Publ. RIMS, Kyoto Univ.
19 (1983), 773-811

Homotopy Classification of Connected Sums of Sphere Bundles over Spheres, III

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By

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Introduction

Let B_i, $i=1, 2, \cdots, k$, be p-sphere bundles over q-spheres $(p, q>1)$. It is understood that each B_i also denotes the total space of the bundle and has the oriented differentiable structure induced from those of the fibre and the base space. We denote the connected sum of B_i, $i=1, 2, \cdots, k$, by $\sharp_{i=1}^{k} B_i$. The necessary and sufficient conditions for such two connected sums of sphere bundles over spheres to be homotopy equivalent were given in [5] and [6]. In [5], we treated with the case that every bundle admits a cross-section, and in [6], we discussed the general case.

As special cases of [5], in the preceding paper [22], we classified the connected sums of p-sphere bundles over q-spheres which admit cross-sections for $(p, q)=(n, n+1)$ $(n\geqq 2)$, $(n-1, n+1)$ $(n\geqq 4)$, and $(n-2, n+1)$ $(n\geqq 6)$, and the results were applied to classify certain manifolds with sufficient connectedness. In this paper, by applying [6], we completely classify the connected sums of p-sphere bundles over q-spheres which do not necessarily have cross-sections up to homotopy equivalence for $(p, q)=(n-1, n+1)$ $(n\geqq 4)$ and $(n-2, n+1)$ $(n\geqq 6)$.

Let $(p, q)=(n-1, n+1)$ $(n\geqq 4)$ or $(n-2, n+1)$ $(n\geqq 6)$. A connected sum $\sharp_{i=1}^{k} B_i$ is called of *type* O if each B_i admits a cross-section, of *type* I if any B_i admits no cross-section, and of *type* (O+I) if there exist B_i admitting a cross-section and B_j admiting no cross-section. These definitions coincide with those defined in [3] and [4] using S_q^2 and Adem's secondary cohomology operation. Hence, the types are homotopically invariant. Furthermore, if s bundles admit cross-sections and t bundles

Communicated by N. Shimada, October 18, 1982.
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admit no cross-sections ($s+t=k$) in $\{B_i;\ i=1, 2, \cdots, k\}$, then s, t are homotopy invariants of $\sharp_{i=1}^{k}B_i$ (see Lemma 1.5).

Since the connected sums of type O have been completely classified up to homotopy equivalence in [22], we classify the connected sums of type I, type (O+I) up to homotopy equivalence in this paper. In other sense, our classifications supplement those of [3] and [4] where the manifolds of type I or type (O+I) were not always completely classified up to diffeomorphism mod θ_m.

In Section 1, we study the classification theorem obtained in [6], freely for the orientation, and represent it by an isomorphism which preserves certain invariants, a pairing, a quadratic form, and a homomorphism. In Section 2—Section 5, we perform certain homotopy theoretical calculations which are needed for our purpose. Our main results classifying the connected sums of type I, type (O+I) up to homotopy equivalence are given in Section 6 and Section 7.

I am grateful to Professor H. Toda for the useful conversation with him and I am most thankful to Professor Y. Nomura for his kind helpful advices.

§ 1. Classification Theorems

Let B be a p-sphere bundle over the q-sphere with the oriented differentiable structure induced from those of the fibre S^p and the base space S^q. We denote the characteristic element of B by $\alpha(B)$. Let ρ: $R^{p+1}\to R^{p+1}$ be the map defined by $\rho(x_1, \cdots, x_p, x_{p+1}) = (x_1, \cdots, x_p, -x_{p+1})$. ρ is considered as an element of O_{p+1}. Let $\bar{\rho}$: $SO_{p+1}\to SO_{p+1}$ be the map defined by $\bar{\rho}(\gamma) = \rho\gamma\rho$, and let $\bar{\rho}_*$: $\pi_{q-1}(SO_{p+1})\to\pi_{q-1}(SO_{p+1})$ be the automorphism induced from $\bar{\rho}$. It is easily seen that if we change the orientation of the total manifold B by exchange of the orientation of S^p, then it is the total space of the bundle with the characteristic element $\bar{\alpha}(B) = \bar{\rho}_*(\alpha(B))$. We express this fact simply by $\alpha(-B) = \bar{\alpha}(B)$. Since ρ is identical on the subspace $R^p\times 0$, we have $\bar{\rho}_*\circ i_* = i_*$, where i_*: $\pi_{q-1}(SO_p)\to\pi_{q-1}(SO_{p+1})$ is induced from the inclusion map. Hence, if A is a p-sphere bundle over the q-sphere which admits a cross-section, then $\alpha(-A) = \alpha(A)$. Thus, there is an orientation preserving diffeo-

morphism between A and $-A$. Let A_i, $i=1, 2, \cdots, k$, p-sphere bundles over q-spheres which admit cross-sections. Then, similarly there is an orientation preserving diffeomorphism between $\sharp_{i=1}^{k} A_i$ and $-(\sharp_{i=1}^{k} A_i)$.

Let $\pi_*: \pi_{q-1}(SO_{p+1}) \to \pi_{q-1}(S^p)$ be the homomorphism induced from the projection $\pi: SO_{p+1} \to S^p = SO_{p+1}/SO_p$. For any element $\kappa \in \pi_{q-1}(SO_{p+1})$, let $\bar{\kappa} = \bar{\rho} \circ \kappa = \bar{\rho}_*(\kappa)$. For any element $\omega \in \pi_{q-1}(S^p)$, we have the homomorphisms

$$\pi_{p+q-1}(S^{q-1}) \xrightarrow{\omega_*} \pi_{p+q-1}(S^p) \xleftarrow{J} \pi_{q-1}(SO_p) \xrightarrow{i_*} \pi_{q-1}(SO_{p+1}),$$

where ω_* is defined by the composition with ω and J is the J-homomorphism. Let $G(\omega) = i_*(J^{-1}(\operatorname{Im} \omega_*))$. (James-Whitehead [9]).

Lemma 1.1. *For any element* $\kappa \in \pi_{q-1}(SO_{p+1})$,

(i) $\pi_*(\bar{\kappa}) = \begin{cases} -\pi_*(\kappa), & \text{if } p \text{ is odd and } 2p > q, \\ \pi_*(\kappa), & \text{if } p \text{ is even}, \end{cases}$

(ii) $G(\pi_*(\bar{\kappa})) = G(\pi_*(\kappa))$.

Proof. Let $e_{p+1} = (0, \cdots, 0, 1) \in R^{p+1}$. Then, for any $x \in S^{q-1}$, $(\pi \circ \bar{\kappa})(x) = \pi(\rho\kappa(x)\rho) = (\rho\kappa(x)\rho)e_{p+1} = \rho(\kappa(x)(-e_{p+1})) = \rho(-\kappa(x)e_{p+1}) = -\rho(\kappa(x)e_{p+1}) = -\rho((\pi \circ \kappa)(x)) = -(\rho \circ \pi \circ \kappa)(x)$. Hence, $\pi \circ \bar{\kappa} = r \circ (-\iota_p) \circ \pi \circ \kappa$, where r is the antipodal map and ι_p is the orientation generator of $\pi_p(S^p)$. Thus, we have

$$\pi \circ \bar{\kappa} = \begin{cases} (-\iota_p) \circ \pi \circ \kappa & (p : \text{odd}) \\ \pi \circ \kappa & (p : \text{even}). \end{cases}$$

Since every element of $\pi_{q-1}(S^p)$ is a suspension element if $2p > q$, we have (i). (ii) is known from the above fact and by Lemma 1.1 of [9].

Let B_i, B_i', $i=1, 2, \cdots, k$, be p-sphere bundles over q-spheres ($2p > q > 1$) and let $W = \natural_{i=1}^{k} \overline{B}_i$, $W' = \natural_{i=1}^{k} \overline{B}_i'$, where $\overline{B}_i$, $\overline{B}_i'$ denote the associated $(p+1)$-disk bundles. Let $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ be the invariants of W, W' respectively defined in [20]. Those are determined from $\partial W = \sharp_{i=1}^{k} B_i$, $\partial W' = \sharp_{i=1}^{k} B_i'$ respectively if $p \neq q-1$, q (Cf. [6]). Let $\bar{\alpha} = \bar{\rho}_* \circ \alpha$, $\varepsilon = \pi_* \circ \alpha$, and $\bar{\varepsilon} = \pi_* \circ \bar{\alpha}$. Then, $G(\varepsilon(x)) = G(\bar{\varepsilon}(x))$ for any $x \in H$ by Lemma 1.1. Define $\bar{\alpha}'$, ε', and $\bar{\varepsilon}'$ similarly for W'. A basis $\{w_1, \cdots,$

$w_k\}$ of H is called admissible if $\phi(w_i, w_j)=0$ for all i, j $(i\neq j)$. If we exchange the orientation of W, then the invariants ϕ, α, and ε are replaced by $-\phi$, $\bar{\alpha}$, and $\bar{\varepsilon}$ respectively. Hence, by Theorem 4 of [6], we have

Theorem 1.2. *Let $q/2<p<q-1$. Then, the connected sums $\sharp_{i=1}^{k}B_i$, $\sharp_{i=1}^{k}B_i'$ are homotopy equivalent if and only if there exist the admissible bases $\{w_1, \cdots, w_k\}$ of H and $\{w_1', \cdots, w_k'\}$ of H' which satisfy the following* (i), (ii) *where it is permitted to replace (α, ε) or (α', ε') by $(\bar{\alpha}, \bar{\varepsilon})$ or $(\bar{\alpha}', \bar{\varepsilon}')$:*

(i) $\varepsilon(w_i)=\varepsilon'(w_i')$, $i=1, 2, \cdots, k$.

(ii) $\{\alpha(w_i)\}=\{\alpha'(w_i')\}$ *in* $\pi_{q-1}(SO_{p+1})/G(\varepsilon(w_i))=\pi_{q-1}(SO_{p+1})/G(\varepsilon'(w_i'))$, $i=1, 2, \cdots, k$.

In some special cases, we can mention Theorem 1.2 without using admissible bases. Let $(p, q)=(n-1, n+1)$, $n\geqq 4$, or $(p, q)=(n-2, n+1)$, $n\geqq 6$. Then, $\pi_{q-1}(S^p)\cong\pi_q(S^{p+1})\cong Z_2$ and $\pi_{q-1}(S^p)$ has the generator

$$\omega=\begin{cases}\eta_{n-1} & \text{if} \quad (p, q)=(n-1, n+1), \quad n\geqq 4,\\ \eta_{n-2}^2 & \text{if} \quad (p, q)=(n-2, n+1), \quad n\geqq 6.\end{cases}$$

Here, η_{n-1} denotes the $(n-3)$-fold suspension of η_2 which is the generator of $\pi_3(S^2)\cong Z$ represented by the Hopf map, and $\eta_{n-2}^2=\eta_{n-2}\circ\eta_{n-1}$.

Let β: $H\to\pi_{q-1}(SO_{p+1})/G(\omega)$ be the map defined by $\beta(x)=\psi\alpha(x)$, where ψ: $\pi_{q-1}(SO_{p+1})\to\pi_{q-1}(SO_{p+1})/G(\omega)$ is the canonical projection. β is a quadratic form with the associated bilinear form $\psi\circ\partial\circ\phi$, where ∂: $\pi_q(S^{p+1})\to\pi_{q-1}(SO_{p+1})$ is the boundary homomorphism, since

$$\alpha(x+y)=\alpha(x)+\alpha(y)+\partial\circ\phi(x, y)$$

by Wall [20]. Let H_0 be a subgroup of H such that $\phi|H_0\times H_0=0$, and let β_0: $H_0\to\pi_{q-1}(SO_{p+1})/G(0)$ be the map defined by $\beta_0(x)=\{\alpha(x)\}$, the coset of $\alpha(x)$. Then, β_0 is a homomorphism since ϕ is trivial on H_0. Similarly define $\bar{\beta}$, $\bar{\beta}_0$ for $\bar{\alpha}$.

We define the invariant β' similarly for $W'=\natural_{i=1}^{k}\overline{B}_i'$ and β_0' when given a subgroup H_0' of H' such that $\phi'|H'\times H'=0$. We also have

$\bar{\beta}'$, $\bar{\beta}'_0$ for $\bar{\alpha}'$.

Theorem 1.3. *Let* $(p, q) = (n-1, n+1)$, $n \geqq 4$, *or* $(p, q) = (n-2, n+1)$, $n \geqq 6$. *Then, the connected sums* $\sharp_{i=1}^{k} B_i$, $\sharp_{i=1}^{k} B'_i$ *have the same oriented homotopy type if and only if there exists an isomorphism* $h: H \to H'$ *satisfying the following conditions*:

(i) $\phi = \phi' \circ (h \times h)$

(ii) $\beta = \beta' \circ h$

(iii) *If rank* $\phi < k$, *there is a direct sum decomposition* $H = H_0 + H_1$ *orthogonal with respect to* ϕ *such that* ϕ *is trivial on* H_0, *the rank of* $\phi | H_1 \times H_1$ *is maximal, and* $\beta_0 = \beta'_0 \circ h$, *where* $H'_0 = h(H_0)$.

If we omit "oriented", (β, β_0) *or* (β', β'_0) *may be replaced by* $(\bar{\beta}, \bar{\beta}_0)$ *or* $(\bar{\beta}', \bar{\beta}'_0)$ *respectively.*

Proof. Let $\sharp_{i=1}^{k} B_i$, $\sharp_{i=1}^{k} B'_i$ have the same oriented homotopy type. Then, there exist admissible bases $\{w_1, \cdots, w_k\}$ of H, $\{w'_1, \cdots, w'_k\}$ of H' satisfying (i), (ii) of Theorem 1.2. By the definition, $\phi(w_i, w_j) = \phi'(w'_i, w'_j) = 0$ if $i \neq j$, and by Theorem 1 of [20], $\phi(w_i, w_i) = E\varepsilon(w_i) = E\varepsilon'(w'_i) = \phi'(w'_i, w'_i)$. Hence, we have an isomorphism $h: H \to H'$ which satisfies (i) by defining $h(w_i) = w'_i$ for $i = 1, 2, \cdots, k$. By (ii) of Theorem 1.2, $\beta(w_i) = \beta'(w'_i)$, $i = 1, 2, \cdots, k$, since $\varepsilon(w_i) = \varepsilon'(w'_i) = 0$ or ω and $G(0) \subset G(\omega)$. β, β' are quadratic forms with the associated bilinear forms $\psi \circ \partial \circ \phi$, $\psi \circ \partial \circ \phi'$ respectively. Therefore, we know (ii) by (i).

