdots\alpha_g^{(p_n)}(T_n).
\end{aligned}
$$

(3) Let $T \in L_1(M, \phi_0)$, $T = u\Delta_{\phi,\phi_0}$. Then by (9.12) and (10.9), $[T]_{\phi_0} = \phi(u)$. Hence $[\alpha_g^{(1)}(T)]_{\phi_0} = \phi\circ\alpha_{g^{-1}}(\alpha_g(u)) = \phi(u) = [T]_{\phi_0}$.

Q.E.D.

Corollary. *Let* $T \in L_p(M, \phi_0)$, $S \in L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$. *Then* $[\alpha_g^{(p)}(T)S]_{\phi_0} = [T\alpha_{g^{-1}}^{(p')}(S)]_{\phi_0}$.

Lemma 13.4. *For a fixed* $T \in L_p(M, \phi_0)$, *the mapping*

$$
(13.6) \qquad \begin{array}{ccl} G & \to & L_p(M, \phi_0) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ g & \mapsto & \alpha_g^{(p)}(T), \quad 1 \leqq p < \infty \end{array}
$$

is norm continuous.

Proof. By the isomorphism $L_2(M, \phi_0) \cong H_{\phi_0}$ and the commutativity of the diagram

$$
(13.7) \qquad \begin{array}{ccc} L_2(M, \phi_0) & \xrightarrow{\alpha_g^{(2)}} & L_2(M, \phi_0) \\ \| \wr & & \| \wr \\ H_{\phi_0} & \xrightarrow{U(g)} & H_{\phi_0}, \end{array}
$$

the assertion holds for $p=2$. Let $T = u\Delta_{\phi,\phi_0} \in L_1(M, \phi_0)$ and $T_1 = u\Delta_{\phi,\phi_0}^{1/2}$, $T_2 = \Delta_{\phi,\phi_0}^{1/2} \in L_2(M, \phi_0)$. Then

$$
\begin{aligned}
(13.8) \qquad \|\alpha_g^{(1)}(T) - T\|_{1,\phi_0} &= \|\alpha_g^{(2)}(T_1)\alpha_g^{(2)}(T_2) - T_1T_2\|_{1,\phi_0} \\
&\leqq \|\alpha_g^{(2)}(T_1) - T_1\|_{2,\phi_0}\|T_2\|_{2,\phi_0} \\
&\qquad + \|\alpha_g^{(2)}(T_1)\|_{2,\phi_0}\|\alpha_g^{(2)}(T_2) - T_2\|_{2,\phi_0} \to 0 \quad \text{as} \quad g \to e\,.
\end{aligned}
$$

Hence we obtain the assertion for $p=1$. Now, we assume $1 < p < \infty$.

Let $T=u\Delta_{\phi,\phi_0}^{1/p}\in L_p(M,\phi_0)$, $S=v\Delta_{\psi,\phi_0}^{1/p'}\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$. We define the function $F(z)$ by

$$(13.9)\qquad F(z)=e^{z^2}(v\Delta_{\psi,\phi_0}^{1-z}[\alpha_g(u)\Delta_{\phi_g,\phi_0}^{z}-u\Delta_{\phi,\phi_0}^{z}]\eta_{\phi_0}(y),\eta_{\phi_0}(y)),$$

where $g\in G$, $y\in M_0^{(1)}$, $z\in\boldsymbol{C}$ satisfying $0\leqq\mathrm{Re}\, z\leqq 1$ and the inner product in the right hand side is in the sense of the linear combination of Notation 3.3. Then the function $F(z)$ is holomorphic on the open strip $\{z: 0<\mathrm{Re}\, z<1\}$ and continuous on its closure. The estimates on one of the boundaries is the following:

$$(13.10)\qquad |F(it)|=e^{-t^2}|(v\Delta_{\psi,\phi_0}^{1-it}[\alpha_g(u)\Delta_{\phi_g,\phi_0}^{it}-u\Delta_{\phi,\phi_0}^{it}]\eta_{\phi_0}(y),n_{\phi_0}(y))|$$
$$\leqq 2e^{-t^2}\psi(1)\leqq 2\psi(1)\quad (0\leqq\|y\|\leqq 1).$$

Hence, if we have the next formula:

$$(13.11)\qquad \lim_{g\to e}\sup_{\psi(1)=1}\sup_{t\in\boldsymbol{R}}|F(1+it)|=0,$$

then due to

$$(13.12)\qquad \|\alpha_g^{(p)}(T)-T\|_{p,\phi_0}=\sup_{\|S\|_{p',\phi_0}=1}|[S(\alpha_g^{(p)}(T)-T]_{\phi_0}|$$
$$=\sup_{\psi(1)=1}e^{-(1/p)^2}|\lim_{y\to 1}F(1/p)|$$
$$\leqq e^{-(1/p)^2}2^{1/p'}(\sup_{\psi(1)=1}\sup_{t\in\boldsymbol{R}}|F(1+it)|)^{1/p},$$

where inequality is due to (13.10) and the three line theorem. Hence we obtain the assertion for $1<p<\infty$. Now, we show (13.11). By the repeated use of Lemma 13.2 and the formula $U(g)^*\eta_{\phi_0,g}(y)=\eta_{\phi_0}(\alpha_{g^{-1}}(y))$,

