

Microlocal Analysis

By

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Since much of the development of microlocal analysis was achieved at Research Institute for Mathematical Sciences, Kyoto University, we want to present a survey on microlocal analysis in celebration of the twentieth birthday of the Institute. As several expositions on microfunctions are now available ([18], [8], [26]), we want to minimize the explanation on microfunctions and concentrate our attention primarily on the theory of microdifferential equations, in particular, on the theory of holonomic systems.^(**)

In this article we restrict our attention to the microlocal analysis in analytic category, and concerning the microlocal analysis in C^∞ -category, we refer the reader to [10], [5], [11] and [27]. We also refer the reader to [20] and the references cited there for the interrelation between the microlocal analysis in C^ω -category and that in C^∞ -category.

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§1. Microfunctions

In order to fix the notations used in later sections, we first recall the defi-

Received February 21, 1983.

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^(**) In the early stage of the development, another terminology “maximally overdetermined system” was used to mean a holonomic system. As it will be explained later, the old terminology manifests the character of the system in question. However, it is somewhat too lengthy, and the new one, which indicates its relation to integrable connections, is now more commonly used in literature.

dition and some basic properties of the sheaf \mathcal{C} of microfunctions. See S–K–K [24] and [18] for the details. Note, however, that the sheaf \mathcal{C} defined in S–K–K [24] corresponds to the sheaf $\gamma_*\mathcal{C}$ in the notations used here.

In general, for a real analytic manifold N and its submanifold M , let T_M^*N and $\widetilde{M}N^* (= (N - M) \sqcup T_M^*N)$ denote the conormal bundle supported by M and the comonoidal transform of N with center M , respectively. (Cf. S–K–K [24], Chap. I, §1.2.) We denote by π the canonical projection from $\widetilde{M}N^*$ to N or its restriction to T_M^*N . To define the sheaf \mathcal{C}_M of microfunctions, we choose a complexification X of M as N . In this case, T_M^*X is canonically isomorphic to $\sqrt{-1}T^*M$. Now, \mathcal{C}_M is, by definition, $\mathcal{H}_{T_M^*X}^n(\pi^{-1}\mathcal{O}_X)^a \otimes_{\omega_{M|X}}$ where n denotes $\dim M$, \mathcal{O}_X denotes the sheaf of holomorphic functions on X , a denotes the antipodal mapping, i.e., $a(x, \sqrt{-1}\xi) = (x, -\sqrt{-1}\xi)$, and $\omega_{M|X}$ denotes the orientation sheaf of M in X , i.e., $\mathcal{H}_M^j(\mathcal{C}_X)$. Here we note that $\mathcal{H}_{T_M^*X}^j(\pi^{-1}\mathcal{O}_X) = 0$ holds for $j \neq n$. In what follows, we denote $T^*M - T_M^*M$ by \dot{T}^*M and $\pi|_{\dot{T}^*M}$ by $\dot{\pi}$. We also denote by γ the projection from $\sqrt{-1}\dot{T}^*M$ to $\sqrt{-1}S^*M$, the pure imaginary co-spherical bundle of M .

We now list up some basic properties of the sheaf \mathcal{C}_M .

(1.1) $\mathcal{C}_M|_M = \mathcal{B}_M$, the sheaf of hyperfunctions.

(1.2) $\mathcal{C}_M|_{\sqrt{-1}\dot{T}^*M} = \gamma^{-1}\gamma_*(\mathcal{C}_M|_{\sqrt{-1}\dot{T}^*M})$.

(1.3) For any point $(x_0, \sqrt{-1}\xi_0)$ in $\sqrt{-1}T^*M$ there exists a canonical surjective homomorphism

$$\text{sp}: \mathcal{B}_{M, x_0} \longrightarrow \mathcal{C}_{M, (x_0, \sqrt{-1}\xi_0)}.$$

(1.4) The following sequence is exact:

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{B}_M \xrightarrow{\text{sp}} \dot{\pi}_*(\mathcal{C}_M|_{\sqrt{-1}\dot{T}^*M}) \longrightarrow 0.$$

Here \mathcal{A}_M denotes the sheaf of real analytic functions on M .

The exact sequence (1.4) means that the sheaf $\mathcal{C}_M|_{\sqrt{-1}\dot{T}^*M}$ describes the detailed structure of singularities of a hyperfunction by dispersing it over the (pure imaginary) cotangent bundle. This fact is really the starting point of microlocal analysis, namely, local analysis on the cotangent bundle.

As we scarcely need any further concrete expression for a section of the sheaf of microfunctions in later sections, at least in an explicit manner, we do not discuss it here. See [18] and [8] for it.

§2. Microdifferential Operators

Let M be a real analytic manifold of dimension n and X its complexification. Then the sheaf \mathcal{D}_X^∞ of linear differential operators on X is, by definition, $\mathcal{H}_{\Delta_X}^n(\Omega_{X \times X}^{(0,n)})$, where Δ_X denotes the diagonal subset of $X \times X$ and $\Omega_{X \times X}^{(0,n)}$ denotes the sheaf of n -forms in the second variable. It immediately follows from the definition that \mathcal{D}_X^∞ acts upon the sheaf \mathcal{B}_M of hyperfunctions. It is then natural to consider the sheaf \mathcal{L}_M defined by $\mathcal{H}_{\sqrt{-1}T_M^*(M \times M)}^0(\mathcal{C}_{M \times M} \otimes_{\mathcal{S}^{M \times M}} v_M)$ so that we may find a sheaf acting upon the sheaf \mathcal{C}_M of microfunctions. Here v_M denotes the sheaf of volume element on the second factor of $M \times M$. One can verify that \mathcal{C}_M is really a left \mathcal{L}_M -Module, and this sheaf \mathcal{L}_M , called the sheaf of microlocal operators, plays an important role in the study of (non-)solvability of linear partial differential equations (e.g. S-K-K [24], Chap. III, § 2.3). However, the sheaf \mathcal{L}_M is too large and not amenable to algebraic manipulations. Hence we introduce another sheaf \mathcal{E}_X^∞ of operators much smaller than \mathcal{L}_M and amenable to algebraic manipulations. Actually we can associate the so-called symbol sequence to each section of \mathcal{E}_X^∞ and important operations on sections of \mathcal{E}_X^∞ such as the composition can be expressed in terms of the associated symbol sequences. (See (2.6) and (2.7) below.)

To define \mathcal{E}_X^∞ , we introduce a complex analogue $\mathcal{C}_{Y|X}^{\mathbf{R}}$ of the sheaf \mathcal{C}_M for a pair (X, Y) of a complex manifold X and its complex submanifold Y . Let d denote the (complex) codimension of Y in X . Then, in parallel with § 1, we find $\mathcal{H}_{T_Y^*X}^j(\pi^{-1}\mathcal{O}_X) = 0$ for $j \neq d$ and define $\mathcal{C}_{Y|X}^{\mathbf{R}}$ by $\mathcal{H}_{T_Y^*X}^d(\pi^{-1}\mathcal{O}_X)^a$. Here π denotes the projection from $\widetilde{YX^*}$ to X . A section of $\mathcal{C}_{Y|X}^{\mathbf{R}}$ is called a real holomorphic microfunction (*) supported by Y .(**) Let us now consider the sheaf $\mathcal{E}_X^{\mathbf{R}}$ of integral operators whose kernel functions are sections of $\mathcal{C}_{\Delta_X|X \times X}^{\mathbf{R}}$, that is, $\mathcal{E}_X^{\mathbf{R}} = \mathcal{C}_{\Delta_X|X \times X}^{\mathbf{R}} \otimes_{\mathcal{O}_{X \times X}} \Omega_{X \times X}^{(0,n)}$, where Δ_X is the diagonal set of $X \times X$ and $n = \dim X$. A section of $\mathcal{E}_X^{\mathbf{R}}$ is called a holomorphic microlocal operator. Note that $\mathcal{E}_X^{\mathbf{R}}|_{T_X^*X} = \mathcal{D}_X^\infty$ holds. It is also easy to see that, if X is a complexification of a real analytic manifold M , then there exists an injection from $\mathcal{E}_X^{\mathbf{R}}|_{T_M^*X}$ to \mathcal{L}_M . Hence a holomorphic microlocal operator defines a microlocal operator. A recent result of Aoki [1], [2] enables us to do the symbol calculus for holomorphic microlocal operators. Although his result is important and interesting,

(*) Although this terminology is not very euphonious, we want to keep the terminology "holomorphic microfunction" for a section of the sheaf $\mathcal{C}_{Y|X}^{\mathbf{R}}$ introduced later ((2.2)).

(**) "Supported by T_Y^*X " "might be more appropriate.

here we discuss the symbol calculus only for a more restricted class of operators, that is, microdifferential operators. To define it we introduce the cotangential projective bundle $P^*X = (T^*X - T_X^*X)/\mathbb{C}^\times$. We denote by $\gamma_{\mathbb{C}}$ the projection from $\hat{T}^*X (= T^*X - T_X^*X)$ to P^*X . Then the sheaf \mathcal{E}_X^∞ of the required operators — which we call microdifferential operators (of infinite order) — is given by

$$(2.1) \quad \begin{cases} \text{(a)} & \mathcal{E}_X^\infty|_{\hat{T}^*X} \stackrel{\text{def}}{=} \gamma_{\mathbb{C}}^{-1} \gamma_{\mathbb{C}*} (\mathcal{E}_X^{\mathbb{R}}|_{\hat{T}^*X}) \\ \text{(b)} & \mathcal{E}_X^\infty|_{T_X^*X} \stackrel{\text{def}}{=} \mathcal{E}_X^{\mathbb{R}}|_{T_X^*X} (= \mathcal{D}_X^\infty). \end{cases}$$

As we will need it later, we define, in parallel with the definition of \mathcal{E}_X^∞ , the sheaf $\mathcal{G}_{Y|X}^\infty$ of holomorphic microfunctions supported by a submanifold Y of X by

$$(2.2) \quad \begin{cases} \text{(a)} & \mathcal{G}_{Y|X}^\infty|_{\hat{T}_Y^*X} = \gamma_{\mathbb{C}}^{-1} \gamma_{\mathbb{C}*} (\mathcal{G}_{Y|X}^{\mathbb{R}}|_{\hat{T}_Y^*X}) \\ \text{(b)} & \mathcal{G}_{Y|X}^\infty|_{Y \times T_X^*X} = \mathcal{G}_{Y|X}^{\mathbb{R}}|_{Y \times T_X^*X}, \end{cases}$$

where $\gamma_{\mathbb{C}}$ denotes the projection from $\hat{T}_Y^*X (= T_Y^*X - Y \times T_X^*X)$ to $P_Y^*X = \hat{T}_Y^*X/\mathbb{C}^\times$.

The sheaf \mathcal{E}_X^∞ thus defined satisfies

$$(2.3) \quad \hat{\pi}_* \mathcal{E}_X^\infty \simeq \mathcal{D}_X^\infty$$

if $\dim X > 1$. (If $\dim X = 1$, \mathcal{D}_X^∞ is a subsheaf of $\hat{\pi}_* \mathcal{E}_X^\infty$.) This means that \mathcal{D}_X^∞ is dispersed over \hat{T}^*X , that is, \mathcal{E}_X^∞ is the microlocalization of \mathcal{D}_X^∞ , and hence we call it the sheaf of microdifferential operators of infinite order. Here we note that S–K–K [24] uses another symbol \mathcal{P}_X to denote \mathcal{E}_X^∞ and calls it the sheaf of pseudo-differential operators. Actually, the correspondence between a microdifferential operator and a symbol sequence stated below shows that, essentially speaking, a section of \mathcal{E}_X^∞ determines an analytic pseudo-differential operator. (Cf. [3], [4].)

