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Solitons and Infinite Dimensional Lie Algebras

By

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Introduction

The theory of soliton equations has been one of the most active branches of mathematical physics in the past 15 years. It deals with a class of non-linear partial differential equations that admit abundant exact solutions. Recent works [1]–[21] shed light to their algebraic structure from a group theoretical view-point. In this paper we shall give a review on these developments, which were primarily carried out in the Research Institute for Mathematical Sciences.

In the new approach, the soliton equations are schematically described as follows. We consider an infinite dimensional Lie algebra and its representation on a function space. The group orbit of the highest weight vector is an infinite

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dimensional Grassmann manifold. Its defining equations on the function space, expressed in the form of differential equations, are then nothing other than the soliton equations. To put it the other way, there is a transitive action of an infinite dimensional group on the manifold of solutions. This picture has been first established by M. and Y. Sato [1], [2] in their study of the Kadomtsev-Petviashvili (KP) hierarchy.

Among the variety of soliton equations, the KP hierarchy is the most basic one in that the corresponding Lie algebra is $\mathfrak{gl}(\infty)$. The present article thus begins with a relatively detailed account for this case (§§1–2). Throughout the paper our description follows the line of the series [3]–[12] with emphasis on representation theoretical aspect. In this connection we refer also to [22]–[26]. The use of the language of free fermions as adopted in [3]–[12] and here was originally inspired by previous studies on Holonomic Quantum Fields [27]–[30]. We find it both natural and expedient, since by considering the representation of the total fermion algebra, of which $\mathfrak{gl}(\infty)$ forms a Lie subalgebra, Hirota's bilinear equations [31] and linear equations of Lax-Zakharov-Shabat come out in a unified way. In the following Section 3–8 we shall show how various types of soliton equations are generated by considering suitable subalgebras of $\mathfrak{gl}(\infty)$ and their representations. Included are the infinite dimensional orthogonal or symplectic Lie algebras (B_∞, C_∞, D_∞) and the Kac-Moody Lie algebras of Euclidean type. In Section 9 and Section 10 we treat two more typical examples of soliton equations, the 2 dimensional Toda lattice and the chiral field, showing further different aspects of our theory. In the appendix we gather lists of bilinear equations of lower degree for the hierarchies mentioned above.

There remain several topics that could not be touched upon in the text: among others, soliton equations related to free fermions on an elliptic curve [9] and the transformation theory for the self-dual Yang-Mills equation [14]–[17], [21]. For these the reader is referred to the original articles.

§1. Fock Representation of $\mathfrak{gl}(\infty)$

Let A be the *Clifford algebra* over $\boldsymbol{C}$ with generators ψ_i, ψ_i^* ($i \in \boldsymbol{Z}$), satisfying the defining relations†)

†) $[X, Y]_+ \underset{\text{def}}{=} XY + YX$.

$$[\psi_i, \psi_j]_+ = 0, \quad [\psi_i, \psi_j^*]_+ = \delta_{ij}, \quad [\psi_i^*, \psi_j^*]_+ = 0.$$

An element of $\mathscr{W} = (\bigoplus_{i \in \boldsymbol{Z}} \boldsymbol{C}\psi_i) \oplus (\bigoplus_{i \in \boldsymbol{Z}} \boldsymbol{C}\psi_i^*)$ will be referred to as a *free fermion*. The Clifford algebra $\boldsymbol{A}$ has a standard representation given as follows. Put $\mathscr{W}_{ann} = (\bigoplus_{i<0} \boldsymbol{C}\psi_i) \oplus (\bigoplus_{i \geqq 0} \boldsymbol{C}\psi_i^*)$, $\mathscr{W}_{cr} = (\bigoplus_{i \geqq 0} \boldsymbol{C}\psi_i) \oplus (\bigoplus_{i<0} \boldsymbol{C}\psi_i^*)$, and consider the left (resp. right) $\boldsymbol{A}$-module $\mathscr{F} = \boldsymbol{A}/\boldsymbol{A}\mathscr{W}_{ann}$ (resp. $\mathscr{F}^* = \mathscr{W}_{cr}\boldsymbol{A}\backslash\boldsymbol{A}$). These are cyclic $\boldsymbol{A}$-modules generated by the vectors $|\mathrm{vac}\rangle = 1 \bmod \boldsymbol{A}\mathscr{W}_{ann}$ or $\langle \mathrm{vac}| = 1 \bmod \cdot \mathscr{W}_{cr}\boldsymbol{A}$, respectively, with the properties

$$(1.1) \qquad \begin{aligned} &\psi_i |\mathrm{vac}\rangle = 0 \quad (i<0), \quad \psi_i^* |\mathrm{vac}\rangle = 0 \quad (i \geqq 0), \\ &\langle \mathrm{vac}| \psi_i = 0 \quad (i \geqq 0), \quad \langle \mathrm{vac}| \psi_i^* = 0 \quad (i<0). \end{aligned}$$

There is a symmetric bilinear form

$$\begin{aligned} \mathscr{F}^* \otimes_{\boldsymbol{A}} \mathscr{F} = \langle \mathrm{vac}| \boldsymbol{A} \otimes_{\boldsymbol{A}} \boldsymbol{A} |\mathrm{vac}\rangle &\longrightarrow \boldsymbol{C} \\ \langle \mathrm{vac}| a \otimes b |\mathrm{vac}\rangle &\longmapsto \langle \mathrm{vac}| a \cdot b |\mathrm{vac}\rangle = \langle ab \rangle \end{aligned}$$

through which $\mathscr{F}^*$ and $\mathscr{F}$ are dual vector spaces. Here $\langle\ \rangle$ denotes a linear form on $\boldsymbol{A}$, called the (*vacuum*) *expectation value*, defined as follows. For $a \in \boldsymbol{C}$ or quadratic in free fermions, set

$$\begin{aligned} &\langle 1 \rangle = 1, \\ &\langle \psi_i \psi_j \rangle = 0, \ \langle \psi_i^* \psi_j^* \rangle = 0, \\ &\langle \psi_i \psi_j^* \rangle = \begin{cases} \delta_{ij} & (i=j<0) \\ 0 & (\text{otherwise}), \end{cases} \qquad \langle \psi_j^* \psi_i \rangle = \begin{cases} \delta_{ij} & (i=j \geqq 0) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

For a general product $w_1 \cdots w_r$ of free fermions $w_i \in \mathscr{W}$, we put

$$(1.2) \qquad \langle w_1 \cdots w_r \rangle = \begin{cases} 0 & (r \ odd) \\ \sum_{\sigma} \mathrm{sgn}\, \sigma \langle w_{\sigma(1)} w_{\sigma(2)} \rangle \cdots \langle w_{\sigma(r-1)} w_{\sigma(r)} \rangle & (r \ even) \end{cases}$$

where σ runs over the permutations such that $\sigma(1) < \sigma(2), \ldots, \sigma(r-1) < \sigma(r)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(r-1)$. The rule (1.2) is known as *Wick's theorem*. We call $\mathscr{F}$, $\mathscr{F}^*$ the *Fock spaces* and the representation of $\boldsymbol{A}$ on them the *Fock representations*.

