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The Cauchy Problem for Hyperbolic Equations with Double Characteristics

By

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§0. Introduction

In the last decade, the theory on the Cauchy problem for hyperbolic equations has developed a little in the analysis of their characters. This article will aim to survey briefly the main point of the advance. Before doing so, we give a short historical remark. The notion of hyperbolic equation began from the characterization of the wave equation. In the present day, however, it comes to be understood as an algebraic and geometric characterization, for symbols of partial differential operators, corresponding to solvability of non characteristic Cauchy problem for them with data of a suitable function space, so called "well posed". Especially it seems to have been considered that the solvability to the space of infinitely differentiable functions called $\mathscr{E}$-well posed is essential since J. Hadamar [5] proposed. In fact I. G. Petrowsky [24] and completely L. Gårding [8] characterized the necessary and sufficient condition for it to be $\mathscr{E}$-well posed in case of constant coefficients. In general, namely, to the equations with variable coefficients, P. D. Lax [14] and S. Mizohata [19] got a necessary condition that the principal part of the equations should be hyperbolic if the Cauchy problem for it is well posed. Nevertheless the generic fact known about the sufficient condition invariant under the change of variables had been essentially only one given by I. G. Petrowsky [25] for a long time. The fact is that the Cauchy problem for a hyperbolic equation with simple characteristics, called strictly hyperbolic, is $\mathscr{E}$-well posed, although it was extended until uniformly symmetrizable ones in case of systems of operators. The case of constant coefficients and the results by A. Lax [13] and by M. Yamaguti [28] have indicated the necessity of clarifying relations between the principal part and the lower

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order terms of operator at the singular points of characteristics. In fact, S. Mizohata and Y. Ohya [21] and [22] show, in the case of characteristics with constant multiplicity two, that the Levi condition, connected with the E. E. Levi's work [15], is necessary and sufficient to be $\mathscr{E}$-well posed, where the Levi condition is that the subprincipal part vanishes identically on the characteristics. This relation is extended in succession by H. Flaschka and G. Strang (necessity) [4] and by J. Chazarain (sufficiency) [1] for operators with heigher constant multiplicities, noted by S. Matsuura [17] to be multi-products of distinct strictly hyperbolic operators. It means that such an operator satisfies the intrisically same type of conditions as ones in case of constant coefficients given by L. Hörmander [6] and justified by L. Svensson [26]. Namely the condition is that the lower order terms are weaker than the principal part. (After them, J. L. Dunn (sufficiency) [3] and S. Wakabayashi (necessity) [27] have checked it valid even for the lower order terms with variable coefficients.) However there existed examples studied by M. Chi [2], O. A. Oleinik [23] and others which had different types of relations from ones supposed in case of constant coefficients. It were V. Ya Ivriĭ and V. M. Petkov [9] who went through these points to get a necessary condition advanced further. Their results refined by L. Hörmander [7] were sharp as enough as a sufficient condition, better than the Petrowsky condition, was conjectured. We here summarize them and sufficient conditions by V. Ya Ivriĭ, L. Hörmander and the author. One of the conclusions for single equations is that the strong hyperbolicity is equivalent to the effective hyperbolicity.

Remark. Throughout this paper, symbols of pseudodifferential operators are the Weyl symbols. An operator $q(x, D)$ with the symbol $q(x, \xi)$ is defined by

$$q(x, D)\phi=(2\pi)^{-n}\int e^{i(x-y)\xi}q((x+y)/2, \xi)\phi(y)dxd\xi .$$

§ 1. Necessary Conditions

We consider a single partial differential operator p of order m on an open set Ω of $\boldsymbol{R}^{n+1}$. The non characteristic Cauchy problem, in other words, the initial value problem, is to find a solution of the equation $pu=f$ on Ω satisfying the initial data on a hypersurface of the derivatives up to $m-1$ of u to the conormal direction of the surface. The solvability of the Cauchy problem according to J. Hadamard [5] consists of two parts of local existence of solutions and local uniqueness. There are some variants to realize it in exact notions. We use here one of stronger definitions in the sense that it is an open condition, namely, if it is solvable at a point, then it is solvable at any point of a neighborhood of it.