In order to define the notion of the symbol sequence of a microdifferential operator, we consider a conic^(*) open subset Ω of T^*X and suppose that a coordinate system (x, ζ) is given there. A sequence $\{p_j(x, \zeta)\}_{j \in \mathbb{Z}}$ of holomorphic functions defined on Ω is called a symbol sequence (of a microdifferential operators) if it satisfies the following conditions (2.4.a)~(2.4.c):

^(*) A subset Ω of T^*X is called conic if it is stable under the action of multiplying cotangent vectors by a non-zero complex number. For a subset of a real cotangent bundle T^*M , multiplication of strictly positive real number is used to define the notion “conic”.

$$(2.4) \left\{ \begin{array}{l} \text{(a) } p_j(x, \xi) \text{ is homogeneous of degree } j \text{ with respect to } \xi, \text{ that is,} \\ \quad \sum_i \xi_i \frac{\partial p_j(x, \xi)}{\partial \xi_i} = j p_j(x, \xi) \text{ holds.} \\ \text{(b) For every } \varepsilon > 0 \text{ and every compact subset } K \text{ of } \Omega, \text{ there exists a} \\ \quad \text{constant } C_{\varepsilon, K} \text{ such that} \\ \quad \quad \sup_K |p_j(x, \xi)| \leq C_{\varepsilon, K} \varepsilon^j / j! \quad (j \geq 0) \\ \text{holds.} \\ \text{(c) For every compact subset } K \text{ of } \Omega, \text{ there exists a constant } R_K \text{ such} \\ \quad \text{that} \\ \quad \quad \sup_K |p_j(x, \xi)| \leq R_K^{-j} (-j)! \quad (j < 0) \\ \text{holds.} \end{array} \right.$$

The most important property of a symbol sequence is that there exists a one-to-one correspondence between the spaces of symbol sequences and the space of sections of \mathcal{E}_X^∞ over Ω . Although the way of assigning a microdifferential operator to a symbol sequence is not unique, we usually assign the Wick product $P(x, D_x) =: \sum_j p_j(x, \xi):$ (the notation used in [2] after the notation used in literature in physics) to a symbol sequence $\{p_j(x, \xi)\}$, namely, all the multiplication operators appear to the left of all the differential operators in $P(x, D_x)$. The notation $\sum_j p_j(x, D_x)$ is also used to denote $: \sum_j p_j(x, \xi):$. This assignment of a microdifferential operator is consistent with the assignment (2.5) below of a microlocal operator \mathcal{X} to a symbol sequence. Note also that, if Ω contains a point $(x, \xi) = (x, 0)$, (2.4.a) entails $p_j = 0$ for $j < 0$. This fact corresponds to (2.1.b).

We now list up some basic properties of symbol sequences.

(2.5) Let $\Phi_\nu(z)$ ($\nu \neq 0, -1, -2, \dots$) denote $\Gamma(\nu)/(-z)^\nu$, where its branch is chosen so that $\Phi_\nu(-1) = \Gamma(\nu)$ and define $\Phi_{-m}(z)$ ($m = 0, 1, 2, \dots$) by $-z^m(\log(-z) - \sum_{j=1}^m 1/j + \gamma)/m!$, where $\gamma = 0.57721\dots$ denotes the Euler constant. Let Ω be a conic complex neighborhood of a point $(x_0, \sqrt{-1}\xi_0)$ in $\sqrt{-1}T^*M$ and define a multi-valued holomorphic function $K(z, w, \zeta)$ by $\sum_j p_j(z, \zeta)\Phi_{n+j}(\langle z-w, \zeta \rangle)$ for each symbol sequence $\{p_j\}_{j \in \mathbb{Z}}$ on Ω . Let $K(x, y, \xi)$ denote the microfunction obtained by taking the boundary value of $K(z, w, \zeta)$ from the domain $\text{Re} \langle z-w, \zeta \rangle < 0$. (See S-K-K [24], Chap. I, § 1 or [18], Chap. 2 for the precise meaning of "taking the boundary value".) Then

$$\frac{1}{(2\pi)^n} \int \left(\int K(x, y, \xi) \omega(\xi) \right) dy$$

determines a microlocal operator \mathcal{K} in a neighborhood of $(x_0, \sqrt{-1}\xi_0)$. Here

$$\omega(\xi) = \sum_{l=1}^n (-1)^{l-1} \xi_l d\xi_1 \wedge \cdots \wedge d\xi_{l-1} \wedge d\xi_{l+1} \wedge \cdots \wedge d\xi_n.$$

Note that the origin of the above representation is the celebrated result of John [12] on the plane wave decomposition of the δ -function.

- (2.6) Let $P = : \sum_j p_j(x, \xi) :$ and $Q = : \sum_k q_k(x, \xi) :$ be microdifferential operators defined in a neighborhood of (x_0, ξ_0) . Then their composite $P \circ Q$ is a well-defined microdifferential operator and it has the form $: \sum_l r_l(x, \xi) :$, where $\{r_l\}_{l \in \mathbb{Z}}$ is the symbol sequence defined by

$$r_l(x, \xi) = \sum_{l=j+k-|\alpha|} \frac{1}{\alpha!} D_\xi^\alpha p_j(x, \xi) D_x^\alpha q_k(x, \xi),$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n (*)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$D_\xi^\alpha = \partial^{|\alpha|} / \partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}, \text{ etc.}$$

- (2.7) Let P be a microdifferential operator and $\{p_j(x, \xi)\}_{j \in \mathbb{Z}}$ the corresponding symbol sequence in a coordinate system (x) . Let (\tilde{x}) be another coordinate system and denote by $(\tilde{x}, \tilde{\xi})$ the corresponding coordinate system on T^*X , i.e., $\tilde{\xi}_j = \sum_{k=1}^n (\partial x_k / \partial \tilde{x}_j) \xi_k$. Let $\{p_j(\tilde{x}, \tilde{\xi})\}_{j \in \mathbb{Z}}$ be the symbol sequence corresponding to P in the coordinate system $(\tilde{x}, \tilde{\xi})$. Then we have

$$\tilde{p}_k(\tilde{x}, \tilde{\xi}) = \sum_I \frac{1}{v! \alpha_1! \cdots \alpha_v!} \langle \tilde{\xi}, D_{\tilde{x}}^{\alpha_1} \tilde{x} \rangle \cdots \langle \tilde{\xi}, D_{\tilde{x}}^{\alpha_v} \tilde{x} \rangle D_{\tilde{\xi}}^\alpha p_j(x, \xi),$$

where $I = \{(j, v, \alpha_1, \dots, \alpha_v, \alpha) \in \mathbb{Z} \times \mathbb{Z}_+ \times (\mathbb{Z}_+^n)^v \times \mathbb{Z}_+^n; |\alpha_1|, \dots, |\alpha_v| \geq 2, k = j + v - |\alpha_1| - \cdots - |\alpha_v| \text{ and } \alpha = \alpha_1 + \cdots + \alpha_v\}$. Here $\langle \tilde{\xi}, D_{\tilde{x}}^\alpha \tilde{x} \rangle$ is, by definition, $\sum_{j=1}^n \tilde{\xi}_j D_{\tilde{x}}^\alpha \tilde{x}_j$. In particular, if $p_j(x, \xi) = 0 (j > m)$, then we have

$$\begin{aligned} \tilde{p}_j(\tilde{x}, \tilde{\xi}) &= 0 (j > m), \tilde{p}_m(\tilde{x}, \tilde{\xi}) = p_m(x, \xi) \text{ and } \tilde{p}_{m-1}(\tilde{x}, \tilde{\xi}) \\ &= p_{m-1}(x, \xi) + \frac{1}{2} \sum_{i,j,k} \tilde{\xi}_k \frac{\partial^2 \tilde{x}_k}{\partial x_i \partial x_j} \frac{\partial^2}{\partial \xi_i \partial \xi_j} p_m(x, \xi). \end{aligned}$$

- (2.8) It follows from (2.7) that the property (F_m) “a microdifferential operator P has the form $: \sum_{j \leq m} p_j(x, \xi) :$ ” is independent of the choice of a local

(*) We denote by \mathbb{Z}_+ the set of non-negative integers.

coordinate system. We denote by $\mathcal{E}_X(m)$ the subsheaf of \mathcal{E}_X^∞ consisting of sections satisfying the property (F_m) . A section of $\mathcal{E}_X(m)$ is called a microdifferential operator of order (at most) m . The union $\bigcup_m \mathcal{E}_X(m)$ is called the sheaf of microdifferential operators of finite order and we denote it by \mathcal{E}_X . We also denote $\mathcal{E}_X|_{T^*X}$ (resp., $\mathcal{E}_X(m)|_{T^*X}$) by \mathcal{D}_X (resp., $\mathcal{D}_X(m)$). For a microdifferential operator P of order at most m , $p_m(x, \xi)$ is well-defined by (2.7) (i.e., independent of the choice of a local coordinate system) and we call it the principal symbol of P . We denote it by $\sigma_m(P)$. By assigning $\sigma_m(P)$ to P in $\mathcal{E}_X(m)$, we obtain an isomorphism between $\mathcal{E}_X(m)/\mathcal{E}_X(m-1)$ and $\mathcal{O}_{T^*X}(m)$, the sheaf of holomorphic functions on T^*X which are homogeneous of degree m with respect to the fiber coordinate ξ .

(2.9) If P in $\mathcal{E}_X(m)_{(x_0, \xi_0)}$ satisfies

$$\sigma_m(P)(x_0, \xi_0) \neq 0,$$

then there exists its inverse P^{-1} in $\mathcal{E}_X(-m)_{(x_0, \xi_0)}$.

§ 3. Quantized Contact Transformation

Egorov [6] observed that one can associate a transformation of pseudo-differential operators to each homogeneous canonical transformation so that the commutation relations and orders of pseudo-differential operators may be preserved and that their principal symbols may be transformed according to the homogeneous canonical transformations. To formulate his observation in our context, we employ holonomic systems of microdifferential equations. In S-K-K [24], such a transformation of operators is called a quantized contact transformation. Its counterpart in C^∞ -category is the theory of Fourier integral operators which Hörmander [10] developed.

Before beginning the discussion on the quantized contact transformation, we review some elementary facts about the sheaf of microdifferential operators and systems of microdifferential equations.

- (3.1) The sheaf \mathcal{E}_X is Nötherian (from the left and the right), namely,
- (a) \mathcal{E}_X is coherent as a left \mathcal{E}_X -Module and as a right \mathcal{E}_X -Module.
 - (b) The stalk $\mathcal{E}_{X,x}$ is a left (and right) Nötherian ring for each point x in X .

(c) For each open subset U of X , a sum of left (and right) coherent $\mathcal{E}_X|_U$ -Ideals is also coherent.

(3.2) \mathcal{D}_X and $\mathcal{E}_X(0)$ are Nötherian in the above sense.

(3.3) \mathcal{E}_X is flat over $\mathcal{E}_X(0)$ and it is flat also over $\pi^{-1}\mathcal{D}_X$.

(3.4) $\mathcal{E}_X^\infty(\mathcal{D}_X^\infty, \mathcal{D}_X, \text{resp.})$ is faithfully flat over $\mathcal{E}_X(\mathcal{D}_X, \mathcal{O}_X, \text{resp.})$.

