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On the Cohomology of Algebraic K-Spectra over Finite Fields

By

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§ 1. Introduction

In [5], D. Quillen showed that the classifying space of the algebraic K-theory of a finite field $\boldsymbol{F}_q$ is the homotopy fibre of $1-\psi^q$: $BU\to BU$ where $q=p^d$ for some prime p and ψ^q is the Adams operation. And by Fiedorowicz-Priddy [4] we know that the map $1-\psi^q$ is an infinite loop map with coefficient $\boldsymbol{Z}[1/p]$. So we can regard the spectrum which represents algebraic K-theory for $\boldsymbol{F}_q$ is a homotopy fibre of $1-\psi^q$.

Let $\boldsymbol{bu}$ be the spectrum which represents the (-1)-connected complex K-theory. Let l be a prime number, and Λ be either the ring $\hat{\boldsymbol{Z}}_l$ of l-adic integers or the ring $\boldsymbol{Z}_{(l)}$ of localized at l. We can introduce coefficient Λ into any spectrum $\boldsymbol{X}$ by setting

$$\boldsymbol{X}_\Lambda=\boldsymbol{M}\Lambda\wedge\boldsymbol{X}$$

where $\boldsymbol{M}\Lambda$ is the Moore spectrum for the group Λ. Let ϕ: $\boldsymbol{bu}_\Lambda\to\boldsymbol{bu}_\Lambda$ be a (degree 0) map of spectra. Then the purpose of this paper is to determine the mod l cohomology group of the homotopy fibre of ϕ over the mod l Steenrod algebra. Especially when $\phi=1-\psi^q$, the homotopy fibre is the spectrum representing the algebraic K-theory for $\boldsymbol{F}_q$ localized at a prime l.

Our main theorems are stated in Section 5.

The paper is organized as follows:
In Section 2 we recall the Adams splitting $\boldsymbol{bu}_\Lambda\simeq\boldsymbol{g}_0\vee\cdots\vee\boldsymbol{g}_{l-2}$. In Section 3 we state certain properties of a spectral sequence associated with the Postnikov system of a spectrum. In the next section we consider the

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homotopy fibre of ϕ: $\boldsymbol{g}_0 \to \boldsymbol{g}_0$. In Section 5 we state our main theorem and applications to the algebraic K-theory.

Throughout this paper we use the following notations: for a spectrum $\boldsymbol{X}$, $\boldsymbol{X}(m, n)$ denotes that term in the Postnikov system of $\boldsymbol{X}$ whose homotopy groups π_r are the same as those of $\boldsymbol{X}$ for $m \leq r \leq n$ and zero for other values of r. If $m = n$, then $\boldsymbol{X}(m, n)$ is simply denoted by $\boldsymbol{X}(n)$. We write $\boldsymbol{EM}(\pi, m)$ for the Eilenberg-MacLane spectrum of type (π, m).

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§ 2. Splitting of $\boldsymbol{bu}_\Lambda$

First we recall the splitting of $\boldsymbol{bu}_\Lambda$. (cf. [1]). We can write

$$\boldsymbol{bu}_\Lambda \simeq \boldsymbol{g}_0 \vee \boldsymbol{g}_1 \vee \cdots \vee \boldsymbol{g}_{l-2}$$

where the homotopy groups of $\boldsymbol{g}_j$ $(0 \leq j \leq l-2)$ are

$$\pi_n(\boldsymbol{g}_j) \cong \begin{cases} \Lambda & n \geq 0 \text{ and } n \equiv 2j \pmod{2l-2} \\ 0 & \text{otherwise.} \end{cases}$$

By i_j: $\boldsymbol{g}_j \to \boldsymbol{bu}_\Lambda$ and p_j: $\boldsymbol{bu}_\Lambda \to \boldsymbol{g}_j$ we denote the canonical inclusion and projection respectively. Then i_j and p_j induce isomorphisms of homotopy groups on degree $2n$ such that $n \equiv j \pmod{l-1}$.

Let ϕ: $\boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$ be a map of spectra. We put $\phi_j = p_j \circ \phi \circ i_j$ $(0 \leq j \leq l-2)$ and $\phi' = (\phi_0, \cdots, \phi_{l-2}) : \boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$. Then we have

Proposition 2.1. *ϕ' is homotopic to ϕ.*

Proof. By Corollary 6.4.8 of Adams [2], we know that ϕ and ϕ' are homotopic if and only if they have the same actions on the homotopy groups $\pi_*(\boldsymbol{bu}_\Lambda)$. The construction of ϕ' implies that ϕ and ϕ' have the same actions on homotopy groups. This completes the proof.