Definition 1.1. The Cauchy problem for p is said to be well posed at a point $x^{\sim}$ with respect to a non characteristic direction $\theta \neq 0$ if there exist a neighborhood Ω of $x^{\sim}$ and an infinitely differentiable function ϕ satisfying $\phi(x^{\sim})=0$ and $d\phi(x^{\sim})=\theta$ such that the following statements $(E)_t$ and $(U)_t$ hold for any small t.

$(E)_t$ For every f belonging to $C_0^{+\infty}(\Omega)$ there is a distribution u belonging to $\mathscr{E}'(\Omega)$ and satisfying the equation $pu=f$ on Ω_t, where Ω_t is the set of x in Ω such that $\phi(x)<t$.

$(U)_t$ If u belonging to $\mathscr{E}'(\Omega)$ satisfies $pu=0$ on Ω_t, then u vanishes identically on Ω_t.

By P. D. Lax [14] and S. Mizohata [19], we have known that the principal part p_m of p is hyperbolic if the Cauchy problem for p is well posed as follows. (It is proved under weaker conditions of well-posedness.)

Theorem 1.1 (P. D. Lax and S. Mizohata). *If the Cauchy problem for p is well posed at $x^{\sim}$ with respect to the direction $\theta \neq 0$ such that $p_m(x^{\sim}, \theta) \neq 0$, then*

$$p_m(x, \zeta\theta+\xi)=0$$

has only real roots in ζ for any real vector ξ and for any x of a neighborhood of $x^{\sim}$.

We can find in L. Hörmander [7] the above result to imply that

$$p_{m(\beta)}^{(\alpha)}(x^{\sim}, \zeta^{\sim}\theta+\xi^{\sim})=0$$

for any derivatives in (ξ, x) of order (α, β) up to $|\alpha|+|\beta|<r$ if

$$p_m(x^{\sim}, \zeta\theta+\xi^{\sim})=0$$

has the roots $\zeta^{\sim}$ of multiplicity r. So we have the following theorem by V. Ya Ivriĭ and V. M. Petkov [9].

Theorem 1.2 (V. Ya Ivriĭ and V. M. Petkov). *Suppose that $p_m(x^{\sim}, \zeta\theta+\xi^{\sim})=0$ has the real root $\zeta^{\sim}$ of multiplicity r. If the Cauchy problem for p is well posed at $x^{\sim}$ with respect to the direction $\theta \neq 0$ $(p_m(x^{\sim}, \theta) \neq 0)$ and has the finite propagation property, then it holds for the lower order term p_s of order s that*

$$p_{s(\beta)}^{(\alpha)}(x^{\sim}, \zeta^{\sim}\theta+\xi^{\sim})=0$$

for any derivative in (ξ, x) *of order* (α, β) *up to*

$$|\alpha|+|\beta|+2(m-s)<r.$$

Remark. The finite propagation property means a stronger condition for the uniqueness such that u belonging to $\mathscr{E}'(\Omega)$ and satisfying $pu=0$ on $\Omega(\varepsilon, x)$ should vanish on $\Omega(\varepsilon, x)$, for any small and positive ε and for any x in Ω, where $\Omega(\varepsilon, x)$ is the set of x' in Ω such that $\phi(x)-\phi(x')>\varepsilon|x-x'|$.

Therefore we can define the localized operator of p.

Definition 1.2. Let p_{loc} be called the localization of p if it is well defined for any point $(x^{\sim}, \xi^{\sim})$ such that

$$p_{loc}(x, \xi)=\lim_{\nu\to\infty}\nu^{-2m+r}p(x^{\sim}+\nu^{-1}x, \nu^2(\xi^{\sim}+\nu^{-1}\xi))$$

at the r-ple characteristic point $(x^{\sim}, \xi^{\sim})$ of p_m.

Remark. The case of double characteristics does not need the results of the above theorem.

At the non characteristic point of p_m, p_{loc} is non zero constant. At the simple characteristic point, it defines a non singular vector field of real constant coefficients. These two cases have no other information from the well-posedness of the Cauchy problem for p. At the double characteristic point, p_{loc} is a polynomial in (x, ξ) of order 2. If we denote the homogeneous part in (x, ξ) of order j by $p_{loc(j)}$, then $p_{loc(2)}$ is defined from the Hesse matrix of p_m, $p_{loc(1)}$ is identically zero and $p_{loc(0)}$ is equal to $p_{m-1}(x^{\sim}, \xi^{\sim})$. $p_{loc(2)}$ is characterized by the fundamental (Hamilton) matrix $\mathscr{F}$ defined as