(3.5) Let

$$(3.5.a) \quad \mathcal{M}' \xrightarrow{\varphi} \mathcal{M} \xrightarrow{\psi} \mathcal{M}''$$

be a complex of coherent \mathcal{E}_X -Modules defined on an open subset of T^*X . Let $\mathcal{M}'_0, \mathcal{M}_0$ and \mathcal{M}''_0 be coherent sub- $\mathcal{E}_X(0)$ -Modules of $\mathcal{M}', \mathcal{M}$ and \mathcal{M}'' which generate them as \mathcal{E}_X -Modules respectively, and suppose that $\varphi(\mathcal{M}'_0)$ is contained in \mathcal{M}_0 and that $\psi(\mathcal{M}_0)$ is contained in \mathcal{M}''_0 . Define the associated symbol sequence

$$(3.5.b) \quad \bar{\mathcal{M}}' \xrightarrow{\bar{\varphi}} \bar{\mathcal{M}} \xrightarrow{\bar{\psi}} \bar{\mathcal{M}}''$$

by denoting by $\bar{\mathcal{M}}$ (resp., $\bar{\mathcal{M}}'$ and $\bar{\mathcal{M}}''$) the \mathcal{O}_{T^*X} -Module $\mathcal{M}_0/\mathcal{E}_X(-1)\mathcal{M}_0$ (resp., $\mathcal{M}'_0/\mathcal{E}_X(-1)\mathcal{M}'_0$ and $\mathcal{M}''_0/\mathcal{E}_X(-1)\mathcal{M}''_0$). If the symbol sequence (3.5.b) is exact, then (3.5.a) is exact. Furthermore,

$$(3.5.c) \quad \mathcal{M}'_0 \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M}''_0$$

is then also exact.

These algebraic properties of the sheaf \mathcal{E}_X etc. are basic in our treatment of systems of (micro)differential equations.

In what follows, a system of microdifferential (resp., linear differential) equations of finite order^(*) means, by definition, a left^(**) coherent \mathcal{E}_X -Module (resp., \mathcal{D}_X -Module) \mathcal{M} . The support of \mathcal{E}_X -Module \mathcal{M} (or $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ if \mathcal{M} is a \mathcal{D}_X -Module) is called the characteristic variety of \mathcal{M} and denoted by $\text{Ch}(\mathcal{M})$. A very important and fundamental result on the geometric structure of $\text{Ch}(\mathcal{M})$ is:

(3.6) $\text{Ch}(\mathcal{M})$ is an involutory subvariety of T^*X . (S-K-K [24], Chap. II, § 5.3. See also [9], [21] and [7].)

^(*) In this article we do not discuss general systems of (micro)differential equations of infinite order, but restrict ourselves to the study of the so-called admissible system, i.e., an \mathcal{E}_X^∞ -Module of the form $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}$ for some left coherent \mathcal{E}_X -Module \mathcal{M} .

^(**) If there is no fear of confusion, we omit the adjective "left".

It immediately follows from (3.6) that

$$(3.7) \quad \text{codim Ch}(\mathcal{M}) \leq \dim X.$$

If the equality in (3.7) holds (and hence $\text{Ch}(\mathcal{M})$ is Lagrangian), we call \mathcal{M} a holonomic system (or a maximally overdetermined system). One important property of a holonomic system is that its solution space is finite-dimensional even locally. ([13], [15]) This suggests that one might study the structure of a function by the holonomic system that it satisfies, if such a system exists. This viewpoint was first advocated by Sato in 1960 (and probably also by Gel'fand around the same time). In later sections we show some typical examples where such "holonomic approach" is successful.

We first prepare some terminologies: Let \mathcal{M} be a system with one unknown function, namely $\mathcal{M} = \mathcal{E}_X u$ for a generator u with a defining relation $\mathcal{I}u = 0^{(*)}$. Then, the symbol $\text{Ideal}^{(**)} \bar{\mathcal{I}}$ is, by definition, the coherent Ideal $\mathcal{O}_{T^*X}(\mathcal{I} \cap \mathcal{E}(0) / \mathcal{I} \cap \mathcal{E}(-1))$ of \mathcal{O}_{T^*X} . When $\bar{\mathcal{I}}$ is reduced, we say that \mathcal{M} is simple, and we call u a non-degenerate section of the simple system \mathcal{M} .

Now, using the notion of a simple holonomic system, we obtain the following

Theorem 3.1. *Let X and Y be complex manifolds of the same dimension and let Λ be a locally closed non-singular homogeneous Lagrangian submanifold of $T^*(X \times Y)$. Suppose that*

$$(3.8) \quad \Lambda \cap (X \times T^*Y) = \Lambda \cap (T^*X \times Y) = \emptyset$$

and that the natural projections $p: \Lambda \rightarrow T^*X$ and $q: \Lambda \rightarrow T^*Y$ are open embedding. Let $\mathcal{M} = \mathcal{E}_{X \times Y} K$ be a simple holonomic system whose characteristic variety is Λ . Then, by assigning PK (resp., QK) to P (resp., Q) for P in \mathcal{E}_X^∞ (resp., Q in \mathcal{E}_Y^∞), we find an isomorphism from \mathcal{E}_X^∞ to $p_*(\mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} \mathcal{M})$ (resp. from \mathcal{E}_Y^∞ to $q_*(\mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} \mathcal{M})$). Furthermore, if we denote by φ the map

$$(q|_\Lambda) \circ (p|_\Lambda)^{-1}: p(\Lambda) \longrightarrow q(\Lambda),$$

we find an anti-isomorphism Φ from the \mathbb{C} -Algebra $\varphi^{-1}(\mathcal{E}_Y^\infty|_{q(\Lambda)})$ to the \mathbb{C} -Algebra $\mathcal{E}_X^\infty|_{p(\Lambda)}$ by assigning $P(x, D_x)$ in $\mathcal{E}_X^\infty|_{p(\Lambda)}$ to $Q(y, D_y)$ in $\mathcal{E}_Y^\infty|_{q(\Lambda)}$ so that $P(x, D_x)K(x, y) = Q(y, D_y)K(x, y)$ may hold. This anti-isomorphism preserves the order of microdifferential operators, namely, $\Phi(\varphi^{-1}\mathcal{E}_Y(m)|_{q(\Lambda)})$

(*) That is, $\mathcal{M} \cong \mathcal{E}_X / \mathcal{I}$.

(**) Using (2.9) we can easily verify that the characteristic variety of the system \mathcal{M} coincides with the set of zeros of the symbol $\text{Ideal} \bar{\mathcal{I}}$.

coincides with $\mathcal{E}_X(m)|_{p(\Lambda)}$ for every integer m . In particular, $\Phi(\varphi^{-1}\mathcal{E}_Y|_{q(\Lambda)})$ coincides with $\mathcal{E}_X|_{p(\Lambda)}$.

Since Φ in Theorem 3.1 is anti-isomorphism (i.e., $\Phi(Q_2Q_1)=\Phi(Q_1)\Phi(Q_2)$), we make a composition of Φ with another special anti-isomorphism $*$ defined below so that we may find a \mathbf{C} -Algebra isomorphism. To define the anti-isomorphism $*$, we assume that X is an open subset of \mathbf{C}^n and choose

$$\mathcal{E}_{X \times X} / \left(\sum_{i=1}^n (\mathcal{E}_{X \times X}(x_i - y_i) + \mathcal{E}_{X \times X}(\partial/\partial x_i + \partial/\partial y_i)) \right)$$

as \mathcal{M} in Theorem 3.1. In this case the map φ reduces to the antipodal map a , i.e., $\varphi(x, \zeta) = (x, -\zeta)$. In this situation $\Phi(Q)$ is denoted by Q^* , and Q^* is called the adjoint operator of Q .

Now, returning to the general situation discussed in Theorem 3.1, we define ψ by $a \circ \varphi$ and $\Psi: \psi^{-1}(\mathcal{E}_Y^\infty|_{a^{-1}q(\Lambda)}) \rightarrow \mathcal{E}_X^\infty|_{p(\Lambda)}$ by $\Phi \circ *$. Then it is clear that ψ is a homogeneous canonical transformation and Ψ is a \mathbf{C} -Algebra isomorphism. We call Ψ (or the pair (ψ, Ψ)) a quantized contact transformation (with kernel function K). Note that, for Q in $\mathcal{E}_Y(m)$, $\sigma_m(\Psi(Q)) = \sigma_m(Q) \circ \psi$ holds, namely, Ψ preserves the principal symbol. We can further prove the following

Theorem 3.2. *For any homogeneous canonical transformation ψ , we can locally find a quantized contact transform (ψ, Ψ) .*

We have so far discussed the quantized contact transformation in the complex domain. If we further suppose that X (resp., Y) is a complexification of a real analytic manifold M (resp., N) and that Λ is real, then making use of a microfunction solution $K(x, y)$ of the system \mathcal{M} , we can define an integral transformation $\mathcal{K}: (\Psi|_{\sqrt{-1}\dot{T}^*M})^{-1}\mathcal{C}_N \rightarrow \mathcal{C}_M$ by $\int K(x, y)u(y)dy$. This transformation \mathcal{K} is an isomorphism if $K(x, y)$ never vanishes on $\Lambda \cap \sqrt{-1}\dot{T}^*(M \times N)$. Therefore we obtain

Theorem 3.3. *For any real homogeneous canonical transformation Ψ , we can locally find a quantized contact transformation (ψ, Ψ) and an isomorphism $\mathcal{K}: (\Psi|_{\sqrt{-1}\dot{T}^*M})^{-1}\mathcal{C}_N \rightarrow \mathcal{C}_M$ so that*

$$\mathcal{K}(Qu) = \Psi(Q)(\mathcal{K}(u))$$

holds for Q in \mathcal{E}_Y^∞ and u in \mathcal{C}_N .

§4. Simple Holonomic Systems

Before beginning the discussion on general holonomic systems, we study the structure of simple holonomic systems. Let us first fix a non-degenerate section u of a simple holonomic system $\mathcal{M} = \mathcal{E}_X/\mathcal{I}$ and denote $\text{Ch}(\mathcal{M})$ by A . We will first define some invariants associated with the section u on the generic points of A , so that they may determine the structure of \mathcal{E}_X -Module \mathcal{M} there. Since we consider the problem microlocally, we assume from the first that A is non-singular. An important invariant $\sigma_A(u)$ of u , called the principal symbol of u along A , is defined as follows:

For $P = : \sum_j p_j(x, \xi) \in \mathcal{E}_X(m)$ satisfying $p_m|_A = 0$, let $L_P^{(m-1)}$, or L_P for short, denotes the following first order linear differential operator on A :

$$(4.1) \quad L_P^{(m-1)} = H_{p_m}(x, \xi) + \left(p_{m-1}(x, \xi) - \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial^2 p_m(x, \xi)}{\partial x_i \partial \xi_i} \right) \right).$$

Here H_{p_m} denotes the Hamiltonian vector field associated with p_m , that is, $H_{p_m}(f) = \{p_m, f\} = \sum_{i=1}^n \left(\frac{\partial p_m}{\partial \xi_i} \frac{\partial f}{\partial x_i} - \frac{\partial p_m}{\partial x_i} \frac{\partial f}{\partial \xi_i} \right)$. In what follows, let $\Omega_A^{\otimes 1/2}$ (resp., $\Omega_A^{\otimes 1/2} \otimes \Omega_X^{\otimes (-1/2)}$) denote a line bundle L such that $L^{\otimes 2}$ is isomorphic to Ω_A (resp., $\Omega_A \otimes \Omega_X^{\otimes (-1)}$), where Ω_A and Ω_X denote the sheaf of n -forms on A and X , respectively. As such a line bundle does not exist globally in general, all the equations among the sections of $\Omega_A^{\otimes 1/2}$ or $\Omega_A^{\otimes 1/2} \otimes \Omega_X^{\otimes (-1/2)}$ are understood up to a constant multiple. Now, let us define $\tilde{L}_P: \Omega_A^{\otimes 1/2} \rightarrow \Omega_A^{\otimes 1/2}$ by

$$(4.2) \quad \tilde{L}_P(s) = \frac{1}{2s} L_v(s^2) + \varphi s \quad (s \in \Omega_A^{\otimes 1/2}),$$

where $v = H_{p_m}$, $\varphi = p_{m-1} - \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial^2 p_m}{\partial x_i \partial \xi_i} \right)$ and $L_v(s^2)$ denotes the Lie derivative of s^2 along the vector field v . If we fix a non-vanishing section $\sqrt{\lambda}$ of $\Omega_A^{\otimes 1/2}$, the action of \tilde{L}_P is explicitly given by

$$(4.3) \quad \tilde{L}_P(f\sqrt{\lambda}) = \left(v(f) + \frac{1}{2} \frac{L_v(\lambda)}{\lambda} + \varphi f \right) \sqrt{\lambda}.$$