Consider the set of finite linear combinations of quadratic elements

$$\mathfrak{g} = \{ \sum a_{ij} \psi_i \psi_j^* \in \boldsymbol{A} \mid a_{ij} \in \boldsymbol{C} \}.$$

Using (1.1) we may verify the commutation relation

$$(1.3) \qquad \begin{aligned} &[\sum a_{ij} \psi_i \psi_j^*, \sum a'_{ij} \psi_i \psi_j^*] = \sum a''_{ij} \psi_i \psi_j^* \\ &a''_{ij} = \sum_k a_{ik} a'_{kj} - \sum_k a'_{ik} a_{kj}, \end{aligned}$$

and hence $\mathfrak{g}$ is a Lie algebra. In fact, (1.3) shows that it is isomorphic to the Lie algebra of infinite matrices $(a_{ij})_{i,j\in\mathbf{Z}}$ having finite number of non-zero entries. As a Lie algebra, $\mathfrak{g}$ is generated by

$$(1.4) \qquad e_i=\psi_{i-1}\psi_i^*, \quad f_i=\psi_i\psi_{i-1}^*, \quad h_i=\psi_{i-1}\psi_{i-1}^*-\psi_i\psi_i^*$$

along with $\psi_0\psi_0^*$. These are analogous to the Chevalley basis in the theory of finite dimensional Lie algebras. The Dynkin diagram for $\mathfrak{g}$ is thus an infinite chain.

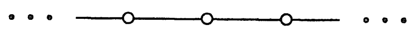


Fig. 1. Dynkin diagram for $\mathfrak{g}$.

Let us extend the Lie algebra $\mathfrak{g}$ to include certain infinite linear combinations of the form

$$(1.5) \qquad X=\sum_{i,j\in\mathbf{Z}} a_{ij}:\psi_i\psi_j^*:\,, \quad :\psi_i\psi_j^*: \underset{\text{def}}{=} \psi_i\psi_j^*-\langle\psi_i\psi_j^*\rangle\,.$$

For the moment assume the sum (1.5) to be finite, so that $X\in\mathfrak{g}\oplus\mathbf{C}\cdot 1$. The commutation relation for X's then takc the form

$$(1.6) \qquad [\sum a_{ij}:\psi_i\psi_j^*:, \sum a'_{ij}:\psi_i\psi_j^*:]=\sum a''_{ij}:\psi_i\psi_j^*:+c\cdot 1\,,$$

with a''_{ij} given by (1.3) and

$$c=\sum_{i<0,j\geq 0} a_{ij}a'_{ji}-\sum_{i\geq 0,j<0} a_{ij}a'_{ji}\,.$$

The action of X on $\mathscr{F}$, $\mathscr{F}^*$ read as

$$(1.7) \qquad \begin{aligned} X\cdot a\,|\,\mathrm{vac}\rangle &=(\mathrm{ad}\,X(a))\,|\,\mathrm{vac}\rangle+a\cdot\sum_{i\geq 0>j} a_{ij}\psi_i\psi_j^*\,|\,\mathrm{vac}\rangle \\ \langle\mathrm{vac}\,|\,a\cdot X &=-\langle\mathrm{vac}\,|\,(\mathrm{ad}\,X(a))+\langle\mathrm{vac}\,|\sum_{j\geq 0>i} a_{ij}\psi_i\psi_j^*\cdot a\,. \end{aligned}$$

Here $\mathrm{ad}\,X\in\mathrm{End}_{\mathbf{C}}(A)$ is by definition

$$(1.8) \qquad \begin{aligned} &\mathrm{ad}\,X(w_1\cdots w_r)=\sum_{i=1}^{r} w_1\cdots\mathrm{ad}\,X(w_i)\cdots w_r, \quad w_i\in\mathscr{W}\,, \\ &\mathrm{ad}\,X(\psi_j)=\sum_{i\in\mathbf{Z}}\psi_i a_{ij}, \quad \mathrm{ad}\,X(\psi_j^*)=-\sum_{i\in\mathbf{Z}}\psi_i^* a_{ji}\,. \end{aligned}$$

Consider infinite matrices $(a_{ij})_{i,j\in\mathbf{Z}}$ satisfying the condition

(1.9) there exists an N such that $\quad a_{ij}=0 \quad$ for $\quad |i-j|>N$.

Under this condition, the operations (1.6)–(1.8) still make sense. We then *define* the Lie algebra $\mathfrak{gl}(\infty)$ to be the vector space

$$\mathfrak{gl}(\infty)=\{\sum_{i,j\in\mathbf{Z}} a_{ij}:\psi_i\psi_j^*:\,|\,(a_{ij}) \text{ satisfies (1.9)}\}\oplus\mathbf{C}\cdot 1$$

equipped with the Lie bracket (1.6), where now $:\psi_i\psi_j^*:$ is regarded as an abstract symbol, and 1 as a central element. In accordance with the classification theory of Lie algebras, we shall also use the notation A_∞ to signify $\mathfrak{gl}(\infty)$. The considerations above show that $\mathfrak{g}\subset\mathfrak{gl}(\infty)$, and that we have a representation $\mathfrak{gl}(\infty)\to\mathrm{End}_{\boldsymbol{C}}(\mathscr{F})$. The latter is a reducible one, for there exists other than 1 a central element $H_0=\sum :\psi_i\psi_i^*:$ which acts non-trivially on $\mathscr{F}$. Since $\mathrm{ad}\, H_0(\psi_i)=\psi_i$, $\mathrm{ad}\, H_0(\psi_i^*)=-\psi_i^*$ and $H_0|\mathrm{vac}\rangle=0=\langle\mathrm{vac}|H_0$, we have the eigenspace decompositions $A=\oplus A_l$, $\mathscr{F}=\oplus\mathscr{F}_l$ and $\mathscr{F}^*=\oplus\mathscr{F}_l^*$, with the eigenvalue l running over the integers. An element $a\in A$ (resp. $v\in\mathscr{F}$ or $\mathscr{F}^*$) is said to have *charge* l if $a\in A_l$ (resp. $v\in\mathscr{F}_l$ or $\mathscr{F}_l^*$). In other words, $a\in A$ has charge l if it is a linear combination of monomials $\psi_{i_1}\cdots\psi_{i_r}\psi_{j_1}^*\cdots\psi_{j_s}^*$ with $r-s=l$, and similarly for $\mathscr{F}$ and $\mathscr{F}^*$. Note that $\mathscr{F}_l^*$ and $\mathscr{F}_{l'}$ are orthogonal unless $l=l'$. The representations

$$\rho_l\colon \mathfrak{gl}(\infty)\longrightarrow \mathrm{End}_{\boldsymbol{C}}(\mathscr{F}_l)$$

turn out to be irreducible. Put

$$(1.10)\qquad \Psi_l^*=\begin{cases}\psi_{-1}\cdots\psi_l & (l<0)\\ 1 & (l=0)\\ \psi_0^*\cdots\psi_{l-1}^* & (l>0)\end{cases},\qquad \Psi_l=\begin{cases}\psi_l^*\cdots\psi_{-1}^* & (l<0)\\ 1 & (l=0)\\ \psi_{l-1}\cdots\psi_0 & (l>0).\end{cases}$$

Then the vectors $\langle l|=\langle\mathrm{vac}|\Psi_l^*$, $|l\rangle=\Psi_l|\mathrm{vac}\rangle$ give the highest weight vectors:

$$e_i|l\rangle=0,\ h_i|l\rangle=\delta_{il}|l\rangle\qquad\text{for all}\quad i.$$

We have $\mathscr{F}_l^*=\langle l|A_0$, $\mathscr{F}_l=A_0|l\rangle$ and $\langle l|l\rangle=1$. If we introduce an automorphism ι_l of $\mathfrak{gl}(\infty)$ by

$$(1.11)\qquad \iota_l(\psi_i)=\psi_{i-l},\quad \iota_l(\psi_i^*)=\psi_{i-l}^*,$$

we have $\rho_l\cong\rho_0\circ\iota_l$. Thus ρ_l are all equivalent to each other. We note also that

$$(1.12)\qquad \langle l|a|l'\rangle=\langle l-m|\iota_m(a)|l'-m\rangle$$

holds for any l, l', m and any $a\in A$.

