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Convergence of Martingales on a Riemannian Manifold

By

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Abstract

When the scalar quadratic variation of a martingale on a Riemannian manifold is finite almost surely, then the martingale converges almost surely in the one-point compactification of the manifold. A partial converse due to Zheng Wei-an is also proved. No curvature conditions on the manifold are required.

§ 0. Introduction

Let M be an n-dimensional differential manifold with a Riemannian metric g, and let Γ be a metric connection for M. We consider a stochastic process X on M with continuous trajectories, and such that the image of X under every C^2 function is a real-valued semimartingale. For a moment think of M as an oblong ball resting on the plane E, with X_0 being the point of contact. A well-known procedure in stochastic differential geometry is the so-called *stochastic development,* in which, by means of the metric connection Γ, the manifold is rolled along the plane E without slipping, such that X_t is the point of contact at each time t. If we imagine the path of X as being traced in ink, a 'developed' process Z is printed onto E.

The process X is called a Γ-martingale if Z is a local martingale on E. The purpose of this paper is to establish two results about Γ-martingales: first, that on the set where the scalar quadratic variation of Z is finite the process (X_t) tends almost surely to a limit X_∞ in the one-point compactification of M; secondly, that on the set where X_∞ exists and lies in M itself, the scalar quadratic variation of Z is finite. The first

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result is due to the author [3], the second to Zheng [8]. The remarkable aspect of these results is that no curvature assumptions are made about the manifold; M does not even have to be complete.

The proofs of these results make use of stochastic calculus on principal fibre bundles (Darling [3]). The first proof exploits the Ito formula for the image of X under C^∞ functions f on M with compact support. The second proof uses an immersion argument and a subtle lemma about continuous semimartingales.

§ 1. Basic Stochastic Notations

When Y is a real-valued stochastic process, then Y_∞^* denotes $\sup_t |Y_t|$; if T is a stopping-time, then Y^T is the process $Y_t^T = Y_{t\wedge T}$. If Y is a continuous semimartingale with a local martingale part N in the standard decomposition, then the scalar quadratic variation process of Y is written $\langle Y, Y\rangle$; of course $\langle Y, Y\rangle_t = \langle N, N\rangle_t$. When Y is $\boldsymbol{R}^n$-valued with components $(Y_t^1, \cdots, Y_t^n)$, then $\langle Y, Y\rangle_t$ denotes $\sum_i \langle Y^i, Y^i\rangle_t$. The matrix-valued differential $((d\langle Y^i, Y^j\rangle_t))$ will be abbreviated to $(dY\otimes dY)_t$.

A Stratonovitch differential is denoted by $\circ$, as in $L_t\circ dY_t$. However '$\circ$' is also used to denote composition of mappings, as in $\varphi\circ X$.

Suppose X is a process with values in a differential manifold M, and (W, φ) is a chart for M. If K is a real-valued bounded predictable process which vanishes outside the random set $\{X\in W\}$, and if $\varphi\circ X$ is a semimartingale in R^n, then it makes sense to refer to

$$(1)\qquad \int_0^t K_s d(\varphi(X_s)), \quad \text{denoted } \int_0^t K_s dX_s^\varphi,$$

and

$$\int_0^t K_s(d(\varphi\circ X)\otimes d(\varphi\circ X))_s, \quad \text{denoted } \int_0^t K_s(dX\otimes dX)_s^\varphi.$$

The last expression could also be described as the $n\times n$ matrix with $(i, j)^{th}$ entry

$$\int_0^t K_s d\langle X^i, X^j\rangle_s,$$

where $X^i = i^{th}$ co-ordinate process of $\varphi\circ X$.

§ 2. Constructions and Formulae from Stochastic Differential Geometry

A good reference for this section is Meyer [4], but the notation is more in the spirit of Darling [3].

1. Let M be a smooth n-dimensional manifold; for brevity, we shall often denote the model space by $\boldsymbol{E}$ instead of $\boldsymbol{R}^n$. A *semimartingale on* M will mean a process X with almost surely continuous paths on M, whose image under every C^2 function from M to $\boldsymbol{R}$ is a real-valued semimartingale. (The definition is due to Schwartz [6].)

