Publ. RIMS, Kyoto Univ.
19 (1983), 765-771

On the Connective K-Homology Groups of the Classifying Spaces BZ/p^r

By

Shin HASHIMOTO*

§ 0. Introduction

Let p be a prime number and Z/p^r be a cyclic group of order p^r. We choose BZ/p^r, the classifying space of Z/p^r, as the colimit of the lens spaces $L^n(p^r)$. It is well known that $\widetilde{K}^{2n}(BZ/p^r) = \bigoplus Z_p^\wedge$ by Atiyah [1], Atiyah-Segal [2], and $\widetilde{K}_{2n-1}(BZ/p^r) = \bigoplus Z/p^\infty$ by Vick [9] (and the groups of other degrees are trivial).

We consider the connective K-theory k. Using the Atiyah-Hirzebruch spectral sequence, connective K-(co)homology group of BZ/p^r is a subgroup of periodic K-(co)homology group of BZ/p^r. And the Atiyah Hirzebruch spectral sequence also shows that $k_{2n-1}(BZ/p^r)$ is a finite group for any n. Moreover, the Atiyah-Hirzebruch spectral sequence determines its order.

The purpose of this paper is to calculate the additive structure of $k_*(BZ/p^r)$.

In Section 1 we interpret the group $k_{2n-1}(BZ/p^r)$ by the group $K(L^n(p^r))$. In Section 2 we use the Gysin sequence to determine the generators and the relations of $k_{2n-1}(BZ/p^r)$. In Section 3 we give the explicit structure of $k_{2n-1}(BZ/p^r)$.

The author expresses his hearty gratitude to Professors S. Araki, Z. Yosimura and A. Kono for their invaluable suggestions.

§ 1. Connective K-Theory and Classifying Spaces

Let Z/m be a cyclic group of order m, which is naturally a sub-

Communicated by N. Shimada, October 16, 1982.
* Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558, Japan.

group of $S^1 \subset C$. We consider the standard lens space

$$L^n(m) = S^{2n+1}/(Z/m)$$

where S^{2n+1} is a unit sphere in C^{n+1}. Then, $L^n(m)$ is a CW-complex with the cell decomposition $L^n(m) = \bigcup_{j=0}^{2n+1} e^j$. We denote the $2n$-skeleton of $L^n(m)$ by

$$L_0^n(m) = \{[z_0, \cdots, z_n] \in L^n(m) \,;\, z_n \text{ is real } \geqq 0\}.$$

Naturally, $L^n(m)$ is a subcomplex of $L^{n+1}(m)$. We denote $L^\infty(m) = \operatorname{colim}_n L^n(m)$, which is a classifying space of Z/m as is well known.

We consider the complex K-homology theory K_* and its associated connective theory $K[0, \infty)_* = k_*$.

Lemma 1.1. [3] *For any CW-complex X, there hold natural isomorphisms*

$$\tilde{k}_q(X) = \operatorname{Im}(i_* : \tilde{K}_q(X^{(q)}) \to \tilde{K}_q(X^{(q+1)}))$$

where i is the inclusion of $X^{(q)}$, the q-skeleton of X, to $X^{(q+1)}$.

The cofibration

$$L^{n-1}(m) \xrightarrow{i} L_0^n(m) \xrightarrow{p} S^{2n}$$

induces an exact sequence

$$\tilde{K}_{2n-1}(L^{n-1}(m)) \xrightarrow{i_*} \tilde{K}_{2n-1}(L_0^n(m)) \xrightarrow{p_*} \tilde{K}_{2n-1}(S^{2n}) = 0\,.$$

So we have

Corollary 1.2. $\tilde{k}_{2n-1}(BZ/m) = \tilde{K}_{2n-1}(L_0^n(m))$.

As is proved in [10], there holds the universal coefficient exact sequence for K-cohomology theory

$$(1.3) \qquad 0 \to \operatorname{Ext}(\tilde{K}_{*-1}(X), Z) \to \tilde{K}^*(X) \to \operatorname{Hom}(\tilde{K}_*(X), Z) \to 0\,.$$

Using the Atiyah-Hirzebruch spectral sequence, we have

Lemma 1.4. (i) $\widetilde{K}_{2n}(L_0^n(m))=0,$

(ii) $\widetilde{K}_{2n-1}(L_0^n(m))$ *is a finite group of order* m^n.