For $n\in\boldsymbol{Z}$, set

$$H_n=\sum_{i\in\boldsymbol{Z}} :\psi_i\psi_{i+n}^*:\ \in\mathfrak{gl}(\infty).$$

We have then the commutation relation

$$[H_m, H_n]=n\delta_{m+n,0}\cdot 1,$$

which shows that H_n $(n\neq 0)$ and 1 span a Heisenberg subalgebra $\mathscr{H}$ in $\mathfrak{gl}(\infty)$.

This fact enables us to construct explicit realizations of the abstract settings above in terms of polynomials in infinitely many variables $x=(x_1, x_2,\ldots)$. An element $X \in \mathfrak{gl}(\infty)$ is called *locally nilpotent* if for any $v \in \mathcal{F}$ there exists an N such that $X^N v=0$. Suppose $n>0$. Then H_n is locally nilpotent. Moreover, for any $v \in \mathcal{F}$ there exists an M such that $H_n v=0$ for $n>M$. Hence we can define the action of the Hamiltonian

$$H(x)=\sum_{n=1}^{\infty} x_n H_n,$$

and moreover that of $e^{H(x)}$ on $\mathcal{F}$. We remark that, by using $H(x)|\mathrm{vac}\rangle=0$, it is sometimes useful to write $e^{H(x)}a|\mathrm{vac}\rangle$ as $a(x)|\mathrm{vac}\rangle$, where $e^{H(x)}ae^{-H(x)}$ is the formal time evolution of $a \in A$.

Example.

$$e^{H(x)}\psi_i e^{-H(x)}=\sum_{\nu=0}^{\infty} \psi_{i-\nu}p_\nu(x)=\psi_i+x_1\psi_{i-1}+\left(x_2+\frac{1}{2}x_1^2\right)\psi_{i-2}+\cdots,$$

$$e^{H(x)}\psi_i^* e^{-H(x)}=\sum_{\nu=0}^{\infty} \psi_{i+\nu}^* p_\nu(-x)=\psi_i^*-x_1\psi_{i+1}^*+\left(-x_2+\frac{1}{2}x_1^2\right)\psi_{i+2}^*+\cdots$$

where the polynomials $p_\nu(x)$ are defined by the generating function

$$\sum_{\nu=0}^{\infty} p_\nu(x)k^\nu=\exp\left(\sum_{n=1}^{\infty} x_n k^n\right). \tag{1.13}$$

We have thus $e^{H(x)}\psi_1|\mathrm{vac}\rangle=(\psi_1+x_1\psi_0)|\mathrm{vac}\rangle$, and so forth.

Theorem 1.1. *Let V_l denote copies of the polynomial algebra $\mathbf{C}[x]$. Then the following map*

$$\begin{aligned} \mathcal{F}=\oplus\mathcal{F}_l &\longrightarrow V=\oplus V_l \\ a|\mathrm{vac}\rangle &\longmapsto \oplus\langle l|e^{H(x)}a|\mathrm{vac}\rangle \end{aligned} \tag{1.14}$$

is an isomorphism of vector spaces.

The Fock representation of A also has a realization in the right hand side of (1.14). Consider the following linear differential operators of infinite order, called the *vertex operators*

$$\begin{aligned} X(k)&=\exp\left(\sum_{n=1}^{\infty} x_n k^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial x_n}k^{-n}\right) \\ X^*(k)&=\exp\left(-\sum_{n=1}^{\infty} x_n k^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial x_n}k^{-n}\right). \end{aligned} \tag{1.15}$$

The coefficients $X_i\left(x, \frac{\partial}{\partial x}\right)$, $X_i^*\left(x, \frac{\partial}{\partial x}\right)$ of the expansion $X(k)=$

$\sum_{i\in\mathbf{Z}} X_i\left(x, \frac{\partial}{\partial x}\right)k^i$, $X^*(k)=\sum_{i\in\mathbf{Z}} X_i^*\left(x, \frac{\partial}{\partial x}\right)k^{-i}$ are well defined linear operators on $\mathbf{C}[x]$. In terms of $p_\nu(x)$ in (1.13), we have $X_i\equiv X_i(x, \partial/\partial x)=\sum_{\nu\geqq 0} p_{\nu+i}(x)p_\nu(-\tilde{\partial})$ with $\tilde{\partial}=(\partial_1, \partial_2/2,\ldots, \partial_n/n,\ldots)$ and $\partial_\nu=\partial/\partial x_\nu$. For example,

$$X_{-1}=-\partial_1+x_1\left(-\frac{1}{2}\partial_2+\frac{1}{2}\partial_1^2\right)+\left(x_2+\frac{1}{2}x_1^2\right)\left(-\frac{1}{3}\partial_3+\frac{1}{2}\partial_1\partial_2-\frac{1}{6}\partial_1^3\right)+\cdots$$

$$X_0=1+x_1(-\partial_1)+\left(x_2+\frac{1}{2}x_1^2\right)\left(-\frac{1}{2}\partial_2+\frac{1}{2}\partial_1^2\right)$$
$$+\left(x_3+x_1x_2+\frac{1}{6}x_1^3\right)\left(-\frac{1}{3}\partial_3+\frac{1}{2}\partial_1\partial_2-\frac{1}{6}\partial_1^3\right)+\cdots$$

$$X_1=x_1+\left(x_2+\frac{1}{2}x_1^2\right)(-\partial_1)+\left(x_3+x_1x_2+\frac{1}{6}x_1^3\right)\left(-\frac{1}{2}\partial_2+\frac{1}{2}\partial_1^2\right)+\cdots$$

Replacing x_ν by $-x_\nu$ and ∂_ν by $-\partial_\nu$ we obtain expressions for $X^*_{-i}(x, \partial/\partial x)$.