$$(13.13)\qquad (v\Delta_{\psi,\phi_0}^{-it}\alpha_g(u)\Delta_{\phi_g,\phi_0}^{1+it}\eta_{\phi_0}(y),\eta_{\phi_0}(y))$$
$$=(\Delta_{\phi_g,\phi_0}^{1/2}\eta_{\phi_0}(y),$$
$$\Delta_{\phi_g,\phi_0}^{1/2}(D\phi_g:D\phi_0)_{-t}\sigma_{-t}^{\phi_0}\circ\alpha_g(u^*)(D\psi:D\phi_0)_{-t}^*v^*\eta_{\phi_0}(y))$$
$$=(y^*v(D\psi:D\phi_0)_{-t}\sigma_{-t}^{\phi_0}\circ\alpha_g(u)(D\phi_g:D\phi_0)_{-t}^*\xi(\phi_g),y^*\xi(\phi_g))$$
$$=(yy^*v\Delta_{\psi,\phi_0}^{-it}U(g)uU(g)^*\Delta_{\phi_g,\phi_{0,g}}^{it}U(g)\xi(\phi),U(g)\xi(\phi))$$
$$=(yy^*v\Delta_{\psi,\phi_0}^{-it}U(g)u\Delta_{\phi,\phi_0}^{it}\xi(\phi),U(g)\xi(\phi)).$$

It follows that

$$(13.14)\quad |F(1+it)| = e^{1-t^2}|(v\Delta_{\psi,\phi_0}^{-it}[\alpha_g(u)\Delta_{\phi_g,\phi_0}^{1+it} - u\Delta_{\phi,\phi_0}^{1+it}]\eta_{\phi_0}(y), \eta_{\phi_0}(y))|$$
$$= e^{1-t^2}|(yy^*v\Delta_{\psi,\phi_0}^{-it}U(g)u\Delta_{\phi,\phi_0}^{it}\xi(\phi), U(g)\xi(\phi))$$
$$-(yy^*v\Delta_{\psi,\phi_0}^{-it}u\Delta_{\phi,\phi_0}^{it}\xi(\phi), \xi(\phi))|$$
$$= e^{1-t^2}|(yy^*v\Delta_{\psi,\phi_0}^{-it}U(g)u\Delta_{\phi,\phi_0}^{it}\xi(\phi), (U(g)-1)\xi(\phi))$$
$$+(yy^*v\Delta_{\psi,\phi_0}^{-it}(U(g)-1)u\Delta_{\phi,\phi_0}^{it}\xi(\phi), \xi(\phi))|$$
$$\leqq e\|\xi(\phi)\|\,\|(U(g)-1)\xi(\phi)\|$$
$$+e^{1-t^2}\|(U(g)-1)u\Delta_{\phi,\phi_0}^{it}\xi(\phi)\|\,\|\xi(\phi)\|.$$

The first term of the right hand side of (13.14) tends to zero uniformly in t if g tends to unit e due to the strong continuity of U. Due to the existence of converging factor e^{1-t^2}, we may consider the convergence of the second term for t in compact set. In that case, due to the continuity of $t\mapsto u\Delta_{\phi,\phi_0}^{it}\xi(\phi)$, the strong continuity of U, and the compactness of the set in which t varies, the second term also tends to zero uniformly in t if g tends to e. This shows (13.11) and completes the proof of the assertion. Q.E.D.

Remark 13.5. If we assume that the reference weight ϕ_0 is relatively G-invariant i.e. there exists a continuous positive character χ: $G\to\boldsymbol{R}_+$ such that

$$(13.15)\quad \phi_0\circ\alpha_g(x) = \chi(g)\phi_0(x),\quad x\in M_+,\ g\in G$$

(see [28]). Then the modular action $\sigma_t^{\phi_0}$ and α_g commute and the Tomita algebra $\mathfrak{A}$ is α-invariant. In this situation the unitary representation U of G on H_{ϕ_0} by (13.3) satisfies

$$(13.16)\quad U(g)\eta_{\phi_0}(y) = \chi(g)^{-1/2}\eta_{\phi_0}(\alpha_g(y)),\quad y\in N_{\phi_0},\ g\in G.$$

Furthermore, $D_{\phi_0}^\infty$ is U-invariant and the followings hold.

(1) $U(g)\Delta_{\phi,\phi_g}^z U(g)^* = \chi(g)^z\Delta_{\phi_g,\phi_0}^z$, $z\in\mathbb{C}$, $g\in G$.

(2) $\alpha_g^{(p)}(T) = \chi(g)^{-1/p}U(g)TU(g)^*$, $g\in G$, $T\in L_p(M,\phi_0)$.

(3) The following diagram is commutative.

$$(13.17)\quad \begin{array}{ccc} D_{\phi_0}^\infty & \xrightarrow{U_p(g)} & D_{\phi_0}^\infty \\ \downarrow T_p & & \downarrow T_p \\ L_p(M,\phi_0) & \xrightarrow{\alpha_g^{(p)}} & L_p(M,\phi_0), \end{array}$$

where $U_p(g)=\chi(g)^{(1/2)-(1/p)}U(g)$.

Proof. (1) follows from Lemma 13.2, (13.15) and the homogenity of relative modular operators.

(2) follows from (1) and Lemma 13.3 (1).