As a corollary of the above proposition we can easily show

Corollary 2.2. *The homotopy fibre of ϕ: $\boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$ is a wedge sum of the homotopy fibres of ϕ_j: $\boldsymbol{g}_j \to \boldsymbol{g}_j$ $(0 \leq j \leq l-2)$.*

Remark 2.3. By the Bott periodicity theorem $\boldsymbol{g}_0 \simeq \Omega^{2j}\boldsymbol{g}_j$. So if we write $\boldsymbol{F}_{\phi_j}$ and $\boldsymbol{F}_{\Omega^{2j}\phi_j}$ for the homotopy fibre of

$$\phi_j\colon\ \boldsymbol{g}_j \to \boldsymbol{g}_j\,,$$

and

$$\Omega^{2j}\phi_j\colon\ \boldsymbol{g}_0 \to \boldsymbol{g}_0\,,$$

respectively, then we have

$$\boldsymbol{F}_{\phi_j} \simeq \sum{}^{2j} \boldsymbol{F}_{\Omega^{2j}\phi_j}\,,$$

and

$$H^*(\boldsymbol{F}_{\phi_j}) \cong H^{*-2j}(\boldsymbol{F}_{\Omega^{2j}\phi_j})\,.$$

Henceforth, from now on we will concentrate our attention to the case of $j=0$; that is the homotopy fibre of $\phi\colon\ \boldsymbol{g}_0 \to \boldsymbol{g}_0$.

§ 3. Spectral Sequence Associated with Postnikov System

We write A for the mod l Steenrod algebra. We have $Q_0 = \beta_l$ and $Q_1 = \mathcal{P}^1\beta_l - \beta_l\mathcal{P}^1$, as usual; if $p=2$ we write Q_1 as $Sq^1Sq^2 + Sq^2Sq^1$. The graded module $\sum^m M$ is defined by regarding M so that an element of degree n in M appears as an element of degree $n+m$ in M. For any spectrum $\boldsymbol{X}$, we put $H^*(\boldsymbol{X}) = H^*(\boldsymbol{X};\ \boldsymbol{Z}/l)$.

For a spectrum $\boldsymbol{X}$, we introduce a spectral sequence to compute $H^*(\boldsymbol{X})$ as in [3]. We filter $\boldsymbol{X}$ by considering its Postnikov system, but for our purpose its filtration is a little different from [3]. The E_1-term of our spectral sequence is

$$\bigoplus_n H^*(\boldsymbol{X}(2n-1, 2n))\,.$$

Assume $\boldsymbol{X} = \boldsymbol{g}_0$ in Section 2. Then

$$\boldsymbol{g}_0(2n-1, 2n) \simeq \begin{cases} \boldsymbol{EM}(\Lambda, 2n) & n \geq 0 \text{ and } n \equiv 0 \pmod{l-1} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H^*(\boldsymbol{EM}(\Lambda, 2n)) \cong \sum{}^{2n} A/AQ_0\,.$$

Thus the E_1-term of the spectral sequence for $\boldsymbol{g}_0$ is

$$\bigoplus_{n\geq 0}\sum^{2n(l-1)} A/AQ_0\,.$$

By the dimensional reason, the differential d_r $(1\leq r<l-1)$ is trivial and $E_1=E_2=\cdots=E_{l-1}$. We know $d_{l-1}(a)=aQ_1$ where $a\in\sum^{2k(l-1)}A/AQ_0$ (cf. [3]), whence the E_{l-1}-term of the spectral sequence is a long exact sequence

$$\cdots\longrightarrow\sum^{4(l-1)}A/AQ_0\xrightarrow{Q_1}\sum^{2(l-1)}A/AQ_0\xrightarrow{Q_1}A/AQ_0\,.$$

So we have $H^*(\boldsymbol{g}_0)\cong E_\infty\cong E_l\cong A/(AQ_0+AQ_1)$.