Theorem 1.2. *Define $\hat{X}_i$, $\hat{X}_i^*\in \mathrm{End}_{\mathbf{C}}(V)$ by the formulas*

$$\hat{X}_i\colon V_l \longrightarrow V_{l+1}, f_l(x) \longmapsto X_{i-l}\left(x, \frac{\partial}{\partial x}\right)f_l(x),$$
$$\hat{X}_i^*\colon V_l \longrightarrow V_{l-1}, f_l(x) \longmapsto X^*_{i-l+1}\left(x, \frac{\partial}{\partial x}\right)f_l(x).$$

Then $\hat{X}_i$, $\hat{X}_i^*$ ($i\in\mathbf{Z}$) generate in $\mathrm{End}_{\mathbf{C}}(V)$ a Clifford algebra isomorphic to $\mathbf{A}$, and (1.14) *gives an $\mathbf{A}$-module isomorphism with the identification*

$$\psi_i=\hat{X}_i,\quad \psi_i^*=\hat{X}_i^*. \tag{1.16}$$

In particular, the representation ρ_l: $\mathfrak{gl}(\infty)\to\mathrm{End}_{\mathbf{C}}(V_l)$ is realized as

$$\rho_l(:\psi_i\psi_j^*:)=Z_{i-l,j-l}\left(x, \frac{\partial}{\partial x}\right)+\delta_{ij}\theta_l(i)$$

where

$$\theta_l(i)=\langle\psi_{i-l}\psi^*_{i-l}\rangle-\langle\psi_i\psi_i^*\rangle$$
$$=\begin{cases} 1 & (\text{if } 0\leqq i\leqq l-1) \\ -1 & (\text{if } l\leqq i\leqq -1) \\ 0 & (\text{otherwise}), \end{cases}$$

and $Z_{ij}\left(x, \frac{\partial}{\partial x}\right)$ is given by the generating function

(1.17)
$$Z(p, q)=\frac{q}{p-q}\left(\exp\left(\sum_{n=1}^\infty (p^n-q^n)x_n\right)\exp\left(-\sum_{n=1}^\infty \frac{1}{n}(p^{-n}-q^{-n})\frac{\partial}{\partial x_n}\right)-1\right)$$
$$=\sum_{i,j\in\mathbf{Z}} Z_{ij}\left(x, \frac{\partial}{\partial x}\right)p^iq^{-j}.$$

Note that the formula (1.17) for $p=q$ gives

$$\rho_0(H_n)=\sum_{i\in Z} Z_{i\,i+n}\left(x, \frac{\partial}{\partial x}\right)=\begin{cases} \dfrac{\partial}{\partial x_n} & (n>0) \\ 0 & (n=0) \\ -nx_{-n} & (n<0). \end{cases}$$

Example. We write $Z_{ij}=Z_{ij}\left(x, \dfrac{\partial}{\partial x}\right)$ and $\partial_\nu=\dfrac{\partial}{\partial x_\nu}$.

$$\begin{aligned} Z_{-1,-1}=&-x_1\partial_1+\left(-\frac{1}{4}x_1^2-\frac{1}{2}x_2\right)\partial_2+\left(\frac{3}{4}x_1^2-\frac{1}{2}x_2\right)\partial_1^2 \\ &+\left(-\frac{1}{18}x_1^3-\frac{1}{3}x_1x_2-\frac{1}{3}x_3\right)\partial_3+\left(\frac{1}{6}x_1^3+x_1x_2-\frac{1}{2}x_3\right)\partial_1\partial_2 \\ &+\left(-\frac{5}{18}x_1^3+\frac{1}{3}x_1x_2-\frac{1}{6}x_3\right)\partial_1^3+\cdots \end{aligned}$$

$$Z_{-1,0}=\partial_1+(-x_1)\partial_1^2+(-x_2)\partial_1\partial_2+\frac{1}{2}x_1^2\partial_1^3+\cdots$$

$$Z_{-1,1}=\frac{1}{2}\partial_2+\frac{1}{2}\partial_1^2+\left(-\frac{1}{2}x_1\right)\partial_1\partial_2+\left(-\frac{1}{2}x_1\right)\partial_1^3+\cdots$$

$$\begin{aligned} Z_{0,-1}=&x_1+(-x_1^2)\partial_1+(-x_1x_2)\partial_2+\frac{1}{2}x_1^3\partial_1^2+\left(-\frac{1}{36}x_1^4-\frac{2}{3}x_1x_3-\frac{1}{3}x_2^2\right)\partial_3 \\ &+x_1^2x_2\partial_1\partial_2+\left(-\frac{5}{36}x_1^4-\frac{1}{3}x_1x_3+\frac{1}{3}x_2^2\right)\partial_1^3+\cdots \end{aligned}$$

$$\begin{aligned} Z_{0,0}=&x_1\partial_1+\left(-\frac{1}{4}x_1^2+\frac{1}{2}x_2\right)\partial_2+\left(-\frac{3}{4}x_1^2-\frac{1}{2}x_2\right)\partial_1^2 \\ &+\left(\frac{1}{18}x_1^3-\frac{1}{3}x_1x_2+\frac{1}{3}x_3\right)\partial_3+\left(\frac{1}{6}x_1^3-x_1x_2-\frac{1}{2}x_3\right)\partial_1\partial_2 \\ &+\left(\frac{5}{18}x_1^3+\frac{1}{3}x_1x_2+\frac{1}{6}x_3\right)\partial_1^3+\cdots \end{aligned}$$

$$\begin{aligned} Z_{0,1}=&\frac{1}{2}x_1\partial_2+\frac{1}{2}x_1\partial_1^2+\left(-\frac{1}{6}x_1^2+\frac{1}{3}x_2\right)\partial_3 \\ &+\left(-\frac{1}{2}x_1^2\right)\partial_1\partial_2+\left(-\frac{1}{3}x_1^2-\frac{1}{3}x_2\right)\partial_1^3+\cdots \end{aligned}$$

$$\begin{aligned} Z_{1,-1}=&\frac{1}{2}x_1^2+x_2+\left(-\frac{1}{2}x_1^3-x_1x_2\right)\partial_1 \\ &+\left(\frac{1}{24}x_1^4-\frac{1}{2}x_1^2x_2-\frac{1}{2}x_1x_3-\frac{1}{2}x_2^2\right)\partial_2 \\ &+\left(\frac{5}{24}x_1^4+\frac{1}{2}x_1^2x_2+\frac{1}{2}x_1x_3-\frac{1}{2}x_2^2\right)\partial_1^2 \\ &+\left(-\frac{1}{72}x_1^5+\frac{1}{18}x_1^3x_2-\frac{1}{3}x_1^2x_3-\frac{1}{6}x_1x_2^2-\frac{1}{3}x_1x_4-\frac{2}{3}x_2x_3\right)\partial_3 \\ &+\left(-\frac{1}{48}x_1^5+\frac{5}{12}x_1^3x_2+\frac{1}{4}x_1^2x_3+\frac{3}{4}x_1x_2^2+\frac{1}{2}x_1x_4-\frac{1}{2}x_2x_3\right)\partial_1\partial_2 \end{aligned}$$

$$+\left(-\frac{7}{144}x_1^5-\frac{5}{36}x_1^3x_2-\frac{5}{12}x_1^2x_3+\frac{5}{12}x_1x_2^2-\frac{1}{6}x_1x_4+\frac{1}{6}x_2x_3\right)\partial_1^3+\cdots$$

$$Z_{1,0}=\left(\frac{1}{2}x_1^2+x_2\right)\partial_1+\left(-\frac{1}{6}x_1^3+\frac{1}{2}x_3\right)\partial_2+\left(-\frac{1}{3}x_1^3-x_1x_2-\frac{1}{2}x_3\right)\partial_1^2$$
$$+\left(\frac{1}{24}x_1^4-\frac{1}{6}x_1^2x_2-\frac{1}{6}x_2^2+\frac{1}{3}x_4\right)\partial_3$$
$$+\left(\frac{5}{48}x_1^4-\frac{1}{4}x_1^2x_2-\frac{1}{2}x_1x_3-\frac{3}{4}x_2^2-\frac{1}{2}x_4\right)\partial_1\partial_2$$
$$+\left(\frac{5}{48}x_1^4+\frac{5}{12}x_1^2x_2+\frac{1}{2}x_1x_3-\frac{1}{12}x_2^2+\frac{1}{6}x_4\right)\partial_1^3+\cdots$$

$$Z_{1,1}=\left(\frac{1}{4}x_1^2+\frac{1}{2}x_2\right)(\partial_2+\partial_1^2)+\left(-\frac{1}{9}x_1^3+\frac{1}{3}x_3\right)\partial_3+\left(-\frac{1}{4}x_1^3-\frac{1}{2}x_1x_2\right)\partial_1\partial_2$$
$$+\left(-\frac{5}{36}x_1^3-\frac{1}{2}x_1x_2-\frac{1}{3}x_3\right)\partial_1^3+\cdots.$$