Let rank $\phi < k$. We may assume that $\varepsilon(w_i) = \varepsilon'(w'_i) = 0$ for $1 \leqq i \leqq s$ $(1 \leqq s \leqq k)$ and $\varepsilon(w_i) = \varepsilon'(w'_i) = \omega$ for $s+1 \leqq i \leqq k$. Let H_0, H_1 be the subgroups of H generated by $\{w_1, \cdots, w_s\}$, $\{w_{s+1}, \cdots, w_k\}$ respectively. Then, by (ii) of Theorem 1.2, (iii) is clear since β_0, β'_0 are homomorphisms.

Conversely, assume (i), (ii), and (iii). $\phi: H \times H \to Z_2 \cong \pi_q(S^{p+1})$ has a unique representation to a diagonal matrix

$$\begin{pmatrix} 0 & \cdots & s & & O \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \cdots & t \\ & & & & \ddots & \\ O & & & & & 1 \end{pmatrix}$$

since originally H has an admissible basis, where $t=\operatorname{rank}\phi$, $s+t=k$. Let $(w_1, \cdots, w_s, w_{s+1}, \cdots, w_k\}$ be the basis of H which gives the above representation of ϕ. Here, we may assume that $\{w_1, \cdots, w_s\}$, $\{w_{s+1}, \cdots, w_k\}$ genetate H_0, H_1 respectively. In fact, since H_0, H_1 are orthogonal with respect to ϕ, the representations of $\phi|H_0\times H_0$, $\phi|H_1\times H_1$ give the representation of ϕ. By Lemma 1.1 of [3], $\phi|H_1\times H_1$ must be represented by a $(t\times t)$-matrix

$$\begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}$$

since its rank is maximal. Let $\{w_{s+1}, \cdots, w_k\}$ be the basis of H_1 which gives such representation of $\phi|H_1\times H_1$ and let $\{w_1, \cdots, w_s\}$ be any basis of H_0. Then, $\{w_1, \cdots, w_s, w_{s+1}, \cdots, w_k\}$ give the above representation of ϕ.

Let $w'_i=h(w_i)$, $i=1, 2, \cdots, k$. $\{w'_1, \cdots, w'_k\}$ is an admissible basis of H' by (i) and $\{w'_1, \cdots, w'_s\}$, $\{w'_{s+1}, \cdots, w'_k\}$ generate $H'_0=h(H_0)$, $H'_1=h(H_1)$ respectively. Since $E\varepsilon(w_i)=\phi(w_i, w_i)=\phi'(w'_i, w'_i)=E\varepsilon'(w'_i)$, we have

$$\varepsilon(w_i)=\varepsilon'(w'_i)=\begin{cases} 0 & (1\leqq i\leqq s) \\ \omega & (s+1\leqq i\leqq k). \end{cases}$$

By (iii), $\{\alpha(w_i)\}=\{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(0)$ for $i=1, 2, \cdots, s$, and by (ii), $\{\alpha(w_i)\}=\{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(\omega)$ for $i=s+1, \cdots, k$. Hence, $\{\alpha(w_i)\}=\{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(\varepsilon(w_i))=\pi_{q-1}(SO_{p+1})/G(\varepsilon'(w'_i))$ for $i=1, 2, \cdots, k$. Thus, there exists an orientation preserving homotopy equivalence between $\sharp_{i=1}^k B_i$ and $\sharp_{i=1}^k B'_i$ by Theorem 1.2.

This completes the proof.

Remark. Let $\sharp_{i=1}^k B_i$, $\sharp_{i=1}^k B'_i$ have the same oriented homotopy type. In the above proof of the necessity, we may adopt arbitrary admissible basis $\{w_1, \cdots, w_k\}$ of H by Assertion 4 in Section 4 of [6].

In Theorem 1.3, it may happen that $G(\omega)=\{0\}$ for some values of n. In this case, β, β' coincide with α, α' respectively. Therefore, Theorem 1.3 induces that if $\sharp_{i=1}^k B_i$ has the oriented homotopy type of

$\sharp_{i=1}^{k}B_i'$, then $(H;\ \phi, \alpha)$, $(H';\ \phi', \alpha')$ are isomorphic, and hence $\natural_{i=1}^{k}\overline{B}_i$ is orientation preservingly diffeomorphic to $\natural_{i=1}^{k}\overline{B}_i'$ by Theorem 2 of [20]. Thus, we have

Corollary 1.4. *Under the condition of Theorem* 1.3, *if* $G(\omega) = \{0\}$, *the following three are equivalent*:

(i) $\sharp_{i=1}^{k}B_i$ *is homotopy equivalent to* $\sharp_{i=1}^{k}B_i'$.

(ii) $\natural_{i=1}^{k}\overline{B}_i$ *is diffeomorphic to* $\natural_{i=1}^{k}\overline{B}_i'$.

(iii) $\sharp_{i=1}^{k}B_i$ *is diffeomorphic to* $\sharp_{i=1}^{k}B_i'$.

Remark. In the above, it is easily seen from Theorem 1.3 that if all B_i, B_i' admit cross-sections then the condition $G(\omega) = \{0\}$ can be replaced by $G(0) = \{0\}$. If J: $\pi_{q-1}(SO_{p+1}) \to \pi_{p+q}(S^{p+1})$ is monomorphic, then $G(0) = \{0\}$.

We have defined the *type* of $\sharp_{i=1}^{k}B_i$ in the introduction. We have

Lemma 1.5. *Let* $(p, q) = (n-1, n+1)$, $n \geqq 4$, *or* $(n-2, n+1)$, $n \geqq 6$. *The type of a connected sum* $\sharp_{i=1}^{k}B_i$ *is homotopically invariant. Furthermore, if s bundles admit cross-sections and t bundles admit no cross-sections* $(s+t=k)$ *in* $\{B_i;\ i=1, 2, \cdots, k\}$, *then s, t are homotopy invariants of* $\sharp_{i=1}^{k}B_i$.

Proof. Let $W = \natural_{i=1}^{k}\overline{B}_i$ and let $(H;\ \phi, \alpha)$ be the invariant system. Let $\{w_1, \cdots, w_k\}$ be the basis of H represented by zero cross-sections of $\overline{B}_i$, $i=1, 2, \cdots, k$. Then, B_i admits a cross-section if and only if $\phi(w_i, w_i) = 0$ since $\phi(w_i, w_i) = E\pi_*\alpha(w_i) = E\pi_*\alpha(B_i)$. So, there exists a representation of ϕ: $H \times H \to Z_2 \cong \pi_q(S^{p+1})$ by a matrix

$$\begin{pmatrix} 0 & \cdots & s & & O \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \cdots & t \\ & & & & \ddots & \\ O & & & & & 1 \end{pmatrix}$$

under a suitable admissible basis. Since $t = \operatorname{rank}\phi$, $s = k - t$, the numbers

s, t are independent of the choice of the admissible basis. By Proposition 1 of [6], ϕ is a homotopy invariant of $\partial W=\sharp_{i=1}^{k}B_i$. In fact, ϕ is isomorphic to a certain bilinear map defined by using S_q^2 or Adem's secondary cohomology operation. (See Theorem 8.3 of [3] or p. 730 of [4], but be careful of symbols.) Hence, the numbers s, t are homotopy invariant.

To classify the connected sums of p-sphere bundles over q-spheres up to homotopy equivalence for such values of (p, q) as above, we must compute $G(\eta_{n-1})$ $(n\geqq 4)$ and $G(\eta_{n-2}^2)$ $(n\geqq 6)$. The following Section 2–Section 5 are devoted for the computation.

§ 2. Some Lemmas in the Case $n=8j$ $(j>0)$

In this section, we investigate the homotopy groups of rotation groups and certain relations with the homotopy groups of spheres in order to calculate $G(\eta_{n-1})$, $G(\eta_{n-2}^2)$ when $n=8j$ $(j>0)$. We have the following Diagram 1, which is commutative and will be used to calculate $G(\eta_{n-1})$, $G(\eta_{n-2}^2)$ $(n=8j,\ j>0)$ in Section 4, 5 respectively. In the diagram, $J^{(i)}$: $\pi_n(SO_{n+2-i})\to\pi_{2n+2-i}(S^{n+2-i})$, $i=0, 1, \cdots, 5$, denote J-homomorphisms and the horizontal maps between the homotopy groups of spheres are the suspension homomorphisms, which we denote by $E^{(i)}$, $i=1, 2, \cdots, 5$, or simply by E if there is no confusion. Those maps between the homotopy groups of rotation groups are the homomorphisms induced from inclusions, which we denote simply by i_* if there is no confusion.

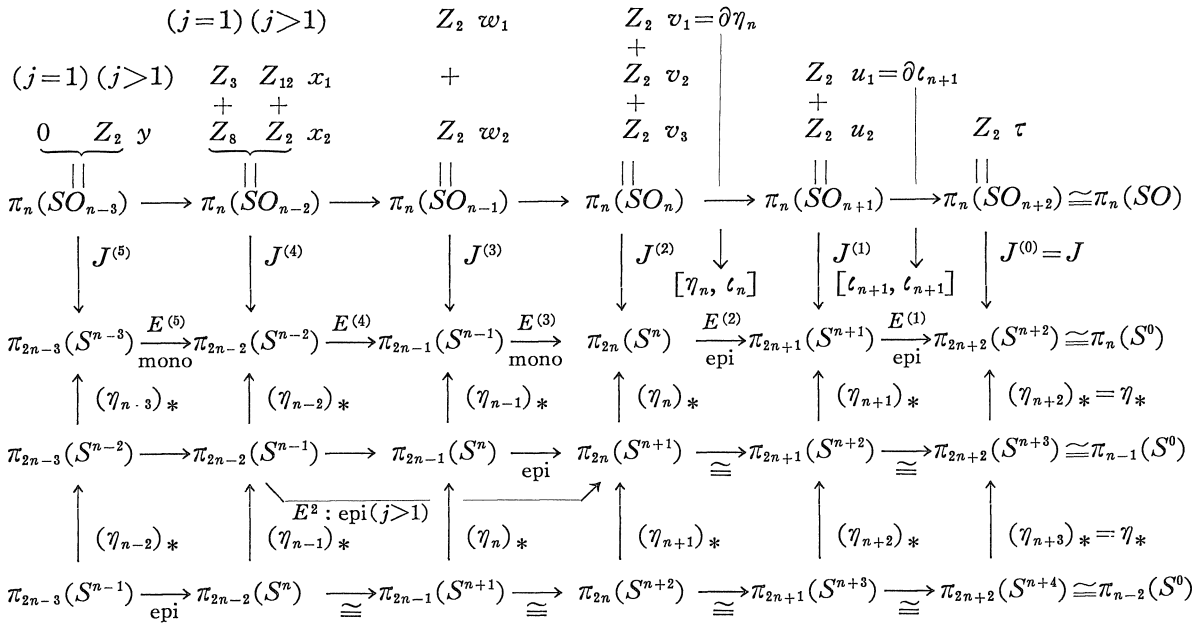
The homotopy groups of rotation groups are known by [11], that is, we have the sequence

$$(*)\qquad 0\to\pi_{8j+1}(V_{m,m-8j+i})\to\pi_{8j}(SO_{8j-i})\to\pi_{8j}(SO_m)\to 0$$

which is exact and splits if $i\leqq 4$, $j\geqq 2$ or $i\leqq 1$, $j=1$, where m is sufficiently large, and the homotopy groups of Stiefel manifolds are known by [15].

Let $j>1$. Since $\pi_{8j+1}(V_{m,m-8j+4})=0$, $\pi_{8j}(SO_{8j-4})$ is isomorphic to $\pi_{8j}(SO_m)\cong Z_2$. So, $\pi_{8j}(SO_{8j-4})$ has the unique non-zero element z. Let u_2, v_3, w_2, x_2, and y be the elements of $\pi_{8j}(SO_{8j-i})$ according as $i=-1, 0,$

Diagram 1 $(n=8j,\ j>0)$



1, 2, 3 which correspond with z under the homomorphism $\pi_{8j}(SO_{8j-4}) \to \pi_{8j}(SO_{8j-i})$ induced from the inclusion. Other generators and the correspondence are known from those of $\pi_{8j+1}(V_{m,m-8j+i})$ (Cf. [15]). We denote such generators by u_1, $\{v_1, v_2\}$, w_1, and x_1 according as $i=-1, 0, 1, 2$. In the diagram, generators correspond horizontally. Those which have no corresponding generators go to zero elements.

Let $j=1$. Then, by [17] and [18], we know the following table:

$\pi_8(SO_6)$	$\pi_8(SO_7)$	$\pi_8(SO_8)$	$\pi_8(SO_9)$	$\pi_8(SO_{10})$
	$Z_2\ \tilde{\rho}\circ\eta_7$	$Z_2\ i_*^{7,8}(\tilde{\rho}\circ\eta_7)$		
		+		
$Z_3\ \beta_1$	+	$Z_2\ \tilde{\sigma}\circ\eta_7$	$Z_2\ i_*^{8,9}(\tilde{\sigma}\circ\eta_7)$	
+		+	+	$\searrow$
$Z_8\ \xi$	$Z_2\ i_*^{6,7}\xi$	$Z_2\ i_*^{6,8}\xi$	$Z_2\ i_*^{6,9}\xi$	$Z_2\ i_*^{6,10}\xi$

The correspondence of generators is represented in a similar way except the indicated one. $i_*^{p,q}$ is the homomorphism induced from the inclusion $i^{p,q}: SO_p \to SO_q$. β_1, ξ are the generators of $\pi_8(SO_6) \cong \pi_8(S^5) \cong Z_3 + Z_8$ corresponding to those of $\pi_8(S^5)$ by the isomorphism induced from the projection. $\tilde{\rho}: S^7 \to SO_7$ and $\tilde{\sigma}: S^7 \to SO_8$ are defined by $\tilde{\rho}(c)c' = c \cdot c' \cdot \bar{c}$ for $c \in S^7$, $c' \in S^6 \subset S^7$ and by $\tilde{\sigma}(c)c' = c \cdot c'$ for $c, c' \in S^7$ respectively, where c, c' are Cayley numbers. For convenience, put $x_1 = \beta_1$, $x_2 = \xi$, $w_1 = \tilde{\rho}\circ\eta_7$, $w_2 = i_*^{6,7}\xi$, $v_1 = i_*^{7,8}(\tilde{\rho}\circ\eta_7)$, $v_2 = \tilde{\rho}\circ\eta_7 + i_*^{6,8}\xi$, $v_3 = i_*^{6,8}\xi$, $u_1 = i_*^{8,9}(\tilde{\sigma}\circ\eta_7) + i_*^{6,9}\xi$, $u_2 = i_*^{6,9}\xi$, and $\tau = i_*^{6,10}\xi$. Then, we have the generators which correspond similarly as in the case $j>1$. Hence, in this meaning, the correspondence of generators in Diagram 1 holds for $j=1$.