$$\sigma(X, \mathscr{F}X)=p_{loc(2)},$$

where $X=(x, \xi)$ and σ is the canonical two form $\sum d\xi^j \wedge dx_j$. The hyperbolicity yields the properties of the fundamental matrix $\mathscr{F}$ that the eigenvalues of $\mathscr{F}$ are only on the pure real and pure imaginary axes and the non zero real eigenvalues, if they exist, are only one by one on the positive part and the negative part of the real axis, respectively. If we define the positive trace $\mathrm{Tr}^{\sim}\mathscr{F}$ of $\mathscr{F}$ by

$$\mathrm{Tr}^{\sim}\mathscr{F}=2^{-1}\sum|\mathrm{Im}\,\lambda_j|,$$

where λ_j are all eigenvalues of $\mathscr{F}$, then we are able to conclude the following.

Theorem 1.3 (V. Ya Ivrii, V. M. Petkov and L. Hörmander). *The subprincipal part* p_{m-1} *should be real and bounded as*

$$-2^{-1}\,\mathrm{Tr}^{\sim}\mathscr{F} \le p_{m-1} \le 2^{-1}\,\mathrm{Tr}^{\sim}\mathscr{F}$$

by the positive trace of the fundamental matrix $\mathscr{F}$ *on the singular points of the characteristic set where the fundamental matrix had no non zero real eigenvalue if the Cauchy problem for p were well posed.*

Remarks. 1) The multiplicities of the singular points need not be always double because the subprincipal part p_{m-1} should be zero by virtue of the previous theorem while the fundamental matrix is zero at the point where the multiplicity is not less than triple. V. Ya Ivriĭ and V. M. Petkov have remarked that the finite propagation property is not necessary to obtain this conclusion.

2) We may also call the inequality at the above theorem the Levi condition.

Now we introduce two notations.

Definition 1.3. 1) Let p_m be hyperbolic with respect to the direction $\theta \neq 0$. p_m is said effectively hyperbolic (with respect to the direction $\theta \neq 0$) if the fundamental matrix at any singular point of p_m has non zero real eigenvalues.

2) p_m is said strongly hyperbolic (with respect to the direction $\theta \neq 0$) if the Cauchy problem for p with any arbitrary lower order term is well posed (with respect to the direction $\theta \neq 0$).

Corollary 1.4 (V. Ya Ivriĭ and V. M. Petkov). *If* p_m *is strongly hyperbolic, then* p_m *is effectively hyperbolic.*

V. Ya Ivriĭ, V. M. Petkov and L. Hörmander have completely classified the canonical types of $p_{loc(2)}$ by means of the symplectic transformations to prove the necessity of the Levi condition. They give us good feelings for the structures of hyperbolic operators. So we quote the following from L. Hörmander [7].

Theorem 1.5 (L. Hörmander). *Let* Q *be a quadratic form* $p_{loc(2)}$ *on the symplectic space* $V = \boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1}$ *with* $\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle$ *such that* $p_{loc(2)}(\theta) < 0$ *and* $p_{loc(2)}$ *is hyperbolic with respect to a direction* θ. *Then* V *is a direct sum of symplectic subspaces* W_j *which are mutually orthogonal with respect to the symplectic form and* Q.

Every subspace W_j *is of one of the following types*:

I) *W is two dimensional with coordinates* (x, ξ), $\sigma((x, \xi), (y, \eta)) = \xi y - x\eta$ *and* Q *has one of the following forms in* W_j: a) $Q = \mu(x^2 + \xi^2)$, $\mu > 0$ b) $Q = \mu(x^2 - \xi^2)$, $\mu > 0$ c) $Q = \xi^2$, d) $Q = -\xi^2$, e) $Q = 0$.

II) *W is four dimensional with coordinates* (x_1, x_2, ξ_1, ξ_2) *and* $\sigma((x, \xi),$

$(y, \eta)) = \langle \xi, y \rangle - \langle x, \eta \rangle$ *and* $Q = -\xi_0^2 + 2\xi_0\xi_1 + x_1^2$.