In general we have

$$Z_{ij}=\sum_{\mu,\nu\geqq 0}\langle\psi_{i+\mu}(x)\psi^*_{j-\nu}(x)\rangle p_\mu(-\tilde\partial)p_\nu(\tilde\partial)-\langle\psi_i\psi_j^*\rangle$$

where

$$\langle\psi_i(x)\psi_j^*(x)\rangle=\begin{cases}0 & (j\geqq 0)\\ \delta_{ij} & (i<0,\ j<0)\\ \sum_{\nu=0}^{-j-1}p_\nu(-x)p_{i-j-\nu}(x) & (i\geqq 0,\ j<0).\end{cases}$$

In order to see these correspondences (1.14), (1.15) more explicitly, let us introduce a convenient basis of $\boldsymbol{C}[x]$. Let Y be a Young diagram of signature $(f_1,\ldots,f_m)$, $f_1\geqq\cdots\geqq f_m$ (Fig. 2).

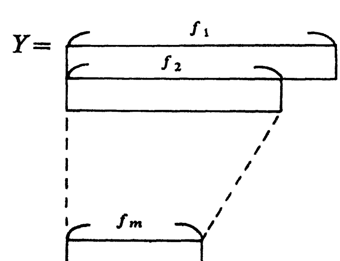


Fig. 2. Young diagram of signature $(f_1,\ldots,f_m)$.

The following polynomial is called the *Schur function* [32] attached to Y:

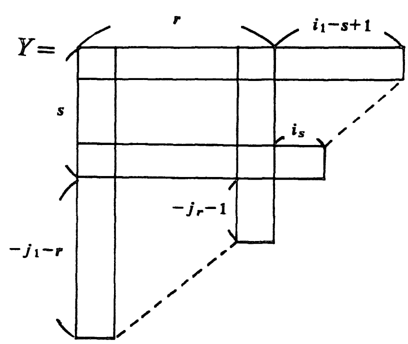
$$\chi_Y(x)=\det\left(p_{f_i-i+j}(x)\right)_{1\leqq i,j\leqq m},$$

where $p_\nu(x)$ are given by (1.13). As is well known, the characters of the general linear group $GL(N)$ $(N\geqq m)$ are given by Schur functions. Namely if ρ_Y denotes the irreducible representation corresponding to Y, we have $\operatorname{tr}\rho_Y(g)=\chi_Y(x)$ with $g\in GL(N)$ and $\nu x_\nu=\operatorname{tr} g^\nu$. When Y runs over all the diagrams, the set of Schur functions provides a basis of $C[x]$.

For $j_1<\cdots<j_r<0\leqq i_s<\cdots<i_1$, the following formula is valid:

$$\langle l|e^{H(x)}\psi^*_{j_1}\cdots\psi^*_{j_r}\psi_{i_s}\cdots\psi_{i_1}|0\rangle =\delta_{l+r,s}(-)^{j_1+\cdots+j_r+(r-s)(r-s+1)/2}\chi_Y(x) \tag{1.18}$$

where Y is given by



Since the vectors $\psi^*_{j_1}\cdots\psi^*_{j_r}\psi_{i_s}\cdots\psi_{i_1}|0\rangle$ $(j_1<\cdots<j_r<0\leqq i_s<\cdots<i_1)$ give a basis of $\mathscr{F}$, the isomorphism (1.14) is evident from this formula.

The action of the vertex operators (1.17) also admits simple description in terms of Schur polynomials. In what follows, we extend the edges of a Young diagram as in Fig. 3 and assign to them a numbering by integers:

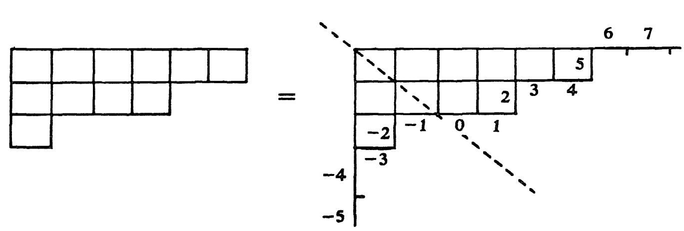


Fig. 3. Numbering of edges of a Young diagram.

By virtue of (1.18), the left multiplication by $:\psi_i\psi^*_j:$ is translated to give the following rule for the action of $Z_{ij}\left(x\frac{\partial}{\partial x}\right)$.

$i=j$

$$Z_{ii}\left(x, \frac{\partial}{\partial x}\right)\chi_Y(x)=\begin{cases} \chi_Y(x) & (i\geqq 0 \text{ and } i \text{ is vertical}) \\ -\chi_Y(x) & (i<0 \text{ and } i \text{ is horizontal}) \\ 0 & (\text{otherwise}), \end{cases}$$

$i\neq j$

$$Z_{ij}\left(x, \frac{\partial}{\partial x}\right)\chi_Y(x)=\begin{cases} (-)^{v-1}\chi_{Y'}(x) & (i \text{ is horizontal and } \\ & \quad j \text{ is vertical}) \\ 0 & (\text{otherwise}). \end{cases}$$

Here Y' signifies the diagram obtained by removing (if $i<j$) or inserting (if $i>j$) the hook corresponding to the pair (i, j), and v is the vertical length of the hook.

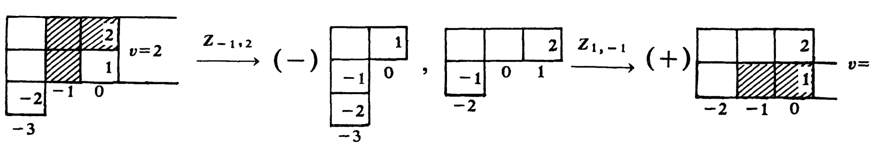


Fig. 4. Action of $Z_{ij}\left(x, \frac{\partial}{\partial x}\right)$.

Note in particular that the action of h_i in (1.4) is simultaneously diagonalized in the basis $\chi_Y(x)$. If we introduce the homogeneous degree by

$$\deg x_\nu=\nu, \ \deg\frac{\partial}{\partial x_\nu}=-\nu$$

then $\deg\chi_Y(x)=\#\{\text{plaquettes of } Y\}$ and $\deg Z_{ij}\left(x, \frac{\partial}{\partial x}\right)=i-j$.