Using the simplicity assumption on \mathcal{M} , we can then prove that the system of differential equations

$$(4.4) \quad \tilde{L}_P s = 0 \quad (P \in \mathcal{I})$$

locally admits one and exactly one non-zero solution s_0 in $\Omega_A^{\otimes 1/2}$, up to a constant multiple. The principal symbol $\sigma_A(u)$ of u is, by definition, $s_0 \otimes \sqrt{dx}^{-1} (\in \Omega_A^{\otimes 1/2})$

$\otimes \Omega_X^{\otimes(-1/2)}$). The factor \sqrt{dx}^{-1} is to make the notion of principal symbol independent of the choice of the local coordinate system in X . The principal symbol of u thus defined is homogeneous with respect to the fiber coordinate ξ . The homogeneous degree is called the *order* of u and denoted by $\text{ord}_A(u)$. Here we note that the order is dependent on the choice of a generator u of \mathcal{M} . Actually, $\text{ord}_A(u)$ may be shifted by an integer if we change u to another non-degenerate section of \mathcal{M} . See Theorem 4.1 below. Although the definition of $\text{ord}_A(u)$ requires to know the principal symbol, it can be readily calculated according to the formula (4.5) below, if we can find^(*) $P = \sum_{j \leq 1} p_j$ in \mathcal{S} such that

$$(4.5) \quad \begin{aligned} dp_1 &\equiv \sum_{i=1}^n \xi_i dx_i \pmod{\mathcal{S}\bar{\Omega}^1} . (**) \\ \text{ord}_A(u) &= \left(p_0(x, \xi) - \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial^2 p_1(x, \xi)}{\partial x_i \partial \xi_i} \right) \right) \Big|_A . \end{aligned}$$

Now, the following Theorem 4.1 shows how the order of a non-degenerate section of a simple holonomic \mathcal{E}_X -Module \mathcal{M} determines the structure of \mathcal{M} at a generic point of $\text{Ch}(\mathcal{M})$.

Theorem 4.1. *Let Λ be a connected and homogeneous Lagrangian submanifold (i.e., without singularities) of T^*X .*

(i) *Let $\mathcal{E}_X u$ and $\mathcal{E}_X v$ denote simple holonomic systems with the same characteristic variety Λ . Then*

$$(4.6) \quad \mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X u, \mathcal{E}_X v) \cong \begin{cases} 0 & \text{if } \text{ord}_A u \not\equiv \text{ord}_A v \pmod{\mathbb{Z}} \\ C_A & \text{if } \text{ord}_A u \equiv \text{ord}_A v \pmod{\mathbb{Z}} . \end{cases}$$

That is, $\mathcal{E}_X u$ and $\mathcal{E}_X v$ are locally isomorphic if and only if $\text{ord}_A u - \text{ord}_A v$ is an integer.

(ii) *Let $\mathcal{E}_X u$ be a simple holonomic system with support Λ and let v be another non-degenerate section of $\mathcal{E}_X u$. Then the difference m of $\text{ord}_A v$ and $\text{ord}_A u$ is an integer and there exists an invertible microdifferential operator P of order m such that $v = Pu$ holds.*

(iii) *Let $\mathcal{E}_X u$ be a simple holonomic system with support Λ . Then, through a quantized contact transformation, $\mathcal{E}_X u$ is locally isomorphic to $\mathcal{E}_{\mathbb{C}^n} f$, where f satisfies the following system of differential equations considered near $(0; dx_1) \in T^*\mathbb{C}^n$.*

^(*) A general theory tells us that such an operator P really exists.

^(**) Here Ω^1 denotes the sheaf of holomorphic 1-forms on T^*X .

$$(4.7) \quad \begin{cases} \left(x_1 \frac{\partial}{\partial x_1} + \left(\alpha + \frac{1}{2}\right)\right)f = 0 \\ \frac{\partial f}{\partial x_j} = 0 \quad (j=2, \dots, n). \end{cases}$$

Here α denotes $\text{ord}_A u$.

Remark 4.2. As the canonical form (4.7) in the above indicates, the order of u describes, so to speak, the “microlocal monodromic structure”.

Thus the microlocal structure of a simple holonomic system is fairly simple at non-singular points of its characteristic variety. On the other hand, a Hartogs’ type result for systems of microdifferential equations ([17], Chap. I, § 2) entails that the structure of a holonomic system \mathcal{M} is determined by $\mathcal{M}|_{\text{Ch}(\mathcal{M})-Z}$, if $\text{codim}_{\text{Ch}(\mathcal{M})} Z \geq 2$. In particular, if $\text{Ch}(\mathcal{M})$ has the form $A_1 \cup A_2$ with Lagrangian submanifolds A_1 and A_2 such that $\text{codim}_{A_j}(A_1 \cap A_2) \geq 2$ ($j=1, 2$), then there exist holonomic systems \mathcal{M}_j ($j=1, 2$) such that $\text{Ch}(\mathcal{M}_j) = A_j$ ($j=1, 2$) and that \mathcal{M} is isomorphic to the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$. Hence it is an important and interesting problem to study the structure of a holonomic system \mathcal{M} whose characteristic variety is reducible and some of whose irreducible components intersect along subvarieties of codimension 1 in $\text{Ch}(\mathcal{M})$. We have a satisfactory answer to this problem when \mathcal{M} satisfies the following condition (4.8).

(4.8) \mathcal{M} is a simple holonomic system and $\text{Ch}(\mathcal{M})$ has the form $A_1 \cup A_2$ with Lagrangian submanifolds A_j ($j=1, 2$) of \dot{T}^*X , where $A_1 \cap A_2$ is also a (non-singular) manifold with $\dim(A_1 \cap A_2)$ being equal to $\dim X - 1$ and $T_p(A_1 \cap A_2) = T_p A_1 \cap T_p A_2$ (*) holds for every point p in $A_1 \cap A_2$.

When condition (4.8) is satisfied, a suitable quantized contact transformation (φ, Φ) brings \mathcal{M} to the form $\mathcal{E}_{\mathbb{C}^n} f$ with f satisfying the following system of equations considered near $(0; dx_1) \in T^*\mathbb{C}^n$:

$$(4.9) \quad \begin{cases} \left(x_1 \frac{\partial}{\partial x_1} + \left(\alpha_1 + \frac{1}{2}\right)\right)f = 0 \\ \left(x_2 \frac{\partial}{\partial x_2} + \left(\alpha_1 - \alpha_2 + \frac{1}{2}\right)\right)f = 0 \\ \frac{\partial f}{\partial x_j} = 0 \quad (j=3, \dots, n). \end{cases}$$

Here $\alpha_j = \text{ord}_{A_j} u$ ($j=1, 2$) with $\varphi(A_1) = \{x_1 = 0, \xi_2 = \dots = \xi_n = 0\}$ and $\varphi(A_2)$

(*) Here $T_p(A_1 \cap A_2)$ etc. denote the tangent space of $A_1 \cap A_2$ etc. at p .

$$= \{x_1 = x_2 = 0, \xi_3 = \dots = \xi_n = 0\}.$$

Using the above canonical form (4.9), we can prove the following

Theorem 4.3. *Let \mathcal{M} and $\Lambda_j (j=1, 2)$ be as above. Then the following three conditions are mutually equivalent:*

(4.10) *There exists a non-zero coherent \mathcal{E}_X -sub-Module \mathcal{M}_1 of \mathcal{M} which is supported by Λ_1 .*

(4.11) *There exists a coherent \mathcal{E}_X -Module \mathcal{M}_2 supported by Λ_2 and an \mathcal{E}_X -homomorphism $\phi: \mathcal{M} \rightarrow \mathcal{M}_2$ that is surjective.*

(4.12)
$$x_2 - \alpha_1 - \frac{1}{2} \in \mathbf{Z}_+ \quad (= \{0, 1, 2, \dots\})$$

This theorem is most effectively used in calculating b -functions for a relative invariant of a prehomogeneous vector space. See [25] for details.

§5. Operations on Systems

As we have emphasized in §3, we want to analyze a function by studying the structure of the holonomic system that it solves. It is then important to know how such an operation as restriction etc. is defined for a holonomic system. We formulate them as some functorial properties associated with a holomorphic map $f: Y \rightarrow X$. We present the formulation for the general case, that is, without restricting ourselves to holonomic systems. The formulation is based on some auxiliary sheaves $\mathcal{E}_Y^{\varphi} \rightarrow_X$ etc., which will be defined below.

Let us first suppose that Y is a submanifold of X with codimension d and that f is the imbedding map. Let us denote by I the defining Ideal of Y . Then

$$\varinjlim_v \mathcal{E}_{\mathcal{O}_X}^j(\mathcal{O}_X/I^v, \mathcal{O}_X) = 0 \quad (j \neq d)$$

holds, and the remaining cohomology group $\varinjlim_v \mathcal{E}_{\mathcal{O}_X}^d(\mathcal{O}_X/I^v, \mathcal{O}_X)$ can be endowed with a natural structure of left \mathcal{D}_X -Module. We denote it by $\mathcal{B}_{Y|X}$. There exists a canonical injective \mathcal{O}_Y -homomorphism from $\Omega_Y \otimes_{\mathcal{O}_Y} \Omega_X^{\otimes -1}$ to $\mathcal{B}_{Y|X}$, which we denote by ι . Let $\delta_{Y|X}$ denote $\iota(dy \otimes (dx dy)^{-1})$. If X is an open subset of \mathbf{C}^n and Y is defined by $\{x \in X \subset \mathbf{C}^n; x_1 = \dots = x_d = 0\}$, then we can verify that $\delta_{Y|X}$ is the modulo class [1] (up to a non-vanishing constant multiple) of the left \mathcal{D}_X -Module

$$\mathcal{D}_X / \left(\sum_{j=1}^d \mathcal{D}_X x_j + \sum_{j=d+1}^n \mathcal{D}_X \partial / \partial x_j \right).$$

Furthermore we find

$$(5.1) \quad \mathcal{E}_{Y|X}^\infty = \mathcal{E}_X^\infty \delta_{Y|X}.$$

In view of the relation (5.1), we define another sheaf $\mathcal{E}_{Y|X}$ by $\mathcal{E}_X \delta_{Y|X}$.

Now, using these sheaves $\mathcal{E}_{Y|X}^\infty$ and $\mathcal{E}_{Y|X}$, we define the required sheaves $\mathcal{E}_{Y \rightarrow X}^\infty$ etc. First let us identify Y with the graph Δ_f of f in $Y \times X$, and, accordingly $T_Y^*(Y \times X)$ with $Y \times_X T^*X$. Then $\mathcal{E}_{Y \rightarrow X}^\infty$ and $\mathcal{E}_X^\infty \leftarrow_Y$ are, by definition, sheaves on $Y \times_X T^*X$ given by $\mathcal{E}_{\Delta_f|Y \times X}^\infty \otimes_{\mathcal{O}_X} \Omega_X = \mathcal{E}_{Y|Y \times X}^\infty \otimes_{\mathcal{O}_X} \Omega_X$ and $\mathcal{E}_{\Delta_f|Y \times X}^\infty \otimes_{\mathcal{O}_Y} \Omega_Y = \mathcal{E}_{Y|Y \times X}^\infty \otimes_{\mathcal{O}_Y} \Omega_Y$, respectively. Note that $\mathcal{E}_X^\infty \xrightarrow{\text{id}}_X$ is nothing but \mathcal{E}_X^∞ . In exactly the same manner we define $\mathcal{E}_Y \rightarrow_X$ and $\mathcal{E}_X \leftarrow_Y$ by using $\mathcal{E}_{Y|Y \times X}$ in place of $\mathcal{E}_{Y|Y \times X}^\infty$. In what follows, we often abbreviate $\mathcal{E}_{Y \rightarrow X}^\infty$ etc. to $\mathcal{E}_{Y \rightarrow X}$ etc. We also use the notation $1_{Y \rightarrow X}$ or, symbolically, $\delta(x-f(y))dy$ to denote $(\delta_{\Delta_f|Y \times X})dy$.

