



Geometry — *Brody hyperbolicity and homotopy*, by SIMONE BORGHESI and GIUSEPPE TOMASSINI*, communicated on 14 May 2010.

ABSTRACT. — The paper’s aim is to develop a theory in which the concept of Brody hyperbolicity of a complex space (cfr. [2]) is interpreted in terms of homotopy-theoretic structures. We contend that this interplay will be particularly useful if implemented by applying homotopy-theoretical techniques and constructions to get information on hyperbolic spaces. Imitating the construction of homotopy groups, we will define *holotopy groups* that will be able to tell apart different complex structures. From our point of view, the most important feature of these groups is that they vanish in a certain range if evaluated on a Brody hyperbolic complex space (see Theorem 4.1), providing therefore a way to reduce the proof of non hyperbolicity of a complex space to the existence of a nonzero holotopy class in these groups.

KEY WORDS: Hyperbolic spaces, simplicial sheaves, homotopical algebra.

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1. INTRODUCTION

The paper’s aim is to develop a theory in which the concept of Brody hyperbolicity of a complex space (cfr. [2]) is interpreted in terms of homotopy-theoretic structures, therefore opening an entirely new link between two, a priori completely different, branches of mathematics. Based on our current understanding, we contend that this interplay will be particularly useful if implemented by applying homotopy-theoretical techniques and constructions to get information on hyperbolic spaces rather than following the opposite path. Imitating the construction of homotopy groups, we will define *holotopy groups* that will be able to tell apart different complex structures. From our point of view, the most important feature of these groups is that they vanish in a certain range if evaluated on a Brody hyperbolic complex space (see Theorem 4.1), providing therefore a way to reduce the proof of non hyperbolicity of a complex space to the existence of a nonzero holotopy class in these groups. This suggests the relevance of finding techniques to compute classes in such groups. We expect them to follow from general results concerning abstract homotopy theory of model categories. In our manuscript, we get some results which relate holotopy groups with homotopy groups; for instance, by general nonsense, we show that suitably defined fibrations induce long exact sequences of holotopy groups and construct a *topological realization functor* (see Section 5) which induces an homomorphism from the

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holotopy groups of a complex space to the homotopy groups of the underlying topological space. Such homomorphism in general is neither injective nor surjective and its image are the homotopy classes which have a “complex geometric origin”.

We define holotopy groups as the Hom pointed sets of pointed morphisms between objects which play the role of spheres and a complex space X in an appropriate category \mathcal{H}_\bullet . Abstract homotopy theory prescriptions require the initial category to have certain structures that the category \mathcal{S} of complex spaces with holomorphic functions does not have. Hence we perform suitable adjustments and extensions in order to obtain a category $\Delta^{op} \mathcal{F}_T(\mathcal{S})$ which is endowed of such structures and we can appropriately localize. The idea is to “add” the inverse to the canonical map $p : \mathbb{C} \rightarrow \text{pt}$ along with all its base changed maps by specifying an appropriate model structure on $\Delta^{op} \mathcal{F}_T(\mathcal{S})$, using the techniques developed by Morel-Voevodsky in [6]. The associated homotopy category \mathcal{H} is the analogue of the (unstable) homotopy category \mathcal{T} of topological spaces (whose objects are compactly generated locally Hausdorff topological spaces) where homotopy groups $\pi_n(X, x)$ may be described as $\text{Hom}_{\mathcal{T}_\bullet}((S^n, pt), (X, x))$, \mathcal{T}_\bullet being the pointed version of \mathcal{T} and S^n the n -th dimensional sphere. The fact that any class in $\pi_n(X, x)$ may be represented by a continuous function $(S^n, pt) \rightarrow (X, x)$ does no longer hold for holotopy groups: there are not enough morphisms in $\Delta^{op} \mathcal{F}_T(\mathcal{S})$ to represent all the classes in holotopy groups. To get holotopy groups as quotient sets of $\text{Hom}_{\mathcal{H}^{olo}}$ sets, we must replace X with a *weakly equivalent* object $\tilde{X} \in \Delta^{op} \mathcal{F}_T(\mathcal{S})$, which in general will not be a complex space, satisfying a certain condition. The key remark is that such condition is the Brody hyperbolicity of \tilde{X} , when \tilde{X} happens to be a complex space (see Corollary 3.1). Because of this, we will call *Brody hyperbolic* any object in $\Delta^{op} \mathcal{F}_T(\mathcal{S})$ satisfying such property. The correspondence $X \rightsquigarrow \tilde{X}$ is functorial and satisfies a universal property among hyperbolic objects, hence \tilde{X} can be considered as an *hyperbolic model* of X and as such will be called in our paper.