It is sometimes useful to consider the generating sums of free fermions

(1.19) $$\psi(k)=\sum_{i\in Z}\psi_i k^i, \ \psi^*(k)=\sum_{i\in Z}\psi_i^* k^{-i}.$$

Their time evolutions take the simple form

(1.20) $$e^{H(x)}\psi(k)e^{-H(x)}=e^{\xi(x,k)}\psi(k), \ e^{H(x)}\psi^*(k)e^{-H(x)}=e^{-\xi(x,k)}\psi^*(k)$$

$$\xi(x, k)=\sum_{n=1}^{\infty}x_n k^n.$$

The vertex operators (1.15), (1.17) give their realizations in a suitable completion of $\boldsymbol{C}[x]$. Although (1.19) do not belong to A, the inner product of $\langle l|\psi(k)$ or $\langle l|\psi^*(k)$ with an element of $\mathscr{F}$ does make sense. Using Wick's theorem, we can verify the following formulas

$$(1.21)\qquad \begin{aligned}\langle l|\psi(k)e^{H(x)}a|0\rangle &= k^{l-1}\langle l-1|e^{H(x-\epsilon(k^{-1}))}a|0\rangle \\ \langle l|\psi^*(k)e^{H(x)}a|0\rangle &= k^{-l}\langle l+1|e^{H(x+\epsilon(k^{-1}))}a|0\rangle\end{aligned}$$

for any $a|0\rangle \in \mathscr{F}$, where

$$(1.22)\qquad \epsilon(k^{-1}) = \left(\frac{1}{k}, \frac{1}{2k^2}, \dots, \frac{1}{nk^n}, \dots\right).$$

For example, put $l=1$ and $a=\psi(p)\psi^*(q)$. By virtue of (1.20) and Wick's theorem, the left hand side of the first equation in (1.21) becomes

$$\begin{aligned}&e^{\xi(x,p)-\xi(x,q)}\langle 0|\psi_0^*\psi(k)\psi(p)\psi^*(q)|0\rangle\\ &= e^{\xi(x,p)-\xi(x,q)}q(k-p)/(p-q)(k-q)\\ &= e^{\xi(x-\epsilon(k^{-1}),p)-\xi(x-\epsilon(k^{-1}),q)}\langle 0|\psi(p)\psi^*(q)|0\rangle\\ &= \langle 0|e^{H(x-\epsilon(k^{-1}))}\psi(p)\psi^*(q)|0\rangle .\end{aligned}$$

The realization (1.16) of ψ_i, ψ_i^* in terms of the vertex operators (1.15) is a consequence of (1.21).

§2. τ Functions and the KP Hierarchy

We now focus our attention to the representation of the group corresponding to the Lie algebra in Section 1, and its relation to soliton theory.

Let $\mathscr{V} = \bigoplus_{i\in \mathbf{Z}} \mathbf{C}\psi_i$, $\mathscr{V}^* = \bigoplus_{i\in \mathbf{Z}} \mathbf{C}\psi_i^*$, and consider the multiplicative group in the Clifford algebra

$$(2.1)\qquad \mathbf{G} = \{g \in A_0 | {}^{\exists}g^{-1},\ g\mathscr{V}g^{-1} = \mathscr{V},\ g\mathscr{V}^*g^{-1} = \mathscr{V}^*\}.$$

The corresponding Lie algebra is nothing but $\mathfrak{g}\oplus \mathbf{C}\cdot 1$. The Fock representation gives rise to a representation of $\mathbf{G}$ on $\mathscr{F}_l$. We shall be concerned with the $\mathbf{G}$-orbit of the highest weight vector $|l\rangle$: $\mathbf{M}_l = \mathbf{G}|l\rangle \subset \mathscr{F}_l$. For each $v \in \mathbf{M}_l$, let $\mathscr{V}_v \subset \mathscr{V}$ denote the linear subspace $\{\psi \in \mathscr{V} \mid \psi v = 0\}$. By the correspondence $v \bmod \mathbf{C}^\times \leftrightarrow \mathscr{V}_v$, $\mathbf{M}_l/\mathbf{C}^\times$ can be identified with the collection of linear subspaces $\{\mathscr{V}_v\}$ in $\mathscr{V}$, which is a (infinite-dimensional) Grassmann manifold. We remark that $\mathbf{M}_l$ is stable under the action of

$$(2.2)\qquad g = e^{X_1}\cdots e^{X_k},\ X_1,\dots, X_k \in \mathfrak{gl}(\infty)$$

provided X_i's are locally nilpotent.

Now fix an integer l, and consider the realization ρ_l of $\mathbf{M}_l$ as polynomials

$$(2.3)\qquad \tau_l(x; g) = \langle l|e^{H(x)}g|l\rangle,\quad g\in \mathbf{G}.$$

We call a polynomial $\tau(x)$ a τ *function* if it is representable in the form (2.3) for some g. (Since ρ_l are all equivalent, this definition is actually independent of l.) As is well known, in the finite dimensional case, a Grassmann manifold is realized as an intersection of quadrics in a projective space. In the present case, we may write down an analogue of these quadratic defining equations (the Plücker relations).

Theorem 2.1. *A polynomial $\tau(x)$ is a τ function if and only if it satisfies the bilinear identity*

$$\oint e^{\xi(x-x',k)}\tau(x-\in(k^{-1}))\tau(x'+\in(k^{-1}))\frac{dk}{2\pi i}=0 \qquad \text{for any} \quad x, x' \tag{2.4}$$

where $\in(k^{-1})$ is given by (1.22),

$$\xi(x,k)=\sum_{n=1}^{\infty}k^n x_n,$$

and the integration is taken along a small contour at $k=\infty$ so that $\oint dk/2\pi ik=1$.

Let us sketch how to derive (2.4). By the definition (2.1), there exist $\alpha_{ij}\in \boldsymbol{C}$ such that $g\psi_j g^{-1}=\sum_{i\in\boldsymbol{Z}}\psi_i\alpha_{ij}$ and $g^{-1}\psi_j^* g=\sum_{i\in\boldsymbol{Z}}\psi_i^*\alpha_{ji}$ hold. This implies

$$\sum_{i\in\boldsymbol{Z}}\psi_i gv\otimes\psi_i^* gv'=\sum_{i\in\boldsymbol{Z}}g\psi_i v\otimes g\psi_i^* v'\in\mathscr{F}\otimes\mathscr{F}$$

for any $v, v'\in\mathscr{F}$. With the choice $v=v'=|l\rangle$ the right hand side becomes 0, since either $\psi_i|l\rangle=0$ or $\psi_i^*|l\rangle=0$. Applying $e^{H(x)}\otimes e^{H(x')}$ and taking the inner product with $\langle l+1|\otimes\langle l-1|$ we get

$$\begin{aligned}0&=\sum_{i\in\boldsymbol{Z}}\langle l+1|e^{H(x)}\psi_i g|l\rangle\,\langle l-1|e^{H(x')}\psi_i^* g|l\rangle\\&=\oint\langle l+1|e^{H(x)}\psi(k)g|l\rangle\,\langle l-1|e^{H(x')}\psi^*(k)g|l\rangle\frac{dk}{2\pi ik}\\&=\oint e^{\xi(x-x',k)}\langle l+1|\psi(k)e^{H(x)}g|l\rangle\,\langle l-1|\psi^*(k)e^{H(x')}g|l\rangle\frac{dk}{2\pi ik}.\end{aligned}$$

Here we have used the time evolutions (1.20) for $\psi(k)$, $\psi^*(k)$. Finally, using the formula (1.21) we arrive at (2.4).