The exact sequence (1.3) and Lemma 1.4 (i) imply

Corollary 1.5. $\mathrm{Ext}(\widetilde{K}_{2n-1}(L_0^n(m)), Z)\cong\widetilde{K}^{2n}(L_0^n(m))$.

And by Lemma 1.4 (ii), we have

Corollary 1.6. $\widetilde{K}_{2n-1}(L_0^n(m))\cong\mathrm{Ext}(\widetilde{K}_{2n-1}(L_0^n(m)), Z)$.

The composition of the isomorphisms of Corollaries 1.2, 1.5 and 1.6 gives

Proposition 1.7. $\tilde{k}_{2n-1}(BZ/m)\cong\widetilde{K}^{2n}(L_0^n(m))$.

The additive structure of $\widetilde{K}^{2n}(L_0^n(m))$ is determined in [5] and [6] when $m=p^r$, where p is a prime. Since the isomorphism of Corollary 1.6 is not natural, Proposition 1.7 does not determine the generators of $\tilde{k}_{2n-1}(BZ/m)$. We determine the generators in the next section.

Remark 1.8. It is natural to consider the group $\tilde{k}^*(BZ/p^r)$, where k^* is the connective K-cohomology theory. But it is easy to see that $\tilde{k}^{2n}(BZ/p^r)=\bigoplus Z_p^{\wedge}$ which is a subgroup of $\widetilde{K}^{2n}(BZ/p^r)$ with a finite cokernel. This cokernel is $\widetilde{K}'^{2n}(BZ/p^r)$, where $K'^*=K(-\infty,-1]^*$ is the 0-coconnective K-cohomology theory. And $\widetilde{K}'^{2n}(BZ/p^r)$ is isomorphic to $\tilde{k}_{2n-3}(BZ/p^r)$. This isomorphism is obtained in two ways as follows: (i) $\widetilde{K}(-\infty,-1]^{2n}(X)\cong\mathrm{Im}(\widetilde{K}^{2n}(X^{(2n)})\to\widetilde{K}^{2n}(X^{(2n-1)}))$ and the cofibration $L^{n-1}(p^r)\to L_0^n(p^r)\to S^{2n}$ induces a surjection $\widetilde{K}^{2n}(L_0^n(p^r))\to\widetilde{K}^{2n}(L^{n-1}(p^r))$, so $\widetilde{K}'^{2n}(BZ/p^r)\cong\widetilde{K}^{2n}(L^{n-1}(p^r))\cong\tilde{k}_{2n-3}(BZ/p^r)$;
(ii) The natural isomorphism

$$\widetilde{K}(-\infty,-1]^*(X)=\widetilde{K}(-\infty,0]^{*-2}(X)$$

and the universal coefficient exact sequence

$$0\to\mathrm{Ext}(\tilde{k}_{*-1}(X), Z)\to\widetilde{K}(-\infty,0]^*(X)\to\mathrm{Hom}(\tilde{k}_*(X), Z)\to 0$$

leads us to the required isomorphism.

As the dual of these facts, $\widetilde{K}'_{2n-1}(BZ/p^r) = \oplus Z/p^\infty$ and there holds an exact sequence

$$0 \to \tilde{k}_*(BZ/p^r) \to \widetilde{K}_*(BZ/p^r) \to \widetilde{K}'_*(BZ/p^r) \to 0 .$$

§ 2. Gysin Sequence

Let h^* be a complex oriented multiplicative cohomology theory and h_* be its associated homology theory. For any complex vector bundle ξ, we denote the euler class of ξ in h^*-theory by $\chi(\xi)$. Let η be the canonical line bundle over CP^∞ and $x=\chi(\eta)$, the h^*-euler class of η. Then, $h^*(CP^\infty) = h^*(point)[[x]]$ ([4]), and $\chi(\eta^m) = [m]_F x$, the m-times of x in the sense of the formal group F of h^*.