2. BASIC CONSTRUCTIONS

2.1. Sheaves and simplicial objects: the categories $\mathcal{F}_T(\mathcal{S})$ and $\Delta^{op} \mathcal{F}_T(\mathcal{S})$

Let \mathcal{S} be the category of complex spaces (or schemes of finite type over a noetherian scheme B). We would like to create a category from \mathcal{S} where the complex space \mathbb{C} becomes isomorphic to a point in a compatible way to how we get the unstable homotopy category from the category of *topological spaces*. From the point of view of homotopy theory, \mathcal{S} has some problems: it is not closed under (finite) colimits and, when a colimit does exist in \mathcal{S} , its underlying set may be too different from the colimit of the underlying sets or even topological spaces. Moreover, no useful model structure on \mathcal{S} to achieve this task is known. We therefore follow the route set forth by Morel and Voevodsky in [6] and let \mathcal{S}_T be the site of complex spaces with the strong topology (or that of schemes of finite type over a noetherian scheme B of finite dimension, endowed with a Grothendieck topology

which is weaker or as fine as the quasi compact flat topology). We denote by $\mathcal{F}_T(\mathcal{S})$ the category sheaves of sets on \mathcal{S}_T where morphisms are maps of sheaves of sets. Let $\mathbf{Y}(X) := \text{Hom}_{\mathcal{S}}(-, X)$. The functorial equality

$$\text{Hom}_{\mathcal{S}}(A, B) = \text{Hom}_{\text{Fun}(\mathcal{S}^{\text{op}}, \text{Sets})}(\mathbf{Y}(A), \mathbf{Y}(B))$$

is known as Yoneda Lemma. The Yoneda embedding is a faithfully full functor $\mathbf{Y} : \mathcal{S} \hookrightarrow \text{Fun}(\mathcal{S}^{\text{op}}, \text{Sets})$ and $\text{Hom}_{\mathcal{S}}(-, X)$ is a sheaf for the topology T .

The category $\mathcal{F}_T(\mathcal{S})$ is complete and cocomplete i.e. has limits and colimits. In particular it possesses two canonical objects: an initial sheaf \emptyset , the sheaf that associates the empty set to any element of the site, except the sheaf that associates the empty set to any element of the site, except for the initial object of the site \mathcal{S}_T to which it associates the one point set and the final sheaf, which we will denote as pt .

In the sequel we will work with the category $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ of simplicial objects in $\mathcal{F}_T(\mathcal{S})$ (cfr. [8] for the basic properties of this subject). A *simplicial object* \mathcal{X} in $\mathcal{F}_T(\mathcal{S})$ is a sequence $\{\mathcal{X}_i\}_{i \geq 0}$ of objects of $\mathcal{F}_T(\mathcal{S})$ with a sequence $\partial_i^n : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1}$ of morphisms for $n \geq 1, i = 0, 1, \dots, n$ called *faces* and a sequence $\sigma_i^n : \mathcal{X}_n \rightarrow \mathcal{X}_{n+1}$ of morphisms for $n \geq 0, i = 0, 1, \dots, n$ called *degenerations*, satisfying appropriate compatibility conditions (cfr. [8]).

A *morphism* $f : \mathcal{X} \rightarrow \mathcal{Y}$ of two simplicial objects $\mathcal{X} = \{\mathcal{X}_i\}_{i \geq 0}, \mathcal{Y} = \{\mathcal{Y}_i\}_{i \geq 0}$ of $\mathcal{F}_T(\mathcal{S})$ is a sequence $\{f_i\}_{i \geq 0}$ of morphisms $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$ commuting with the face and boundary operators. In $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ we can retrieve the site \mathcal{S}_T fully faithfully embedded as the *simplicially constant* objects represented by $X \in \mathcal{S}_T$: given $X \in \mathcal{S}_T$ we will denote by the same symbol the *constant* (or *discrete*) simplicial object defined as $\mathcal{X}_i = X, \partial_i^n = \sigma_i^n = \text{Id}_X$, for every i, n .