Note. Since the sequence (*) is exact if $i \leqq 2$, $j=1$ and splits if $i \leqq 1$, $j=1$, we can take the generators in a similar way as in the case $j>1$, and in fact we know the above relations except the one for v_2. The aim of the above definition is to clarify the operation of $\pi_0(O_8)$.

Lemma 2.1. *In Diagram* 1,

(i) $E^{(1)}$, $E^{(2)}$ *are epimorphic.*

(ii) $E^{(3)}$, $E^{(5)}$ *are monomorphic.*

(iii) $E^2: \pi_{2n-2}(S^{n-1}) \to \pi_{2n}(S^{n+1})$ *is epimorphic* $(j>1)$.

Proof. (i) is the well known fact (Cf. [5], p. 28).

(ii) Since $i_*: \pi_n(SO_{n-1}) \to \pi_n(SO_n)$ $(n=8j,\ j>0)$ is monomorphic, $\partial: \pi_{n+1}(S^{n-1}) \to \pi_n(SO_{n-1})$ must be trivial. So, $P=[\ ,\iota_{n-1}]: \pi_{n+1}(S^{n-1}) \to \pi_{2n-1}(S^{n-1})$ is trivial since $P=-J^{(3)}\circ\partial$. Hence, $E^{(3)}$ is monomorphic, and similarly $E^{(5)}$, where the sequence

$$\pi_q(S^p) \xrightarrow{P} \pi_{p+q-1}(S^p) \xrightarrow{E} \pi_{p+q}(S^{p+1})$$

is exact if $2p>q-1$ (Cf. P. 198 of [8] and (7.7) of [7]).

(iii) Let n be even and $n\neq 2, 4, 8$. Then, the sequence

$$\pi_{2n-2}(S^{n-1}) \xrightarrow{E} \pi_{2n-1}(S^n) \xrightarrow{H} 2Z \to 0$$

is exact and splits since $H[\iota_n, \iota_n]=2$. So,

$$\pi_{2n-1}(S^n) = ([\iota_n, \iota_n]) \oplus E\pi_{2n-2}(S^{n-1}).$$

$E: \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$ is epimorphic and Ker $E=([\iota_n, \iota_n])$. Hence, we know that $E^2\pi_{2n-2}(S^{n-1}) = \pi_{2n}(S^{n+1})$.

Lemma 2.2. *$J^{(i)}$, $i=0, 1, \cdots, 5$, are monomorphic for $n=8j$, $j>0$.*

Proof. $J^{(0)}$ is monomorphic by [1]. Since $[\iota_{n+1}, \iota_{n+1}]\neq 0$ $(n\neq 0, 2, 6)$ and $J^{(1)}u_1 = (J^{(1)}\circ\partial)\iota_{n+1} = [\iota_{n+1}, \iota_{n+1}]$, $J^{(1)}$ is monomorphic from the diagram. $J^{(2)}$ is monomorphic on the subgroup generated by $\{v_2, v_3\}$, and $J^{(2)}v_1 = J^{(2)}(\partial\eta_n) = [\eta_n, \iota_n]$. Since $[\eta_n, \iota_n]\neq 0$ $(n=4k)$ by [2], $J^{(2)}$ is also monomorphic from the diagram. $J^{(3)}$ is monomorphic since $i_*: \pi_{8j}(SO_{8j-1}) \to \pi_{8j}(SO_{8j})$ is monomorphic. If $n=8$, $J^{(4)}$ is monomorphic from the following diagram which is commutative up to sign:

$$\begin{array}{ccc} \pi_8(SO_6) & \xrightarrow[\cong]{\pi_*} & \pi_8(S^5) \\ \downarrow J^{(4)} & & \cong \downarrow E^6 \\ \pi_{14}(S^6) & \xrightarrow{H} & \pi_{14}(S^{11}), \end{array}$$

where $\pi: SO_6 \to S^5$ is the projection and H is the Hopf homomorphism.

Let $n=8j$ $(j>1)$. The exact sequence

$$\begin{array}{ccccc} \pi_{n+1}(S^{n-2}) & \xrightarrow{\partial} & \pi_n(SO_{n-2}) & \xrightarrow{i_*} & \pi_n(SO_{n-1}) \\ \| & & \| & & \| \\ Z_3+Z_8 & & Z_{12}\ x_1 & & \\ \alpha_1\ \ \nu & & + & & \\ & & Z_2\ \ x_2 & & Z_2\ w_2 \\ & & & & + \\ & & & & Z_2\ w_1 \end{array}$$

implies that $\mathrm{Im}\,\partial = \mathrm{Ker}\,i_* \cong Z_{12}$, $\partial\alpha_1 \neq 0$, and hence $\partial\alpha_1$ is of order 3, where α_1, ν denote the generators of order 3 and 8 respectively. So that $\partial\nu$ must be of order 4, and $\mathrm{Ker}\,i_* \cong Z_3 + Z_4$ and is generated by $\partial\alpha_1$, $\partial\nu$. On the other hand, $[\nu, \iota_{n-2}] = -J^{(4)}\partial\nu$ shows that order $[\nu, \iota_{n-2}] \leqq 4$, and $H[\nu, \iota_{n-2}] = \pm 2E^{n-3}\nu$ (Cf. [21] (5.32)) shows that order $[\nu, \iota_{n-2}] \geqq 4$. Hence, $[\nu, \iota_{n-2}]$ is of order 4. Similarly, $H[\alpha_1, \iota_{n-2}] = \pm E^{n-3}\alpha_1$ shows that $[\alpha_1, \iota_{n-2}]$ is of order 3. Thus the diagram induces that $J^{(4)}$ is monomorphic for $n = 8j$ $(j>1)$. Since $i_*: \pi_{8s}(SO_{8s-3}) \to \pi_{8s}(SO_{8s-2})$ is monomorphic, it is clear that $J^{(5)}$ is monomorphic. This completes the proof.

Remark 1. If $n = 8j$ $(j>1)$, $i_*: \pi_n(SO_{n-4}) \to \pi_n(SO_{n-3})$ is isomorphic. Hence, $J^{(6)}$ is also monomorphic for $n = 8j$ $(j>1)$.

Remark 2. It holds that $\partial\nu$ is of order 4 for $n = 8$, and therefore $[\nu, \iota_6]$ is of order 4. In fact, we have an exact sequence

$$\begin{array}{ccccc} \pi_9(S^6) & \xrightarrow{\partial} & \pi_8(SO_6) & \xrightarrow{i_*} & \pi_8(SO_7) \\ \| & & \| & & \| \\ Z_3\ \alpha_1 & & Z_3\ \beta_1 & & \\ + & & + & & \\ Z_8\ \nu & & Z_8\ \xi & & Z_2\ w_2 \\ & & & & + \\ & & & & Z_2\ w_1 \end{array}$$

and it shows that $\partial\nu = \pm 2\xi$ (Cf. Diagram 1).

Lemma 2.3. (Cf. Nomura [13]). *Let* $n = 8j$ $(j>0)$. *Then,*

(i) $[\iota_{n+1}, \iota_{n+1}]$ *does not belong to* $\mathrm{Im}\,(\eta_{n+1})_*$.

(ii) $[\eta_n, \iota_n]$ *does not belong to* $\mathrm{Im}\,(\eta_n)_*$.

Proof. (i) Let $[\iota_{n+1}, \iota_{n+1}] = (\eta_{n+1})_*\beta$ for some $\beta \in \pi_{2n+1}(S^{n+2})$. We know that $[\iota_{n+1}, \iota_{n+1}] = J^{(1)}u_1 = J^{(1)}(i_*v_2) = E^{(2)}(J^{(2)}v_2)$ by Diagram 1. Let $\beta = E\gamma$ for certain $\gamma \in \pi_{2n}(S^{n+1})$. Then, $(\eta_{n+1})_*\beta = E^{(2)}((\eta_n)_*\gamma)$ and therefore $E^{(2)}(J^{(2)}v_2) = E^{(2)}((\eta_n)_*\gamma)$. Hence, $J^{(2)}v_2 - (\eta_n)_*\gamma \in \mathrm{Ker}\,E^{(2)} = ([\eta_n, \iota_n]) \cong Z_2$. We note that $H[\eta_n, \iota_n] = 0$ since $[\eta_n, \iota_n] = J^{(2)}v_1 = J^{(2)}(i_*w_1) = E^{(3)}(J^{(3)}w_1)$. Similarly $H((\eta_n)_*\gamma) = 0$, but $H(J^{(2)}v_2) \neq 0$ by the commutative diagram

$$\begin{array}{ccc} \pi_n(SO_n) & \xrightarrow{\pi_*} & \pi_n(S^{n-1}) \\ \downarrow J^{(2)} & & \cong \downarrow E^n \\ \pi_{2n}(S^n) & \xrightarrow{H} & \pi_{2n}(S^{2n-1}) \end{array}$$

since v_2 is not in the image of i_*: $\pi_n(SO_{n-1}) \to \pi_n(SO_n)$. This yields a contradiction.

(ii) Let $n>8$, and let $[\eta_n, \iota_n] = (\eta_n)_*\gamma$ for some $\gamma \in \pi_{2n}(S^{n+1})$. Since E^2: $\pi_{2n-2}(S^{n-1}) \to \pi_{2n}(S^{n+1})$ is epimorphic, there exists $\delta \in \pi_{2n-1}(S^n)$ such that $\delta = E\varepsilon$, $\varepsilon \in \pi_{2n-2}(S^{n-1})$, and $E\delta = \gamma$. Then, $E^{(3)}((\eta_{n-1})_*\delta) = (\eta_n)_*\gamma$ and $[\eta_n, \iota_n] = E^{(3)}(J^{(3)}w_1)$ by the diagram. So that, $J^{(3)}w_1 = (\eta_{n-1})_*\delta$ since $E^{(3)}$ is monomorphic by Lemma 2.1. Here, $(\eta_{n-1})_*\delta = E^{(4)}((\eta_{n-2})_*\varepsilon)$ and w_1 is not in the image of i_*: $\pi_n(SO_{n-2}) \to \pi_n(SO_{n-1})$. Hence $H((\eta_{n+1})_*\delta) = 0$ but $H(J^{(3)}w_1) \neq 0$ similarly as in (i). This is a contradiction.

Let $n=8$. By Toda [19], it is known that $\pi_{16}(S^8) \cong Z_2+Z_2+Z_2+Z_2$ and is generated by $\sigma_8\circ\eta_{15}$, $E\sigma'\circ\eta_{15}$, $\bar{\nu}_8$, and ε_8, where we retain the symbols of [19]. It is also known that $\operatorname{Ker} E^{(2)} \cong Z_2$ and is generated by $E\sigma'\circ\eta_{15}$. Since the non-zero element $[\eta_8, \iota_8]$ belongs to $\operatorname{Ker} E^{(2)}$, we know that $[\eta_8, \iota_8] = E\sigma'\circ\eta_{15}$. On the other hand, $\pi_{16}(S^9) \cong Z_{16}+Z_3+Z_5$ and the 2-component is generated by σ_9. But $\eta_8\circ\sigma_9 = E\sigma'\circ\eta_{15}+\bar{\nu}_8+\varepsilon_8$ by (7.4) of [19], and hence $[\eta_8, \iota_8]$ does not belong to $\operatorname{Im}(\eta_8)_*$.

§ 3. Some Lemmas in the Case $n=8j+1$ $(j>0)$

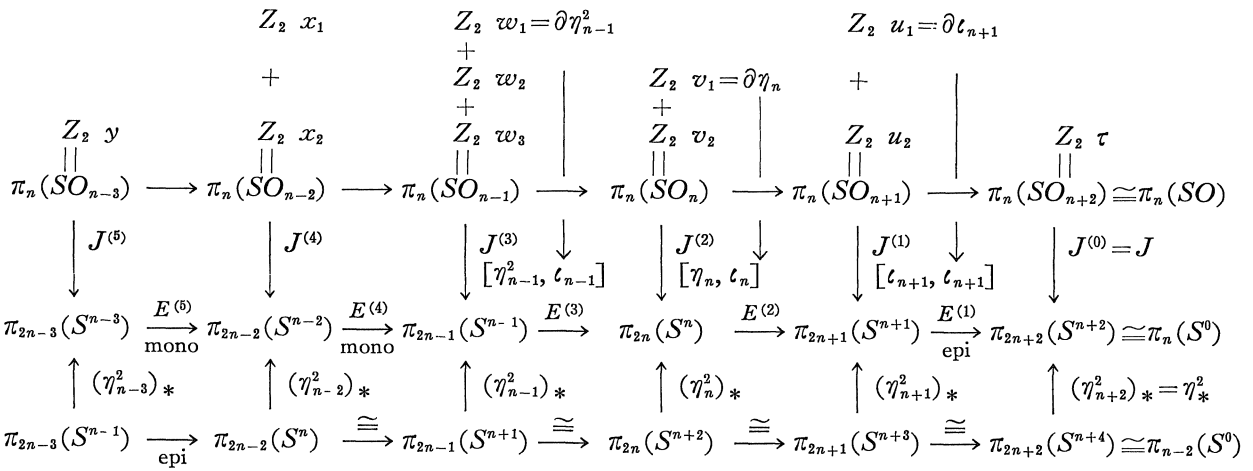
To calculate $G(\eta_{n-2}^2)$ when $n=8j+1$ $(j>0)$, we study the homotopy groups of rotation groups and certain relations with homotopy groups of spheres. We have the following Diagram 2, where we keep the notations in Section 2 and the diagram is written in a similar way.

The homotopy groups of rotation groups are known by the following sequence

$$(**) \qquad 0 \to \pi_{8j+2}(V_{m,m-8j+i}) \to \pi_{8j+1}(SO_{8j-i}) \to \pi_{8j+1}(SO_m) \to 0$$

which is exact and splits for $i \leqq 3$, $j \geqq 2$ or $i \leqq 2$, $j=1$, where m is sufficiently large ([11]). Let $j>1$. Since $\pi_{8j+2}(V_{m,m-8j+3}) = 0$ by [15], $\pi_{8j+1}(SO_{8j-3})$ is isomorphic to $\pi_{8j+1}(SO_m) \cong Z_2$ and has a unique non-zero element z. Denote the element $i_*^{8j-3,8j-i}(z)$ by u_2, v_2, w_3, x_2, and y according

Diagram 2 $(n=8j+1, j>0)$



as $i=-2$, -1, 0, 1, and 2. The other generators and the correspondence are known from those of $\pi_{8j+2}(V_{m,m-8j+i})$ by Paechter [15]. Let u_1, v_1, $\{w_1, w_2\}$, and x_1 be such generators according as $i=-2$, -1, 0, and 1. Although the generators are denoted by similar symbols as in Section 2, we use those since it seems that there is no confusion.