Proposition 2.1. [7] *There holds the Gysin exact sequence*

$$\begin{aligned} \cdots \to h_{*-1}(CP^\infty) \to \tilde{h}_*(BZ/m) \to \tilde{h}_*(CP^\infty) \\ \xrightarrow{\mu_m} h_{*-2}(CP^\infty) \to \tilde{h}_{*-1}(BZ/m) \to \tilde{h}_{*-1}(CP^\infty) \to \cdots \end{aligned}$$

where $\mu_m(a) = \chi(\eta^m) \cap a$.

Now we restrict h^* to the periodic K-theory and the connective K-theory. Let $u \in K^{-2}(point)$ be the Bott element. Then, for a complex line bundle ξ the K^*-euler class $\chi(\xi)$ is defined to be $u^{-1}([\xi]-1)$. Thus $\chi(\eta^m) = u^{-1}((1+ux)^m - 1) = \sum_{i=1}^{m} \binom{m}{i} u^{i-1}x^i$. Since $K^*(CP^\infty) = K^*(point)[[x]]$, we use the duality and have

Lemma 2.2. (i) $K_{2n+1}(CP^\infty) = 0$,

(ii) $K_{2n}(CP^\infty) = Z\langle \beta_{i,j};\ i+j=n,\ i \geqq 0 \rangle$

where $\beta_{i,j}$ *is the dual base of* $u^{-i}x^j$, *i.e.*, $\langle u^{-i}x^j, \beta_{r,s} \rangle = \delta_{i,r} \cdot \delta_{j,s}$.

Using the Atiyah-Hirzebruch spectral sequence, we have

Corollary 2.3. $k_*(CP^\infty)$ *is a subgroup of* $K_*(CP^\infty)$ *and* $k_{2n}(CP^\infty)$

is spanned by $\{\beta_{i,j};\ i+j=n,\ i,j\geqq 0\}$.

So we have the commutative diagram with horizontal exact sequences

$$\text{(2.4)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{k}_{2n}(CP^\infty) & \xrightarrow{\mu_m} & k_{2n-2}(CP^\infty) & \xrightarrow{\pi} & \tilde{k}_{2n-1}(BZ/m) & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \longrightarrow & \tilde{K}_{2n}(CP^\infty) & \xrightarrow{\mu_m} & K_{2n-2}(CP^\infty) & \xrightarrow{\pi} & \tilde{K}_{2n-1}(BZ/m) & \longrightarrow & 0\,. \end{array}$$

We compute the map μ_m in K-homology theory with the Kronecker product.

$$\begin{aligned}\langle\mu_m(\beta_{n-j,j}),\, u^{-n+1+k}x^k\rangle &= \langle\beta_{n-j,j},\, \chi(\eta^m)\cup u^{-n+1+k}x^k\rangle \\ &= \langle\beta_{n-j,j},\, \sum_{i=1}^{m}\binom{m}{i}u^{-n+k+i}x^{k+i}\rangle.\end{aligned}$$

So we have

Lemma 2.5. $\mu_m(\beta_{n-j,j})=\sum_{i=1}^{m}\binom{m}{i}\beta_{n-j+i-1,j-i}$.

Because of the diagram (2.4) the identity in Lemma 2.5 for K-homology theory applies also to k-homology theory.

We put $\pi(\beta_{n-j-1,j})=B_j$ for $0\leqq j\leqq n-1$, and $B_j=0$ for $j<0$. Then

Lemma 2.6. $k_{2n-1}(BZ/m)=Z\langle B_0, B_1, \cdots, B_{n-1}\rangle/\langle\sum_{i=1}^{m}\binom{m}{i}B_{j-i}\rangle$, *where* $0\leqq j\leqq n$.

Lemma 2.7 (N. Mahammed [8]). $K(L^n(m))=Z[x]/\langle(x+1)^m-1,\, x^{n+1}\rangle$.

We define a module homomorphism

$$d\colon K(L^n(m))\to Z\oplus\tilde{k}_{2n-1}(BZ/m)$$

by

$$d(1)=1\in Z,$$

$$d(x^j)=B_{n-j}\in k_{2n-1}(BZ/m).$$

Then, it is easy to show

Lemma 2.8. *The homomorphism d is a well defined map and it is an isomorphism.*

§ 3. Main Theorem

The group $K(L^n(p^r))$ is calculated by Kobayashi, Sugawara [11] and Fujii, Kobayashi, Shimomura, Sugawara [5] when $p=2$, and Kobayashi, Murakami, Sugawara [6] when p is an odd prime. We use the process of their calculations and have the main theorem.