2.2. Simplicial localization

The following will give $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a model structure in the sense of Quillen ([7]).

A morphism $f : \mathcal{G} \rightarrow \mathcal{F}$ of simplicial sheaves is a *weak equivalence* if for every point x of a complex space or a scheme over B , $f_x : \mathcal{G}_x \rightarrow \mathcal{F}_x$ is a weak equivalence of simplicial sets (\mathcal{G}_x and \mathcal{F}_x being the respective stalks over x of \mathcal{F} and \mathcal{G}).

An injective morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be a *simplicial cofibration*.

A *lifting* in a commutative square of morphisms

$$(1) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{q} & \mathcal{X} \\ \downarrow j & & \downarrow f \\ \mathcal{B} & \xrightarrow{r} & \mathcal{Y} \end{array}$$

is a morphism $h : \mathcal{B} \rightarrow \mathcal{X}$ which makes the diagram commutative. In such situation we say that j has the *left lifting property* with respect to f and f has the *right lifting property* with respect to j .

A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called a *fibration* if all diagrams (1) admit a lifting, for all *acyclic cofibrations* j (cofibration and weak equivalence simultaneously).

An object \mathcal{X} of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$

- 1) is called *cofibrant* if $\emptyset \rightarrow \mathcal{X}$ is a cofibration;
- 2) is called *fibrant* if $\mathcal{X} \rightarrow \text{pt}$ is a fibration.

The classes of weak equivalences, cofibrations and fibrations give $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a structure of *simplicial model category* as shown in [5]. Under these assumptions, there exists a localization \mathcal{H}_s of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ with respect of the weak equivalences.

The same constructions as for $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ can be performed for the *pointed category* $\Delta^{\bullet,\text{op}}\mathcal{F}_T(\mathcal{S})$ associated to $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ obtaining a homotopy category $\mathcal{H}_{s,\bullet}$. $\Delta^{\bullet,\text{op}}\mathcal{F}_T(\mathcal{S})$ is the category $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ whose objects are the pairs (\mathcal{X}, x) where $\mathcal{X} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ and $x : \text{pt} \rightarrow \mathcal{X}$ is a morphism; a morphism of pairs $(\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ is a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that $f \circ x = y$. Let us fix some notation. If \mathcal{X} and \mathcal{Y} are pointed simplicial sheaves the sheaf $\mathcal{X} \vee \mathcal{Y}$ is, by definition, the colimit of

$$\begin{array}{ccc} \text{pt} & \longrightarrow & \mathcal{X} \\ \downarrow & & \\ \mathcal{Y} & & \end{array}$$

pointed by the image of pt .

The *pointed simplicial sheaf* $\mathcal{X} \wedge \mathcal{Y}$ is defined by $\mathcal{X} \times \mathcal{Y} / \mathcal{X} \vee \mathcal{Y}$.

The *simplicial pointed constant sheaf* S_s^1 is defined by $\Delta[1] / \partial\Delta[1]$ where $\partial\Delta[1]$ is the simplicial subsheaf of $\Delta[1]$ consisting of the union of the images of the face morphisms of $\Delta[1]$. For $n \in \mathbb{N}$ we set $S_s^n = S_s^1 \wedge \dots \wedge S_s^1$.

3. HYPERBOLICITY

3.1. Affine localization

Starting from \mathcal{H}_s we give $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a new model structure whose weak equivalences contain $p : \mathbb{C} \rightarrow \text{pt}$, and are in a sense the “smallest” class including all the base changelements of p , as well. Such weak equivalences which are written in terms of morphisms in \mathcal{H}_s are based on the following notion of \mathbb{C} -locality.

A simplicial sheaf $\mathcal{X} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ is said to be *\mathbb{C} -local* if the projection $\mathcal{Y} \times \mathbb{C} \rightarrow \mathcal{Y}$ induces a bijection of sets

$$\text{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \rightarrow \text{Hom}_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

for every $\mathcal{Y} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$.