Let $j=1$, and let ξ be the generator of the 2-component of $\pi_8(SO_6)\cong Z_8+Z_8$ defined in Section 2. Put $y=\xi\circ\eta_8$. Then, by [17], [18], and [14], we have the following table:

$\pi_9(SO_6)$	$\pi_9(SO_7)$	$\pi_9(SO_8)$	$\pi_9(SO_9)$	$\pi_9(SO_{10})$	$\pi_9(SO_{11})$
	$Z_2\ \tilde{\rho}\circ\eta_7\circ\eta_8$	$Z_2\ i_*^{7,8}(\tilde{\rho}\circ\eta_7\circ\eta_8)$ +			
	+	$Z_2\ \tilde{\sigma}\circ\eta_7\circ\eta_8$ +	$Z_2\ i_*^{8,9}(\tilde{\sigma}\circ\eta_7\circ\eta_8)$ +	$Z\ (T)$ +	
$Z_2\ y$	$Z_2\ i_*^{6,7}y$	$Z_2\ i_*^{6,8}y$	$Z_2\ i_*^{6,9}y$	$Z_2\ i_*^{6,10}y$	$Z_2\ i_*^{6,11}y$

The correspondence of generators is represented in a similar way as in Section 2 except the indicated one. Let $x_1=\tilde{\rho}\circ\eta_7\circ\eta_8$, $x_2=i_*^{6,7}y$, $w_1=i_*^{7,8}(\tilde{\rho}\circ\eta_7\circ\eta_8)$, $w_2=\tilde{\rho}\circ\eta_7\circ\eta_8+i_*^{6,8}y$, $w_3=i_*^{6,8}y$, $v_1=i_*^{8,9}(\tilde{\rho}\circ\eta_7\circ\eta_8)+i_*^{6,9}y$, $v_2=i_*^{6,9}y$, $u_1=(T)$, $u_2=i_*^{6,10}y$, and $\tau=i_*^{6,11}y$. Then, we have the generators which correspond similarly as in the case $j>1$.

Note. Since $\pi_{10}(V_{m,m-6})=0$ by [15], $\pi_9(SO_6)$ is isomorphic to $\pi_9(SO_m)\cong Z_2$ by the sequence (**), and has a unique non-zero element y. Since the sequence (**) is exact and splits if $i\leqq 2$, $j=1$, we can take the generators for $j=1$ quite similarly as in the case $j>1$. In fact, we have the above relations except the one for w_2. The aim of the above definition is also to clarify the operation of $\pi_0(O_8)$.

Lemma 3.1. *In Diagram 2, $E^{(4)}$ and $E^{(5)}$ are monomorphic.*

Proof. The proof is similar to that of (ii) of Lemma 2.1.

Lemma 3.2. *$J^{(i)}$, $i=0, 1, \cdots, 6$, are monomorphic for $n=8j+1$, $j>0$.*

Proof. $J^{(0)}$ is monomorphic by [1], and in a similar way as in the

proof of Lemma 2.2, $J^{(i)}$, $i=1,2,3$, are monomorphic (Cf. Proposition 2.1 of [22]). Since $i_*^{n-2,n-1}$, $i_*^{n-3,n-2}$ $(n=8j+1,\ j>0)$ are monomorphic, so are $J^{(4)}$, $J^{(5)}$. In the exact sequence

$$\pi_{n+1}(S^{n-4}) \xrightarrow{\partial} \pi_n(SO_{n-4}) \xrightarrow{i_*^{n-4,n-3}} \pi_n(SO_{n-3}),$$

$\pi_{n+1}(S^{n-4})=0$ if $n>10$, and $\pi_9(SO_5)=0$. Hence, $i_*^{n-4,n-3}$ $(n=8j+1,\ j>0)$ is monomorphic, and therefore $J^{(6)}$ is monomorphic for $j>0$.

Lemma 3.3. (Cf. Nomura [13]). *Let* $n=8j+1$ $(j>0)$. *Then,*

(i) $[\iota_{n+1}, \iota_{n+1}]$ *does not belong to* $\mathrm{Im}(\eta_{n+1}^2)_*$.

(ii) $[\eta_n, \iota_n]$ *does not belong to* $\mathrm{Im}(\eta_n^2)_*$.

(iii) $[\eta_{n-1}^2, \iota_{n-1}]$ *does not belong to* $\mathrm{Im}(\eta_{n-1}^2)_*$.

Proof. (i) holds trivially since $[\iota_{n+1}, \iota_{n+1}]$ (n: odd) has the infinite order. (ii) and (iii) are known by Diagram 2 in a similar way as in Lemma 2.3, so we omit the precise description.

§ 4. Calculation of $G(\eta_{n-1})$

Let π_*: $\pi_n(SO_n)\to\pi_n(S^{n-1})$ be the homomorphism induced from the projection π: $SO_n\to S^{n-1}$. It is known that $\pi_*=0$ for $n=4j-1$ $(j\geqq 3)$, $4j+1$ $(j\geqq 1)$, and $4j+2$ $(j\geqq 1)$ by Lemma 2.2 of [3]. Hence, there exist $(n-1)$-sphere bundles over $(n+1)$-spheres $(n\geqq 4)$ which admit no cross-sections only for $n=7$, $8j$ $(j\geqq 1)$, and $8j+4$ $(j\geqq 0)$. Furthermore, if $n=8j+4$ $(j\geqq 0)$ the connected sum consisting of $(n-1)$-sphere bundles over $(n+1)$-spheres $(n\geqq 4)$ and containing such bundles is unique up to diffeomorphism by Theorem 4.4 and Theorem 5.3 of [3]. Therefore, we may calculate $G(\eta_{n-1})$ $(n\geqq 4)$ only for $n=7$ and $8j$ $(j\geqq 1)$. Here, $G(\eta_{n-1})=i_*(J^{(3)-1}(\mathrm{Im}(\eta_{n-1})_*))$, and the groups and the homomorphisms are as follows:

$$\pi_{2n-1}(S^n) \xrightarrow{(\eta_{n-1})_*} \pi_{2n-1}(S^{n-1}) \xleftarrow{J^{(3)}} \pi_n(SO_{n-1}) \xrightarrow{i_*} \pi_n(SO_n).$$

Proposition 4.1. $G(\eta_6)=120Z$, *where* $G(\eta_6)$ *is the subgroup of* $\pi_7(SO_7)=Z$.

Proof. The above homotopy groups are given as follows:

$$\begin{array}{ccccccc} \pi_{13}(S^7) & \xrightarrow{(\eta_6)_*} & \pi_{13}(S^6) & \xleftarrow{J^{(3)}} & \pi_7(SO_6) & \xrightarrow{i_*} & \pi_7(SO_7) \\ \| & & \| & & \| & & \| \\ Z_2 & & Z_{60} & & Z & & Z \end{array}$$

Here, $\pi_{13}(S^7)$ is generated by $\nu_7^2=\nu_7\circ\nu_{10}$ and $\eta_6\circ\nu_7=0$ by (5.9) of [19]. Hence, $(\eta_6)_*$ is trivial. $J^{(3)}$ is epimorphic by Proposition 2.1 of [22]. i_* carries the generator of $\pi_7(SO_6)$ to twice the generator of $\pi_7(SO_7)$ up to sign (Cf. [17], [18]). In fact, we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_7(SO_6) & \xrightarrow{i_*} & \pi_7(SO_7) & \xrightarrow{\pi_*} & \pi_7(S^6) & \longrightarrow & 0\,, \\ & & \| & & \| & & \| & & \\ & & Z & & Z & & Z_2 & & \end{array}$$

where $\pi_8(S^6)=Z_2$ and $\pi_6(SO_6)=0$. Thus, $G(\eta_6)=i_*(J^{(3)-1}(\mathrm{Im}\,(\eta_6)_*))$ $=i_*(\mathrm{Ker}\,J^{(3)})=i_*(60Z)=120Z$.

Proposition 4.2. $G(\eta_7)=\{0\}$.

Proof. The related homotopy groups are given as follows (Cf. [19]):

$$\begin{array}{ccccc} \pi_{15}(S^8) & \xrightarrow{(\eta_7)_*} & \pi_{15}(S^7) & \xleftarrow{J^{(3)}} & \pi_8(SO_7) \\ \| & & \| & & \| \\ Z+Z_8+(Z_3+Z_5) & & Z_2+Z_2+Z_2 & & Z_2+Z_2 \\ \sigma_8,\ E\sigma' & & \sigma'\circ\eta_{14},\ \bar{\nu}_7,\ \varepsilon_7 & & w_1,\ w_2\,, \end{array}$$

where the symbols written below the groups denote the generators. In the proof of (ii) of Lemma 2.3, we observed that $[\eta_8,\iota_8]=E\sigma'\circ\eta_{15}$, a generator of $\pi_{16}(S^8)$. Hence, by Diagram 1, $E^{(3)}J^{(3)}(w_1)=J^{(2)}i_*(w_1)=$ $J^{(2)}(v_1)=[\eta_8,\iota_8]=E\sigma'\circ\eta_{15}$. Since $E^{(3)}$ is monomorphic by Lemma 2.1, we know that $J^{(3)}(w_1)=\sigma'\circ\eta_{14}$. It is known that $J^{(4)}(x_2)=\bar{\nu}_6$ by Kachi [10]. Therefore, $J^{(3)}(w_2)=J^{(3)}i_*(x_2)=E^{(4)}J^{(4)}(x_2)=E^{(4)}(\bar{\nu}_6)=\bar{\nu}_7$. On the other hand, $\eta_7\circ\sigma_8=\sigma'\circ\eta_{14}+\bar{\nu}_7+\varepsilon_7$ and $\eta_6\circ\sigma'=4\bar{\nu}_6$ by (7.4) of [19]. So that, $\eta_7\circ E\sigma'=E(\eta_6\circ\sigma')=E(4\bar{\nu}_6)=4\bar{\nu}_7=0$. Thus, we know that $\mathrm{Im}\,J^{(3)}\cap$ $\mathrm{Im}\,(\eta_7)_*=\{0\}$ and therefore $G(\eta_7)=\{0\}$ since $J^{(3)}$ is monomorphic.

Proposition 4.3. *If* $j>1$, $G(\eta_{8j-1})\cong Z_2$ *and is generated by* v_3,

where $G(\eta_{8j-1})$ *is the subgroup of* $\pi_{8j}(SO_{8j}) \cong Z_2+Z_2+Z_2$ *generated by* v_1, v_2, *and* v_3.

The proof is given by the following assertions, where we assume that $n=8j$ $(j>1)$. Throughout the proof, we should confer Diagram 1.

Assertion 1. $J^{(3)}(w_1) \notin \mathrm{Im}\,(\eta_{n-1})_*$.

Proof. If $J^{(3)}(w_1) \in \mathrm{Im}\,(\eta_{n-1})_*$, then $[\eta_n, \iota_n] = J^{(2)}(v_1) = J^{(2)}i_*(w_1) = E^{(3)}J^{(3)}(w_1) \in \mathrm{Im}\,(\eta_n)_*$ by Diagram 1. But, $[\eta_n, \iota_n] \notin \mathrm{Im}\,(\eta_n)_*$ by (ii) of Lemma 2.3.

Assertion 2. $J^{(3)}(w_2) \in \mathrm{Im}\,(\eta_{n-1})_*$ *if and only if* $J(\tau) \in \mathrm{Im}\,\eta_*$.

Proof. By Diagram 1, it is clear that $J^{(3)}(w_2) \in \mathrm{Im}(\eta_{n-1})_*$ induces $J(\tau) \in \mathrm{Im}\,\eta_*$. Assume that $J(\tau)=\eta_*(\alpha)$ for some $\alpha \in \pi_{n-1}(S^0)=\pi_{2n+2}(S^{n+3})$. Let $\alpha=E\beta$, $\beta=E\gamma$, $\gamma=E\delta$, and $\delta=E\varepsilon$. Since $E^{(1)}(J^{(1)}u_2-(\eta_{n+1})_*\beta) = 0$, $J^{(1)}u_2-(\eta_{n+1})_*\beta \in \mathrm{Ker}\,E^{(1)} = ([\iota_{n+1}, \iota_{n+1}]) \cong Z_2$. If $J^{(1)}u_2-(\eta_{n+1})_*\beta = [\iota_{n+1}, \iota_{n+1}]$, $E^{(2)}J^{(2)}v_3 - E^{(2)}(\eta_n)_*\gamma = E^{(2)}J^{(2)}v_2$ by Diagram 1. So, $J^{(2)}v_3 - (\eta_n)_*\gamma - J^{(2)}v_2 \in \mathrm{Ker}\,E^{(2)} = ([\eta_n, \iota_n]) \cong Z_2$. Now, taking the Hopf invariant of both sides, it yields a contradiction. In fact, $HJ^{(2)}v_3 = H(\eta_n)_*\gamma = H[\eta_n, \iota_n] = 0$ since $J^{(2)}v_3 = E^{(3)}J^{(3)}w_2$, $(\eta_n)_*\gamma = E^{(3)}(\eta_{n-1})_*\delta$, and $[\eta_n, \iota_n] = E^{(3)}J^{(3)}w_1$. But, $HJ^{(2)}v_2 \neq 0$ as is seen in the proof of (i) of Lemma 2.3. Thus, $J^{(1)}u_2 = (\eta_{n+1})_*\beta$.

Since $E^{(2)}(J^{(2)}v_3 - (\eta_n)_*\gamma) = 0$, $J^{(2)}v_3 - (\eta_n)_*\gamma \in \mathrm{Ker}\,E^{(2)} = ([\eta_n, \iota_n]) \cong Z_2$. If $J^{(2)}v_3 - (\eta_n)_*\gamma = [\eta_n, \iota_n]$, then $E^{(3)}J^{(3)}w_2 - E^{(3)}(\eta_{n-1})_*\delta = E^{(3)}J^{(3)}w_1$. Since $E^{(3)}$ is monomorphic by Lemma 2.1, we have $J^{(3)}w_2 - (\eta_{n-1})_*\delta = J^{(3)}w_1$. Then, by taking the Hopf invariant of both sides, we have a contradiction. In fact, $HJ^{(3)}w_2 = H(\eta_{n-1})_*\delta = 0$ since $J^{(3)}w_2$, $(\eta_{n-1})_*\delta$ are suspension elements $(j>1)$. But, $HJ^{(3)}w_1 \neq 0$ since w_1 is not in the image of $i_*: \pi_n(SO_{n-2}) \to \pi_n(SO_{n-1})$. Here, we are refering to the following diagram which is commutative up to sign:

$$
(***) \qquad \begin{array}{ccc} \pi_n(SO_r) & \xrightarrow{\pi_*} & \pi_n(S^{r-1}) \\ \downarrow J & & \downarrow E^r \\ \pi_{n+r}(S^r) & \xrightarrow{H} & \pi_{n+r}(S^{2r-1}), \end{array}
$$

where $2r>n+2$ and π: $SO_r\to S^{r-1}=SO_r/SO_{r-1}$ is the projection.