We define an integer $t(i)=t(i,r,n)$ as follows. Let $N=\min\{n, p^r-1\}$. For an integer i, satisfying $1\leqq i\leqq N$, we put $i=d+p^s$ for $0\leqq d<p^s(p-1)$, and $n-p^s+1=a_{s,n}p^s(p-1)+b_{s,n}$ such that $0\leqq b_{s,n}<p^s(p-1)$. Let

$$\bar{a}=\bar{a}(i,n)=\begin{cases} a_{s,n}+1 & \text{if } d<b_{s,n} \\ a_{s,n} & \text{if } d\geqq b_{s,n}. \end{cases}$$

Then we define $t(i)=p^{r-s-1+\bar{a}}$, and an element of $k_{2n-1}(BZ/p^r)$, $B(i)=B(i,n)$ by

$$B(i)=\begin{cases} \displaystyle\sum_{k=0}^{p^s}\left(\sum_{t=0}^{s}\sum_{j=0}^{p^t}(-1)^{p^t-j}p^{(p^t-1)\bar{a}}\binom{p^t}{j}\binom{jp^{s-t}}{k}\right)B_{n-k-d} \\ \quad \text{if } b_{s,n}\leqq d<b_{s,n}+p^s-1 \text{ or } d<b_{s,n}-p^s(p-2)-1, \\ \displaystyle\sum_{k=1}^{p_s}\binom{p^s}{k}B_{n-k-d} \qquad \text{otherwise.} \end{cases}$$

Theorem 3.1.

$$\tilde{k}_{2n-1}(BZ/p^r)=\sum_{i=1}^{N}Z/t(i)\langle B(i)\rangle.$$

Remark 3.2. The Adams operations of the generators of $K(L^n(p^r))$ ([5], [6]) are computable. And it is also true for the generators of $K'^*(BZ/p^r)$ in Remark 1.8.

We consider the case of $k_*(BZ/p^r)$. The map π of Section 2 is a composition of Thom isomorphism and the boundary homomorphism, and Thom isomorphism do not commute with the Adams operation. So the Adams operations on the generators of Theorem 3.1 is hard to compute in this way.

References

[1] Atiyah, M. F., Characters and cohomology of finite groups, *Publ. Math. I. H. E. S.*, **9** (1961), 23-64.

[2] Atiyah, M. F. and Segal, G. B., Equivariant K-theory and completions, *J. Differential Geometry*, **3** (1969), 1-18.

[3] Dold, A., Lectures on general cohomology, *Nordic Summer School in Math.*, Aarhus, 1968.

[4] ———, Chern classes in general cohomology, *Symposia Math., Inst. Naz. de alta math.*, **5** (1970), 385-410.

[5] Fujii, K., Kobayashi, T. and Shimomura, K., KO-groups of lens spaces modulo powers of two, *Hiroshima Math. J.*, **8** (1978), 469-489.

[6] Kobayashi, T., Murakami, S. and Sugawara, M., Note on J-groups of lens spaces, *Hiroshima Math. J.*, **7** (1977), 387-409.

[7] Landweber, P. S., Cobordism and classifying spaces, Algebraic topology, *Proc. of Symposia in pure Math.*, **22** (1971), 125-129.

[8] Mahammed, N., A propos de la K-théorie des espaces lenticulaires, *C. R. Acad. Sc. Paris*, **271** (1970), 639-642.

[9] Vick, J. W., Pontryagin duality in K-theory, *Proc. Amer. Math. Soc.*, **24** (1970), 611-616.

[10] Yosimura, Z., Universal coefficient sequences for cohomology theories of CW-spectra, *Osaka J. Math.*, **12** (1975), 305-323.

[11] Kobayashi, T, and Sugawara, M., Note on KO-rings of lens spaces mod 2^r, *Hiroshima Math. J.*, **8** (1978), 85-90.