A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called:

- 1) a *\mathbb{C} -weak equivalence* if, for every \mathbb{C} -local simplicial sheaf $\mathcal{Z} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$

$$f^* : \text{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Z}) \rightarrow \text{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{Z})$$

is a bijection;

- 2) a \mathbb{C} -cofibration if it is injective;
- 3) a \mathbb{C} -fibration if all diagrams (1) admit a lifting, where j is any \mathbb{C} -cofibration and \mathbb{C} -weak equivalence.

An object \mathcal{X} of $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ is called

- 1) \mathbb{C} -fibrant if the canonical morphism $\mathcal{X} \rightarrow \text{pt}$ is an affine fibration;
- 2) \mathbb{C} -cofibrant if $\emptyset \rightarrow \mathcal{X}$ a \mathbb{C} -cofibration.

The structures listed above endow $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ of a model structure, which will be called \mathbb{C} -model structure or affine model structure (cfr. [6, Theorem 3.2]). The localized category with respect of the \mathbb{C} -weak equivalences is denoted as \mathcal{H} and its pointed version as \mathcal{H}_* .

REMARK 3.1. The affine localization functor $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{H}$ factors as

$$\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \mathcal{H}_s \rightarrow \mathcal{H},$$

where the first functor is the simplicial localization and the second is the identity on objects. However, the functor $\mathcal{H}_s \rightarrow \mathcal{H}$ is not an equivalence of categories. The same classes of pointed morphisms give $\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ a model structure.

As a particular case of [7, Proposition 4'] we have the following result: if $j : \mathcal{Y} \rightarrow \mathcal{X}$ is a \mathbb{C} -cofibration then, for every simplicial pointed sheaf \mathcal{Z} , the morphism j induces long exact sequence of pointed sets and groups

$$\begin{aligned} (2) \quad \text{Hom}_{\mathcal{H}_*}(\mathcal{Y}, \mathcal{Z}) &\xleftarrow{j^*} \text{Hom}_{\mathcal{H}_*}(\mathcal{X}, \mathcal{Z}) \xleftarrow{\pi^*} \text{Hom}_{\mathcal{H}_*}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}) \\ &\xleftarrow{\quad} \text{Hom}_{\mathcal{H}_*}(\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \xleftarrow{j^*} \text{Hom}_{\mathcal{H}_*}(\mathcal{X} \wedge S_s^1, \mathcal{Z}) \\ &\xleftarrow{\pi^*} \text{Hom}_{\mathcal{H}_*}(\mathcal{X}/\mathcal{Y} \wedge S_s^1, \mathcal{Z}) \dots \end{aligned}$$

where \mathcal{X}/\mathcal{Y} is the object making the following square cocartesian:

$$(3) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{j} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \text{pt} & \longrightarrow & \mathcal{X}/\mathcal{Y}. \end{array}$$

3.2. Hyperbolic simplicial sheaves

\mathbb{C} -local simplicial sheaves \mathcal{X} are said to be hyperbolic. A hyperbolic resolution of \mathcal{X} is a morphism of simplicial sheaves $r : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ where $\tilde{\mathcal{X}}$ is a hyperbolic simplicial sheaf and r is an affine weak equivalence.

A hyperbolic resolution functor is a pair (\mathfrak{R}, r) where \mathfrak{R} is a functor

$$\Delta^{\text{op}}\mathcal{F}_T(\mathcal{S}) \rightarrow \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$$

and r is a natural transformation $\text{Id} \rightarrow \mathfrak{I}$ such that every morphism $\mathcal{X} \rightarrow \mathfrak{I}(\mathcal{X})$ is a hyperbolic resolution. We have the following, fundamental

THEOREM 3.1. *There exists a hyperbolic resolution functor $(\mathfrak{I}p, r)$ with the following properties:*

- 1) for every $\mathcal{X} \in \Delta^{\text{op}}\mathcal{F}_T(\mathcal{S})$ the simplicial sheaf $\mathfrak{I}p(\mathcal{X})$ is hyperbolic and (simplicially) fibrant;
- 2) r is a \mathbb{C} -equivalence and a cofibration;
- 3) let $\mathcal{H}_{s, \mathbb{C}}$ be the full subcategory in \mathcal{H}_s of \mathbb{C} -local (hyperbolic) objects. $\mathfrak{I}p$ sends a \mathbb{C} -weak equivalence to a simplicial weak equivalence, hence it induces a functor $L : \mathcal{H}_s \rightarrow \mathcal{H}_{s, \mathbb{C}}$ that factors as $\mathcal{H}_s \rightarrow \mathcal{H} \rightarrow \mathcal{H}_{s, \mathbb{C}}$, where the first functor is the identity on objects;
- 4) the canonical immersion $I : \mathcal{H}_{s, \mathbb{C}} \hookrightarrow \mathcal{H}_s$ is a right adjoint of L .