The *bilinear identity* can be further rewritten into a series of non-linear differential equations for $\tau(x)$. They are described by Hirota's bilinear differential operator ([31])

$$P(D)f\cdot g\underset{\text{def}}{=}P\Big(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\dots\Big)(f(x+y)g(x-y))\,|_{y_1=y_2=\dots=0}$$

where $P(D)$ is a polynomial in $D=(D_1, D_2,\dots)$. In fact, with a change of variables $x\to x+y$, $x'\to x-y$, (2.4) is brought to the form

$$(2.5)\qquad (\sum_{j=0}^{\infty} p_j(-2y)p_{j+1}(\tilde{D})\exp(\sum_{n=1}^{\infty} y_nD_n))\tau\cdot\tau=0 \qquad \text{for any} \quad y$$

with $\tilde{D}=(D_1, D_2/2,\dots, D_n/n\cdots)$. For instance the coefficient of y_1^4 in (2.5) gives an equation

$$(2.6)\qquad (D_1^4+3D_2^2-4D_1D_3)\tau\cdot\tau=0,$$

or in terms of the dependent variable $u=2\frac{\partial^2}{\partial x_1^2}\log\tau$

$$(2.7)\qquad 3\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(-4\frac{\partial u}{\partial x_3}+6u\frac{\partial u}{\partial x_1}+\frac{\partial^3 u}{\partial x_1^3}\right)=0.$$

Equation (2.7) (resp. (2.6)) is a most typical example of soliton equations, known as the *Kadomtsev-Petviashvili* (*KP*) *equation* ([33]) in the ordinary form (resp. the bilinear form ([34])). The whole system of non-linear equations (2.5) is termed the (bilinear) *KP hierarchy*.

More generally, we have similar bilinear identities corresponding to an arbitrary pair of vertices (l, l') of the Dynkin diagram (Fig. 1):

$$(2.4)_{l,l'}\qquad \oint e^{\xi(x-x',k)}k^{l-l'}\tau_l(x-\epsilon(k^{-1});g)\tau_{l'}(x'+\epsilon(k^{-1});g)\frac{dk}{2\pi i}=0$$
$$\text{for any}\quad x, x', l, l' \quad\text{with}\quad l\geqq l'.$$

These are sometimes called the $(l-l')$-th modified KP hierarchies. The simplest examples are $(l-l'=1)$

$$(D_1^2-D_2)\tau_{l+1}\cdot\tau_l=0$$
$$(D_1^3+3D_1D_2-4D_3)\tau_{l+1}\cdot\tau_l=0,$$

which lead to

$$-\frac{\partial v}{\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}+\left(\frac{\partial v}{\partial x_1}\right)^2+u=0,$$
$$3\frac{\partial^2 v}{\partial x_2^2}+6\frac{\partial v}{\partial x_2}\frac{\partial^2 v}{\partial x_1^2}-4\frac{\partial^2 v}{\partial x_1\partial x_3}+\frac{\partial^4 v}{\partial x_1^4}-6\left(\frac{\partial v}{\partial x_1}\right)^2\frac{\partial^2 v}{\partial x_1^2}=0,$$

where $u=2\frac{\partial^2}{\partial x_1^2}\log\tau_l$ and $v=\log\tau_{l+1}/\tau_l$. Explicit forms of bilinear equations of higher order are listed in the Appendix 1.

Originally the KP hierarchy was introduced in connection with an auxiliary linear system of equations ([2])

$$(2.8)\qquad \frac{\partial}{\partial x_n}w=B_n\left(x,\frac{\partial}{\partial x_1}\right)w \qquad (n=1,2,\ldots)$$
$$B_n\left(x,\frac{\partial}{\partial x_1}\right)=\left(\frac{\partial}{\partial x_1}\right)^n+\sum_{\nu=0}^{n-2}u_{n\nu}(x)\left(\frac{\partial}{\partial x_1}\right)^\nu.$$

Their integrability condition

$$\left[\frac{\partial}{\partial x_m}-B_m\left(x,\frac{\partial}{\partial x_1}\right),\frac{\partial}{\partial x_n}-B_n\left(x,\frac{\partial}{\partial x_1}\right)\right]=0 \qquad (m,n=1,2,\ldots)$$

leads to non-linear equations for the coefficients $u_{n\nu}(x)$, which constitute the KP hierarchy in the ordinary form. For example (2.7) follows by the choice $m=2$, $n=3$. Let us show that these linear equations (2.8) also are derived from the bilinear identity (2.4).

With l and g fixed, we set ($\tau(x)=\tau_l(x;g)$)

$$w(x,k)=\langle l+1|e^{H(x)}\psi(k)g|l\rangle/k^l\tau(x)$$
$$w^*(x,k)=\langle l-1|e^{H(x)}\psi^*(k)g|l\rangle/k^{1-l}\tau(x).$$

From the proof of Theorem 2.1, we have

$$(2.9)\qquad w(x,k)=e^{\xi(x,k)}\tau(x-\epsilon(k^{-1}))/\tau(x)$$
$$w^*(x,k)=e^{-\xi(x,k)}\tau(x+\epsilon(k^{-1}))/\tau(x).$$

The bilinear identity now reads

$$(2.10)\qquad \oint w(x,k)w^*(x',k)\frac{dk}{2\pi i}=0 \qquad \text{for any}\quad x,x'.$$

Theorem 2.2. *Suppose we have formal series of the form*

$$w(x,k)=e^{\xi(x,k)}\Big(1+\sum_{j=1}^{\infty}w_j(x)k^{-j}\Big)$$
$$w^*(x,k)=e^{-\xi(x,k)}\Big(1+\sum_{j=1}^{\infty}w_j^*(x)k^{-j}\Big)$$

that satisfy the identity (2.4). *Then the following are valid.*

(i) *There exists a function $\tau(x)$, unique up to a constant multiple, such that $w(x,k)$ and $w^*(x,k)$ are expressed as* (2.9).

(ii) *$\tau(x)$ solves the KP hierarchy.*

(iii) *$w(x,k)$ and $w^*(x,k)$ solve the linear equations*

$$\frac{\partial}{\partial x_n}w=B_n\left(x,\frac{\partial}{\partial x_1}\right)w,\quad -\frac{\partial}{\partial x_n}w^*=B_n^*\left(x,\frac{\partial}{\partial x_1}\right)w^*$$

where $B_n\left(x,\frac{\partial}{\partial x_1}\right)$ *is of the form* (2.8), *and* $B_n^*\left(x,\frac{\partial}{\partial x_1}\right)=\left(-\frac{\partial}{\partial x_1}\right)^n+$

$\sum_{\nu=0}^{n-2}\left(-\frac{\partial}{\partial x_1}\right)^{\nu} u_{n\nu}(x)$ *is its formal adjoint operator.*

For explicit computation of $B_n\left(x, \frac{\partial}{\partial x_1}\right)$, the following bilinear identity is available:

$$\tau_l(x')v_l(x;p)=\oint\frac{dk}{2\pi i}k^{l-l'-1}e^{\xi(x-x',k)}\tau_l(x-\epsilon(k^{-1}))v_{l'}(x'+\epsilon(k^{-1});p)$$

$$0=\oint\frac{dk}{2\pi i}k^{l-l'+1}e^{\xi(x-x',k)}v_l(x-\epsilon(k^{-1});p)\tau_{l'}(x') \qquad (l\geqq l')$$

where

$$v_l(x;p)=\langle l+1|e^{H(x)}\psi(p)g|l\rangle/p^l=\tau_l(x)w_l(x;p).$$

Note that the latter equation is the same as $(2.4)_{l,l'}$. Hence the pair $(v_l(x;p), \tau_l(x))$ satisfies the modified KP equations for $(\tau_{l+1}(x), \tau_l(x))$. Rewriting these we obtain linear equations for $w_l(x;p)=v_l(x;p)/\tau_l(x)$

$$\frac{\partial}{\partial x_2}w_l=\left(\frac{\partial^2}{\partial x_1^2}+2\frac{\partial^2}{\partial x_1^2}\log\tau_l\right)w_l$$

$$\frac{\partial}{\partial x_3}w_l=\left(\frac{\partial^3}{\partial x_1^3}+3\frac{\partial^2}{\partial x_1^2}\log\tau_l\frac{\partial}{\partial x_1}+3\frac{\partial^3}{\partial x_1^3}\log\tau_l+3\frac{\partial^2}{\partial x_1\partial x_2}\log\tau_l\right)w_l\cdots.$$

From the viewpoint of soliton theory, the framework of Sections 1–2 provides with a method to construct solutions to the KP hierarchy as well.