Let $\phi\colon \boldsymbol{g}_0\to\boldsymbol{g}_0$ be a map of spectra. Let $a_{n(l-1)}$ $(n\geq 0)$ be an element of Λ such that $\phi_*\colon \pi_{2n(l-1)}(\boldsymbol{g}_0)\to\pi_{2n(l-1)}(\boldsymbol{g}_0)$ is a multiplication by $a_{n(l-1)}$. Then we have the following:

Proposition 3.1. *$a_0\in l\Lambda$ if and only if $a_{n(l-1)}\in l\Lambda$ for any $n\geq 0$.*

Proof. By induction on n we need only show that $a_{n(l-1)}\in l\Lambda$ if and only if $a_{(n+1)(l-1)}\in l\Lambda$. Since the spectral sequence considered above has a naturality for maps of spectra, we have a commutative diagram:

$$\begin{array}{ccc}
H^*(\boldsymbol{g}_0(2(n+1)(l-1))) & \xrightarrow{d_{l-1}} & H^*(\boldsymbol{g}_0(2n(l-1))) \\
\downarrow \phi^*_{(n+1)(l-1)} & & \downarrow \phi^*_{n(l-1)} \\
H^*(\boldsymbol{g}_0(2(n+1)(l-1))) & \xrightarrow{d_{l-1}} & H^*(\boldsymbol{g}_0(2n(l-1))),
\end{array}$$

where we write $\phi_{n(l-1)}$ for the restriction of ϕ on $\boldsymbol{g}_0(2n(l-1))$ $(n\geq 0)$. Here $\phi^*_{n(l-1)}$ is trivial if $a_{n(l-1)}\in l\Lambda$ and an isomorphism if $a_{n(l-1)}\notin l\Lambda$ for any $n\geq 0$. Since d_{l-1} is not trivial, $\phi^*_{n(l-1)}=0$ if and only if $\phi^*_{(n+1)(l-1)}=0$ and the result follows.

As a corollary of the above proposition we have

Corollary 3.2. *Let $\phi\colon \boldsymbol{bu}_\Lambda\to\boldsymbol{bu}_\Lambda$ be a map and a_n be an element of Λ such that $\phi_*\colon \pi_{2n}(\boldsymbol{bu}_\Lambda)\to\pi_{2n}(\boldsymbol{bu}_\Lambda)$ is a multiplication by a_n. Then $a_n\in l\Lambda$ for some $n\geq 0$ if and only if $a_{n+m(l-1)}\in l\Lambda$ for any $m\geq 0$.*

§ 4. Cohomology of the Homotopy Fibre of ϕ: $\boldsymbol{g}_0 \to \boldsymbol{g}_0$

In this section we consider the spectral sequence for the homotopy fibre of ϕ: $\boldsymbol{g}_0 \to \boldsymbol{g}_0$.

Let $a_{n(l-1)}$ be as in previous section.

For the case of $a_0 \notin l\Lambda$, $\phi_*: \pi_*(\boldsymbol{g}_0) \to \pi_*(\boldsymbol{g}_0)$ is an isomorphism by Proposition 3.1. So ϕ is a homotopy equivalence and the homotopy fibre of ϕ is trivial. Henceforth in this section we assume $a_0 \in l\Lambda$, that is $a_{n(l-1)} \in l\Lambda$ for any $n \geq 0$.

First recall the following lemma which can be easily proved by standard connectivity argument:

Lemma 4.1. *Let $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ be spectra and $\boldsymbol{x} \xrightarrow{i} \boldsymbol{y} \xrightarrow{j} \boldsymbol{z}$ a fibration. For $m \leq n$ we have that $\boldsymbol{x}(m, n) \xrightarrow{i} \boldsymbol{y}(m, n) \xrightarrow{j} \boldsymbol{z}(m, n)$ is a fibration if and only if*

(i) $\pi_n(\boldsymbol{x}) \xrightarrow{i_*} \pi_n(\boldsymbol{y})$ *is a monomorphism and*

(ii) $\pi_m(\boldsymbol{y}) \xrightarrow{j_*} \pi_m(\boldsymbol{z})$ *is an epimorphism.*

We write $\boldsymbol{f\phi}$ for the homotopy fibre of ϕ: $\boldsymbol{g}_0 \to \boldsymbol{g}_0$. Then we have fibrations:

$$\boldsymbol{f\phi} \xrightarrow{i} \boldsymbol{g}_0 \xrightarrow{\phi} \boldsymbol{g}_0, \tag{4.2}$$

and

$$\Sigma^{-1}\boldsymbol{g}_0 \xrightarrow{\delta} \boldsymbol{f\phi} \xrightarrow{i} \boldsymbol{g}_0. \tag{4.3}$$