Furthermore, $\mathcal{H}_{s, \mathbb{C}}$ is a category equivalent to \mathcal{H} .

This is a consequence of the Bousfield framework [1] for localizing model categories. For more details about this result as stated here see [6].

Given $X = \mathcal{X} \in \mathcal{F}_T$, $\mathfrak{I}p(X)$ is the hyperbolic simplicial sheaf associated to the simplicially constant sheaf X .

The morphism $r : \mathcal{X} \rightarrow \mathfrak{I}p(\mathcal{X})$ is universal in the category \mathcal{H} (respectively in the category \mathcal{H}_s): for any hyperbolic object \mathcal{Y} and morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ in \mathcal{H} (respectively in the category \mathcal{H}_s), there exists a unique morphism $\tilde{f} : \mathfrak{I}p(\mathcal{X}) \rightarrow \mathcal{Y}$ in \mathcal{H}_s factoring f as $\tilde{f} \circ r$.

As a consequence we get that if X and Y are sheaves with Y hyperbolic and $f : X \rightarrow Y$ is a morphism of sheaves, then the composition $\tilde{f} \circ r$ is a morphism of sheaves and the commutativity of the diagram

$$(4) \quad \begin{array}{ccc} X & \xrightarrow{r} & \mathfrak{I}p(X) \\ \downarrow f & \swarrow \tilde{f} & \\ Y & & \end{array}$$

is in the category of sheaves, i.e. it is strictly commutative and not only “up to homotopy” in \mathcal{H}_s .

3.3. Hyperbolicity and Brody hyperbolicity

The concept of hyperbolicity as introduced above is the same as Brody hyperbolicity for a simplicial sheaves represented by a complex space X . This is a consequence of the following crucial

THEOREM 3.2. *A sheaf $X \in \mathcal{F}_T(\mathcal{S})$ is hyperbolic if and only if the projection $U \times \mathbb{C} \rightarrow U$ induces a bijection*

$$\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(U, X) \rightarrow \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(U \times \mathbb{C}, X)$$

for every object $U \in \mathcal{S}_T$. Moreover, under this hypothesis, for every $Y \in \mathcal{F}_T(\mathcal{S})$ there exists a bijection

$$(5) \quad \text{Hom}_{\mathcal{H}}(Y, X) \cong \text{Hom}_{\mathcal{H}_s}(Y, X) \cong \text{Hom}_{\mathcal{F}_T(\mathcal{S})}(Y, X).$$

COROLLARY 3.1. *Let X be a complex space, C a closed complex subspace of X . Then X is hyperbolic modulo C in the sense of Brody (cfr. [4]) if and only if X/C is a hyperbolic sheaf.*

If \mathcal{X} is a simplicial sheaf, Y, Y' hyperbolic complex spaces such that

$$\mathfrak{Sp}(\mathcal{X}) = [Y]_{\mathcal{H}} = [Y']_{\mathcal{H}}.$$

then Y' and Y are isomorphic complex spaces. In particular, if X is a complex space and $\mathfrak{Sp}(X)$ is represented by a hyperbolic complex space Y , then Y is unique up to biholomorphisms. In general we cannot hope to have hyperbolic complex spaces in the class of a simplicial sheaf or even of a complex space: in the next section we will show that $\mathfrak{Sp}(\mathbb{P}^n)$ cannot be \mathbb{C} -weakly equivalent to a hyperbolic complex space.

In some cases, we can extend some results known for hyperbolic complex spaces to hyperbolic sheaves: e.g. it can be proved that if F is a hyperbolic sheaf then

$$\text{Hom}_{\mathcal{F}_T(\mathcal{S})}(\mathbb{C}\mathbb{P}^n, F) = F(\text{pt})$$

for any $n \geq 1$. In other words, any sheaf map from $\mathbb{C}\mathbb{P}^n$ to a hyperbolic sheaf F must be constant.