(3) Let $\zeta\in D_{\phi_0}^{\infty}$ be of the form (12.29) and $S\in L_{p'}(M,\phi_0)$, $p^{-1}+(p')^{-1}=1$. Then by Corollary after Lemma 13.3,

$$
\begin{aligned}
(13.18)\qquad [S\alpha_g^{(p)}(T_p(\zeta))]_{\phi_0} &= [\alpha_{g^{-1}}^{(p')}(S)T_p(\zeta)]_{\phi_0}\\
&=\lim_{y\to 1}(\chi(g)^{1/p'}U(g)^*SU(g)[\sum_{k=1}^{n}a_k(\Delta_{\phi_0}^{1/(2p)}x_k^{(1)*})^*\\
&\quad\times\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(x_k^{(2)})}]\eta_{\phi_0}(y),\eta_{\phi_0}(y))\\
&=\lim_{y\to 1}(S\sum_{k=1}^{n}\chi(g)^{(1/p')-1}a_k(\Delta_{\phi_0}^{1/(2p)}\alpha_g(x_k^{(1)})^*)^*\\
&\quad\times\overline{(\Delta_{\phi_0}^{1/(2p)}\sigma_{i/p}^{\phi_0}(\alpha_g(x_k^{(2)}))}))\eta_{\phi_0}(\alpha_g(y)),\eta_{\phi_0}(\alpha_g(y)))\\
&=[ST_p(U_p(g)\zeta)]_{\phi_0}.
\end{aligned}
$$

Hence we obtain (13.17). Q.E.D.

§ 14. Proof of Theorems

Theorem 1 (1) is shown in Lemma 4.3 (also see Definition 2.2) for $p=\infty$, Lemma 5.5 for $2\leqq p<\infty$, Lemma 7.1 for $p=1$ and Lemma 8.4 for $1<p<2$. (2) for $1\leqq p\leqq 2$ is shown in Lemmas 6.2 and 8.3 (2). (2) for $2<p<\infty$ follows from Lemma 5.5, (2) for $1<p<2$, and the reflexivity which is implied by the uniform convexity of (3). (3) is shown in Lemmas 8.1 and 8.2.

Theorem 2 (1), (2) for $1\leqq p<\infty$ follows from Lemmas 4.1, 4.2 and (2.6). The case $p=\infty$ for (1), (2) follows from the usual polar decomposition and Theorem 3 (1). (3) follows from Lemma 4.1, (2.6) and the uniqueness of the polar decomposition of closed operators. (4) is shown in the Remark after Lemma 4.2.

Theorem 3 (1) is shown in Lemma 4.3 and (2) is shown in Lemma 7.1. (3) is shown in Lemma 7.2 and (7.5) in the proof of Lemma

7.2.

Theorem 4 is shown in Corollary of Lemma 10.1.

Theorem 5 (1) is (10.10). (2) is shown in Lemma 10.3, (2.9) and Theorem 2 (3).

Theorem 6 (1), (2) are shown in Lemma 11.3. (3) is shown as follows;

$$(14.1) \qquad \begin{aligned}[] [T_1\cdots T_k, T_{k+1}\cdots T_n]_{\phi_0} &= \langle T_1\cdots T_k, (T_{k+1}\cdots T_n)^*\rangle_{\phi_0} \\ &= \langle T_{k+1}\cdots T_n, (T_1\cdots T_k)^*\rangle_{\phi_0} \\ &= \langle T_1\cdots T_k T_{k+1}\cdots T_n, 1\rangle_{\phi_0} \\ &= [T_1\cdots T_n, 1]_{\phi_0}\end{aligned}$$

where we used Remark after Lemma 9.2 for the second equality. (4) is a consequence of Notation 10.4 and Lemma 11.3. (5) follows from (2.4), (2.9), Lemma 11.3 (4) and $\|1\|_{\infty,\phi_0}=1$.

Proof of Corollary. Follows directly from the method of [9].

Theorem 7 (1) is obtained in Lemmas 9.2 and 9.3. And two equations of (2) can be identified with (9.6) and (9.18) if polar decompositions of T and T' are submitted into explicit definitions (9.2), (9.16) of $J_p(\tilde{\phi}_0, \phi_0)$ and $\tau_p(\tilde{\phi}_0, \phi_0)$, and by the fact that $[\,,\,]$ is transformed into $\langle\,,\,\rangle$ by (2.9).

(3) for J is (11.1). (3) for τ follows from (3) for J and Lemma 9.3.

Theorem 8 (1) is shown in Lemma 12.3 for $p=1, 2, \infty$ and in Lemma 12.5 for $1<p<\infty$. (2) is shown in Lemma 12.8. (3) and (4) are shown in Lemma 12.6.

§ 15. Discussions

The L_p-space $L_p(M, \phi_0)$ we constructed in this paper is isomorphic to that of L_p-spaces developed by Connes-Hilsum [12, 16], Kosaki [19, 20] and Terp [29]. Actually, our L_p-space is identical to $L_p(M, \phi_0(J\cdot J))$ in the notation of Hilsum [16] due to the equality $\frac{d\phi}{d\phi_0(J\cdot J)} = \Delta_{\phi,\phi_0}$ where $\phi\in M_*^+$ and ϕ_0 is the fixed faithful normal semifinite weight.