2. Let η be a first-order differential form ('1-form') on M. If (W, φ) is a chart for M, then η has the local representation $(\varphi(x), a(x)) \in \boldsymbol{E} \times \boldsymbol{E}^*$ at each x in W, where $\eta = \varphi^* a$ on W (meaning that $\eta(x)(v) = a(x)(T_x\varphi(v))$ for v in T_xM). On the random set $\{X \in W\}$, the following differential makes sense and is intrinsic (for the notation, see Section 1):

$$a(X_t)\,dX_t^\varphi + \frac{1}{2} Da(X_t)\,(dX \otimes dX)_t^\varphi .$$

Consequently there is a unique real-valued process Y with $Y_0 = 0$ such that dY_t equals the last expression on $\{X \in W\}$, for each chart (W, φ). The process Y is called the *Stratonovitch integral* of the 1-form η along the semimartingale X, and we usually write

$$Y = (S) \int_X \eta, \quad Y_t = (S) \int_{X_0^t} \eta$$

(Meyer would omit the symbol (S).)

3. Let p: $P \to M$ be a principal fibre bundle with group G, which is a sub-bundle of the bundle of linear frames. Hence for each x in M and each u in $p^{-1}(x)$, u is a linear isomorphism from E into T_xM. Let ω be a connection 1-form on P. Various authors have shown (Meyer [4, p. 80], Darling [3, pp. 30–34], Shigekawa [7]) that given a semimartingale on M and an initial frame U_0 in $p^{-1}(X_0)$, there is a unique

semimartingale U on P, called the *horizontal lift of X to P through ω,* satisfying the equations:

$$(S)\int_U \omega = 0,\quad p(U_t) = X_t\,. \tag{2}$$

For a given connection 1-form ω and a given initial frame U_0, the *stochastic development of X into E* is the E-valued semimartingale Z defined by:

$$Z = (S)\int_U \theta \tag{3}$$

where θ is the canonical 1-form on P, namely the E-valued 1-form defined by:

$$\theta(u)(\xi) = u^{-1}(T_u p(\xi)),\quad u\in P,\ \xi\in T_u P\,. \tag{4}$$

The usefulness of the horizontal lift and the stochastic development is that they allow formulas related to the process X on M to be written down in 'absolute' terms, without reference to any system of co-ordinates on M. For example, let $f\colon M\to \boldsymbol{R}$ be a smooth function. Then $df(X_s)\in T^*_{X_s}M$. Let $(e_1, \cdots, e_n)$ be an orthonormal basis for E, and write $Z_t = (Z_t^1, \cdots, Z_t^n)$ with respect to this basis. Then $U_s(e_i)$ is a tangent vector at X_s for each i, and so $df(X_s)(U_s(e_i))$ is real-valued. Likewise $\nabla df(X_s)(U_s(e_i), U_s(e_j))$ is real-valued, where ∇ is the covariant derivative induced by ω (assuming P is the linear or the orthonormal frame bundle). Versions of the following 'Ito formula' have been given by many authors (e.g. Meyer [4], Bismut [1]) but we give the version appearing in Darling [3, p. 24];

$$f(X_t) - f(X_0) = \int_0^t (df(X_s)\circ U_s(e_i))\,dZ_s^i + \frac{1}{2}\int_0^t \nabla df(X_s)(U_s(e_i), U_s(e_j))\,d\langle Z^i, Z^j\rangle_s\,. \tag{5}$$

In fact the use of a basis for E is not necessary, and we may abbreviate to:

$$f(X_t) - f(X_0) = \int_0^t (df(X_s)\circ U_s)\,dZ_s + \frac{1}{2}\int_0^t \nabla df(X_s)(UdZ\otimes UdZ)_s\,. \tag{6}$$

4. Suppose (M, g) is a Riemannian manifold and P is the bundle of orthonormal frames. For the sake of easy comparison with Meyer [4, p. 64], we shall work in local co-ordinates (x^i). Take an orthonormal basis $(e_1, \cdots, e_n)$ for E and write $U_s(e_k)$ as $U_k^i(s) D_i$ in the tangent space to M at X_s. Then equation (5) implies

$$dX_s^i = U_k^i(s)\, dZ_s^k + \{\text{terms of bounded variation}\}.$$

Hence

$$g_{ij}(X_s)\, d\langle X^i, X^j\rangle_s = g_{ij}(X_s)\, U_k^i(s)\, U_m^j(s)\, d\langle Z^k, Z^m\rangle_s$$
$$= \sum_m d\langle Z^m, Z^m\rangle_s$$

by the orthonormality of the frames U_s. It follows that

$$g_{ij}(X_s)\, d\langle X^i, X^j\rangle_s = d\langle Z, Z\rangle_s \tag{7}$$

and we can define the scalar quadratic variation of X (with respect to the metric g) by: $\langle X, X\rangle_t = \langle Z, Z\rangle_t$.