4. HOLOTOPY GROUPS

Let \mathcal{S}_T denote the site of complex spaces endowed with the strong topology. A simplicial object of \mathcal{S}_T is, by definition, a *simplicial complex space*. A rather natural modification of the definition of homotopy enables us to attach to every simplicial sheaf on \mathcal{S}_T two families $\{\pi_{i,j}^{\text{par}}(\mathcal{X})\}_{i,j}$, $\{\pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X})\}_{m,n}$ of sets which, for positive simplicial degrees, have a canonical group structure and are invariant under biholomorphisms. We will use these groups in Section 6.

The holotopy groups are refinements of homotopy groups in a precise sense: there are homomorphisms from them to the homotopy groups of the topological realizations (see Section 5) and there are holomorphic functions between complex spaces whose topological realizations are homotopy equivalences, while not being isomorphisms on holotopy groups. A simple example is the embedding $D^* \hookrightarrow C^*$ of the punctured disk in the punctured complex line: the holotopy groups in positive degrees of D^* are all zero since it is hyperbolic (see Theorem 4.1), whereas $\pi_{1,0}^{\text{par}}(C^*, x)$ contains a nontrivial holotopy class, namely the identity (see just below the definition of this holotopy group).

It would be interesting to find such examples in the compact case.

Define the *parabolic circle* S_{par}^1 by

$$S_{\text{par}}^1 = \mathbb{C}/(0 \amalg 1),$$

and the *hyperbolic circle* $S_{\text{iper}}^1(z_1, z_2)$ by

$$S_{\text{iper}}^1(z_1, z_2) = D/(z_1 \amalg z_2)$$

where $D \subset \mathbb{C}$ is the unit disc and $z_1 \neq z_2$ two points of D . (The quotients defining parabolic and hyperbolic circles are taken in the category $\mathcal{F}_T(\mathcal{S})$, even though the set theoretic quotients have a complex structure (cfr. [3])). Set $S_{\text{par}}^n, S_{\text{iper}}^n(z_1, z_2)$ the sheaves $S_{\text{par}}^1 \wedge \cdots \wedge S_{\text{par}}^1, S_{\text{iper}}^1(z_1, z_2) \wedge \cdots \wedge S_{\text{iper}}^1(z_1, z_2)$ respectively.

Given a simplicial sheaf \mathcal{X} on \mathcal{S}_T the sets

$$(6) \quad \pi_{i,j}^{\text{par}}(\mathcal{X}, x) = \text{Hom}_{\mathcal{H}_\bullet}((\mathbb{C}^*)^{\wedge j} \wedge S_{\text{par}}^{i-j}, (\mathcal{X}, x)) \quad \text{for } i \geq j \geq 0,$$

$$(7) \quad \pi_{n,m}^{\text{iper}}(z_1, z_2)(\mathcal{X}, x) = \text{Hom}_{\mathcal{H}_\bullet}(S_{\text{iper}}^n(z_1, z_2) \wedge S_{\text{par}}^m, (\mathcal{X}, x)) \quad \text{for } n, m \geq 0,$$

are called respectively *parabolic holotopy pointed sets* of \mathcal{X} (or groups in the case they are) and *hyperbolic holotopy pointed sets* of \mathcal{X} (or groups in the case they are).

As a consequence of the long exact sequence (2), we can prove that the sets $\pi_{i,j}^{\text{par}}, \pi_{n,m}^{\text{iper}}$ have a canonical group structure for $i > j > 0$ and $m > 0$.

From our point of view, the most important feature of these groups is that they vanish in a certain range if evaluated on a Brody hyperbolic complex space:

THEOREM 4.1. *Let X be a hyperbolic sheaf. Then the groups $\pi_{i,j}^{\text{par}}(X, x), \pi_{n,m}^{\text{iper}}(X, x)$ vanish for $i - j > 0$ and any $m > 0$.*

REMARK 4.1. To relate holotopy groups of a complex space X with morphisms in $\Delta^{op} \mathcal{F}_T(\mathcal{S})$ it is necessary to replace X with its hyperbolic model $\mathfrak{I}p(X)$. Then $\pi_{i,j}(X, x)$ will be a quotient of the set $\text{Hom}_{\Delta^{op} \mathcal{F}_T(\mathcal{S})}(\mathbf{S}^{i,j}, \mathfrak{I}p(X))$, where $\mathbf{S}^{i,j}$ is a pointed model of the relevant sphere.