As we have already mentioned, the result stated in this paper is in some sense a straightforward generalization of the results obtained in our previous paper [8]. So, the fundamental idea and the main tools are analogous to [8]. The only non trivial part is the density argument and the method of taking limit (for example, Lemma 3.4). If ϕ_0 is bounded then $1\in N_{\phi_0}$ and $\eta_{\phi_0}(y) = y\xi(\phi_0)$, $y\in N_{\phi_0}$. Hence the density argument as well as the process of taking limit is not necessary. So we have the previous L_p-theory through the mapping $T = u\Delta_{\phi,\phi_0}^{1/p} \mapsto T\xi(\phi_0) = u\Delta_{\phi,\phi_0}^{1/p}\xi(\phi_0)$. On the other hand, if M is not σ-finite, a cyclic and separating vector is not available and the method of [8] is not directly applicable. In the present approach, we first realize the L_p-space as the linear space of closed linear operators acting on the standard representation Hilbert space. We also find out that the L_p-space can be obtained as the completion of a certain vector subspace $D_{\phi_0}^\infty$ of a Tomita algebra associated with ϕ_0 with respect to the L_p-norm.

Appendix A

In this section, we extend the notion of positive cones for a faithful normal semifinite weight and show their properties which are more or less known.

Let M be a von Neumann algebra with a faithful normal semifinite weight ϕ_0. For $0\leqq\alpha\leqq 1/2$, we define the set,

$$\tilde{V}_{\phi_0}^\alpha \equiv \{\Delta_{\phi_0}^\alpha x^*\eta_{\phi_0}(x) : x\in M_0\}, \tag{A.1}$$

where M_0 is the set of all entire analytic elements of $N_{\phi_0}^*\cap N_{\phi_0}$ with respect to the modular action $\sigma_t^{\phi_0}$. We define the positive cone $V_{\phi_0}^\alpha$, $0\leqq\alpha\leqq 1/2$, to be the closure of $\tilde{V}_{\phi_0}^\alpha$ in H_{ϕ_0}. Note that $\mathfrak{A} = \eta_{\phi_0}(N_{\phi_0}^*\cap N_{\phi_0})$ is an achieved Hilbert algebra and $\mathfrak{A}_0 = \eta_{\phi_0}(M_0)$ is a maximal Tomita algebra equivalent to $\mathfrak{A}$. By the argument of Perdrizet and Haagerup ([23], [15]), $\mathcal{P}^{\sharp}_{\phi_0}=V^0_{\phi_0}$, $\mathcal{P}^{\natural}_{\phi_0}=V^{1/4}_{\phi_0}$ and $\mathcal{P}^{\flat}_{\phi_0}=V^{1/2}_{\phi_0}$. Furthermore, $\mathcal{P}^{\sharp}_{\phi_0}$ and $\mathcal{P}^{\flat}_{\phi_0}$ are dual cones each other, and $\mathcal{P}^{\natural}_{\phi_0}$ is selfdual.

Lemma A.1. (1) *$V^{\alpha}_{\phi_0}$ is a pointed weakly closed convex cone invariant under $\Delta^{it}_{\phi_0}$, $t\in\boldsymbol{R}$.*

(2) *$V^{\alpha}_{\phi_0}\subset D(\Delta^{(1/2)-2\alpha}_{\phi_0})$ and $J_{\phi_0}\Delta^{(1/2)-2\alpha}_{\phi_0}\xi=\xi$ for any $\xi\in V^{\alpha}_{\phi_0}$.*

(3) *$\Delta^{\alpha}_{\phi_0}V^0_{\phi_0}$ is a dense subset of $V^{\alpha}_{\phi_0}$.*

(4) *$J_{\phi_0}V^{\alpha}_{\phi_0}=V^{(1/2)-\alpha}_{\phi_0}$.*

(5) *$V^{\alpha}_{\phi_0}$ and $V^{(1/2)-\alpha}_{\phi_0}$ are the dual cone of each other.*

Proof. (2) Let $\zeta=\Delta^{\alpha}_{\phi_0}x^*\eta_{\phi_0}(x)$ with $x\in M_0$. Then,

$$J_{\phi_0}\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta=J_{\phi_0}\Delta^{(1/2)-\alpha}_{\phi_0}\eta_{\phi_0}(x^*x)=\Delta^{\alpha}_{\phi_0}J_{\phi_0}\Delta^{1/2}_{\phi_0}\eta_{\phi_0}(x^*x)=\Delta^{\alpha}_{\phi_0}\eta_{\phi_0}(x^*x)=\zeta\,. \tag{A.2}$$

It follows that $\widetilde{V}^{\alpha}_{\phi_0}\subset D(\Delta^{(1/2)-2\alpha}_{\phi_0})$ and any element in $\widetilde{V}^{\alpha}_{\phi_0}$ is pointwise fixed under $J_{\phi_0}\Delta^{(1/2)-2\alpha}_{\phi_0}$. Let $\zeta\in V^{\alpha}_{\phi_0}$. There exists a sequence $\{\zeta_n\}\subset\widetilde{V}^{\alpha}_{\phi_0}$ which converges to ζ; $J_{\phi_0}\zeta_n$ converges to $J_{\phi_0}\zeta$ and $\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta_n=J_{\phi_0}\zeta_n$ by (A.2). By the closedness of $\Delta^{(1/2)-2\alpha}_{\phi_0}$, it follows that $\zeta\in D(\Delta^{(1/2)-2\alpha}_{\phi_0})$ and $\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta=J_{\phi_0}\zeta$. This shows the assertion.