Considering the homotopy exact sequence of (4.2) we have

Lemma 4.4. *For $n \geq 0$*

$$\pi_{2n(l-1)}(\boldsymbol{f\phi}) \cong \begin{cases} \Lambda & \text{if } a_{n(l-1)} = 0, \\ \{0\} & \text{if } a_{n(l-1)} \neq 0, \end{cases}$$

$$\pi_{2n(l-1)-1}(\boldsymbol{f\phi}) \cong \begin{cases} \Lambda & \text{if } a_{n(l-1)} = 0, \\ \Lambda / a_{n(l-1)}\Lambda & \text{if } a_{n(l-1)} \neq 0, \end{cases}$$

and all the other homotopy groups of $\boldsymbol{f\phi}$ are trivial.

Next, we apply the spectral sequence to the fibration (4.3). Since $\sum^{-1}\boldsymbol{g}_0(2n, 2n-1) \to \boldsymbol{f\phi}(2n, 2n-1) \to \boldsymbol{g}_0(2n, 2n-1)$ is a fibration by Lemma 4.1 and f^* is trivial, we have a commutative diagram

(4.5)
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum^{4l-4}A/AQ_0 & \xrightarrow{i^*} & H^*(\boldsymbol{f\phi}(4l-5, 4l-4)) & \xrightarrow{\delta} & \sum^{4l-5}A/AQ_0 & \longrightarrow & 0 \\
 & & \downarrow Q_1 & & \downarrow d_{l-1} & & \downarrow Q_1 & & \\
0 & \longrightarrow & \sum^{2l-2}A/AQ_0 & \xrightarrow{i^*} & H^*(\boldsymbol{f\phi}(2l-3, 2l-2)) & \xrightarrow{\delta} & \sum^{2l-3}A/AQ_0 & \longrightarrow & 0 \\
 & & \downarrow Q_1 & & \downarrow d_{l-1} & & \downarrow Q_1 & & \\
0 & \longrightarrow & A/AQ_0 & \xrightarrow{i^*} & H^*(\boldsymbol{f\phi}(-1, 0)) & \xrightarrow{\delta} & \sum^{-1}A/AQ_0 & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are short exact and vertical sequences are the E_{l-1} terms of the spectral sequences. Then we have

Lemma 4.6. *With notation as above*

$$H^*(\boldsymbol{f\phi}) \cong \mathrm{Coker}\,(H^*(\boldsymbol{f\phi}(2l-3, 2l-2)) \xrightarrow{d_{l-1}} H^*(\boldsymbol{f\phi}(-1, 0))).$$

Proof. Since the left and right vertical sequences in diagram (4.5) are exact, so is the middle sequence by considering the homology long exact sequence of the three chain complexes. Hence the spectral sequence for $\boldsymbol{f\phi}$ becomes trivial after the differential d_{l-1}, and this proves the lemma.

Next, we consider the two fibrations:

$$\boldsymbol{f\phi}(2l-3, 2l-2) \xrightarrow{i} \boldsymbol{EM}(A, 2l-2) \xrightarrow{a_{l-1}} \boldsymbol{EM}(A, 2l-2),$$

and

$$\boldsymbol{f\phi}(-1, 0) \xrightarrow{i} \boldsymbol{EM}(A, 0) \xrightarrow{a_{l-1}} \boldsymbol{EM}(A, 0),$$

which induce bottom two short exact sequences in diagram (4.5). Let x and x' be generators of $H^*(\boldsymbol{EM}(A, 0)) \cong A/AQ_0$ and $H^*(\boldsymbol{EM}(A, 2l-2)) \cong \sum^{2l-2}A/AQ_0$ respectively such that $d_{l-1}(x') = Q_1 x$. Then we can set generators of $H^*(\boldsymbol{f\phi}(2l-3, 2l-2))$ and $H^*(\boldsymbol{f\phi}(-1, 0))$ as follows:

(i) $u', v' \in H^*(\boldsymbol{f\phi}(2l-3, 2l-2))$ such that $i^*(x') = u'$ and $\delta(v') = x'$

(ii) $u, v \in H^*(\boldsymbol{f\phi}(-1, 0))$ such that $i^*(x) = u$ and $\delta(v) = x$.

Put $\alpha_{n(l-1)}$ $(n \geq 0)$ be the element in $\boldsymbol{Z}/l$ such that $a_{n(l-1)} \equiv \alpha_{n(l-1)} l$ (mod l^2). We know that $\beta_{a_{l-1}}(v') = u'$ and $\beta_{a_0}(v) = u$ if $a_{l-1} \neq 0$ and $a_0 \neq 0$ (cf. for example [5]). So we have $\beta_l v' = \alpha_{l-1} u'$ and $\beta_l v = \alpha_0 u$. Note that this holds even when $a_0 = 0$ and $a_{l-1} = 0$.