5. THE TOPOLOGICAL REALIZATION FUNCTOR

The objects in \mathcal{H} can be compared with the topological spaces, objects of the (unstable) homotopy category \mathcal{H}^{top} of topological spaces (i.e. the localization of the category of topological spaces with respect to the usual weak equivalences). Indeed we can prove that there exists a functor $t^{olo} : \mathcal{H} \rightarrow \mathcal{H}^{top}$ which extends the functor associating the underlying topological space to a complex space. In the algebraic case it extends the corresponding functor which associates to an algebraic variety over \mathbb{C} the topological space of its Zariski closed points. More precisely, let $r : \mathcal{S} \rightarrow \mathcal{T}$ be the functor sending a complex space to its underlying topological space. Then t^{olo} is a functor satisfying the following properties:

- (1) if $X \in \Delta^{op} \mathcal{F}_T(\mathcal{S})$ is a simplicial set, then the class $t^{olo}(X)$ can be represented by the geometric realization $|X|$;

- (2) if F is the sheaf $\text{Hom}_{\mathcal{S}}(-, X)$, where $X \in \mathcal{S}$, then $t^{olo}(F)$ can be represented by $r(X)$;
- (3) t^{olo} commutes with direct products and homotopy colimits.

In particular,

$$t^{olo}(S_{\text{par}}^n) \cong t^{olo}(S_{\text{iper}}^n) \cong S^n$$

and

$$t^{olo}((\mathbb{C}^*)^{\wedge j} \wedge S_{\text{par}}^{i-j}) \cong t^{olo}((\mathbb{C}^*)^{\wedge j} \wedge S_{\text{iper}}^{i-j}) \cong S^i.$$

Therefore, for any complex space X , we obtain homomorphisms $\pi_{i,j}(X, x) \rightarrow \pi_i(r(X), r(x))$ induced by the topological realization functor. Such homomorphisms are not injective nor surjective, in general, however, sometimes they are useful to show that some holotopy class is nonzero.

6. SOME APPLICATIONS

In this last section we are going to consider few applications of the theory developed so far. We will begin with examples of complex spaces that are not \mathbb{C} -weakly equivalent to any complex hyperbolic space.

We will say that a complex space is *weakly hyperbolic* if it is \mathbb{C} -weakly equivalent to a Brody hyperbolic complex space.

A preliminary result is the following:

LEMMA 6.1. *The pointed simplicial sheaf $\mathbb{C}^* \wedge S_{\text{par}}^1$ is canonically weakly equivalent to $\mathbb{C}\mathbb{P}^1$.*

Then we can prove that

THEOREM 6.1. *For any $n > 0$, $\mathbb{C}\mathbb{P}^n$ is not weakly hyperbolic. In other words, $\mathbb{C}\mathbb{P}^n$ cannot be represented in \mathcal{H} by a Brody hyperbolic complex space.*

PROOF. In view of Theorem 4.1, it is sufficient to show that

$$\pi_{2,1}^{\text{par}}(\mathbb{C}\mathbb{P}^n, \infty) = \text{Hom}_{\mathcal{H}_*}(\mathbb{C}^* \wedge S_{\text{par}}^1, (\mathbb{C}\mathbb{P}^n, \{\infty\})) \neq 0$$

or equivalently, by Lemma 6.1, that

$$\text{Hom}_{\mathcal{H}_*}(\mathbb{P}^1, (\mathbb{C}\mathbb{P}^n, \{\infty\})) \neq 0.$$

Our candidate to represent a nonzero class is the canonical embedding $i: \mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^n$.