(3) By (2), $V^0_{\phi_0}\subset D(\Delta^{1/2}_{\phi_0})$ and hence, the expression $\Delta^{\alpha}_{\phi_0}V^0_{\phi_0}$ is well defined. By definition, $\Delta^{\alpha}_{\phi_0}\widetilde{V}^0_{\phi_0}=\widetilde{V}^{\alpha}_{\phi_0}$ is a dense subset of $V^{\alpha}_{\phi_0}$. Let $\zeta\in V^0_{\phi_0}$. There exists a sequence $\{\zeta_n\}\subset\widetilde{V}^0_{\phi_0}$ which converges to ζ. Again by (2), $\Delta^{1/2}_{\phi_0}\zeta=J_{\phi_0}\zeta$ and $\Delta^{1/2}_{\phi_0}\zeta_n=J_{\phi_0}\zeta_n$ hence $\Delta^{1/2}_{\phi_0}\zeta_n$ converges to $\Delta^{1/2}_{\phi_0}\zeta$. Due to the inequality $\|\Delta^{\alpha}_{\phi_0}\xi\|\leqq\|\Delta^{1/2}_{\phi_0}\xi\|^{2\alpha}\|\xi\|^{1-2\alpha}$, $\Delta^{\alpha}_{\phi_0}\zeta_n$ converges to $\Delta^{\alpha}_{\phi_0}\zeta$. Due to $\Delta^{\alpha}_{\phi_0}\zeta_n\in V^{\alpha}_{\phi_0}$, we conclude that $\Delta^{\alpha}_{\phi_0}V^0_{\phi_0}\subset V^{\alpha}_{\phi_0}$.

(4) Let $\zeta=\Delta^{\alpha}_{\phi_0}x^*\eta_{\phi_0}(x)$ with $x\in M_0$. Then,

$$J_{\phi_0}\zeta=J_{\phi_0}\Delta^{\alpha}_{\phi_0}x^*\eta_{\phi_0}(x)=\Delta^{(1/2)-\alpha}_{\phi_0}J_{\phi_0}\Delta^{1/2}_{\phi_0}\eta_{\phi_0}(x^*x)=\Delta^{(1/2)-\alpha}_{\phi_0}\eta_{\phi_0}(x^*x)\in V^{(1/2)-\alpha}_{\phi_0}\,. \tag{A.3}$$

It follows $J_{\phi_0}\widetilde{V}^{\alpha}_{\phi_0}\subset V^{(1/2)-\alpha}_{\phi_0}$ and hence $J_{\phi_0}V^{\alpha}_{\phi_0}\subset V^{(1/2)-\alpha}_{\phi_0}$. We replace α by $(1/2)-\alpha$ and multiply J_{ϕ_0} on both sides and get $V^{(1/2)-\alpha}_{\phi_0}\subset J_{\phi_0}V^{\alpha}_{\phi_0}$. Hence we have the assertion.

(5) Let $\zeta_1=\Delta_{\phi_0}^{\alpha}x^*\eta_{\phi_0}(x)$, $\zeta_2=\Delta_{\phi_0}^{(1/2)-\alpha}y^*\eta_{\phi_0}(y)$ with $x, y\in M_0$. Then,

$$(\text{A. }4)\qquad \begin{aligned}(\zeta_1, \zeta_2) &= (\Delta_{\phi_0}^{\alpha}x^*\eta_{\phi_0}(x), \Delta_{\phi_0}^{(1/2)-\alpha}y^*\eta_{\phi_0}(y)) \\ &= (\Delta_{\phi_0}^{1/2}x^*\eta_{\phi_0}(x), y^*\eta_{\phi_0}(y)) \\ &= (J_{\phi_0}x^*\eta_{\phi_0}(x), y^*\eta_{\phi_0}(y)) \\ &= (J_{\phi_0}\eta_{\phi_0}(x), J_{\phi_0}xJ_{\phi_0}y^*\eta_{\phi_0}(y)) \\ &= (yJ_{\phi_0}\eta_{\phi_0}(x), J_{\phi_0}xJ_{\phi_0}\eta_{\phi_0}(y)) \\ &= (yJ_{\phi_0}\eta_{\phi_0}(x), yJ_{\phi_0}\eta_{\phi_0}(x))\geqq 0\,,\end{aligned}$$

where the last equality is due to $J_{\phi_0}xJ_{\phi_0}\eta_{\phi_0}(y)=yJ_{\phi_0}\eta_{\phi_0}(x)$ for $x, y\in N_{\phi_0}$. It follows that $\zeta_1\in V_{\phi_0}^{\alpha}$, $\zeta_2\in V_{\phi_0}^{(1/2)-\alpha}$ implies $(\zeta_1, \zeta_2)\geqq 0$, and hence,

$$(\text{A. }5)\qquad (V_{\phi_0}^{\alpha})^{\circ}\equiv\{\xi\in H_{\phi_0}: (\xi, \zeta)\geqq 0 \quad \text{for any } \zeta\in V_{\phi_0}^{\alpha}\}\supset V_{\phi_0}^{(1/2)-\alpha}.$$