Now we can calculate d_{l-1} images of u' and v'.

Lemma 4.7. *We have*

(i) $d_{l-1}(u') = Q_1 u$ *and*

(ii) $d_{l-1}(v') = -\alpha_{l-1} \mathcal{P}^1 u + \beta_l \mathcal{P}^1 v$.

Proof. Commutativity $d_{l-1} \circ i = i \circ d_{l-1}$ implies (i). For (ii) we put $d_{l-1}(v') = \xi \mathcal{P}^1 u + \eta \beta_l \mathcal{P}^1 v$ since $\mathcal{P}^1 \beta_l v = \alpha_0 \mathcal{P}^1 u$. Then we have

$$Q_1 x = d_{l-1}(x') = d_{l-1} \circ \delta(v') = -\delta \circ d_{l-1}(v')$$
$$= -\delta(\xi \mathcal{P}^1 u + \eta \beta_l \mathcal{P}^1 v) = -\eta \beta_l \mathcal{P}^1 v = \eta Q_1 v$$

in A/AQ_0. So, $\eta = 1$. On the other hand

$$\alpha_{l-1} Q_1 u = d_{l-1}(\alpha_{l-1} u') = d_{l-1}(\beta_l v')$$
$$= \xi \beta_l \mathcal{P}^1 u = -\xi Q_1 u$$

in A/AQ_0. Hence $\xi = -\alpha_{l-1}$. This completes the proof.

To state our main theorem in this section, we need following definition:

Definition 4.8. Let ξ and η be any elements in $\boldsymbol{Z}/l$. By $M(\xi, \eta)$ we denote the A-module generated by u and v $(\deg(u) = 0$ and $\deg(v) = -1)$ with the relations as follows:

$$\beta_l u = 0, \qquad Q_1 u = 0,$$
$$\beta_l v = \xi u \quad \text{and} \quad \beta_l \mathcal{P}^1 v = \eta \mathcal{P}^1 u.$$

By Lemma 4.6 and 4.7 we have

Theorem 4.9. *Let ϕ: $\boldsymbol{g}_0 \to \boldsymbol{g}_0$ be a map of spectra whose action on the homotopy group $\pi_{2n(l-1)}(\boldsymbol{g}_0)$ is multiplication by $a_{n(l-1)} \in l\Lambda$ $(n \geq 0)$. Let $\alpha_{n(l-1)} \in \boldsymbol{Z}/l$ be such that $a_{n(l-1)} \equiv \alpha_{n(l-1)} l$ (mod l^2). Then the cohomology of the homotopy fibre $\boldsymbol{f\phi}$ is $M(\alpha_0, \alpha_{l-1})$.*

Remark. As A-module $M(\xi, \eta)$ is as follows:

(i) $M(0, 0) \cong A/(AQ_0 + AQ_1) \oplus \sum^{-1} A/(AQ_0 + AQ_1)$,

(ii) if $\eta \neq 0$ then
$M(0, \eta) \cong (A/(AQ_0 + AQ_1) \oplus \sum^{-1} A/AQ_0)/A((\mathcal{P}^1, \beta_l \mathcal{P}^1))$,

(iii) if $\xi \neq 0$ then
$M(\xi, 0) \cong \sum^{-1} A/(A\beta_l \mathcal{P}^1 + A\beta_l \mathcal{P}^1 \beta_l)$ and

(iv) *if* $\xi \neq 0$ and $\eta \neq 0$ then

$$M(\xi, \eta) \cong \sum^{-1} A/A\left(\frac{\eta}{\xi} \mathcal{P}^1 \beta_l - \beta_l \mathcal{P}^1\right).$$

§ 5. Statement and Proof of Main Theorem

In this section we go back to the study of ϕ: $\boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$.
First we state our main theorem.