Now, denoting by ρ_f and $\varpi_f^{(*)}$ the projections from $T_Y^*(Y \times X)$ to T^*Y and T^*X , respectively, we find that the sheaf $\mathcal{E}_{Y \rightarrow X}^\infty$ is $(\rho^{-1}\mathcal{E}_Y^\infty, \varpi^{-1}\mathcal{E}_X^\infty)$ -bi-Module, and that the sheaf $\mathcal{E}_X^\infty \leftarrow_Y$ is a $(\varpi^{-1}\mathcal{E}_X^\infty, \rho^{-1}\mathcal{E}_Y^\infty)$ -bi-Module. In particular, when Y is a submanifold of $X = \mathbb{C}^n$ defined by $\{x \in \mathbb{C}^n; x_1 = \dots = x_d = 0\}$, then $\mathcal{E}_{Y \rightarrow X}^\infty$ is isomorphic to $\mathcal{E}_X^\infty / (\sum_{j=1}^d x_j \mathcal{E}_X^\infty)$ as a right \mathcal{E}_X^∞ -Module, and $\mathcal{E}_X^\infty \leftarrow_Y$ is isomorphic to $\mathcal{E}_X^\infty / (\sum_{j=1}^d \mathcal{E}_X^\infty x_j)$ as a left \mathcal{E}_X^∞ -Module. These properties hold without any change if we replace $\mathcal{E}_{Y \rightarrow X}^\infty$ etc. by $\mathcal{E}_{Y \rightarrow X}$ etc. Using these notions, we formulate the non-characteristic condition as follows:

(5.2) Let \mathcal{M} be a coherent \mathcal{E}_X -Module defined on an open subset U of T^*X and let W be an open subset of T^*Y . If ρ_f restricted to $\varpi_f^{-1}(\text{Supp } \mathcal{M}) \cap \rho_f^{-1}(W)$ is a finite map, i.e., a proper map with finite fiber, then f is said to be *non-characteristic* with respect to \mathcal{M} (over W).

If f is non-characteristic with respect to \mathcal{M} over W , then

$$(5.3) \quad \mathcal{F}_{\rho_f^* \varpi_f^{-1} \mathcal{E}_X}(\mathcal{E}_{Y \rightarrow X}, \varpi^{-1} \mathcal{M}) = \mathcal{F}_{\rho_f^* \varpi_f^{-1} \mathcal{E}_X}(\mathcal{E}_{Y \rightarrow X}, \varpi^{-1}(\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M})) = 0 \quad (i \neq 0)$$

holds on $\rho^{-1}(W)$.

The surviving object $\rho_*(\mathcal{E}_{Y \rightarrow X} \otimes_{\varpi^{-1} \mathcal{E}_X} \varpi^{-1} \mathcal{M})$ is a coherent \mathcal{E}_Y -Module on W , and it is denoted by $f^* \mathcal{M}$. Furthermore $\mathcal{E}_Y^\infty \otimes f^* \mathcal{M}$ is isomorphic to $\rho_*(\mathcal{E}_{Y \rightarrow X}^\infty \otimes_{\varpi^{-1} \mathcal{E}_X^\infty} (\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}))$. When Y is a submanifold of X and f is the imbedding, we sometimes denote $f^* \mathcal{M}$ by \mathcal{M}_Y and call it a tangential system of \mathcal{M} (along Y). When \mathcal{M} is a coherent right \mathcal{E}_X -Module, $f^* \mathcal{M}$ is defined by

(*) If there is no fear of confusions, we omit the suffix f .

replacing $\mathcal{E}_{Y \rightarrow X} \otimes \varpi^{-1} \mathcal{M}$ etc. by $\varpi^{-1} \mathcal{M} \otimes \mathcal{E}_{X \leftarrow Y}$ etc.

In parallel with the above definition, we define $f^* \mathcal{N}$ for a coherent right \mathcal{E}_Y -Module \mathcal{N} by $\varpi_* (\rho^{-1} \mathcal{N} \otimes_{\rho^{-1} \mathcal{E}_Y} \mathcal{E}_{Y \rightarrow X})$. It is a coherent \mathcal{E}_X -Module on $U \subset T^*X$ if

(5.4) ϖ_f restricted to $\rho_f^{-1}(\text{Supp } \mathcal{N}) \cap \varpi_f^{-1}(U)$ is a finite map.

The condition (5.4) is a reasonable and convenient one outside the zero-section $T^*_X X$. It is, however, too restrictive at the zero-section in that the finiteness assumption is seldom satisfied there. The following result resolves this trouble:

(5.5) Let \mathcal{N} be a coherent \mathcal{D}_Y -Module. Suppose that f is projective and that there exists a coherent \mathcal{O}_Y -sub-Module \mathcal{N}_0 of \mathcal{N} which generates \mathcal{N} as a \mathcal{D}_Y -Module. Then

(5.5.a) $\int_{\text{def}}^i \mathcal{N} = R^i f_* (\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N})$ is a coherent \mathcal{D}_X -Module for every i ,

and

(5.5.b) $\text{Ch} \left(\int^i \mathcal{N} \right) \subset \varpi \rho^{-1}(\text{Ch } \mathcal{N})$.

In application, the conditions in (5.5) are usually easy to verify.

Using the notion of induced systems, we obtain the following

Theorem 5.1. *Let X be \mathbb{C}^n and let \mathcal{M} and \mathcal{N} be coherent \mathcal{E}_X -Modules whose characteristic varieties are contained in $\{(x, \xi) \in T^*X; \xi_1 = 0\}$. Then we have the following isomorphism for every j :*

(5.6) $\text{Ext}_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N})|_Y \simeq \text{Ext}_{\mathcal{E}_Y}^j(\mathcal{M}_Y, \mathcal{E}_Y^\infty \otimes_{\mathcal{E}_Y} \mathcal{N}_Y)$,

where $Y = \{x \in X; x_1 = 0\}$.

We want to emphasize that the use of microdifferential operators of infinite order is crucial in Theorem 5.1; similar results cannot be expected if only operators of finite order are used.

We note that the involutory character of $\text{Ch}(\mathcal{M})$ (see (3.6)) immediately follows from Theorem 5.1.

Now, the classical theory of Jacobi tells us that an involutory manifold can be microlocally brought to a simple canonical form by a contact transformation. With the aid of this classical result, Theorem 5.1 entails the following decisive

result on the structure of systems of microdifferential equations:

Theorem 5.2. *Let \mathcal{M} be a coherent \mathcal{E}_X -Module satisfying the following conditions:*

(5.7)
$$\mathcal{E}_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X) = 0 \quad \text{for } j \neq d.$$

(5.8) $\text{Ch}(\mathcal{M})$ is regular at p in $\text{Ch}(\mathcal{M})$, that is,

(5.8.a) $\text{Ch}(\mathcal{M})$ is non-singular near p

and

(5.8.b) $(\omega|_{\text{Ch}(\mathcal{M})})(p) \neq 0$ for the canonical 1-form ω .

Then, through a quantized contact transformation (φ, Φ) , $\mathcal{E}_X^\infty \otimes \mathcal{M}$ is isomorphic to a direct summand of a direct sum of finite copies of partial de Rham system $\mathcal{M}_0 = \mathcal{E}_{\mathbb{C}^n}^\infty / \left(\sum_{j=1}^d \mathcal{E}_{\mathbb{C}^n}^\infty \frac{\partial}{\partial x_j} \right)$ with $\varphi(p) = (0; 0, \dots, 0, 1) \in T^*\mathbb{C}^n$. In particular, \mathcal{M} is isomorphic to $\Phi(\mathcal{M}_0)^m$ at generic points of $\text{Ch}(\mathcal{M})$, where m is a non-negative integer.

Note that (5.8.b) implies $d < \dim X$, thus eliminating the possibility that $\text{Ch}(\mathcal{M})$ is Lagrangian. This is one of the reasons why the study of holonomic systems requires special attention.

We end this section by presenting the rule how the order behaves when we apply the functors f^* and f_* .

Theorem 5.3. *Let $f: Y \rightarrow X$ be a holomorphic map. Let \mathcal{M} be a simple holonomic system with characteristic variety $\Lambda \subset \dot{T}^*X$ and let u be a non-degenerate section of \mathcal{M} . Assume that ρ is transversal to Λ and $\varpi|_{\rho^{-1}(\Lambda)}$ is an imbedding. Then $f^*\mathcal{M}$ is a simple holonomic \mathcal{E}_Y -Module, and $1_{Y \rightarrow X} \otimes u$ is a non-degenerate section of $f^*\mathcal{M}$, and its order is equal to the order of u .*

Theorem 5.4. *Let $f: Y \rightarrow X$ be a holomorphic map. Let \mathcal{N} be a right holonomic \mathcal{E}_Y -Module with characteristic variety $\Lambda \subset \dot{T}^*Y$, and let u be a non-degenerate section of \mathcal{N} . Assume that ϖ is transversal to Λ and that $\rho|_{\varpi^{-1}(\Lambda)}$ is an imbedding. Then $f_*\mathcal{N}$ is a simple holonomic \mathcal{E}_X -Module, and $(dx)^{-1} \otimes u \otimes 1_{Y \rightarrow X}$ is a non-degenerate section of $f_*\mathcal{N}$. Its order is $\text{ord}((dy)^{-1} \otimes u) - \frac{1}{2}(\dim Y - \dim X)$.^(*)*

^(*) Here dx and dy denote the nowhere vanishing section of $\Omega_{X/\text{def}}^{\dim X} Q_X^{\dim X}$ and $\Omega_{Y/\text{def}}^{\dim Y} Q_Y^{\dim Y}$, respectively. The factors $(dy)^{-1}$ and $(dx)^{-1}$ make $(dy)^{-1} \otimes u$ etc. a section of a left-Module, anti-isomorphic to the corresponding right-Module.

§6. Regular Holonomic Systems

As indicated by the canonical form (4.7), a simple holonomic system considered at a generic point of its characteristic variety is, essentially speaking, an ordinary differential equation with regular singularities. In view of the fact that linear ordinary differential equations with regular singularities was one of the central subjects in classical analysis, establishing a solid basis of the theory of holonomic systems with regular singularities is crucially important for making use of holonomic systems in application. This is done most neatly in the framework of microlocal analysis. We refer the reader to [17] for the details of the subject given here. Although we have used the terminology “holonomic systems with R.S.” in [17], we use in this article another terminology “regular holonomic systems (or \mathcal{E}_X -Modules or \mathcal{D}_X -Modules, if we need to specify)”. In order to define the notion of regular holonomic systems, we first introduce the notion of a system with regular singularities along an involutory subvariety V of \hat{T}^*X . We do not assume that V is non-singular, but we assume that V is homogeneous with respect to the fiber coordinate of T^*X . Let I_V denote the sheaf of holomorphic functions on \hat{T}^*X which vanish on V , and $I_V(m)$ denote $I_V \cap \mathcal{O}_{\hat{T}^*X}(m)$. Defining \mathcal{S}_V by $\{P \in \mathcal{E}_X(1); \sigma_1(P) \in I_V(1)\}$, we define \mathcal{E}_V by the sub-Algebra of \mathcal{E}_X generated by \mathcal{S}_V . We denote by $\mathcal{E}_V(m)$ the sheaf $\mathcal{E}_V \mathcal{E}_X(m)$ ($= \mathcal{E}_X(m) \mathcal{E}_V$). It is then easy to verify that \mathcal{E}_V is Nötherian (in the sense of (3.1)). Using the sheaf \mathcal{E}_V , we introduce the following

Definition 6.1. ([19], [17]). Let V be an involutory subvariety of \hat{T}^*X and let \mathcal{M} be a coherent \mathcal{E}_X -Module defined on an open subset Ω of \hat{T}^*X . Then \mathcal{M} is said to be with regular singularities along V if \mathcal{M} satisfies one of the following three mutually equivalent conditions:

- (6.1) For each point p in Ω , there exist an open neighborhood U of p and an \mathcal{E}_V -sub-Module \mathcal{M}_0 of \mathcal{M} which is defined on U so that the following two conditions are satisfied:
- (6.1.a) \mathcal{M}_0 is $\mathcal{E}(0)$ -coherent,
 - (6.1.b) $\mathcal{M} = \mathcal{E}_X \mathcal{M}_0$ holds on U .
- (6.2) For every open subset U of Ω and every coherent $\mathcal{E}(0)$ -sub-Module \mathcal{L} of \mathcal{M} that is defined on U , $\mathcal{E}_V \mathcal{L}$ is $\mathcal{E}(0)$ -coherent.
- (6.3) For every open subset U of Ω and every coherent \mathcal{E}_V -sub-Module \mathcal{N} of \mathcal{M} that is defined on U , \mathcal{N} is $\mathcal{E}(0)$ -coherent.