To prove the converse inclusion, we assume $\xi\in(V_{\phi_0}^{\alpha})^{\circ}$. Then $(\xi, \zeta)\geqq 0$ for any $\zeta\in\tilde{V}_{\phi_0}^{\alpha}$. By definition, $\tilde{V}_{\phi_0}^{\alpha}$ is $\Delta_{\phi_0}^{it}$-invariant and hence, $(\Delta_{\phi_0}^{it}\xi, \zeta)\geqq 0$ for any $\zeta\in\tilde{V}_{\phi_0}^{\alpha}$. If we put,

$$(\text{A. }6)\qquad \xi_n=(n/\pi)^{1/2}\int_{\boldsymbol{R}}e^{-nt^2}\Delta_{\phi_0}^{it}\xi\, dt\,,$$

then ξ_n is an entire analytic element of Δ_{ϕ_0} and $(\xi_n, \zeta)\geqq 0$ for any $\zeta\in\tilde{V}_{\phi_0}^{\alpha}$ and hence, $(\Delta_{\phi_0}^{\alpha}\xi_n, \zeta)\geqq 0$ for any $\zeta\in\tilde{V}_{\phi_0}^{0}$. By the density of $\tilde{V}_{\phi_0}^{0}$ in $V_{\phi_0}^{0}=\mathcal{P}_{\phi_0}^{\sharp}$ and the dual relation between $\mathcal{P}_{\phi_0}^{\sharp}$ and $\mathcal{P}_{\phi_0}^{\flat}=V_{\phi_0}^{(1/2)}$, $\Delta_{\phi_0}^{\alpha}\xi_n\in V_{\phi_0}^{(1/2)}$. By (4), $J_{\phi_0}\Delta_{\phi_0}^{\alpha}\xi_n\in V_{\phi_0}^{0}$ and hence $J_{\phi_0}\xi_n=\Delta_{\phi_0}^{\alpha}J_{\phi_0}\Delta_{\phi_0}^{\alpha}\xi_n\in V_{\phi_0}^{\alpha}$. Again by (4), $\xi_n\in J_{\phi_0}V_{\phi_0}^{\alpha}=V_{\phi_0}^{(1/2)-\alpha}$. By (A. 6), ξ_n converges to ξ. Since $V_{\phi_0}^{(1/2)-\alpha}$ is closed, we conclude $\xi\in V_{\phi_0}^{(1/2)-\alpha}$.

(1) By (5), we can see easily that $V_{\phi_0}^{\alpha}$ is weakly closed and convex. The $\Delta_{\phi_0}^{it}$-invariance of $V_{\phi_0}^{\alpha}$ is immediate. To see that $V_{\phi_0}^{\alpha}$ is pointed, assume $\xi\in V_{\phi_0}^{\alpha}\cap\{-V_{\phi_0}^{\alpha}\}$. Then by (5), $\xi\perp V_{\phi_0}^{(1/2)-\alpha}$. The linear span of $V_{\phi_0}^{(1/2)-\alpha}$ contains the linear span of $\tilde{V}_{\phi_0}^{(1/2)-\alpha}$, so it contains $\{\eta_{\phi_0}(xy): x, y\in M_0\}$ and hence, $V_{\phi_0}^{(1/2)-\alpha}$ is total in H_{ϕ_0}. It follows $\xi=0$ and $V_{\phi_0}^{\alpha}$ is pointed. Q.E.D.

Appendix B

In this section, we discuss the polar decomposition in $D(\Delta_{\phi_0}^{(1/2)-2\alpha})$ in terms of the positive cone $V_{\phi_0}^{\alpha}$, $0\leqq\alpha\leqq 1/2$ (see Appendix A for the positive cones) for a faithful normal semifinite weight ϕ_0 on a von Neumann algebra M. The results are similar to but weaker than that obtained in [8] due to the fact that we don't assume the boundedness of ϕ_0.

Theorem B.1. *Let $\zeta\in D(\Delta_{\phi_0}^{(1/2)-2\alpha})$, $0\leqq\alpha\leqq 1/2$. Then there exist a partial isometry $u\in M$ (resp. $u'\in M'\equiv J_{\phi_0}MJ_{\phi_0}$) and $|\zeta|_\alpha\in V_{\phi_0}^{\alpha}$ such that $\zeta=u|\zeta|_\alpha$ and $u^*u=s^M(|\zeta|_\alpha)$ (or equivalently $uu^*=s^M(\zeta)$) (resp. $\zeta=u'|\zeta|_\alpha$ and $u'^*u'=s^{M'}(|\zeta|_\alpha)$ (or equivalently $u'u'^*=s^{M'}(\zeta)$)).*

The proof of the theorem is divided into several steps. We consider the involution operator,

$$J_\alpha\equiv J_{\phi_0}\Delta_{\phi_0}^{(1/2)-2\alpha}. \tag{B.1}$$

(Note that $J_p^{\phi_0}$ discussed in Theorem 8 (3) is the restriction of this J_α, $\alpha=1/(2p)$ to $D_{\phi_0}^{\infty}$.) By (B.1) and $J_{\phi_0}\Delta_{\phi_0}^{\lambda}J_{\phi_0}=\Delta_{\phi_0}^{-\lambda}$, $\lambda\in\mathbb{R}$, the domain $D(\Delta_{\phi_0}^{(1/2)-2\alpha})$ is J_α-invariant. For $\zeta\in D(\Delta_{\phi_0}^{(1/2)-2\alpha})$, we define two operators T_0 and R_0 as follows;

$$T_0\eta_{\phi_0}(y)\equiv J_{\phi_0}\sigma_{-i[(1/2)-2\alpha]}^{\phi_0}(y^*)J_{\phi_0}\zeta, \tag{B.2}$$

$$R_0\eta_{\phi_0}(y)\equiv J_{\phi_0}\sigma_{-i[(1/2)-2\alpha]}^{\phi_0}(y^*)J_{\phi_0}J_\alpha\zeta, \quad y\in M_0, \tag{B.3}$$

where M_0 is the set of all entire analytic elements of $N_{\phi_0}\cap N_{\phi_0}^*$ with respect to the modular action $\sigma_t^{\phi_0}$. By (B.2) and (B.3), $D(T_0)=D(R_0)=\eta_{\phi_0}(M_0)$ and hence T_0, R_0 are densely defined operators.