Exactly one of the indefinite cases Ib), Id) *and* II) *occurs, and the decomposition is unique apart from the order of the spaces* W_j, *that is, the number of spaces of different types and the numbers* μ *are uniquely determind.*

According to the above theorem, $p_{loc(2)}$ can be classified into three types including $\mu(x^2 - \xi^2)$, $-\xi^2$ or $-\xi_0^2 + 2\xi_0\xi_1 + x_1^2$. An effectively hyperbolic operator corresponds to the case of $\mu(x^2 - \xi^2)$, and the other cases need the Levi condition. Therefore we can give easily typical examples of hyperbolic operators with double characteristics.

Examples.

1) $$-\xi_0^2 + x_0^2(\xi_1^2 + \xi_2^2) + a^2x_1^2(\xi_1^2 + \xi_2^2) + b^2\xi_1^2 \quad \text{on} \quad \boldsymbol{R}^3$$

is an effectively hyperbolic operator.

2) $$-\xi_0^2 + a^2x_1^2(\xi_1^2 + \xi_2^2) + b^2\xi_1^2 \quad \text{on} \quad \boldsymbol{R}^3$$

and

$$-\xi_0^2 + 2\xi_0\xi_1 + x_1^2(\xi_2^2 + \xi_3^2) + a^2x_2^2(\xi_2^2 + \xi_3^2) + b^2\xi_2^2 \quad \text{on} \quad \boldsymbol{R}^4$$

need the Levi conditions bounded by $2^{-1}\mathrm{Tr}^{\sim}\mathscr{F} = |ab|$ at $\xi_0 = b\xi_1 = ax_1 = 0$ and at $\xi_0 = \xi_1 = b\xi_2 = x_1 = ax_2 = 0$, respectively.

§2. Sufficient Conditions

At the previous section, it has been explained that hyperbolic operators with double characteristic sets are classified into three types. One called effectively hyperbolic has required no limit for lower order terms, and the others have needed some restrictions. We approach to these types from a view point of sufficiency. We restrict the operators treated here to ones of second order for the sake of simplicity because operators with double characteristic set are essentially equivalent to ones of second order with respect to types of characteristic sets.

First, we deal with effectively hyperbolic cases. O. A. Oleinik [23] and others raise to us an idea that the Cauchy problems for effectively hyperbolic operators might be proved to be well posed without any other restrictive condition, because their results have proved so for typical examples. The author has answered to the conjecture by finding a standard type of effectively hyperbolic operators. The combination of the author [11] and [12] yields the following theorem.

Definition 2.1. The Cauchy problem for p is said to be $\mathcal{E}$-well posed if it is well posed in the sense of Definition 1.1 and if any solution u there is infinitely differentiable.

Theorem 2.1. *Let p be a partial differential operator of second order. If the principal part p_2 is effectively hyperbolic with respect to dx_0, then the Cauchy problem for p is $\mathcal{E}$-well posed with respect to dx_0. The solution u satisfies the estimates, by Sobolev norms on $x_0 < T$, that $\|u\|_s \leq C_s \|f\|_{s+l}$ with some constant l independent of s, where the support of the datum f lay on a bounded domain.*

This result combines with the result by V. Ya Ivriĭ and V. M. Petkov to yield an assertion.

Corollary 2.2. *Let p_2 be a principal part of a partial differential operator of second order. p_2 is strongly hyperbolic if and only if p_2 is effectively hyperbolic.*

We mention a standard type of effectively hyperbolic operators, which helps us to understand them.

Theorem 2.3. *Any effectively hyperbolic p_2 with infinitely differentiable coefficients is written locally as*

$$p_2 = -(\xi_0 - \Lambda_1)(\xi_0 - \Lambda_0) + b_2,$$

where Λ_0, Λ_1 and b_2 are infinitely differentiable functions in (x_0, x, ξ) of homogeneous order 1, 1 *and* 2 *in ξ, respectively, such that*

$$b_2 \geq 0,$$

$$\{\xi_0 - \Lambda_0, \xi_0 - \Lambda_1\} > 0 \quad at \quad \Lambda_0 - \Lambda_1 = b_2 = 0$$

and

$$\{\xi_0 - \Lambda_0, b_2\} + cb_2 = 0$$

with an infinitely differentiable function c in (x_0, x, ξ).