Thus, $J^{(2)}v_3=(\eta_n)_*\gamma$, and therefore $E^{(3)}J^{(3)}w_2=E^{(3)}(\eta_{n-1})_*\delta$. Since $E^{(3)}$ is monomorphic, we have $J^{(3)}w_2=(\eta_{n-1})_*\delta$. This completes the proof.

Assertion 3. $J^{(3)}(w_1+w_2)\notin \mathrm{Im}\,(\eta_{n-1})_*$.

Proof. If $J^{(3)}(w_1+w_2)\in \mathrm{Im}\,(\eta_{n-1})_*$, then $J(\tau)\in \mathrm{Im}\,\eta_*$ by Diagram 1. Hence, $J^{(3)}(w_2)\in \mathrm{Im}\,(\eta_{n-1})_*$ by Assertion 2. Therefore, $J^{(3)}(w_1)=J^{(3)}(w_1+w_2)-J^{(3)}(w_2)\in \mathrm{Im}\,(\eta_{n-1})_*$. This contradicts to Assertion 1.

Proof of Proposition 4.3. By Mahowald [12], it is known that $J(\tau)$ is in the image of η_*. In fact, the element in his notation $\rho_j\in \pi_{8j-1}(S^0)$ belongs to J-image, and so $\rho_j\circ\eta\in\pi_{8j}(S^0)$ belongs to J-image. Since stably $\rho_j\circ\eta=\eta\circ\rho_j$, we know that $J(\tau)=\eta\circ\rho_j$. Thus, by Assertions 1, 2, and 3, we have $J^{(3)-1}(\mathrm{Im}\,(\eta_{n-1})_*)=(w_2)\cong Z_2$. Therefore, $G(\eta_{n-1})=(v_3)\cong Z_2$. This completes the proof.

§ 5. Calculation of $G(\eta^2_{n-2})$

Let π_*: $\pi_n(SO_{n-1})\to\pi_n(S^{n-2})$ be the homomorphism induced from the projection π: $SO_{n-1}\to SO_{n-1}/SO_{n-2}=S^{n-2}$. Then, by Lemma 2.2 of [4], $\pi_*=0$ for $n=4j-1$ $(j\geqq 3)$ and $4j+2$ $(j\geqq 1)$. Hence, there exist $(n-2)$-sphere bundles over $(n+1)$-spheres $(n\geqq 6)$ which admit no cross-sections for $n=7$, $8j$, $8j+1$, $8j+4$, and $8j+5$, $j>0$. Furthermore, if $n=8j+4$ $(j>0)$ or $8j+5$ $(j>0)$, the connected sum consisting of $(n-2)$-sphere bundles over $(n+1)$-spheres $(n\geqq 6)$ and containing such bundles is unique up to diffeomorphism by Theorems 4.2, 4.4, 6.2, and 6.4 of [4]. Therefore, we may calculate $G(\eta^2_{n-2})$ only for $n=7$, $8j$ $(j>0)$, and $8j+1$ $(j>0)$. Here, $G(\eta^2_{n-2})=i_*(J^{(4)-1}(\mathrm{Im}\,(\eta^2_{n-2})_*))$, and the groups and the homomorphisms are as follows:

$$\pi_{2n-2}(S^n)\xrightarrow{(\eta^2_{n-2})_*}\pi_{2n-2}(S^{n-2})\xleftarrow{J^{(4)}}\pi_n(SO_{n-2})\xrightarrow{i_*}\pi_n(SO_{n-1}).$$

Proposition 5.1. *$G(\eta^2_5)=60Z$, where $G(\eta^2_5)$ is the subgroup of $\pi_7(SO_6)=Z$.*

Proof. The above homotopy groups are given as follows:

$$\begin{array}{ccccccc} \pi_{12}(S^7) & \xrightarrow{(\eta_5^2)_*} & \pi_{12}(S^5) & \xleftarrow{J^{(4)}} & \pi_7(SO_5) & \xrightarrow{i_*} & \pi_7(SO_6). \\ \| & & \| & & \| & & \| \\ 0 & & Z_{30} & & Z & & Z \end{array}$$

By [22], $J^{(4)}$ is epimorphic. By [17] and [18], i_* maps the generator of $\pi_7(SO_5)$ to twice the generator of $\pi_7(SO_6)$ up to sign. In fact, we have the following exact sequence:

$$\begin{array}{ccccccc} \pi_7(SO_5) & \xrightarrow{i_*} & \pi_7(SO_6) & \xrightarrow{\pi_*} & \pi_7(S^5) & \longrightarrow & \pi_6(SO_5). \\ \| & & \| & & \| & & \| \\ Z & & Z & & Z_2 & & 0 \end{array}$$

Hence, $G(\eta_5^2) = i_*(\mathrm{Ker}\, J^{(4)}) = i_*(30Z) = 60Z$.

Proposition 5.2. *Let* $n=8j$ $(j>0)$. *Then,* $G(\eta_{n-2}^2) = \{0\}$.

The proof is given by the following assertions. Throughout the proof, we should confer Diagram 1.

Assertion 1. *For any non-zero element* $x \in (x_1) \subset \pi_n(SO_{n-2})$, $J^{(4)}(x) \notin \mathrm{Im}\,(\eta_{n-2}^2)_*$, *where* $n=8j$, $j>0$.

Proof. Let $J^{(4)}(x) = (\eta_{n-2}^2)_*\varepsilon$ for some $\varepsilon \in \pi_{2n-2}(S^n)$ and let $\varepsilon = E\zeta$. Then, $J^{(4)}(x) = E^{(5)}(\eta_{n-3}^2)_*\zeta$ by Diagram 1. If we take the Hopf invariant of both sides, it yields a contradiction by the diagram (***) in Section 4 since x is not in the image of i_*: $\pi_n(SO_{n-3}) \to \pi_n(SO_{n-2})$.

Assertion 2. $J^{(4)}(x_2) \in \mathrm{Im}\,(\eta_{n-2}^2)_*$ *if and only if* $J(\tau) \in \mathrm{Im}\,\eta_*^2$, *where* $n=8j$, $j>1$.

Proof. By Diagram 1, it is clear that $J^{(4)}(x_2) \in \mathrm{Im}\,(\eta_{n-2}^2)_*$ induces $J(\tau) \in \mathrm{Im}\,\eta_*^2$. Let $J(\tau) = \eta_*^2\alpha$, for some $\alpha \in \pi_{n-2}(S^0) \cong \pi_{2n+2}(S^{n+4})$, and let $\alpha = E\beta$, $\beta = E\gamma$, $\gamma = E\delta$, $\delta = E\varepsilon$, and $\varepsilon = E\zeta$. Then, by a quite similar way to the proof of Assertion 2 in Proposition 4.3, we know that $J^{(3)}(w_2) = (\eta_{n-1}^2)_*\delta$. Hence, $J^{(4)}(x_2) - (\eta_{n-2}^2)_*\varepsilon \in \mathrm{Ker}\, E^{(4)} = \mathrm{Im}\, P$ which is generated by $[\alpha_1, \iota_{n-2}]$, $[\nu_{n-2}, \iota_{n-2}]$, where α_1, ν_{n-2} are the generators of $\pi_{n+1}(S^{n-2})$

$\cong Z_3+Z_8$ of order 3 and order 8 respectively. So, $J^{(4)}(x_2)-(\eta^2_{n-2})_*\varepsilon = b[\nu_{n-2}, \iota_{n-2}]$, where b is an integer. Here, $J^{(4)}(x_2)=E^{(5)}J^{(5)}(y)$ $(j>1)$ and $(\eta^2_{n-2})_*\varepsilon=E^{(5)}(\eta^2_{n-3})_*\zeta$. Then, by taking the Hopf invariant of both sides, $0=bH[\nu_{n-2}, \iota_{n-2}]=\pm 2bE^{n-3}\nu_{n-2}=\pm 2b\nu_{2n-5}$ (Cf. (5.32) of [21]). Hence, $b\equiv 0 \pmod 4$. In the proof of Lemma 2.2, it is known that $[\nu_{n-2}, \iota_{n-2}]$ is of order 4. Thus, we know that $J^{(4)}(x_2)=(\eta^2_{n-2})_*\varepsilon$.

Assertion 3. *For any non-zero element* $x\in(x_1)\subset\pi_n(SO_{n-2})$, $J^{(4)}(x+x_2)\notin \mathrm{Im}\,(\eta^2_{n-2})_*$, *where* $n=8j$, $j>1$.

Proof. If $J^{(4)}(x+x_2)\in \mathrm{Im}\,(\eta^2_{n-2})_*$, then $J(\tau)\in \mathrm{Im}\,\eta^2_*$ by Diagram 1. Hence, $J^{(4)}(x_2)\in \mathrm{Im}\,(\eta^2_{n-2})_*$ by Assertion 2. Therefore, $J^{(4)}(x)=J^{(4)}(x+x_2)-J^{(4)}(x_2)\in \mathrm{Im}\,(\eta^2_{n-2})_*$. This contradicts to Assertion 1. So, $J^{(4)}(x+x_2)\notin \mathrm{Im}\,(\eta^2_{n-2})_*$.

Proof of Proposition 5.2. Firstly, assume that $j>1$. It is known that $J(\tau)=\eta\rho_j$ for certain $\rho_j\in\pi_{8j-1}(S^0)$ (Cf. Mahowald [12]). Then, $J(\tau)$ is not in the image of η^2_*. For, if $J(\tau)=\eta^2\alpha$ for some $\alpha\in\pi_{8j-2}(S^0)$, $\eta^2\rho_j=\eta(\eta\rho_j)=\eta J(\tau)=\eta^3\alpha=4\nu\alpha$. But $\eta^2\rho_j$ generates the image of J: $\pi_{8j+1}(SO)\to\pi_{8j+1}(S^0)$ (Cf. Mahowald [12]), where J is monomorphic and the image is a direct summand of $\pi_{8j+1}(S^0)$ by Adams [1]. Therefore, this yields a contradiction. Thus, $J(\tau)\notin \mathrm{Im}\,\eta^2_*$, and by Assertion 2, $J^{(4)}(x_2)\notin \mathrm{Im}\,(\eta^2_{8j-2})_*$. Hence, by Assertion 1, 3, $G(\eta^2_{8j-2})=\{0\}$ for $j>1$.

Let $j=1$. We remember that $\pi_*\colon \pi_8(SO_6)\to\pi_8(S^5)$ is an isomorphism and x_1, x_2 correspond to α_1, ν_5, respectively (See § 2). Let $J^{(4)}(ax_1+bx_2)=(\eta^2_6)_*\varepsilon$ for some $\varepsilon\in\pi_{14}(S^8)$. Since $\varepsilon=E\zeta$ for some $\zeta\in\pi_{13}(S^7)$, $(\eta^2_6)_*\varepsilon=E^{(5)}(\eta^2_5)_*\zeta$. Hence, $0=HJ^{(4)}(ax_1+bx_2)=E^6\pi_*(ax_1+bx_2)=a\alpha_1(11)+b\nu_{11}$. So that, $a\equiv 0 \pmod 3$, $b\equiv 0 \pmod 8$, and therefore, $ax_1+bx_2=0$. Thus, $J^{(4)-1}(\mathrm{Im}\,(\eta^2_6)_*)=\{0\}$, and so $G(\eta^2_6)=\{0\}$. This completes the proof of the proposition.

Proposition 5.3. (i) $G(\eta^2_7)=\{0\}$. (ii) *If* $j>1$, $G(\eta^2_{8j-1})\cong Z_2$ *and is generated by* w_3, *where* $G(\eta^2_{8j-1})$ *is the subgroup of* $\pi_{8j+1}(SO_{8j})\cong Z_2+Z_2+Z_2$ *generated by* w_1, w_2, *and* w_3.

The proof is given by the following assertions, where we assume that $n=8j+1$ $(j>0)$. Throughout the proof, we should confer Diagram 2.

Assertion 1. $J^{(4)}(x_1) \notin \mathrm{Im}\,(\eta^2_{n-2})_*$.

Proof. Let $J^{(4)}(x_1) = (\eta^2_{n-2})_*\varepsilon$ for some $\varepsilon \in \pi_{2n-2}(S^n)$. Since $\varepsilon = E\zeta$ for some $\zeta \in \pi_{2n-3}(S^{n-1})$, $J^{(4)}(x_1) = E^{(5)}(\eta^2_{n-3})_*\zeta$. Then, taking the Hopf invariant of both sides, there arise a contradiction by the diagram (***) in Section 4, since x_1 is not in the image of i_*: $\pi_n(SO_{n-3}) \to \pi_n(SO_{n-2})$.

Assertion 2. $J^{(4)}(x_2) \in \mathrm{Im}\,(\eta^2_{n-2})_*$ *if and only if* $J(\tau) \in \mathrm{Im}\,\eta^2_*$.

Proof. By Diagram 2, it is clear that $J^{(4)}(x_2) \in \mathrm{Im}\,(\eta^2_{n-2})_*$ induces $J(\tau) \in \mathrm{Im}\,\eta^2_*$. Let $J(\tau) = \eta^2\alpha$ for some $\alpha \in \pi_{n-2}(S^0)$ and let $\alpha = E\beta$, $\beta = E\gamma$, $\gamma = E\delta$, $\delta = E\varepsilon$, and $\varepsilon = E\zeta$. Then, $J^{(1)}(u_2) - (\eta^2_{n+1})_*\beta \in \mathrm{Ker}\,E^{(1)} = ([\iota_{n+1}, \iota_{n+1}]) \cong Z$. Considering the order, $J^{(1)}(u_2) - (\eta^2_{n+1})_*\beta = 0$. Hence, $J^{(2)}(v_2) - (\eta^2_n)_*\gamma \in \mathrm{Ker}\,E^{(2)} = ([\eta_n, \iota_n]) \cong Z_2$. Let $J^{(2)}(v_2) - (\eta^2_n)_*\gamma = [\eta_n, \iota_n]$. Since $J^{(2)}(v_2) = E^{(3)}J^{(3)}(w_3)$, $(\eta^2_n)_*\gamma = E^{(3)}(\eta^2_{n-1})_*\delta$, and $[\eta_n, \iota_n] = E^{(3)}J^{(3)}(w_2)$, $J^{(3)}(w_3) - (\eta^2_{n-1})_*\delta - J^{(3)}(w_2) \in \mathrm{Ker}\,E^{(3)} = ([\eta^2_{n-1}, \iota_{n-1}]) \cong Z_2$. Here, $J^{(3)}(w_3) = E^{(4)}J^{(4)}(x_2)$, $(\eta^2_{n-1})_*\delta = E^{(4)}(\eta^2_{n-2})_*\varepsilon$, $[\eta^2_{n-1}, \iota_{n-1}] = E^{(4)}J^{(4)}(x_1)$, and w_2 is not in the image of i_*: $\pi_n(SO_{n-2}) \to \pi_n(SO_{n-1})$. Hence, considering the Hopf invariant of both sides, there arise a contradiction by the diagram (***) in Section 4. Thus, $J^{(2)}(v_2) = (\eta^2_n)_*\gamma$.