Lemma B.2. *T_0 and R_0 are preclosed and*

$$T^*\supset R, \quad R^*\supset T \tag{B.4}$$

where T and R are the closures of T_0 and R_0 respectively.

Proof. It is sufficient to prove that for any $y_1y_2\in M_0$,

$$(T_0\eta_{\phi_0}(y_1), \eta_{\phi_0}(y_2))=(\eta_{\phi_0}(y_1), R_0\eta_{\phi_0}(y_2)). \tag{B.5}$$

By definitions of T_0 and R_0, both sides of (B.5) are computed as follows;

$$(\mathrm{B}.6)\qquad (T_0\eta_{\phi_0}(y_1), \eta_{\phi_0}(y_2)) = (J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y_1^*)J_{\phi_0}\zeta, \eta_{\phi_0}(y_2))$$
$$= (\zeta, J_{\phi_0}\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y_1)J_{\phi_0}\eta_{\phi_0}(y_2))$$
$$= (\zeta, J_{\phi_0}\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y_1)\eta_{\phi_0}(\sigma^{\phi_0}_{-i/2}(y_2^*)))$$
$$= (\zeta, J_{\phi_0}\Delta^{1/2}_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{i(1-2\alpha)}(y_1)y_2^*))$$
$$= (\zeta, \eta_{\phi_0}(y_2\sigma^{\phi_0}_{-i(1-2\alpha)}(y_1^*))),$$

$$(\mathrm{B}.7)\qquad (\eta_{\phi_0}(y_1), R_0\eta_{\phi_0}(y_2)) = (\eta_{\phi_0}(y_1), J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y_2^*)J_{\phi_0}J_\alpha\zeta)$$
$$= (\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y_2^*)\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta, J_{\phi_0}\eta_{\phi_0}(y_1))$$
$$= (\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta, \sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y_2)\eta_{\phi_0}(\sigma^{\phi_0}_{-i/2}(y_1^*)))$$
$$= (\Delta^{(1/2)-2\alpha}_{\phi_0}\zeta, \eta_{\phi_0}(\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y_2)\sigma^{\phi_0}_{-i/2}(y_1^*)))$$
$$= (\zeta, \Delta^{(1/2)-2\alpha}_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y_2)\sigma^{\phi_0}_{-i/2}(y_1^*)))$$
$$= (\zeta, \eta_{\phi_0}(y_2\sigma^{\phi_0}_{-i(1-2\alpha)}(y_1^*))).$$

Q.E.D.

Lemma B. 3. *T and R are (ϕ_0, p)-measurable operators with $p=1/(2\alpha)$.*

Proof. If we have

$$(\mathrm{B}.8)\qquad T_0J_{\phi_0}\sigma^{\phi_0}_{-2i\alpha}(y)J_{\phi_0}\eta_{\phi_0}(\tilde{y}) = J_{\phi_0}yJ_{\phi_0}T_0\eta_{\phi_0}(\tilde{y})$$

for any $y, \tilde{y}\in M_0$, we get the assertion for T by taking the closure. Two sides of (B. 8) are computed as follows;

$$(\mathrm{B}.9)\qquad T_0J_{\phi_0}\sigma^{\phi_0}_{-2i\alpha}(y)J_{\phi_0}\eta_{\phi_0}(\tilde{y}) = T_0J_{\phi_0}\sigma^{\phi_0}_{-2i\alpha}(y)\eta_{\phi_0}(\sigma^{\phi_0}_{-i/2}(\tilde{y}^*))$$
$$= T_0J_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{-2i\alpha}(y)\sigma^{\phi_0}_{-i/2}(\tilde{y}^*))$$
$$= T_0J_{\phi_0}\Delta^{1/2}_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y)\tilde{y}^*)$$
$$= T_0\eta_{\phi_0}(\tilde{y}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y^*))$$
$$= J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y)\tilde{y}^*)J_{\phi_0}\zeta$$
$$= J_{\phi_0}y\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(\tilde{y}^*)J_{\phi_0}\zeta,$$

$$(\mathrm{B}.10)\qquad J_{\phi_0}yJ_{\phi_0}T_0\eta_{\phi_0}(\tilde{y}) = J_{\phi_0}y\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(\tilde{y}^*)J_{\phi_0}\zeta.$$

If we replace ζ by $J_\alpha\zeta$, we obtain the proof for R. Q.E.D.