Definition 6.2. (i) Let \mathcal{M} be a holonomic \mathcal{E}_X -Module defined on an open subset Ω of \hat{T}^*X . Then \mathcal{M} is said to be a regular holonomic \mathcal{E}_X -Module, or, for short, a regular holonomic system, if the following condition is satisfied:

(6.4) \mathcal{M} is with regular singularities along $\text{Ch}(\mathcal{M})$.

(ii) For a holonomic \mathcal{D}_X -Module \mathcal{M} , we call \mathcal{M} a regular holonomic system if $\mathcal{E}_X \otimes \mathcal{M}$ is a regular holonomic \mathcal{E}_X -Module on \hat{T}^*X .

Note that [17] starts with the definition that \mathcal{M} is a regular holonomic system if it satisfies the following condition:

(6.5) There exists an open subset Ω' of Ω such that $\Omega' \cap \text{Ch}(\mathcal{M})$ is dense in $\text{Ch}(\mathcal{M})$ and that \mathcal{M} is with regular singularities along $\text{Ch}(\mathcal{M})$ in Ω' ,

and later proves that it satisfies the stronger condition (6.4). Although the reasoning of [17] is highly transcendental, we can prove the equivalence of (6.4) and (6.5) in a more algebraic way by using the argument in [7]. Hence, here we have presented the definition in a simpler form as above.

We know ([17], Chap. V) that $f^*\mathcal{M}$ and $f_*\mathcal{M}$ are regular holonomic systems, if so is \mathcal{M} and if f satisfies condition (5.2) and (5.4), respectively.

Also, in parallel with (5.5), $\int^i \mathcal{N}$ is a regular holonomic \mathcal{D}_X -Module for every i , if $f: Y \rightarrow X$ is a projective map and \mathcal{N} is a regular holonomic \mathcal{D}_Y -Module. (In this case the existence of \mathcal{O}_Y -sub-Module \mathcal{N}_0 follows from the assumption that \mathcal{N} is a regular holonomic system.)

Thus the notion of regular holonomic systems enjoys nice functorial properties. The following comparison theorem guarantees that this notion is exactly what we want to have as the generalization to the higher dimension of the notion of linear ordinary differential equations with regular singularities:

Theorem 6.3. ([17], Chap. VI.) *Let \mathcal{M} and \mathcal{N} be regular holonomic \mathcal{E}_X -Modules. Then we have*

$$\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}) = \mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X^{\infty} \otimes_{\mathcal{E}_X} \mathcal{N})$$

and

$$\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}) = \mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \hat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{N}),$$

where $\hat{\mathcal{E}}_X = \varprojlim_k \mathcal{E}_X / \mathcal{E}_X(k)$.

The relation of the notion of regular holonomic systems to the classical theory of ordinary differential equations with regular singularities may become

clearer by the following

Theorem 6.4. ([17], Chap. VI.) *Let \mathcal{M} be a holonomic \mathcal{D}_X -Module. Then the following two conditions are equivalent:*

(6.6) *\mathcal{M} is a regular holonomic system.*

$$(6.7) \quad \text{Ext}_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_{X,x}) \simeq \text{Ext}_{\mathcal{D}_X}^j(\mathcal{M}, \hat{\mathcal{O}}_{X,x})$$

holds for every j and for every x in X . Here $\hat{\mathcal{O}}_{X,x} = \varprojlim_k \mathcal{O}_{X,x}/\mathfrak{m}^k$, where \mathfrak{m} denotes the maximal ideal of the ring $\mathcal{O}_{X,x}$ of germs of holomorphic functions at x .

Now, an important result proved in [17] is Theorem 6.6 to be stated below. It may be regarded as a counterpart of Theorem 5.2 for holonomic systems. In order to state the theorem, we first introduce the following

Definition 6.5. Let \mathcal{M} be a holonomic \mathcal{E}_X -Module defined on an open subset Ω of T^*X . We then define the (pre-)sheaf \mathcal{M}_{reg} by assigning $\mathcal{M}_{reg}(U)$ to each open subset U of Ω as follows:

$\mathcal{M}_{reg}(U) = \{s \in (\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M})(U); \text{ for each point } p \text{ in } U, \text{ there exist a neighborhood } W \text{ of } p \text{ and a coherent Ideal } \mathcal{I} \text{ of } \mathcal{E}_X \text{ defined on } W \text{ so that } \mathcal{I}s = 0 \text{ holds and that } \mathcal{E}_X/\mathcal{I} \text{ is a regular holonomic system}\}$.

A priori, it is not clear whether \mathcal{M}_{reg} is a regular holonomic system or not — actually, even the coherency of \mathcal{M}_{reg} is far from obvious. However, we can show

Theorem 6.6. ([17], Chap. V, Theorem 5.2.1.) *Let \mathcal{M} be a holonomic \mathcal{E}_X -Module defined on a neighborhood of p_0 in T^*X . Then \mathcal{M}_{reg} is a regular holonomic \mathcal{E}_X -Module, and*

$$\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M} = \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_{reg}$$

holds on a neighborhood of p_0 .

Furthermore, if there exists a regular holonomic \mathcal{E}_X -sub-Module \mathcal{N} of $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}$ such that $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M} = \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N}$ holds, then \mathcal{N} is isomorphic to \mathcal{M}_{reg} as an \mathcal{E}_X -Module.

Remark 6.7. If $\text{Ch}(\mathcal{M})$ is non-singular near p_0 , we can describe \mathcal{M}_{reg} concretely as follows:

In this case, by using a quantized contact transformation, we may assume from the first

$$(6.8) \quad \text{Ch}(\mathcal{M}) = \{(x, \xi) \in T^*\mathbf{C}^n; x_1 = 0, \xi_2 = \dots = \xi_n = 0, |x| < \varepsilon, \xi_1 \neq 0\},$$

$$(6.9) \quad p_0 = (0; (1, 0, \dots, 0)) \in T^*\mathbf{C}^n.$$

Then, $\mathcal{M}_{reg} \cong \bigoplus_{\text{finite sum}} \mathcal{M}_{\lambda, m}$ holds near p_0 , where $\mathcal{M}_{\lambda, m}$ ($\lambda \in \mathbf{C}, m \in \{1, 2, \dots\}$) denotes

$$\mathcal{E}_{\mathbf{C}^n} \left/ \left(\mathcal{E}_{\mathbf{C}^n} \left(x_1 \frac{\partial}{\partial x_1} - \lambda \right)^m + \mathcal{E}_{\mathbf{C}^n} \frac{\partial}{\partial x_2} + \dots + \mathcal{E}_{\mathbf{C}^n} \frac{\partial}{\partial x_n} \right) \right.$$

Now, let us restrict our consideration to holonomic \mathcal{D}_X -Modules so that we may consider their \mathcal{O}_X -solutions. Then, through the correspondence between systems and their solution sheaves, we can find an interesting interrelation between an analytic object — (regular) holonomic \mathcal{D}_X -Modules — and a geometric object — constructible sheaves. Let us first recall the definition of a constructible sheaf.

Definition 6.8. A sheaf \mathcal{F} of \mathbf{C} -vector spaces defined on a complex manifold X is called constructible^(*), if it satisfies the following condition (6.10).

(6.10) There exists a decreasing family $\{X_j\}_{j=0,1,2,\dots}$ of closed analytic subsets of X which satisfy the following two conditions:

$$(6.10.a) \quad X = X_0, \bigcap_j X_j = \emptyset.$$

$$(6.10.b) \quad \mathcal{F}|_{X_j - X_{j+1}} \text{ is a locally constant sheaf of finite rank, i.e.,} \\ \mathcal{F}|_{X_j - X_{j+1}} \text{ is locally isomorphic to } \mathbf{C}_{X_j - X_{j+1}}^r \text{ (} r < \infty \text{).}$$

An important result on the structure of solution sheaves of a holonomic \mathcal{D}_X -Module \mathcal{M} is that $\mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_X)$ is a constructible sheaf for every j ([13]). Further we know ([17], Chap. I. § 4) that

$$\mathbf{R}\mathcal{H}om_{\mathbf{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\bullet}, \mathcal{O}_X), \mathcal{O}_X) = \mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} \mathcal{M}^{\bullet}$$

holds for every bounded complex \mathcal{M}^{\bullet} of \mathcal{D}_X -Modules such that $\mathcal{H}^j(\mathcal{M}^{\bullet})$ is holonomic. Thus we can assign constructible sheaves to a holonomic \mathcal{D}_X -Module by considering its solution sheaves, and, at the same time, reconstruct the system from its solution sheaves, if we employ differential operators of infinite order. In order to formulate these correspondences in a more precise way, we introduce several categories. In what follows, we call a \mathcal{D}_X^{∞} -Module \mathcal{N} a

^(*) The terminology “finitistic” is used in [13].

holonomic \mathcal{D}_X^∞ -Module if there locally exists a holonomic \mathcal{D}_X -Module \mathcal{M} such that $\mathcal{N} = \mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M}$ holds. We denote by $\text{Mod}(\mathcal{D}_X)$ (resp., $\text{Mod}(\mathcal{D}_X^\infty)$, $\text{Mod}(X)$) the Abelian category of \mathcal{D}_X -Modules (resp., \mathcal{D}_X^∞ -Modules, the sheaves of \mathbf{C} -vector spaces on X). We also denote by $\text{D}(\mathcal{D}_X)$ (resp., $\text{D}(\mathcal{D}_X^\infty)$, $\text{D}(X)$) the derived category of $\text{Mod}(\mathcal{D}_X)$ (resp., $\text{Mod}(\mathcal{D}_X^\infty)$, $\text{Mod}(X)$). Now the categories in which we are interested are introduced by the following

Definition 6.9. (i) $\text{D}_{rh}^b(\mathcal{D}_X)$ is, by definition, the full subcategory of $\text{D}(\mathcal{D}_X)$ which consists of \mathcal{M}^\bullet in $\text{Ob}(\text{D}(\mathcal{D}_X))$ such that $\mathcal{H}^j(\mathcal{M}^\bullet) = 0$ holds except for finitely many j 's and that $\mathcal{H}^j(\mathcal{M}^\bullet)$ is a regular holonomic \mathcal{D}_X -Module for every j .