Therefore, $J^{(3)}(w_3) - (\eta^2_{n-1})_*\delta \in \mathrm{Ker}\,E^{(3)} = ([\eta^2_{n-1}, \iota_{n-1}]) \cong Z_2$. If $J^{(3)}(w_3) - (\eta^2_{n-1})_*\delta = [\eta^2_{n-1}, \iota_{n-1}]$, then $E^{(4)}J^{(4)}(x_2) - E^{(4)}(\eta^2_{n-2})_*\varepsilon = E^{(4)}J^{(4)}(x_1)$, and since $E^{(4)}$ is monomorphic by Lemma 3.1, $J^{(4)}(x_2) - (\eta^2_{n-2})_*\varepsilon = J^{(4)}(x_1)$. Here, $J^{(4)}(x_2) = E^{(5)}J^{(5)}(y)$, $(\eta^2_{n-2})_*\varepsilon = E^{(5)}(\eta^2_{n-3})_*\zeta$, and x_1 is not in the image of i_*: $\pi_n(SO_{n-3}) \to \pi_n(SO_{n-2})$. Hence, taking the Hopf invariant of both sides, we have a contradiction again. Thus, $J^{(3)}(w_3) = (\eta^2_{n-1})_*\delta$, and so $J^{(4)}(x_2) = (\eta^2_{n-2})_*\varepsilon$ since $E^{(4)}$ is monomorphic. This completes the proof.

Assertion 3. $J^{(4)}(x_1 + x_2) \notin \mathrm{Im}\,(\eta^2_{n-2})_*$.

Proof. If $J^{(4)}(x_1+x_2)\in \mathrm{Im}(\eta^2_{n-2})_*$, then $J(\tau)\in \mathrm{Im}\,\eta^2_*$ by Diagram 2, and therefore $J^{(4)}(x_2)\in \mathrm{Im}(\eta^2_{n-2})_*$ by Assertion 2. Hence, $J^{(4)}(x_1)=J^{(4)}(x_1+x_2)-J^{(4)}(x_2)\in \mathrm{Im}(\eta^2_{n-2})_*$. This contradicts to Assertion 1.

From the above assertions and Diagram 2, we have

Assertion 4. *If* $J(\tau)\notin \mathrm{Im}\,\eta^2_*$, *then* $G(\eta^2_{n-2})=\{0\}$. *If* $J(\tau)\in \mathrm{Im}\,\eta^2_*$, *then* $G(\eta^2_{n-2})=(w_3)\cong Z_2$.

Proof of Proposition 5.3. (i) It is known by Kachi [10] that $J(\tau)=\nu^3$ for the generator $\tau\in\pi_9(SO_{11})\cong Z_2$, where ν^3 is a basis element of $\pi_9(S^0)\cong Z_2+Z_2+Z_2$ generated by ν^3, μ, and $\eta\circ\varepsilon$ (Cf. [19]). On the other hand, $\pi_7(S^0)\cong Z_{16}+Z_3+Z_5$ and the 2-component is generated by σ. But, $\eta^2\circ\sigma=\eta\circ(\eta\circ\sigma)=\eta\circ(\bar\nu+\varepsilon)=\eta\circ\bar\nu+\eta\circ\varepsilon=\nu^3+\eta\circ\varepsilon$ (Cf. [19]). This implies that $J(\tau)\notin \mathrm{Im}\,\eta^2_*$. Therefore, by Assertion 4, we know that $G(\eta^2_7)=\{0\}$.

(ii) It is known by Mahowald [12] that the image of $J\colon \pi_{8j+1}(SO)\to\pi_{8j+1}(S^0)$ is generated by $\eta^2\circ\rho_j$ for $j>1$. So, $J(\tau)\in \mathrm{Im}\,\eta^2_*$ for $j>1$. Hence, $G(\eta^2_{8j-1})=(w_3)\cong Z_2$ by Assertion 4.

This completes the proof of the proposition.

§ 6. Classification of Connected Sums, the First Case

Let $(p,q)=(n-1,n+1)$, $n\geqq 4$ or $(n-2,n+1)$, $n\geqq 6$. A connected sum of p-sphere bundles over q-spheres is of *type* I if and only if it consists only of the bundles which admit no cross-sections, and of *type* (O+I) if and only if it contains both the bundles which admit cross-sections and the bundles which admit no cross-sections (Cf. Lemma 1.5). In this section and the next section, we completely classify the connected sums of type I or type (O+I) up to homotopy equivalence using the results obtained in the previous sections.

Throughout this and next sections, we should confer Table 2 of [3] and [4]. B_α denotes the bundle with the characteristic element α, but we denote the product bundle by A_0. mB_α, mA_0 denote the connected sum of m copies of B_α, A_0 respectively. By Lemma 2.2 of [3] and

Lemma 2.2 of [4], it is known that the connected sum of type I or type (O+I) exists only for $n=7$, $8j$ $(j>0)$, and $8j+4$ $(j\geqq 0)$ if $(p, q)=(n-1, n+1)$, $n\geqq 4$, and for $n=7$, $8j$ $(j>0)$, $8j+1$ $(j>0)$, $8j+4$ $(j>0)$, and $8j+5$ $(j>0)$ if $(p, q)=(n-2, n+1)$, $n\geqq 6$. In this section, we treat with certain cases which are simpler in a sense. The rest is treated in the next section.

Theorem 6.1. *Let* $\sharp_{i=1}^{k}B_{\alpha_i}$, $\sharp_{i=1}^{k}B_{\alpha_i'}$ *be connected sums of type* I *or type* (O+I) *consisting of* 6*-sphere bundles over* 8*-spheres. We assume that if* $\sharp_{i=1}^{k}B_{\alpha_i}$, $\sharp_{i=1}^{k}B_{\alpha_i'}$ *are of type* (O+I), *they have the same number of bundles which admit cross-sections. Then, they are homotopy equivalent if and only if*

(i) $\alpha_1=\pm\alpha_1'$ mod 120, *if* $k=1$,

(ii) *G.C.D.* $(\alpha_1, \cdots, \alpha_k, 120)=$*G.C.D.* $(\alpha_1', \cdots, \alpha_k', 120)$, *if* $k>1$,

where the greatest common divisors are taken as positive integers. Here, every characteristic element belongs to $Z\cong\pi_7(SO_7)$ *and must be even or odd according as the bundle admits a cross-section or admits no cross-section. If the connected sum are of type* (O+I), *then* $k>1$.

Remark 1. In contrast of this theorem, it is implicitly included in Theorem 1 and Theorem 2 of [3] that above $\sharp_{i=1}^{k}B_{\alpha_i}$, $\sharp_{i=1}^{k}B_{\alpha_i'}$ are diffeomorphic if and only if G.C.D.$(\alpha_1, \cdots, \alpha_k)=$G.C.D.$(\alpha_1', \cdots, \alpha_k')$ (>0). Here, if $k=1$, G.C.D.$(\alpha_1)=$G.C.D.(α_1') means that $\alpha_1=\pm\alpha_1'$. (Cf. Theorem 4.1, 5.2, and Corollary 9.4 of [3]).

Remark 2. Let $n=4j-1$ $(j\geqq 2)$, and let $\sharp_{i=1}^{k}A_{\alpha_i}$, $\sharp_{i=1}^{k}A_{\alpha_i'}$ be connected sums of type O consisting of $(n-1)$-sphere bundles over $(n+1)$-spheres. Here, the characteristic elements α_i, α_i', $i=1, 2, \cdots, k$, belong to $Z\cong\pi_{4j-1}(SO_{4j-1})$, and must be even if $j=2$.

The following is implicitly included in Theorem 3 of [22]: They are homotopy equivalent if and only if

(i) $\alpha_1=\pm\alpha_1'$ mod m if $k=1$,

(ii) G.C.D.$(\alpha_1, \cdots, \alpha_k, m)=$G.C.D.$(\alpha_1', \cdots, \alpha_k', m)$ (>0) if $k>1$,

where

$$m=\begin{cases} 120 & (j=2) \\ m(2j) & (j\geqq 3) \end{cases}$$

and $m(2j)$ is the denominator of (the j-th Bernoulli number)$/4j$.

On the other hand, the following is known by [3]: They are diffeomorphic if and only if both of them consist of trivial bundles or G.C.D. $(\alpha_1, \cdots, \alpha_k)=$G.C.D.$(\alpha'_1, \cdots, \alpha'_k)$ (>0). Here, if $k=1$, G.C.D.$(\alpha_1)=$ G.C.D.(α'_1) means that $\alpha_1=\pm\alpha'_1$. (Cf. Theorem 3.1 and Corollary 9.4 of [3]).

In proving the theorem, we note that exchange of orientation has no affection since the operation of $\pi_0(O_7)$ to $\pi_7(SO_7)$ is trivial by 22.4 of [16].

Let $e_1, e_2, \cdots, e_k$ be the basis of $H=H_{n+1}(\natural_{i=1}^{k}\overline{B}_{\alpha_i})$ represented by zero cross-sections, and let $e'_1, e'_2, \cdots, e'_k$ be the basis of $H'=H_{n+1}(\natural_{i=1}^{k}\overline{B}_{\alpha'_i})$ similarly, where $n=7$. Then, $\alpha(e_i)=\alpha_i$, $\alpha'(e'_i)=\alpha'_i$, $i=1, 2, \cdots, k$. Note that α, α' are homomorphisms by Lemma 2.1 of [3]. The following argument is also applicable to the case that $(p, q)=(5, 8)$. So that, it is convenient to put $G(\eta_6)=mZ$. We know already that $m=120$ by Proposition 4.1. If $(p, q)=(6, 8)$ or $(5, 8)$, a p-sphere bundle over the q-sphere admits no cross-section if and only if the characteristic element is odd, by (i) of Lemma 2.2 of [3] or [4].

The proof will be accomplished by the following assertions.

Assertion 1. *Let* $\sharp_{i=1}^{k}B_{\alpha_i}$, $\sharp_{i=1}^{k}B_{\alpha'_i}$ *be of type* I. *Then,* $\sharp_{i=1}^{k}B_{\alpha_i}$ *has the oriented homotopy type of* $\sharp_{i=1}^{k}B_{\alpha'_i}$ *if and only if there exists a unimodular* $k\times k$*-matrix* L *such that* L *is orthogonal* mod 2, *i.e.* $LL^t=E(\mathrm{mod}\ 2)$, *and*

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_k\end{pmatrix}=L\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \vdots\\ \alpha'_k\end{pmatrix} \quad \mathrm{mod}\ m\ .$$

Proof. This is easily seen by considering the matrix representation of h in Theorem 1.3 using the standard admissible bases $\{e_1, \cdots, e_k\}$, $\{e'_1, \cdots, e'_k\}$, where rank $\phi=k$ since the connected sums are of type I.

Assertion 2. *Let c be an odd integer and let $d=G.C.D.(c, m)>0$. Then, kB_c has the oriented homotopy type of kB_d, where $k>1$.*

Proof. Let $c=c_1d$. c_1, d must be odd. There exist the integers a, b such that $ca+mb=d$. Since m is even, a must be odd. By multiplying both sides of the equality by c_1+1, we have

$$c\{a(c_1+1)-1\}+mb(c_1+1)=d.$$

We put

$$P=\begin{pmatrix} a-1 & 1 \\ (a-1)(c_1+1)-1 & c_1+1 \end{pmatrix}$$

and define a $k\times k$-matrix Q by $Q=\mathrm{diag}(P, E_{k-2})$, where E_{k-2} is the unit $(k-2)\times(k-2)$-matrix. Then, Q is unimodular, orthogonal mod 2, and satisfies

$$\begin{pmatrix} d \\ d \\ c \\ \vdots \\ c \end{pmatrix}=Q\begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} \mod m\,.$$

On the other hand, Lemma 4.2 of [3] shows that there exists a unimodular $k\times k$-matrix K which is orthogonal mod 2, satisfying

$$\begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix}=K\begin{pmatrix} d \\ d \\ c \\ \vdots \\ c \end{pmatrix},$$

where G.C.D.$(c, d)=d$ since c is divisible by d. Thus, we have a unimodular $k\times k$-matrix $L=KQ$ which is orthogonal mod 2, satisfying

$$\begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix}=L\begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} \mod m\,.$$

Hence, by Assertion 1, kB_c has the oriented homotopy type of kB_d.

Assertion 3. *Let $\sharp_{i=1}^{k}B_{\alpha_i}$ be of type* I, *and let G.C.D.$(\alpha_1, \cdots, \alpha_k, m)=d$, $k>1$. Then, $\sharp_{i=1}^{k}B_{\alpha_i}$ has the oriented homotopy type of kB_d.*

Proof. Let $c=\text{G.C.D.}(\alpha_1, \cdots, \alpha_k)>0$. Then, by Lemma 4.2 of [3], there exists a unimodular $k\times k$-matrix K which is orthogonal mod 2, satisfying

$$\begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} = K \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

Hence, $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of kB_c by Assertion 1. More strongly, they are diffeomorphic by Theorem 4.1 of [3]. Since $\text{G.C.D.}(c, m)=\text{G.C.D.}(\alpha_1, \cdots, \alpha_k, m)=d$, kB_c has the oriented homotopy type of kB_d by Assertion 2. Hence, $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of kB_d.

Proof of the theorem in type I *case.* (i) is clear from Assertion 1. Let $k>1$. Put $\text{G.C.D.}(\alpha_1, \cdots, \alpha_k, m)=d$, $\text{G.C.D.}(\alpha_1', \cdots, \alpha_k', m')=d'$. If $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of $\sharp_{i=1}^{k} B_{\alpha_i'}$, d is divisible by d' by Assertion 1. Considering L^{-1}, d' is also divisible by d. Therefore, $d=d'$. Conversely, if $d=d'$ Assertion 3 shows that $\sharp_{i=1}^{k} B_{\alpha_i}$, $\sharp_{i=1}^{k} B_{\alpha_i'}$ are of the same oriented homotopy type. This completes the proof of the theorem when the connected sums are of type I.