5. A semimartingale X on M is said to be a *Γ-martingale* if Z is an ($\boldsymbol{R}^n$-valued) local martingale, where Γ refers to the linear connection for M induced by the connection 1-form ω on the bundle of linear or orthonormal frames. In the case where (M, g) is Riemannian, P is the orthonormal frame bundle, (and so Γ is a *metric* connection) X is called a *square-integrable Γ-martingale* if Z is a square-integrable martingale, which implies that $\langle Z, Z\rangle_\infty < \infty$. More information about Γ-martingales may be found in Darling [2], [3].

§ 3. Convergence Theorem for Martingales on a Riemannian Manifold

Theorem A. *We assume that (M, g) is a Riemannian manifold with a metric connection Γ (possibly with torsion), and X is a Γ-martingale on M. Then the limit $X_\infty = \lim_t X_t$ exists almost surely in the one-point compactification (= Alexandroff compactification) $M \cup \{\delta\}$ of M, on the set where $\langle X, X\rangle_\infty$ is finite.*

Theorem B. (*Zheng* [8]). *The assumption are the same as for*

Theorem A. On the set where X_∞ *exists and lies in* M*, we have:* $\langle X, X\rangle_\infty < \infty$ *almost surely.*

Remark. Let (M, g) be a Riemannian manifold on which the Brownian motion B has a finite lifetime almost surely. In [5], Meyer shows how to construct a time-change $t \to \tau_t$ such that if $X_t = B_{\tau_t}$, then X is a square-integrable martingale (with respect to the Levi-Civita connection), and X_∞ is the point at infinity. So the compactification of M really is necessary for the theorem.

The proof of (B) comes in Section 5. The proof of (A) is preceded by a pair of easy lemmas, both of which were noticed by P. A. Meyer.

Lemma 1. *Let* E *be a finite-dimensional inner product space, with dual* E^*. *Let* c *be a bilinear form on* E *and* q *a positive semi-definite symmetric bilinear form on* E^*. *Then with respect to any orthonormal basis for* E,

$$|c_{ij}q^{ij}| \leqq \sum_i q^{ii} \|c\| \tag{8}$$

where

$$\|c\| = \sup\{c(a, a) : \sum_i (a^i)^2 = 1\}.$$

Furthermore if c *is also positive semidefinite, then*

$$c_{ij}q^{ij} \geqq \sum_i q^{ii} \alpha(c) \tag{8'}$$

where $\alpha(c) = \inf\{c(a, a) : \sum_i (a^i)^2 = 1\}$.

Proof. Both sides of (8) are independent of the choice of basis. Take a basis so that q has a diagonal matrix. Then

$$|c_{ij}q^{ij}| = |\sum_i c_{ii}q^{ii}| \leqq (\sum_i q^{ii}) \sup_i |c_{ii}|$$

$$= (\sum_i q^{ii}) \sup_i |c(e_i, e_i)|$$

where the e_i are the basis vectors. The proof (8′) is similar. □

In the next lemma, $C_K^\infty(M)$ denotes the C^∞ functions from M to $\boldsymbol{R}$

with compact support.

Lemma 2. *Let* $C=\{\omega\colon \lim_{t\to\infty} f(X_t(\omega))$ *exists for all* $f\in C_K^\infty(M)\}$. *Then for all* $\omega\in C$, $\lim_{t\to\infty} X_t(\omega)$ *exists in* $M\cup\{\delta\}$.

Proof. Suppose $\omega\in C$. If for some $f\in C_K^\infty(M)$, $\lim_{t\to\infty} f(X_t(\omega))\neq 0$, then $X_t(\omega)$ lies in some compact set K for all sufficiently large t. Since $\lim_{t\to\infty} f(X_t(\omega))$ exists for all C^∞ f with support in K, it follows $\lim_{t\to\infty} X_t(\omega)$ exists in K. On the other hand if $\lim_{t\to\infty} f(X_t(\omega))=0$ for all f, then for every compact set K, $X_t(\omega)$ lies outside K for all sufficiently large t. Therefore $\lim_{t\to\infty} X_t(\omega)$ is the point at infinity. □

Proof of Theorem A. As P. A. Meyer has pointed out, it suffices to treat the case where $\langle X, X\rangle_\infty$ is integrable. "For this case gives us the convergence of the stopped martingale X^{T_n}, where T_n is the stopping-time $\inf\{t\colon \langle X, X\rangle_t > n\}$; on the other hand, on the set $\{\langle X, X\rangle_\infty < \infty\}$ we have $T_n=+\infty$ for n sufficiently large." We suppose henceforward that $\langle X, X\rangle_\infty$ is integrable. By the definition of $\langle X, X\rangle_t$ above, this says that $\langle Z, Z\rangle_\infty$ is integrable.