Proof of Theorem B. 1. First we assume $\alpha>0$. By Lemmas 4.1, B.3 and (B.2), there exist a partial isometry $u\in M$ and a normal semi-finite weight ϕ satisfying $u^*u=s(\phi)$ and $T=u\Delta^{2\alpha}_{\phi,\phi_0}$, i.e.

$$u\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(y)=J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y^*)J_{\phi_0}\zeta\,,\quad y\in M_0\,. \tag{B.11}$$

Next we show that $P\equiv uu^*$ and $Q=s^M(\zeta)$ coincide. Since $Pu=u$, (B.11) in which $\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y^*)$ is replaced by y, implies $J_{\phi_0}yJ_{\phi_0}P\zeta=PJ_{\phi_0}yJ_{\phi_0}\zeta=J_{\phi_0}yJ_{\phi_0}\zeta$. Taking the strong limit $y\to1$, $P\zeta=\zeta$ and we obtain $P\geqq Q$. Conversely, (B.11) and $QJ_{\phi_0}yJ_{\phi_0}\zeta=J_{\phi_0}yJ_{\phi_0}Q\zeta=J_{\phi_0}yJ_{\phi_0}\zeta$, $y\in M_0$ implies $Qu\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(y)=u\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(y)$, $y\in M_0$. By definition (B.2), $\eta_{\phi_0}(M_0)$ is a core for $u\Delta^{2\alpha}_{\phi,\phi_0}$. This implies that $\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(M_0)$ is dense in $s(\phi)H_{\phi_0}$ and we obtain $Qu=u$. Hence $Q\geqq P$. Therefore we obtain $P=Q$.

Next, we show $u^*\zeta\equiv|\zeta|_\alpha\in V^\alpha_{\phi_0}$. By (B.11),

$$\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(y)=J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y^*)J_{\phi_0}|\zeta|_\alpha\,,\quad y\in M_0\,. \tag{B.12}$$

On the other hand,

$$\begin{aligned} &J_{\phi_0}\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y)J_{\phi_0}\eta_{\phi_0}(y)\\ &=yJ_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y))\\ &=yJ_{\phi_0}\Delta^{1/2}_{\phi_0}\eta_{\phi_0}(\sigma^{\phi_0}_{i(1-2\alpha)}(y))\\ &=y\Delta^{1-2\alpha}_{\phi_0}\eta_{\phi_0}(y^*)\\ &=\Delta^{(1/2)-\alpha}_{\phi_0}\sigma^{\phi_0}_{i[(1/2)-\alpha]}(y)\eta_{\phi_0}(\sigma^{\phi_0}_{i[(1/2)-\alpha]}(y)^*),\quad y\in M_0\,. \end{aligned} \tag{B.13}$$

Combining (B.12) and (B.13),

$$\begin{aligned} 0&\leqq(\Delta^{2\alpha}_{\phi,\phi_0}\eta_{\phi_0}(y),\eta_{\phi_0}(y))\\ &=(J_{\phi_0}\sigma^{\phi_0}_{-i[(1/2)-2\alpha]}(y^*)J_{\phi_0}|\zeta|_\alpha,\eta_{\phi_0}(y))\\ &=(|\zeta|_\alpha,J_{\phi_0}\sigma^{\phi_0}_{i[(1/2)-2\alpha]}(y)J_{\phi_0}\eta_{\phi_0}(y))\\ &=(|\zeta|_\alpha,\Delta^{(1/2)-\alpha}_{\phi_0}\tilde{y}\eta_{\phi_0}(\tilde{y}^*)), \end{aligned} \tag{B.14}$$

where $\tilde{y}=\sigma^{\phi_0}_{i[(1/2)-\alpha]}(y)$, $y\in M_0$. By the density of $\Delta^{(1/2)-\alpha}_{\phi_0}\tilde{y}\eta_{\phi_0}(\tilde{y}^*)$ in $V^{(1/2)-\alpha}_{\phi_0}$ and Lemma A.1 (5), we obtain $|\zeta|_\alpha\in V^\alpha_{\phi_0}$.

Now, we give the proof for $\alpha=0$. By Lemmas 4.3, B.3 and (B.2), there exists a closed operator T affiliated with M satisfying

$$T\eta_{\phi_0}(y)=J_{\phi_0}\sigma^{\phi_0}_{-i/2}(y^*)J_{\phi_0}\zeta\,,\quad y\in M_0\,. \tag{B.15}$$

Let $T=u|T|$ be the polar decomposition. Then by the same reason as the case $\alpha>0$, $uu^*=s^M(\zeta)$. Let $|\zeta|_0\equiv u^*\zeta$. Then

$$\text{(B. 16)} \qquad \begin{aligned} 0 &\leqq (|T|\eta_{\phi_0}(y), \eta_{\phi_0}(y)) \\ &= (|\zeta|_0, J_{\phi_0}\sigma^{\phi_0}_{i/2}(y) J_{\phi_0}\eta_{\phi_0}(y)) \\ &= (|\zeta|_0, \Delta^{1/2}_{\phi_0}\sigma^{\phi_0}_{i/2}(y)\eta_{\phi_0}(\sigma^{\phi_0}_{i/2}(y)^*)) \end{aligned}$$

implies $|\zeta|_0\in V^0_{\phi_0}$.

The polar decomposition in terms of M' ($u'\in M'$ instead of $u\in M$) follows from the polar decomposition in terms of M (which we have shown above) for the complementary index $\alpha'=(1/2)-\alpha$, if we use $J_{\phi_0}D(\Delta^{(1/2)-2\alpha}_{\phi_0})=D(\Delta^{(1/2)-2\alpha'}_{\phi_0})$, and Lemma A. 1 (4). Q.E.D.

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