To prove the theorem for connected sums of type (O+I), we must modify the above assertions a little. Since $G(\eta_6)=G(0)$ as is seen in the proof of Proposition 4.1, β and β_0 in Theorem 1.3 take values in the same group. Hence, similarly to Assertion 1, we have

Assertion 4. *Assume that B_{α_i}, $B_{\alpha_i'}$ admit cross-sections for $i=1, 2, \cdots, s$, and admit no cross-sections for $i=s+1, s+2, \cdots, s+t=k$, where $s, t>0$. Then, $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of $\sharp_{i=1}^{k} B_{\alpha_i'}$ if and only if there exists a unimodular $k\times k$-matrix L such that*

$$\text{(i)} \quad L \begin{pmatrix} 0 & 0 \\ 0 & E_t \end{pmatrix} L^t = \begin{pmatrix} 0 & 0 \\ 0 & E_t \end{pmatrix} \mod 2,$$

where E_t is the unit $t\times t$-matrix, and

(ii) $\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_k\end{pmatrix}=L\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \vdots\\ \alpha'_k\end{pmatrix} \mod m$.

In the following, A_0 denotes the product bundle $S^8\times S^6$. sA_0 denotes the connected sum of s copies of A_0.

Assertion 5. *Let c be an odd integer and let $d=G.C.D.(c, m)>0$. Then, $sA_0\sharp tB_c$ has the oriented homotopy type of $sA_0\sharp tB_d$, where $s, t>0$ and $s+t=k$.*

Proof. Firstly, let $t=1$. Let $c=c_1d$. Then, there exist the integers a, b such that $ca+mb=d$. Here, c_1, d, and a must be odd. Multiplying both sides of the equality by c_1, we have

$$c(ac_1-1)=-mbc_1 .$$

we put

$$P=\begin{pmatrix}c_1 & ac_1-1\\ 1 & a\end{pmatrix}$$

and define a $k\times k$-matrix Q by $Q=\mathrm{diag}(E_{s-1}, P)$, where E_{s-1} is the unit $(s-1)\times(s-1)$-matrix. Then, Q is unimodular and satisfies (i) of Assertion 4, and also satisfies the relation

$$\begin{pmatrix}0\\ \vdots\\ 0\\ d\end{pmatrix}=Q\begin{pmatrix}0\\ \vdots\\ 0\\ c\end{pmatrix} \mod m .$$

Hence, by Assertion 4, $sA_0\sharp B_d$ has the oriented homotopy type of $sA_0\sharp B_c$.

If $t>1$, by Assertion 2, there exists a unimodular $t\times t$-matrix L which is orthogonal mod 2, satisfying

$$\begin{pmatrix}d\\ d\\ \vdots\\ d\end{pmatrix}=L\begin{pmatrix}c\\ c\\ \vdots\\ c\end{pmatrix} \mod m .$$

We define the $k\times k$-matrix M by $M=\mathrm{diag}(E_s, L)$. Then, M is unimodular and satisfies (i) of Assertion 4 and the relation

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ d \\ \vdots \\ d \end{pmatrix} = M \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c \\ \vdots \\ c \end{pmatrix} \mod m .$$

Hence, by Assertion 4, $sA_0 \sharp tB_d$ has the oriented homotopy type of $sA_0 \sharp tB_c$.

Assertion 6. *Assume that B_{α_i} admits a cross-section for $i=1, 2, \cdots, s$ and admits no cross-section for $i=s+1, s+2, \cdots, s+t=k$, where $s, t>0$. Let $d=$G.C.D.$(\alpha_1, \cdots, \alpha_k, m)>0$. Then, $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of $sA_0 \sharp tB_d$.*

Proof. By Theorem 5.2 of [3], $\sharp_{i=1}^{k} B_{\alpha_i}$ is orientation preservingly diffeomorphic to $sA_0 \sharp tB_c$, where $c=$G.C.D.$(\alpha_1, \cdots, \alpha_k)>0$. Since G.C.D.$(c, m)=$G.C.D.$(\alpha_1, \cdots, \alpha_k, m)$, the assertion follows from Assertion 5.

Proof of the theorem in type (O+I) *case.* Let $\sharp_{i=1}^{k} B_{\alpha_i}$, $\sharp_{i=1}^{k} B_{\alpha'_i}$ be of type (O+I), and assume that they have respectively s bundles which admit cross-sections. If $\sharp_{i=1}^{k} B_{\alpha_i}$ has the oriented homotopy type of $\sharp_{i=1}^{k} B_{\alpha'_i}$, then d is divisible by d' by Assertion 4, and similarly d' by d. Therefore, $d=d'$. Conversely, if $d=d'$, Assertion 6 shows that $\sharp_{i=1}^{k} B_{\alpha_i}$, $\sharp_{i=1}^{k} B_{\alpha'_i}$ are of the same oriented homotopy type.

This completes the proof of the theorem.

By Theorem 6.1, immediately we have

Corollarly 6.2. *The connected sums of type* I *consisting of k 6-sphere bundles over 8-spheres are classified up to homotopy equivalence as follows*:

(i) *If $k=1$, B_c, $0<c<60$, where c's are odd integers.*

(ii) *If $k>1$, kB_1, kB_3, kB_5, and kB_{15}.*

Here, every characteristic element belongs to $Z \cong \pi_7(SO_7)$.

Corollary 6.3. *The connected sums of type* (O+I) *consisting of* k 6*-sphere bundles over* 8*-spheres of which* s *bundles admit cross-sections are classified up to homotopy equivalence as follows*:

$$sA_0\sharp tB_1,\ sA_0\sharp tB_8,\ sA_0\sharp tB_5,\ \text{and}\ sA_0\sharp tB_{15},$$

where A_0 *is the product bundle* $S^8\times S^6$, $s+t=k$, $s,\ t>0$, *and the characteristic elements belong to* $Z\cong\pi_7(SO_7)$.

Similar argument is applicable to the case $(p, q)=(5, 8)$. We have

Theorem 6.4. *Let* $\sharp_{i=1}^k B_{\alpha_i}$, $\sharp_{i=1}^k B_{\alpha_i'}$ *be connected sums of type* I *or type* (O+I) *consisting of* 5*-sphere bundles over* 8*-spheres. We assume that if* $\sharp_{i=1}^k B_{\alpha_i}$, $\sharp_{i=1}^k B_{\alpha_i'}$ *are of type* (O+I), *they have the same number of bundles which admit cross-sections. Then, they are homotopy equivalent if and only if*

(i) $\alpha_1=\pm\alpha_1'$ mod 60, *if* $k=1$,

(ii) *G.C.D.* $(\alpha_1, \cdots, \alpha_k, 60)=$*G.C.D.* $(\alpha_1', \cdots, \alpha_k', 60)$, *if* $k>1$,

where the greatest common divisors are taken as positive integers. Here, every characteristic element belongs to $Z\cong\pi_7(SO_6)$ *and must be even or odd according as the bundle admits a cross-section or admits no cross-section. If the connected sums are of type* (O+I), *then* $k>1$.

Remark. The remarks corresponding to those of Theorem 6.1 also hold by Theorem 4.1 and 5.1 of [4] (see also p. 731) and by Theorem 3 of [22]. Here, we must put $m=60$ if $j=2$.

Proof of the theorem. Let $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ be the invariant systems of $\natural_{i=1}^k \overline{B}_{\alpha_i}$, $\natural_{i=1}^k \overline{B}_{\alpha_i'}$ respectively. Then, α, α' are homomorphisms by Lemma 2.1 of [4], and by Proposition 5.1, $G(\eta_5^2)=60Z$. Hence, quite similarly as in the proof of Theorem 6.1, we have the theorem. The facts corresponding to those used in the proof of Theorem 6.1 are obtained from [3] and [4]. We must note that the operation of $\pi_0(O_6)$ to $\pi_7(SO_6)$ is also trivial. In fact, the exact sequence

$$\longrightarrow \underset{\|}{\pi_7(SO_5)} \xrightarrow{i_*} \underset{\|}{\pi_7(SO_6)} \xrightarrow{\pi_*} \underset{\|}{\pi_7(S^5)} \longrightarrow \pi_6(SO_5)=0$$
$$\qquad\quad Z \qquad\qquad Z \qquad\qquad Z_2$$

shows that $i_*(\gamma_7)=2\delta_7$ up to sign, where γ_7, δ_7 denote the generators of $\pi_7(SO_5)$, $\pi_7(SO_6)$ respectively. Let ρ_0 be the non-trivial element of $\pi_0(O_6)$. Then, $2\rho_0(\delta_7)=\rho_0(2\delta_7)=\rho_0(i_*\gamma_7)=i_*\gamma_7=2\delta_7$ (Cf. 22.5 of [16]). Hence, we know that $\rho_0(\delta_7)=\delta_7$.

Immediately, we have

Corollary 6.5. *The connected sums of type* I *consisting of* k 5*-sphere bundles over* 8*-spheres are classified up to homotopy equivalence as follows*:

(i) *If* $k=1$, B_c, $0<c<30$, *where* c*'s are odd integers.*

(ii) *If* $k>1$, kB_1, kB_3, kB_5, *and* kB_{15}.

Here, every characteristic element belongs to $Z\cong\pi_7(SO_6)$.

Corollary 6.6. *The connected sums of type* (O+I) *consisting of* k 5*-sphere bundles over* 8*-spheres of which* s *bundles admit cross-sections are classified up to homotopy equivalence as follows:*

$$sA_0\sharp tB_1,\ sA_0\sharp tB_3,\ sA_0\sharp tB_5,\ \text{and}\ sA_0\sharp tB_{15},$$

where A_0 *is the product bundle* $S^8\times S^5$, $s+t=k$, $s, t>0$, *and the characteristic elements belong to* $Z\cong\pi_7(SO_6)$.

To complete our cases, we quote the following from [3] and [4].

Theorem 6.7. *Let* $n=8j+4$ $(j\geqq 0)$. *Then, the connected sums of type* I *or type* (O+I) *consisting of* k $(n-1)$*-sphere bundles over* $(n+1)$*-spheres are unique up to diffeomorphism and can be represented as*

$$sA_0\sharp tB_{(0,1)},\quad s+t=k,$$

where we assume that each connected sum contains just s *bundles* $(0\leqq s<k)$ *which admit cross-sections. The characteristic element*

$(0, 1)$ *belongs to* $Z_2+Z_2\cong\pi_{8j+4}(SO_{8j+4})$.

Theorem 6.8. *The connected sums of type* I *or type* (O+I) *consisting of* k $(n-2)$*-sphere bundles over* $(n+1)$*-spheres are unique up to diffeomorphism in the following cases and can be represented as follows, where we assume that each connected sum contains just* s *bundles* $(0\leqq s<k)$ *which admit cross-sections*:

(i) *If* $n=8j+4$ $(j>0)$, $sA_0\sharp tB_1$, $s+t=k$, *where the characteristic element* 1 *belongs to* $Z_2\cong\pi_{8j+4}(SO_{8j+3})$.

(ii) *If* $n=8j+5$ $(j>0)$, $sA_0\sharp tB_{(0,1)}$, $s+t=k$, *where the characteristic element* $(0, 1)$ *belongs to* $Z_2+Z_2\cong\pi_{8j+5}(SO_{8j+4})$.

§ 7. Classification of Connected Sums, the Second Case

In this section, we classify, up to homotopy equivalence, the connected sums of $(n-1)$-sphere bundles over $(n+1)$-spheres of type I or type (O+I) for $n=8j$ $(j>0)$ and the connected sums of $(n-2)$-sphere bundles over $(n+1)$-spheres of type I or type (O+I) for $n=8j$ $(j>0)$, $8j+1$ $(j>0)$. If $n=8j$ $(j>0)$, an $(n-1)$-sphere bundle over the $(n+1)$-sphere admits no cross-section if and only if the characteristic element is given by $(\varepsilon, 1, \delta)\in Z_2+Z_2+Z_2\cong\pi_{8j}(SO_{8j})$. (Cf. [3], (ii) of Lemma 2.2). An $(n-2)$-sphere bundle over the $(n+1)$-sphere ($n=8j$ or $8j+1$, $j>0$) admits no cross-section if and only if the characteristic elemment is given by $(1, \delta)\in Z_2+Z_2\cong\pi_{8j}(SO_{8j-1})$ if $n=8j$ $(j>0)$, or by $(\varepsilon, 1, \delta)\in Z_2+Z_2+Z_2\cong\pi_{8j+1}(SO_{8j})$ if $n=8j+1$ $(j>0)$. (Cf. [4], (ii), (iii) of Lemma 2.2). We keep the notation in the previous section.

For connected sums of type I, we have frequently the following classification style, which we call the *typical classification style in type* I: There are just two bundles B_a, B_b or just four bundles B_a, $B_{a'}$, B_b, and $B_{b'}$ which have no cross-sections. But, $B_{a'}$ is diffeomorphic to B_a and $B_{b'}$ is diffeomorphic to B_b. Hence, any connected sum consisting of such k bundles is represented as $lB_a\sharp mB_b$ $(l+m=k)$.

(1) kB_a, kB_b are independent of the other connected sums up to diffeomorphism and kB_a is not diffeomorphic to kB_b.

(2) $lB_a\sharp mB_b$, $l'B_a\sharp m'B_b$, where l, m, l', m' are positive and $l+m$

$=l'+m'=k$, are diffeomorphic if and only if $l=l'$ (mod 2) or $m=m'$ (mod 2).

Hence, as independent representatives of classes classified up to diffeomorphism, we can take

(i) kB_a,

(ii) kB_b,

(iii) $(k-1)B_a\sharp B_b$ $(k\geqq 2)$,

(iv) $(k-2)B_a\sharp 2B_b$ $(k\geqq 3)$.

In the above, diffeomorphisms are orientation preserving.

Theorem 7.1. *Let $n=8j$ $(j>0)$. Then, the connected sums of type* I *consisting of k $(n-1)$-sphere bundles over $(n+1)$-spheres are classified as follows*:

(i) *If $j=1$, such two connected sums are homotopy equivalent if and only if they are diffeomorphic, and such connected sums are classified into the typical classification style in type* I.

(ii) *If $j>1$, such connected sums are all homotopy equivalent, and therefore, may be represented as kB_a up to homotopy equivalence.*

Here, $a=(0,1,0)$, $a'=(1,1,0)$, $b=(0,1,1)$, and $b'=(1,1,1)$, which belong to $Z_2+Z_2+Z_2\cong\pi_{8j}(SO_{8j})$, $j>0$.