The Cauchy problem does not admit all symplectic transformation. It causes some complexities in both proofs of necessary parts and sufficient parts. In fact the above standard type also is not always the canonical type. For example, let us consider

$$p_2 = -\tau^2 + (t\eta + \xi)^2 + 2^{-1}x^2\eta^2 .$$

The canonical type is

$$p_2 = -2^{-1}\sigma^2 + s^2\omega^2 + \zeta^2$$

by the symplectic transformation such that $\sigma = 2\tau + x\eta$, $s = t + \xi/\eta$, $\zeta = \tau + x\eta$, $z = t/2 - \xi/(2\eta)$, $\omega = \eta$ and $w = y + x\xi/\eta$. The standard type is, however,

$$p_2 = -\tau^2 + \Lambda_0^2 + b_2 ,$$

$$\Lambda_0 = 2^{-1/2}(t + \xi/\eta - 2^{-1/2}x)|\eta|$$

and

$$b_2 = 2^{-1}(t\eta + \xi + 2^{-1/2}x\eta)^2 ,$$

where they should be properly modified on a conic neighborhood of $\{\eta = 0\}$.

The non effectively hyperbolic cases have not yet conditions as clear as in the effectively hyperbolic cases. One of the reasons is caused by the difficulty of development from pointwise situations of conditions to local situations. There are, however, some cases with additional conditions, which are finally reduced to the following according to V. Ya Ivriĭ [10].

The principal part p_2 is written with the function Λ_0, Λ_1 and b_2 in (x_0, x, ξ) of homogeneous order 1, 1 and 2 in ξ, respectively, as

$$p_2 = -(\xi_0 - \Lambda_1)(\xi_0 - \Lambda_0) + b_2 ,$$

$$b_2 \geq 0$$

and

$$|\{\xi_0 - \Lambda_0, b_2\}| \leq Cb_2$$

with a positive constant C. It is clear that this type of p_2 is not effectively hyperbolic at any singular point of the characteristic set (double characteristic points) if $\{\xi_0 - \Lambda_0, \xi_0 - \Lambda_1\}$ vanishes there. As to this type, the following result is easily obtained by virtue of the inequality due to A. Melin [18].

Theorem 2.4 (V. Ya Ivriĭ). *Let p_2 be defined as the above. If the positive trace $\mathrm{Tr}^{\sim}\mathscr{F}$ of the fundamental matrix $\mathscr{F}$ do not vanish at the double characteristic points and if the subprincipal part p_1 is bounded as*

$$|\mathrm{Re}\, p_1| < 2^{-1}\mathrm{Tr}^{\sim}\mathscr{F}$$

at the double characteristic points and

$$|\mathrm{Im}\, p_1 + 2^{-1}\{\xi_0 - \Lambda_0, \xi_0 - \Lambda_1\} + \theta_0\Lambda_0 + \theta_1\Lambda_1|^2 \leq Cb_2$$

with infinitely differentiable functions θ_j in (x_0, x, ξ) and a constant C, then the Cauchy problem for a partial differential operator $p = p_2 + p_1 + p_0$ is $\mathscr{E}$-well posed. The solution u satisfies the estimates that

$$\|u\|_{s+1/2} \leq C_s \|f\|_s$$

for any Sobolev norm on $x_0 < T$ if the support of datum f is included in a fixed bounded set.

Remark. The examples at the previous section satisfy the conditions of the theorem if the constant ab does not vanish and if the subprincipal parts satisfy them.

Let us apply it to the cases of exactly double characteristics, that is, suppose that the singular points of the characteristic set of p_2 form a submanifold and the Hesse matrices on it are non-degenerate with respect to vectors transversal to the submanifold. It is important to pick out the term $\xi_0 - \Lambda_0$ from p_2. It is possible in the cases that the fundamental matrices are always of type Id) of Theorem 1.5 and null eigenspaces of $\mathscr{F}$ have a constant dimension.

Theorem 2.5 (V. Ya Ivriĭ and L. Hörmander). *Let p be a partial differential operator such that the principal part p_2 is hyperbolic with respect to dx_0, the set of double characteristic points is exactly double, the fundamental matrices on it are always of type* Id) *and the null eigenspaces have a constant dimension on it. If the subprincipal part p_1 satisfies, at the double characteristic point, that*

$$-2^{-1}(1-\varepsilon)\,\mathrm{Tr}^{\sim}\mathscr{F} \leq p_1 \leq 2^{-1}(1-\varepsilon)\,\mathrm{Tr}^{\sim}\mathscr{F}$$

for some positive ε, then the Cauchy problem for p is $\mathscr{E}$-well posed.