Let $f\in C_K^\infty(M)$. By formula (5) we may write

$$f(X_t)-f(X_0)=\int_0^t a_i(s)\,dZ_s^i+\int_0^t b_{ij}(s)\,d\langle Z^i, Z^j\rangle_s\,,$$

where

$$a_i(s)=df(X_s)\circ U_s(e_i)$$

and

$$b_{ij}(s)=\frac{1}{2}\nabla df(X_s)\,(U_s(e_i), U_s(e_j)).$$

Let Q_t denote the first integral on the right (a local martingale) and A_t the second. Since df and ∇df are bounded, and $U_s(e_i)$ is always of unit Riemannian length, there is a constant K such that

$$\sum_i a_i(s)^2 < K, \qquad \text{all} \quad s$$

and

$$\sup_i |b_{ii}(s)| < K, \quad \text{all} \quad s.$$

We shall now show that Q is a square integrable martingale; for apply Lemma 1 with $c_{ij} = a_i(s) a_j(s)$, $q^{ij} = d\langle Z^i, Z^j\rangle_s$, to deduce:

$$\begin{aligned} d\langle Q, Q\rangle_s &= a_i(s) a_j(s)\, d\langle Z^i, Z^j\rangle_s \\ &\leqq \sup\{|c_{ij} v^i v^j| : \sum_i (v^i)^2 = 1\} \sum_i d\langle Z^i, Z^j\rangle_s \\ &= \sup\{(a_i(s) v^i)^2 : \sum_i (v^i)^2 = 1\}\, d\langle Z, Z\rangle_s \\ &\leqq K^2 d\langle Z, Z\rangle_s, \end{aligned}$$

using the Schwartz inequality. The convergence theorem for real-valued square-integrable martingales implies that $Q_\infty = \lim_{t\to\infty} Q_t$ exists a.s.. As for the process A, apply Lemma 1 again with $c_{ij} = b_{ij}(s)$, $q^{ij} = d\langle Z^i, Z^j\rangle_s$ to deduce that

$$|dA_t| \leqq K \sum_i d\langle Z^i, Z^i\rangle_t = K d\langle Z, Z\rangle_t.$$

Hence $A_\infty = \lim_{t\to\infty} A_t$ exists a.s.. By Lemma 2, this completes the proof. $\square$

§ 4. A Result on Semimartingales in $\boldsymbol{R}^n$

W. A. Zheng has proved the following result [8]:

Lemma 3. *Let Y be a real-valued continuous semimartingale, with canonical decomposition $Y = Y_0 + N + A$. We suppose that $dA_t \ll d\langle Y, Y\rangle_t = d\langle N, N\rangle_t$, so that one can write*

$$Y_t = Y_0 + N_t + \int_0^t H_s d\langle N, N\rangle_s.$$

Then on the set

$$C = \{Y_\infty \textit{ exists and is finite, and } \operatorname{ess\,sup}_t |H_t| < \infty\}$$

the limit $\langle N, N\rangle_\infty$ is finite. (The ess $\sup_t$ *is taken with respect to the measure $d\langle N, N\rangle_t(\omega)$).*

We would like to apply the lemma in a more complicated situation. Suppose $Z = (Z^1, \cdots, Z^q)$ is a continuous martingale in $\boldsymbol{R}^q$, and suppose

Y is an $\boldsymbol{R}^n$-valued continuous semimartingale of the form

$$Y_t = Y_0 + N_t + A_t\,,$$

where

$$N_t^i = \int_0^t J_j^i(s)\, dZ_s^j$$

and

$$A_t^i = \int_0^t H_{jk}^i(s)\, d\langle Z^j, Z^k\rangle_s\,.$$

Set $|H_t| = \sup_i \sup_{|x|<1} |H_{jk}^i(t)\, x^j x^k|$ and $H_\infty^* = \operatorname{ess\,sup}_t |H_t|$. (The ess sup may be taken with respect to the measure $d\langle Z, Z\rangle_t$). Regarding $J(s)$ as a linear transformation from $\boldsymbol{R}^q$ to $\boldsymbol{R}^n$, define a random variable

$$\gamma(J) = \inf_t \inf\{e^T J^T(t)\, J(t)\, e\colon\ e \in \boldsymbol{R}^q,\ |e| = 1\}.$$