Proof. Since $G(\eta_7)=\{0\}$ by Proposition 4.2, (i) is obtained from Theorem 4.5 of [3] by Corollary 1.4. Here, we must note that exchange of orientation does not affect the classification in Theorem 4.5 of [3]. In fact, in a way similar to page 117 of [16], we know that the non-trivial element ρ_0 of $\pi_0(O_8)$ operates on $\pi_8(SO_8)$ as $\rho_0(v_1)=v_1$, $\rho_0(v_2)=v_1+v_2$, $\rho_0(v_3)=v_3$, where $v_1=i_*^{7,8}(\tilde{\rho}\circ\eta_7)$, $v_2=\tilde{\sigma}\circ\eta_7+i_*^{6,8}\xi$, $v_3=i_*^{6,8}\tilde{\xi}$ and those are the basis of $\pi_8(SO_8)\cong Z_2+Z_2+Z_2$. (Cf. § 2). Hence, $-B_a=B_{a'}$, $-B_b=B_{b'}$, but $B_{a'}$, $B_{b'}$ are orientation preservingly diffeomorphic to B_a, B_b respectively (Cf. [3] p. 228). Therefore, if $j=1$, the typical classification style in type I is independent of the choice of orientation.

(ii) is obtained also from Theorem 4.5 of [3] by Theorem 1.2 since $G(\eta_{8j-1})=((0,0,1))\cong Z_2$ by Proposition 4.3.

Theorem 7.2. *Let $n=8j$ $(j>0)$. Then, the two connected sums*

of type I *consisting of* k $(n-2)$*-sphere bundles over* $(n+1)$*-spheres are homotopy equivalent if and only if they are diffeomorphic, and such connected sums are classified into the typical classification style of type* I, *where* $a=(1,0)$, $b=(1,1)$ *and those belong to* $Z_2+Z_2\cong\pi_{8j}(SO_{8j-1})$.

Proof. Since $G(\eta_{8j-2}^2)=\{0\}$ by Proposition 5.2, we have the theorem from Theorem 4.3 of [4] by Corollary 1.4. We note that the operation of $\pi_0(O_{8j-1})$ to $\pi_{8j}(SO_{8j-1})$ is trivial by 22.4 of [16].

Theorem 7.3. *Let* $n=8j+1$ $(j>0)$. *Then, the connected sums of type* I *consisting of* k $(n-2)$*-sphere bundles over* $(n+1)$*-spheres are classified as follows*:

(i) *If* $j=1$, *such two connected sums are homotopy equivalent if and only if they are diffeomorphic, and such connected sums are classified into the typical classification style in type* I.

(ii) *If* $j>1$, *such connected sums are all homotopy equivalent, and therefore, may be represented as* kB_a *up to homotopy equivalence.*

Here, $a=(0,1,0)$, $a'=(1,1,0)$, $b=(0,1,1)$, *and* $b'=(1,1,1)$, *which belong to* $Z_2+Z_2+Z_2\cong\pi_{8j+1}(SO_{8j})$, $j>0$.

Proof. Since $G(\eta_7^2)=\{0\}$ by Proposition 5.3, we know (i) from Theorem 4.5 of [4] by Corollary 1.4. Here, if $j=1$, similarly as in the proof of Theorem 7.1, exchange of orientation does not affect the typical classification style in type I. In fact, in a way similar to p. 117 of [16], we have $\rho_0(w_1)=w_1$, $\rho_0(w_2)=w_1+w_2$, and $\rho_0(w_3)=w_3$, where $w_1=i_*^{7,8}(\tilde{\rho}\circ\eta_7\circ\eta_8)$, $w_2=\tilde{\sigma}\circ\eta_7\circ\eta_8+i_*^{6,8}y$, $w_3=i_*^{6,8}y$. (Cf. § 3).

(ii) is obtained from Theorem 4.5 of [4] by Theorem 1.2. Since $G(\eta_{8j-1}^2)=((0,0,1))\cong Z_2$ $(j>1)$ by Proposition 5.3.

For connected sums of type (O+I), the classification is a little complicated. We have frequently the following classification style, which we call the *typical classification style in type* (O+I): There are just two bundles B_a, B_b or just four bundles B_a, $B_{a'}$, B_b and $B_{b'}$ which admit no cross-sections. $B_{a'}$, $B_{b'}$ are diffeomorphic to B_a, B_b respectively.

Hence, any connected sum of type (O+I) consisting of s bundles which admit cross-sections and t bundles which admit no cross-sections ($s+t=k$, $s,\ t>0$) is represented as

$$\sum = (\sharp_{i=1}^{s} A_{\alpha_i}) \sharp (lB_a \sharp mB_b), \quad l+m=t,$$

where each A_{α_i} denotes the bundle which admits a cross-section and has the characteristic element α_i. In our cases $\alpha_i=(\alpha_{i1}, \alpha_{i2}) \in Z_2+Z_2$ or $\alpha_i=(\alpha_{i1}, \alpha_{i2}, \alpha_{i3}) \in Z_2+Z_2+Z_2$.

(a) If the last component of α_i is zero for $i=1, 2, \cdots, s$, then $\sum$ is diffeomorphic to $sA_0 \sharp lB_a \sharp mB_b$, and we have

(1) $sA_0 \sharp tB_a$, $sA_0 \sharp tB_b$ are independent of the other connected sums up to diffeomorphism and $sA_0 \sharp tB_a$ is not diffeomorphic to $sA_0 \sharp tB_b$.

(2) $sA_0 \sharp lB_a \sharp mB_b$, $sA_0 \sharp l'B_a \sharp m'B_b$, where l, m, l', m' are positive and $l+m=l'+m'=t$, are diffeomorphic if and only if $l=l'$ (mod 2) or $m=m'$ (mod 2).

(b) If there exists α_i, $1 \leqq i \leqq s$, such that the last component of α_i is not zero, then $\sum$ is unique up to diffeomorphism and can be represented as

$$\sum = A_c \sharp (s-1) A_0 \sharp tB_a,$$

where $c=(0, 1)$ or $(0, 0, 1)$ according as c belongs to Z_2+Z_2 or $Z_2+Z_2+Z_2$. The case (b) is independent of the case (a) up to diffeomorphism.

Hence, as independent representatives of classes classified up to diffeomorphism, we can take

(i) $sA_0 \sharp tB_a$,
(ii) $sA_0 \sharp tB_b$,
(iii) $sA_0 \sharp (t-1) B_a \sharp B_b \quad (t \geqq 2)$,
(iv) $sA_0 \sharp (t-2) B_a \sharp 2B_b \quad (t \geqq 3)$,
(v) $A_c \sharp (s-1) A_0 \sharp tB_a$.

In the above, diffeomorphisms are orientation preserving.

Theorem 7.4. *Let $n=8j$ $(j>0)$. Then, the connected sums of type* (O+I) *consisting of k $(n-1)$-sphere bundles over $(n+1)$-spheres of which s bundles admit cross-sections are classified as follows*:

(i) *If $j=1$, such two connected sums are homotopy equivalent*

if and only if they are diffeomorphic, and such connected sums are classified into the typical classification style in type (O+I).

(ii) *If* $j>1$, *such connected sums are classified into two classes up to homotopy equivalence: Let* $\sum=\#_{i=1}^{s}A_{\alpha_i}\#lB_a\#mB_b$ $(l+m=t=k-s)$ *be a connected sum of type* (O+I).

(a) *If the last components of* α_i, $i=1, 2, \cdots, s$, *are all zero, then* $\sum$ *is homotopy equivalent to* $sA_0\#tB_a$.

(b) *If there exists a certain* α_t *such that its last component is non-zero, then* $\sum$ *is homotopy equivalent to* $A_c\#(s-1)A_0\#tB_a$.

Here, $a=(0, 1, 0)$, $a'=(1, 1, 0)$, $b=(0, 1, 1)$, $b'=(1, 1, 1)$, *and* $c=(0, 0, 1)$, *which belong to* $Z_2+Z_2+Z_2\cong\pi_{8j}(SO_{8j})$, $j>0$.

Proof. Since $G(\eta_7)=0$ by Proposition 4.2, (i) is obtained from Theorem 5.4 of [3] by Corollary 1.4. Here, exchange of orientation does not affect the typical classification style in type (O+I) as is shown in the proof of Theorem 7.1.

Let $j>1$. By Theorem 5.4 of [3], $\sum$ still has a representation given by one of (i)-(v) in the typical classification style of type (O+I). Then, by Proposition 4.3 and Theorem 1.2, $\sum$ is orientation preservingly homotopy equivalent to one of

(a) $\quad\sum_a=sA_0\#tB_a,\quad s+t=k,$

(b) $\quad\sum_b=A_{(0,0,1)}\#(s-1)A_0\#tB_a,\quad s+t=k.$

As is known from the proof of Theorem 5.4 of [3], (a) arises if the last component of α_i is zero for $i=1, 2, \cdots, s$, and similarly, (b) arises if there exists a certain α_i such that its last component is not zero.

We show that $\sum_b$ is not homotopy equivalent to $\sum_a$. Let $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ be the invariant systems corresponding to (b), (a) respectively, and let $\{u_1, \cdots, u_s; e_1, \cdots, e_t\}$, $\{u_1', \cdots, u_s'; e_1', \cdots, e_t'\}$ be the canonical bases of H, H' respectively. Let H_0 be the subgroup of H generated by $\{u_1, \cdots, u_s\}$. Assume that $\sum_b$ is orientation preservingly homotopy equivalent to $\sum_a$. Then, by Theorem 1.3 and the remark, there exists an isomorphism $h: H\to H'$ such that $\phi=\phi'\circ(h\times h)$, $\beta=\beta'\circ h$, and $\beta_0=\beta_0'\circ h$. Since $J^{(3)}$ and $i_*: \pi_n(SO_{n-1})\to\pi_n(SO_n)$ $(n=8j)$ are monomorphic by Lemma 2.2 and Diagram 1, $G(0)=i_*(\mathrm{Ker}\, J^{(3)})=\{0\}$. Hence, $\beta_0=\alpha|H_0$,

$\beta'_0=\alpha'|h(H_0)$, and therefore, $\alpha=\alpha'\circ h$ on H_0. Let $p_3\colon \pi_{8j}(SO_{8j})\cong Z_2+Z_2+Z_2\to Z_2$ be the projection to the 3rd component, and let $\alpha^{(3)}=p_3\circ\alpha$, $\alpha'^{(3)}=p_3\circ\alpha'$. Then, those are homomorphisms since $\partial\eta_{8j}=(1,0,0)$ by Lemma 2.1 of [3]. Let $h(u_1)=\sum_{i=1}^{s}k_iu'_i+\sum_{j=1}^{t}l_je'_j$. Then, we have

$$1=\alpha^{(3)}(u_1)=\alpha'^{(3)}(h(u_1))=\sum_{i=1}^{s}k_i\alpha'^{(3)}(u'_i)+\sum_{j=1}^{t}l_j\alpha'^{(3)}(e'_j)=0.$$

This is a contradiction. Hence, there is no orientation preserving homotopy equivalence between $\sum_a$ and $\sum_b$.

Furthermore, $-\sum_a$ is orientation preservingly homotopy equivalent to $\sum_a$. In fact, since $\bar{\alpha}'(u'_i)=\bar{\rho}_*\circ\alpha'(u'_i)=0$ for $i=1,2,\cdots,s$ and s, t are homotopy invariants, $-\sum_a$ is represented as $-\sum_a=sA_0\sharp(\sharp_{j=1}^{t}B_{(\varepsilon_j,1,\delta_j)})$. But, every $B_{(\varepsilon_j,1,\delta_j)}$ is orientation preservingly diffeomorphic to $B_{(0,1,0)}$ or $B_{(0,1,1)}$ as is shown in p. 228 of [3]. Hence, we have a representation $-\sum_a=sA_0\sharp lB_{(0,1,0)}\sharp mB_{(0,1,1)}$ for certain l, m. Then, this is orientation preservingly homotopy equivalent to $\sum_a$ as we have seen in the above.

Thus, $\sum_a$ is not homotopy equivalent to $\sum_b$. This completes the proof.

Theorem 7.5. *Let $n=8j$ $(j>0)$. Then, the two connected sums of type* (O+I) *consisting of k $(n-2)$-sphere bundles over $(n+1)$-spheres of which s bundles admit cross-sections are homotopy equivalent if and only if they are diffeomorphic, and such connected sums are classified into the typical classification style of type* (O+I), *where $a=(1,0)$, $b=(1,1)$, and $c=(0,1)$ and those belong to $Z_2+Z_2\cong\pi_{8j}(SO_{8j-1})$.*

Proof. Since $G(\eta^2_{8j-2})=\{0\}$ by Proposition 5.2, we have the theorem from Theorem 5.3 of [4] by Corollary 1.4. Here, the operation of $\pi_0(O_{8j-1})$ to $\pi_{8j}(SO_{8j-1})$ is trivial by 22.4 of [16].

Theorem 7.6. *Let $n=8j+1$ $(j>0)$. Then, the connected sums of type* (O+I) *consisting of k $(n-2)$-sphere bundles over $(n+1)$-spheres of which s bundles admit cross-sections are classified as follows:*

(i) *If $j=1$, such two connected sums are homotopy equivalent if and only if they are diffeomorphic, and such connected sums are*

classified into the typical classification style in type (O+I).

(ii) *If* $j>1$, *such connected sums are classified into two classes up to homotopy equivalence: Let* $\sum=\sharp_{i=1}^{s}A_{\alpha_i}\sharp lB_a\sharp mB_b\,(l+m=t=k-s)$ *be a connected sum of type* (O+I).

(a) *If the last components of* α_i, $i=1, 2, \cdots, s$, *are all zero, then* $\sum$ *is homotopy equivalent to* $sA_0\sharp tB_a$.

(b) *If there exists a certain* α_i *such that its last component is non-zero, then* $\sum$ *is homotopy equivalent to* $A_c\sharp(s-1)A_0\sharp tB_a$.

Here, $a=(0, 1, 0)$, $a'=(1, 1, 0)$, $b=(0, 1, 1)$, $b'=(1, 1, 1)$, *and* $c=(0, 0, 1)$, *which belong to* $Z_2+Z_2+Z_2\cong\pi_{8j+1}(SO_{8j})$, $j>0$.

Proof. The proof is quite similar to that of Theorem 7.4. The operation of $\pi_0(O_8)$ to $\pi_9(SO_8)$ does not affect the typical classification style in type (O+I) as is seen in the proof of Theorem 7.3. Hence, by Proposition 5.3, we have (i). Other corresponding facts in the proof are given by Lemma 3.2, Diagram 2 in Section 3, and Lemma 2.1, Theorem 5.5 of [4].

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