Theorem 5.1. *Let ϕ: $\boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$ be a map of spectra. Let $\boldsymbol{f\phi}$ be the homotopy fibre of ϕ. Then*

$$H^*(\boldsymbol{f\phi}) \cong \bigoplus_{\substack{0 \leq j < l-1 \\ a_j \in l\Lambda}} \sum^{2j} M(\alpha_j, \alpha_{j+l-1}),$$

where a_n is the element of Λ such that ϕ_*: $\pi_{2n}(\boldsymbol{bu}_\Lambda) \to \pi_{2n}(\boldsymbol{bu}_\Lambda)$ is a multiplication by a_n and α_n is an element of $\boldsymbol{Z}/l$ such that $a_n = \alpha_n l$ (mod l^2).

Proof. Put $\phi_j = p_j \circ \phi \circ i_j$ $(0 \leq j \leq l-2)$ as in Section 2. Let $\boldsymbol{f\phi}_j$ be the homotopy fibre of ϕ_j: $\boldsymbol{g}_j \to \boldsymbol{g}_j$. By Corollary 2.2 we have

$$H^*(\boldsymbol{f\phi}) \cong \bigoplus_{0 \leq j < l-1} H^*(\boldsymbol{f\phi}_j).$$

If $a_j \notin l\Lambda$, then by Theorem 4.9 and Remark 2.3, we have

$$H^*(\boldsymbol{f\phi}_j) \cong \sum^{2j} M(\alpha_j, \alpha_{j+l-1}).$$

This proves the theorem.

Next, we will consider a special case of ϕ. Let F_q be a finite field of order q such that l does not divide q. Put $\phi=1-\psi^q$ where ψ^q is the Adams operation. Then the action of ϕ_* on the homotopy group $\pi_{2n}(\boldsymbol{bu}_\Lambda)$ is the multiplication by $1-q^n$. Let r be the least integer ≥ 1 such that $1\equiv q^r \pmod{l}$. Let ρ be the element of Z/l such that $1-q^r=\rho l \pmod{l^2}$.

Then as a corollary of Theorem 5.1 we have

Corollary 5.2. *If we write $f\psi^q$ for the homotopy fibre of $1-\psi^q$: $\boldsymbol{bu}_\Lambda \to \boldsymbol{bu}_\Lambda$, then*

$$H^*(f\psi^q)\cong \bigoplus_{0\leq kr<l-1} \Sigma^{2kr} M\left(k\rho, \left(k+\frac{l-1}{r}\right)\rho\right).$$

Proof. Since $q^r\equiv 1-\rho l \pmod{l^2}$, we have $q^{kr}\equiv 1-k\rho l \pmod{l^2}$. Thus $a_{kr}=1-q^{kr}\equiv k\rho l$ and $\alpha_{kr}=k\rho$. If r does not divide n, then $a_n\notin l\Lambda$. So, Theorem 5.1 implies Corollary 5.2.

By Bott periodicity and the fact that $f\psi^q(2n-1, \infty)$ is the homotopy fibre of $1-\psi^q$: $\boldsymbol{bu}_\Lambda(2n, \infty)\to \boldsymbol{bu}_\Lambda(2n, \infty)$, we have

Corollary 5.3. *If $n\geq 0$ then*

$$H^*(f\psi^q(2n-1, \infty))\cong \bigoplus_{n\leq kr<n+l-1} \Sigma^{2kr} M\left(k\rho, \left(k+\frac{l-1}{r}\right)\rho\right).$$

Next, let $\boldsymbol{bo}$ be the spectrum which represents (-1)-connected real K-theory. We write $f\psi^q o$ for the homotopy fibre of $1-\psi^q$: $\boldsymbol{bo}_\Lambda\to \boldsymbol{bo}_\Lambda$, then we have

Corollary 5.4. *If $l\neq 2$ and $n\geq 0$, then*

$$H^*(f\psi^q o(2n-1, \infty))\cong \bigoplus_{n\leq kr'<n+l-1} \Sigma^{2kr'} M\left(k\rho', \left(k+\frac{l-1}{r'}\right)\rho'\right),$$

where r' is the least even integer ≥ 1 such that $q^{r'}\equiv 1 \pmod{l}$ and ρ' is the element of Z/l such that $1-q^{r'}=l\rho' \pmod{l^2}$.

Proof. Recall that $\boldsymbol{bo}_A \simeq \boldsymbol{g}_0 \vee \boldsymbol{g}_2 \vee \cdots \vee \boldsymbol{g}_{l-3}$ and $(1-\psi^q)_*: \pi_{4n}(\boldsymbol{bo}_A) \to \pi_{4n}(\boldsymbol{bo}_A)$ is the multiplication by $1-q^{2n}$. Thus proof of Corollary 5.4 is analogous to the proof of Corollary 5.3.

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