Notice that in terms of $\alpha(\cdot)$ defined in Lemma 1 above,

$$\gamma(J) = \inf_t \alpha(J^T(t)\, J(t))\,.$$

Lemma 4. *Let* $\varepsilon > 0$, *and let the situation be as just described. On the set*

$$C = \{Y_\infty \textit{ exists and is finite, } H_\infty^* < \infty, \textit{ and } \gamma(J) > \varepsilon\}$$

we have: $\langle Z, Z\rangle_\infty < \infty$ *a.s.*.

Proof. A straightforward extension of Lemma 3 to the vector case shows that $\langle N, N\rangle_\infty < \infty$ a.s. on C. Apply Lemma 1, second part, with $c_{ij} = \sum_k J_i^k(t)\, J_j^k(t)$ and $q^{ij} = d\langle Z^i, Z^j\rangle_t$. We obtain:

$$\begin{aligned} d\langle N, N\rangle_t &= \Big(\sum_k J_i^k(t)\, J_j^k(t)\Big)\, d\langle Z^i, Z^j\rangle_t \\ &\geqq \alpha(J^T(t)\, J(t))\, d\langle Z, Z\rangle_t \\ &\geqq \gamma(J)\, d\langle Z, Z\rangle_t \geqq \varepsilon\, d\langle Z, Z\rangle_t\,. \end{aligned}$$

This completes the proof. □

§ 5. Proof of Theorem B

Let ω be the connection 1-form on the orthonormal frame bundle

$O(M)$, associated with the metric connection Γ for (M, g). Given a Γ-martingale X on M, take an initial frame at X_0, and form the corresponding horizontal lift U of X to $O(M)$ through ω, and the stochastic development Z of X into $\boldsymbol{R}^n$, as in Section 2, part 3. By definition of Γ-martingale, Z is a local martingale.

There exists an integer p and an immersion h of M into $\boldsymbol{R}^{n+p}$. Let Y denote the image of X in $\boldsymbol{R}^{n+p}$ under h. It follows from the Ito formula (6) that

$$Y_t = Y_0 + \int_0^t J(s)\,dZ_s + \int_0^t H(s)\,(dZ \otimes dZ)_s\,,$$

where

$$J(s) = (dh(X_s) \circ U_s) \in L(\boldsymbol{R}^n;\ \boldsymbol{R}^{n+p})$$

and

$$H(s) = \frac{1}{2}\nabla dh(X_s)\,(U_s(\cdot), U_s(\cdot)) \in L(\boldsymbol{R}^n, \boldsymbol{R}^n; \boldsymbol{R}^{n+p}).$$

Since h is an immersion, $dh(x): T_xM \to \boldsymbol{R}^{n+p}$ is $1-1$ at each x in M, and is a smooth function of x. Consequently on each compact subset K of M, we can find a c greater than zero such that

$$|dh(x)(v)| > c\,, \quad \text{all } x \in K, \quad \text{all unit vectors } v \text{ in } T_xM.$$

Since U_s is an isometry from $\boldsymbol{R}^n$ to $T_{X_s}M$ for each s, it follows that on the set $Q(K) := \{X_t \text{ lies in } K \text{ for all } t\}$,

(i) $\inf_t \inf \{|J(t, \omega)e| : e \in \boldsymbol{R}^n,\ |e| = 1\} > c$

or in the notation of Lemma 4, $\gamma(J) > c$ on $Q(K)$. Moreover it is easy to see that

(ii) $H^*_\infty < \infty$ on $Q(K)$, in the sense of Lemma 4.

(A minor technical point: if H is not symmetric, replace $H^i_{jk}(s)$ by $\frac{1}{2}(H^i_{kj}(s) + H^i_{jk}(s))$; this does not affect Y).

Let C be the set $\{X_\infty \text{ exists in } M\} = \{Y_\infty \text{ exists in } F\}$. From Lemma 4, it follows that on $C \cap Q(K)$, $\langle Z, Z\rangle_\infty < \infty$ a.s.. But for *every* ω in C, there is some compact K containing the whole trajectory $X.(\omega)$. Hence $\langle Z, Z\rangle_\infty < \infty$ a.s. on C, as desired. □

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