(ii) $\text{D}_h^b(\mathcal{D}_X^\infty)$ is, by definition, the full subcategory of $\text{D}(\mathcal{D}_X^\infty)$ which consists of \mathcal{N}^\bullet in $\text{Ob}(\text{D}(\mathcal{D}_X^\infty))$ such that $\mathcal{H}^j(\mathcal{N}^\bullet) = 0$ holds except for finitely many j 's and that $\mathcal{H}^j(\mathcal{N}^\bullet)$ is a holonomic \mathcal{D}_X^∞ -Module for every j .

(iii) $\text{D}_c^b(X)$ is, by definition, the full subcategory of $\text{D}(X)$ which consists of \mathcal{F}^\bullet in $\text{Ob}(\text{D}(X))$ such that $\mathcal{H}^j(\mathcal{F}^\bullet)$ is zero except for finitely many j 's and that $\mathcal{H}^j(\mathcal{F}^\bullet)$ is a constructible sheaf for every j .

Then we have

Theorem 6.10. ([19], [22].) *The three categories $\text{D}_{rh}^b(\mathcal{D}_X)$, $\text{D}_h^b(\mathcal{D}_X^\infty)$ and $\text{D}_c^b(X)$ are mutually isomorphic.*

In application, it is an important question how to characterize an object in $\text{D}_c^b(X)$ when the corresponding object \mathcal{M}^\bullet in $\text{D}_{rh}^b(\mathcal{D}_X)$ is a single complex. The answer is given by using $\mathcal{F}^\bullet =_{\text{def}} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}^\bullet)$ as follows.

Theorem 6.11. (i) *The following two statements are equivalent:*

(6.11)
$$\mathcal{H}^j(\mathcal{M}^\bullet) = 0 \quad (j > 0)$$

(6.12) *For every integer k , the codimension of each irreducible component of $\text{Supp } \mathcal{H}^k(\mathcal{F}^\bullet)$ is equal to or greater than k .*

(ii) *The following two statements are equivalent:*

(6.13)
$$\mathcal{H}^j(\mathcal{M}^\bullet) = 0 \quad (j < 0)$$

(6.14) *For every integer k , the codimension of each irreducible component of $\text{Supp } \mathcal{E}xt_{\mathbf{C}}^k(\mathcal{F}^\bullet, \mathbf{C}_X)$ is equal to or greater than k .*

In what follows, we say that \mathcal{F}^\bullet in $\text{Ob}(\text{D}_c^b(X))$ is perverse if it satisfies conditions (6.12) and (6.14). Theorem 6.11 guarantees that the corresponding

object \mathcal{M} in $D_{rh}^b(\mathcal{D}_X)$ is then a single complex.

§7. Local Monodromies and the Asymptotic Expansion of Solutions of Regular Holonomic Systems

One of the reasons why ordinary differential equations with regular singularities are important lies in the fact that its local monodromy can be calculated in an algebraic manner. In this section we discuss how this important property of ordinary differential equations with regular singularities can be generalized to regular holonomic systems. Our method is based on the asymptotic expansion in our sense (Theorem 7.2 below. See also [16].) of the solutions of equations of a special type (7.4) given below. One important fact is that, for any section u of any regular holonomic system, we can find an equation of that type which u satisfies. (Theorem 7.3.)

Let us begin our discussion by recalling how the local monodromy is calculated for ordinary differential equations with regular singularities. In order to fix the notations, let us consider the following equation (7.1), which has the origin as its regular singular point.

$$(7.1) \quad \left(\left(t \frac{\partial}{\partial t} \right)^m + a_1(t) \left(t \frac{\partial}{\partial t} \right)^{m-1} + \dots + a_m(t) \right) u(t) = 0.$$

Let $\{\lambda_1, \dots, \lambda_m\}$ be the set of roots of its indicial equation:

$$(7.2) \quad \lambda^m + a_1(0)\lambda^{m-1} + \dots + a_m(0) = 0.$$

For the sake of simplicity, we suppose that $\lambda_j - \lambda_k$ is not an integer if $j \neq k$. Then we know that there exist solutions of (7.1) of the form $t^{\lambda_j} \times$ (holomorphic function) ($j=1, \dots, m$), and hence the local monodromy is the diagonal matrix with $\exp(2\pi\sqrt{-1}\lambda_j)$ as its eigenvalues.

In order to discuss how this fact can be generalized in the higher dimensional case, we first introduce some notations.

Let X be a complex manifold and Y a submanifold of X . We denote by I_Y the defining \mathcal{O}_X -Ideal of Y . Let $F^k(\mathcal{D})$ denote the subsheaf of \mathcal{D}_X given by

$$(7.3) \quad \{P \in \mathcal{D}_X; P I_Y^j \subset I_Y^{j+k} \mathcal{D}_X \text{ holds for every } j\}.$$

Here and in what follows, I_Y^j is, by definition, \mathcal{O}_X for $j \leq 0$.

Proposition 7.1. *$F^k(\mathcal{D})/F^{k+1}(\mathcal{D})$ is isomorphic to the sheaf of homogeneous differential operators on $T_Y X$ of degree k . In particular,*

$\text{gr}_F \cdot (\mathcal{D}) \stackrel{\text{def}}{=} \bigoplus_k F^k(\mathcal{D})/F^{k+1}(\mathcal{D})$ is the sub-Ring of $\mathcal{D}_{\Gamma Y X}$.

Let θ be a vector field tangent to Y such that the canonical action of θ on I_Y/I_Y^2 equals the identity. Thus vector field θ is unique modulo $F^2(\mathcal{D})$. Note that, if we take a local coordinate system (x_1, \dots, x_n) of X such that Y is given by $x_1 = \dots = x_d = 0$, then we can take $\sum_{j \leq d} x_j \frac{\partial}{\partial x_j}$ as θ .

Letting $b(\lambda)$ be a polynomial of degree m and P a section of $F^1(\mathcal{D}) \cap \mathcal{D}_X(m)$, we shall consider the following equation

$$(7.4) \quad b(\theta)u = Pu.$$

Factorizing $b(\lambda)$ as

$$(7.5) \quad b(\lambda) = \prod_{j=1}^N (\lambda - \lambda_j)^{n_j},$$

we suppose

$$(7.6) \quad \lambda_j - \lambda_k \notin \mathbf{Z} \quad \text{for } j \neq k.$$

Then the local behavior of solutions of (7.4) is described by the following theorem.

Theorem 7.2. *Let X be an open subset of \mathbf{C}^n and suppose that Y is given by $\{(x_1, \dots, x_n) \in X; x_1 = \dots = x_d = 0\}$. Suppose further that (7.6) holds. Then there exist $A_{j,v}$ in $F^v(\mathcal{D})$ ($j = 1, \dots, N$ and $v = 0, 1, 2, \dots$) which have the following properties (7.7), (7.8) and (7.9).*

$$(7.7) \quad A_{j,v} = 1 \quad \text{for } v = 0.$$

$$(7.8) \quad [\theta, A_{j,v}] = vA_{j,v} \quad \text{for } j = 1, \dots, N \quad \text{and } v = 0, 1, 2, \dots.$$

(7.9) *Let U be a neighborhood of a point p of Y and let Γ and Γ' be open cones in \mathbf{C}^d with their apices at the origin. Assume that Γ contains $\overline{\Gamma'} - \{0\}$. Then we find the following:*

- (a) *For any holomorphic solution u of (7.4) defined on $(\Gamma \times \mathbf{C}^{n-d}) \cap U$, there exists an open neighborhood U' of p such that u can be expanded on $(\Gamma' \times \mathbf{C}^{n-d}) \cap U'$ to a convergent series $\sum_{j=1}^N \sum_{v=0}^{\infty} u_{j,v}(x)$, where $u_{j,v}$ are holomorphic functions defined on $(\Gamma \times \mathbf{C}^{n-d}) \cap U'$ satisfying $(\theta - \lambda_j - v)^{n_j} u_{j,v} = 0$ and $u_{j,v} = A_{j,v} u_{j,0}$.*
- (b) *Conversely, if $u_{j,0}$ ($j = 1, \dots, N$) are holomorphic functions defined on $(\Gamma \times \mathbf{C}^{n-d}) \times U$ with $(\theta - \lambda_j)^{n_j} u_{j,0} = 0$, then $\sum_{j=1}^N \sum_{v=0}^{\infty} A_{j,v} u_{j,0}$ converges on $(\Gamma' \times \mathbf{C}^{n-d}) \cap U'$ for some U' and it gives a holomorphic solution of (7.4) there.*

Now, using this theorem as an analytic tool, we shall study the local structure of solutions of a regular holonomic \mathcal{D}_X -Module.

Let \mathcal{M} be a coherent \mathcal{D}_X -Module. A filtration $F^\cdot(\mathcal{M})$ of \mathcal{M} is called a good filtration with respect to $F^\cdot(\mathcal{D})$ if there locally exist sections u_j of \mathcal{M} and integers r_j ($j=1, \dots, m$) such that

$$F^k(\mathcal{M}) = \sum_{j=1}^m F^{k-r_j}(\mathcal{D})u_j$$

holds for every k .

Let us denote by \mathcal{R} the category of coherent \mathcal{D}_X -Modules \mathcal{M} such that there locally exist a coherent sub- \mathcal{O}_X -Module \mathcal{F} of \mathcal{M} and a polynomial $b(\lambda)$ of degree m which satisfy

$$(7.10) \quad b(\theta)\mathcal{F} \subset (\mathcal{D}(m) \cap F^1(\mathcal{D}))\mathcal{F}$$

and

$$(7.11) \quad \mathcal{M} = \mathcal{D}_X\mathcal{F}.$$

Then \mathcal{R} is a full abelian subcategory of the category of coherent \mathcal{D}_X -Modules. The importance of the category \mathcal{R} lies in the fact that it is ample enough for our purpose (Theorem 7.3 below) while any object of \mathcal{R} is amenable to the analysis based on the asymptotic expansion in the sense of Theorem 7.2.

Theorem 7.3. *Any regular holonomic \mathcal{D}_X -Module belongs to \mathcal{R} .*

Theorem 7.4. *Let \mathcal{M} be an object of \mathcal{R} . Then we have the following:*

(i) *There exist a good filtration F^\cdot of \mathcal{M} and a non-zero polynomial $b(\lambda)$ which have the following two properties:*

$$(7.12) \quad b(\theta - k)F^k \subset F^{k+1}.$$

(7.13) *The difference of any pair of distinct roots of $b(\lambda)=0$ is not an integer.*

(ii) *A (not graded) $\text{gr}_{F^\cdot}(\mathcal{D})$ -Module $\text{gr}_{F^\cdot}\mathcal{M} \stackrel{\text{def}}{=} \bigoplus_k F^k/F^{k+1}$ does not depend on the choice of a good filtration F^\cdot which satisfies the condition (7.12) for some non-zero polynomial $b(\lambda)$ satisfying the condition (7.13).*

Thanks to (ii) of Theorem 7.4, we may use the notation $\text{gr } \mathcal{M}$ instead of $\text{gr}_{F^\cdot}\mathcal{M}$. The local structure of multi-valued holomorphic solutions of \mathcal{M} is most neatly described by the aid of $\text{gr } \mathcal{M}$. To present the results (Theorem 7.5 and Theorem 7.7 given below), we first introduce the real blowing up $\widetilde{Y}X$ of X with center at Y , that is, $\widetilde{Y}X = X \sqcup T_Y X$ as a set with the suitable topology.

Let $j: X \hookrightarrow \widetilde{Y}X$ be the open embedding and define the sheaf $\tilde{\mathcal{O}}$ by $j_*(\mathcal{O}_X)|_{T_Y X}$. Then our result is stated as follows:

Theorem 7.5. *If a \mathcal{D}_X -Module \mathcal{M} belongs to \mathcal{R} , then*

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{O}}) = \mathbf{R}\mathcal{H}om_{\mathbf{gr}_{F^*}(\mathcal{D})}(\mathbf{gr} \mathcal{M}, \mathcal{O}_{T_Y X})$$

holds.