Let $|\mathbb{C}\mathbb{P}^n|$ be the underlying topological space of $\mathbb{C}\mathbb{P}^n$. The topological realization yields a group homomorphism

$$t: \pi_{2,1}^{\text{par}}(\mathbb{C}\mathbb{P}^n, \infty) \rightarrow \pi_2(|\mathbb{C}\mathbb{P}^n|, \infty).$$

$t^{olo}(i) : \mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^n$ is the canonical inclusion and not null homotopic, since $|\mathbb{C}\mathbb{P}^n|$ is obtained by $|\mathbb{C}\mathbb{P}^1|$ by attaching cells of dimension 4 and above, hence it is a homotopy equivalence up to dimension 2 and in particular

$$t^{olo}(i)_* : \mathbb{Z} = \pi_2(|\mathbb{C}\mathbb{P}^1|, \infty) \rightarrow \pi_2(|\mathbb{C}\mathbb{P}^n|, \infty)$$

is an isomorphism. In conclusion $t[i] \neq 0$, thus $[i] \neq 0 \in \pi_{2,1}^{par}(\mathbb{C}\mathbb{P}^n, \infty)$. □

The following proposition describes a characteristic that a weakly hyperbolic, non Brody hyperbolic complex spaces X must have:

PROPOSITION 6.1. *Let X be a complex space and $p : \tilde{X} \rightarrow X$ a covering holomorphic function. Assume that X is weakly hyperbolic and let $f : \mathbb{C} \rightarrow X$ be a nonconstant holomorphic function. Then for any lifting \tilde{f} of f to \tilde{X} , $\tilde{f}(\mathbb{C})$ contains just one point in each fiber of p or equivalently $p|_{\tilde{f}(\mathbb{C})}$ is a biholomorphism for any such f and \tilde{f} .*

PROOF. Let X be weakly hyperbolic. Assume, by contradiction, that there exist a nonconstant holomorphic function $f : \mathbb{C} \rightarrow X$ and a lifting $\tilde{f} : \mathbb{C} \rightarrow \tilde{X}$ such that $a \neq b \in p^{-1}(x)$, $x \in X$, $a, b \in \tilde{f}(\mathbb{C})$. For the purposes of this proof, we can assume that $\tilde{f}(0) = a$ and $\tilde{f}(1) = b$. Then we have the following commutative diagram:

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow q & & \downarrow p \\ \mathbb{C}/(0 \amalg 1) & \xrightarrow{\alpha} & X \end{array}$$

where α sends the class of $\{0\} \amalg \{1\}$ to $x \in X$. We have that $[\alpha] \neq 0 \in \pi_{1,0}^{par}(X, x)$. Indeed, $[\alpha^{top}] \neq 0 \in \pi_1(X^{top}, x)$. Consider the composition

$$[0, 1] \xrightarrow{g} \mathbb{C}/(0 \amalg 1) \xrightarrow{\alpha^{top}} X^{top},$$

where g is a path from 0 to 1 in \mathbb{C} . If $\alpha^{top} \circ g$ is not homotopic to a constant relatively to $\{0, 1\}$, then α^{top} is not homotopic to a constant. But, by construction, $\alpha^{top} \circ g$ lifts uniquely to a path in \tilde{X}^{top} starting from a and ending at b , hence $\alpha^{top} \circ g$ cannot be homotopic to a constant relatively to $\{0, 1\}$. This shows that $\pi_1(X^{top}, x) \neq 0$ which is absurd since X is weak hyperbolic. □

The Proposition 6.1 in particular implies the following

COROLLARY 6.1. *Any complex space X whose universal covering space is \mathbb{C}^n for some $n \geq 1$, is not weakly hyperbolic.*

PROOF. Let $p : \mathbb{C}^n \rightarrow X$ be the universal covering of X . Let $a \neq b \in p^{-1}(x)$, $x \in X$. A complex line $l \subset \mathbb{C}^n$ passing through a, b provides a holomorphic map $f : \mathbb{C} \rightarrow X$ which does not satisfy the conclusion of Proposition 6.1. □

Knowing that a nonzero holotopy group implies that the complex space is not (weakly) hyperbolic, we may ask if the opposite implication holds, as well. In general the answer is negative; however, by a different rephrasing of the previous proposition, we conclude:

PROPOSITION 6.2. *Let X be a non Brody hyperbolic complex space admitting a covering $p : Y \rightarrow X$ with a fiber $p^{-1}(x)$ intersecting the image of $\mathbb{C} \rightarrow Y$ in at least two distinct points. Then $\pi_{1,0}^{\text{par}}(X, x) \neq 0$.*

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