Remark. We refered to L. Hörmander [7] for the formulation of this theorem. The case of non-nilpotent $\mathscr{F}$ is due to V. Ya Ivriĭ [10]. The case of nilpotent $\mathscr{F}$, that is, $\mathrm{Tr}^{\sim}\mathscr{F} = 0$, is due to L. Hörmander, which needs the improved A. Melin's inequality with the strong bound, so that it implies only the estimate that $\|u\|_s \leq C_s \|pu\|_s$.

The result by O. A. Oleinik [23] should be stated before finishing this section, because it includes many types with double characteristics not discussed in this article and because it has motivated many related works including V. Ya Ivriĭ and V. M. Petkov's after it.

Theorem 2.6 (O. A. Oleinik). *Let p be a partial differential operator such that the principal part p_2 is $\xi_0^2 - b_2(x_0, x, \xi)$, where $b_2 \geq 0$. The Cauchy problem*

$$pu = f \quad at \quad 0 \leq x_0 < T$$

and

$$u\,|_{x_0=0} = u_0,\ (\partial/\partial x_0)u\,|_{x_0=0} = u_1$$

for infinitely differentiable data u_0, u_1 and f with compact supports has a unique infinitely differentiable solution u if the subprincipal part p_1 satisfies

$$\alpha x_0 (p_1|_{\xi_0=0})^2 \leq A b_2 + c(\partial/\partial x_0) b_2$$

at $0 \leq x_0 \leq T$ with some positive constant α and A.

§3. At the End

We have no concrete and general result at multiple characteristic points. Is it possible to get the canonical types, by means of symplectic transformations, of the principal part of the localization p_{loc}, which is a multi-linear form in (x, ξ) and a hyperbolic polynomial with respect to the direction $\theta \neq 0$ as a polynomial in (x, ξ)? In the case of double characteristic points, according to L. Hörmander [7], the necessary conditions are closely related to getting a fundamental solution in tempered distributions of the Cauchy problem for the localization, that is, to finding a solution $F(x, y)$ in $\mathscr{S}'$ of the equation

$$p_{loc}(x, D)F(x, y) = \delta(x - y)$$

and

$$F(x, y) = 0 \quad \text{if} \quad \langle x - y, \theta \rangle < 0.$$

Are there any necessary and sufficient conditions of the above problem for the localization of higher order? The localization defined in this article is not general. For example, the necessary and sufficient condition to be well posed for the equation

$$\tau^2 - t^{2k}\xi^2 + at^l\xi \quad (a \neq 0,\ \theta = dt) \quad \text{on} \quad \boldsymbol{R}^2$$

is that $l \geq k-1$. We have not discussed these cases $(k > 1)$ in this article. The localization of this operator at $t = x = \tau = 0$ and $\xi \neq 0$ should be considered as $\tau^2 - t^{2k}\xi^2$ if $l > k-1$, $\tau^2 + at^l\xi$ if $l < k-1$ and itself if $l = k-1$. It seems effective according to S. Mizohata [20] and T. Mandai [16] that the weight controlling the localization is defined by means of the Newton polygon.

Some questions arise immediately of sufficiency. Under what conditions are the principal part p_2 able to have a decomposition by smooth functions at Theorem 2.4, especially, in the case of type II) at Theorem 1.5? Are there any methods independent of such a decomposition, especially, a method applying to the case of fixed one of non effectively hyperbolic types and non nilpotent fundamental matrices? The case of nilpotent fundamental matrices at Theorem 2.5 means that the manifold of double characteristic points is involutive. The condition that the subprincipal part vanishes there is equivalent to the condition that p is weaker than p_2 in the sense by L. Hörmander and L. Svensson. The results by J. L. Dunn and S. Wakabayashi show us the posibility of a condition independent of change of variables unifying the cases of constant coefficients, constant multiplicity and the above as "involutive" cases. If the establishment of such a notion is possible, it will be one of big conditions for the Cauchy problem.

In this article, we mainly dealt with only cases of conditions stated by means of fundamental matrices. There are many related works after O. A. Oleinik. At References 2, we collected some of them published after V. Ya Ivriĭ and V. M. Petkov without comment. (See the detailed references by V. Ya Ivriĭ and V. M. Petkov for works before them.) It would be our great pleasure if an aspect for farther clarifications would be found from among them.

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