Remark 7.6. For a complex \mathcal{F}^\bullet of sheaves on X , let us define $\nu_Y(\mathcal{F}^\bullet)$ and $\mu_Y(\mathcal{F}^\bullet)$ by $(\mathbf{R}j_*\mathcal{F}^\bullet)|_{T_Y X}$ and $\mathbf{R}\Gamma_{T_Y^* X}(\pi_Y^{-1} \mathcal{F}^\bullet)^a$, respectively. Here $\pi_{Y/X}$ denotes the projection from the comonoidal transform $\widetilde{Y}X^*$ of X to X and a denotes the antipodal map. Then $\nu_Y(\mathcal{F}^\bullet)$ and $\mu_Y(\mathcal{F}^\bullet)$ are related in the following manner:

Let Z be the closed subset $\{(v, \xi) \in T_Y X \times T_Y^* X, \langle v, \xi \rangle \geq 0\}$ of $T_Y X \times T_Y^* X$, and let p and q denote the projections from $T_Y X \times T_Y^* X$ onto $T_Y X$ and $T_Y^* X$, respectively. Then we have

$$(7.14) \quad \mu_Y(\mathcal{F}^\bullet) = \mathbf{R}q_* \mathbf{R}\Gamma_Z(p^{-1} \nu_Y(\mathcal{F}^\bullet))$$

and

$$(7.15) \quad \nu_Y(\mathcal{F}^\bullet) = (\mathbf{R}p_* \mathbf{R}\Gamma_Z(q^{-1} \mu_Y(\mathcal{F}^\bullet)))^a [\text{codim}_{\mathbf{R}} Y].$$

Now, let $\tilde{\theta}$ denote the endomorphism of $\mathbf{gr} \mathcal{M}$ given by assigning $(\theta - k)u$ ($u \in F^k$) to u in F^k . Then $\tilde{\theta}$ is clearly $\mathbf{gr}_{F^*}(\mathcal{D})$ -linear. Furthermore we immediately see that $b(\tilde{\theta}) = 0$ holds for $b(\lambda)$ satisfying (7.12). Hence we can define a $\mathbf{gr}_{F^*}(\mathcal{D})$ -linear automorphism $\exp(2\pi\sqrt{-1}\tilde{\theta})$ on $\mathbf{gr} \mathcal{M}$. On the other hand, for any point v of $T_Y X$, $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{O}})$ is locally constant on $\mathbf{C}^\times v$, and hence we can define the monodromy of $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{O}})$. Then Theorem 7.5 entails the following

Theorem 7.7. *For \mathcal{M} in \mathcal{R} , the monodromy on $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{O}})$ is given by $\mathbf{R}\mathcal{H}om_{\mathbf{gr}_{F^*}(\mathcal{D})}(\exp(2\pi\sqrt{-1}\tilde{\theta}), \mathcal{O}_{T_Y X})$ on $\mathbf{R}\mathcal{H}om_{\mathbf{gr}_{F^*}(\mathcal{D})}(\mathbf{gr} \mathcal{M}, \mathcal{O}_{T_Y X})$.*

We now microlocalize the discussion given so far. (Cf. [16] and [23].)

Let Λ be a (non-singular) Lagrangian submanifold of T^*X and let \mathcal{E}_Λ denote the sub-Ring of \mathcal{E}_X introduced at the beginning of § 6. As was stated there, $\mathcal{E}_\Lambda(m)$ denotes the sheaf $\mathcal{E}_\Lambda \mathcal{E}_X(m) (= \mathcal{E}_X(m) \mathcal{E}_\Lambda)$. Let $\theta = : \theta_1(x, \xi) + \theta_0(x, \xi) + \dots :$ be a section of $\mathcal{E}_X(1)$ which satisfies the following three conditions:

$$(7.16) \quad \theta_1 \equiv 0 \text{ mod } I_\Lambda,$$

$$(7.17) \quad d\theta_1 \equiv -\omega \text{ mod } I_\Lambda \Omega_{T^*X}^1,$$

$$(7.18) \quad \theta_0 \equiv \frac{1}{2} \sum_j \frac{\partial^2 \theta_1}{\partial x_j \partial \xi_j} \text{ mod } I_A.$$

Here I_A denotes the defining Ideal of A and ω denotes the canonical 1-form $\sum_j \xi_j dx_j$. Such an operator θ exists locally and it is unique modulo $\mathcal{E}_A(-1)$.

Now we have the following

Theorem 7.8. *Let \mathcal{M} be a regular holonomic \mathcal{E}_X -Module defined on an open subset of \hat{T}^*X . Then there exist a coherent sub- \mathcal{E}_A -Module \mathcal{M}_0 of \mathcal{M} and a non-zero polynomial $b(\lambda)$ which satisfy the following three conditions:*

$$(7.19) \quad \mathcal{M} = \mathcal{E}_X \mathcal{M}_0.$$

$$(7.20) \quad b(\theta) \mathcal{M}_0 \subset \mathcal{M}_0(-1). \text{ Here and in what follows, } \mathcal{M}_0(k) \text{ denotes } \mathcal{E}_X(k) \cdot \mathcal{M}_0 \text{ for an integer } k.$$

$$(7.21) \quad \text{The difference of any pair of distinct roots of } b(\lambda)=0 \text{ is not an integer. Furthermore, if we denote } \bigoplus_k \mathcal{E}_A(k) / \mathcal{E}_A(k-1) \text{ by } \text{gr}_A \mathcal{E} \text{ and define a } \text{gr}_A(\mathcal{E})\text{-Module } \text{gr}_A \mathcal{M} \text{ by } \bigoplus_k \mathcal{M}_0(k) / \mathcal{M}_0(k-1), \text{gr}_A(\mathcal{M}) \text{ is independent of the choice of } \mathcal{M}_0 \text{ satisfying the above conditions.}$$

In order to microlocalize Theorem 7.5, we fix a simple holonomic \mathcal{E}_X -Module \mathcal{L} whose support is A . Let $\text{gr}^k \mathcal{O}_A$ denote the sheaf of homogeneous functions on A of degree k . Then $\text{gr}_A \mathcal{L}$ is an invertible $\text{gr } \mathcal{O}_A$ -Module. Using this system \mathcal{L} instead of $\tilde{\mathcal{O}}$, we obtain the following theorem.

Theorem 7.9. *Let \mathcal{M} be a regular holonomic \mathcal{E}_X -Module defined on an open subset of \hat{T}^*X . Then we have*

$$(7.22) \quad \mathbf{R} \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) = \mathbf{R} \mathcal{H}om_{\text{gr}_A \mathcal{E}}(\text{gr}_A \mathcal{M}, \text{gr}_A \mathcal{L})$$

and

$$(7.23) \quad \mathbf{R} \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}_X} \mathcal{L}) = \mathbf{R} \mathcal{H}om_{\text{gr}_A \mathcal{E}}(\text{gr}_A \mathcal{M}, \mathcal{O}_A \otimes_{\text{gr} \mathcal{O}_A} \text{gr}_A \mathcal{L}).$$

Remark 7.10. $\text{gr}_A^k \mathcal{E}$ is isomorphic to the sheaf of differential operator endomorphisms of $\text{gr}_A \mathcal{L}$ which are homogeneous of degree k . Hence, if we denote by $\text{gr}^k \mathcal{D}_A$ the sheaf of homogeneous differential operators on A of degree k and if we set $\text{gr } \mathcal{D}_A = \bigoplus_k \text{gr}_k \mathcal{D}_A$, $\text{gr}_A \mathcal{E}$ is isomorphic to $\text{gr}_A \mathcal{L} \otimes_{\text{gr} \mathcal{O}_A} \text{gr } \mathcal{D}_A \otimes_{\text{gr} \mathcal{O}_A} (\text{gr}_A \mathcal{L})^{\otimes -1}$. Hence $\mathcal{F} = (\text{gr}_A \mathcal{L})^{\otimes -1} \otimes_{\text{gr} \mathcal{O}_A} \text{gr}_A \mathcal{M}$ is a \mathcal{D}_A -Module. Further, it is a regular holonomic \mathcal{D}_A -Module. Using this \mathcal{D}_A -Module \mathcal{F} , we can rewrite (7.23) to the following form:

$$(7.24) \quad \mathbf{R} \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}^{\mathbf{R}}) = \mathbf{R} \mathcal{H}om_{\mathcal{D}_A}(\mathcal{F}, \mathcal{O}_A).$$

Here and in what follows, $\mathcal{L}^{\mathbf{R}}$ denotes $\mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}_X} \mathcal{L}$, as usual.

As a special case of Theorem 7.9, we find the following interesting result (Theorem 7.11 below) for a pair of regular holonomic \mathcal{E}_X -Modules \mathcal{M} and \mathcal{N} defined on an open subset of T^*X . In order to obtain the result we choose $T_X^*(X \times X)$ as Λ and identify it with T^*X by the first projection. Then $\Omega_X \otimes_{\mathcal{O}_X} \text{gr}_{\Lambda}(\mathcal{E}_{X \times X}) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ is canonically isomorphic to the sheaf \mathcal{D}_{T^*X} of differential operators on T^*X .

Let us now denote by $\Phi(\mathcal{M}, \mathcal{N})$ the \mathcal{D}_{T^*X} -Module defined by $(\mathcal{O}_{T^*X} \otimes_{\mathcal{E}_X} \Omega_X^{\otimes -1}) \otimes_{\text{gr}_{\Lambda}}(\mathcal{M} \hat{\otimes} \mathcal{N})$. (See S-K-K [24] p. 418 for the definition of the product system $\mathcal{M} \hat{\otimes} \mathcal{N}$.) Then, by choosing $\mathcal{E}_{X|X \times X}$ as \mathcal{L} in Theorem 7.9, we can deduce the following Theorem 7.11 from Theorem 7.9 and Remark 7.10.

Theorem 7.11. (i) $\Phi(\mathcal{M}, \mathcal{N})$ is a regular holonomic \mathcal{D}_{T^*X} -Module. Its characteristic variety is given by $C(\text{Supp } \mathcal{M}, (\text{Supp } \mathcal{N})^a)$, the normal cone of $\text{Supp } \mathcal{M}$ along $(\text{Supp } \mathcal{N})^a$, the antipodal set of $\text{Supp } \mathcal{N}$, if we identify the tangent bundle and the cotangent bundle of T^*X by using the symplectic structure of T^*X .

(ii) $\Phi(\mathcal{M}, \mathcal{N})$ is an exact functor with respect to the first and the second variable.

(iii) $\Phi(\mathcal{N}, \mathcal{M}) = \Phi(\mathcal{M}, \mathcal{N})^a$ and $\Phi(\mathcal{N}^*, \mathcal{M}^*) = \Phi(\mathcal{M}, \mathcal{N})^*$ hold. Here \mathcal{N}^* etc. denote the dual system of \mathcal{N} etc.

(iv) $\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N}, \mathcal{M}^{\mathbf{R}}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_{T^*X}}(\Phi(\mathcal{M}, \mathcal{N}^*), \mathcal{O}_{T^*X})[\dim X]$.

In particular, $\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N}, \mathcal{M}^{\mathbf{R}})[- \dim X]$ is a perverse complex on T^*X .

As an immediate consequence of (iii) and (iv) in the above, we obtain the following

Corollary 7.12.

$$\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}}) = \mathbf{R}\mathcal{H}om_{\mathcal{C}}(\mathbf{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N}, \mathcal{M}^{\mathbf{R}}), \mathcal{C}_{T^*X})[2 \dim X].$$

Remark 7.13. Here we have presented Theorem 7.11 restricting ourselves to the complement of the zero section of the cotangent bundle. However, we can obtain the same result also at the zero section, if we define $\Phi(\mathcal{M}, \mathcal{N})$ using $\text{gr}_F(\mathcal{M} \hat{\otimes} \mathcal{N})$.

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