Example 1. Using the "Chevalley basis" (1.4) we put

$$(2.11) \qquad -r_i=\exp(e_i)\exp(-f_i)\exp(e_i)\in \boldsymbol{G}.$$

By virtue of the action rule of $Z_{ij}\left(x, \frac{\partial}{\partial x}\right)$ on Schur functions, we can verify that $\rho_0(r_i)$ adds one plaquette at $(i-1, i)$-th corner (Fig. 5):

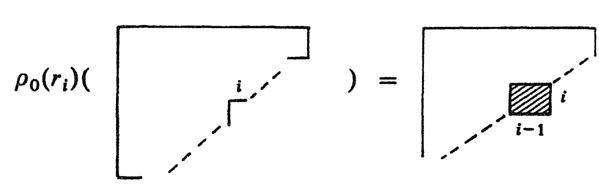


Fig. 5. Action of r_i.

It then follows that all the Schur polynomials are τ functions (hence solve the KP hierarchy ([2])).

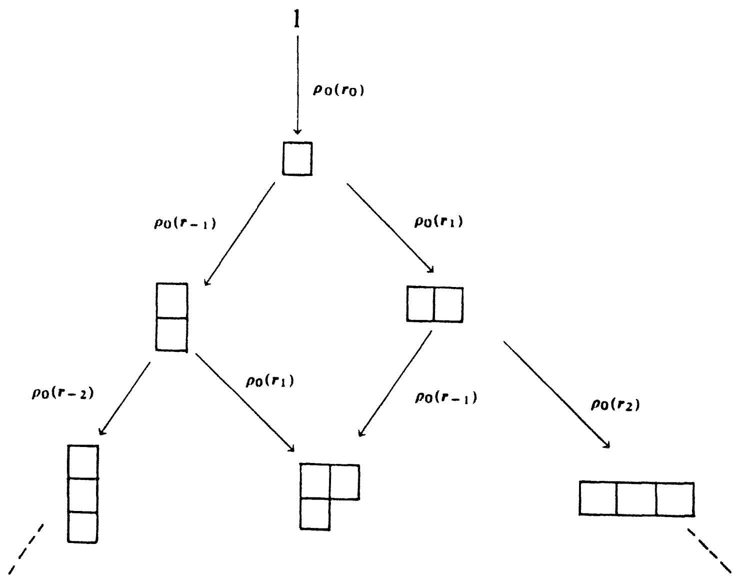


Fig. 6. Generation of Schur polynomials.

Example 2. Given a polynomial $\tau(x)\in \boldsymbol{C}[x]$, let

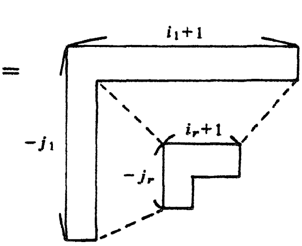
$$\tau(x)=\sum_{Y} c_Y\chi_Y(x)$$

be its expansion in the basis $\{\chi_Y(x)\}$. Then $\tau(x)$ solves the KP hierarchy if and only if the coefficients c_Y are subject to the relation ([2])

$$0=c\begin{pmatrix} i_1\cdots\hat{\imath}_\mu\cdots\hat{\imath}_\nu\cdots i_r \\ j_1\cdots\hat{\jmath}_\mu\cdots\hat{\jmath}_\nu\cdots j_r\end{pmatrix}c\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}-c\begin{pmatrix} i_1\cdots\hat{\imath}_\mu\cdots i_r \\ j_1\cdots\hat{\jmath}_\mu\cdots j_r\end{pmatrix}c\begin{pmatrix} i_1\cdots\hat{\imath}_\nu\cdots i_r \\ j_1\cdots\hat{\jmath}_\nu\cdots j_r\end{pmatrix}$$
$$+c\begin{pmatrix} i_1\cdots\hat{\imath}_\mu\cdots i_r \\ j_1\cdots\hat{\jmath}_\nu\cdots j_r\end{pmatrix}c\begin{pmatrix} i_1\cdots\hat{\imath}_\nu\cdots i_r \\ j_1\cdots\hat{\jmath}_\mu\cdots j_r\end{pmatrix}$$

for all $j_1<\cdots<j_r<0\leqq i_r<\cdots<i_1$ and μ, ν,

where we have put $c\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}=c_Y$ for a Young diagram

$Y=$ 

Up till now we have considered polynomial solutions only. However the bilinear identities (2.4), $(2.4)_{ll'}$, linear equations (2.8) etc. are meaningful for wider class of solutions, which correspond to considering representations of suitable completion of Lie algebras.

Example 3. Put

$$g=\exp\left(\sum_{i=1}^{N} a_i\psi(p_i)\psi^*(q_i)\right).$$

Then the time evolution (1.20) and Wick's theorem give the *N-soliton solution* ([34])

$$(2.12) \qquad \begin{aligned}\tau_0(x;\,g) &= 1+\sum_{i=1}^{N} e^{\eta_i}+\sum_{i<j} c_{ij}e^{\eta_i+\eta_j}+\cdots \\ &= \sum_{r=0}^{N}\sum_{i_1<\cdots<i_r}\prod_{\mu<\nu} c_{i_\mu i_\nu} e^{\eta_{i_1}+\cdots+\eta_{i_r}},\end{aligned}$$

where

$$\eta_i=\xi(x,\,p_i)-\xi(x,\,q_i)+\log\left(a_i\frac{q_i}{p_i-q_i}\right),$$

$$c_{ij}=\frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}=\frac{\langle\psi(p_i)\psi^*(q_i)\psi(p_j)\psi^*(q_j)\rangle}{\langle\psi(p_i)\psi^*(q_i)\rangle\langle\psi(p_j)\psi^*(q_j)\rangle}.$$

In terms of the vertex operator (1.17), we can write (2.12) as

$$\tau_0(x;\,g)=\prod_{i=1}^{N}\exp\left(a_i\left(Z(p_i,\,q_i)+\frac{q_i}{p_i-q_i}\right)\right)\cdot 1.$$

If we formally let $N\to\infty$, we get a "general solution"

$$g=\exp\left(\iint a(p,\,q)\psi(p)\psi^*(q)dpdq\right)$$

$$\begin{aligned}\tau(x)=\sum_{r=0}^{\infty}\frac{1}{r!}\int\cdots\int \det\left(\frac{q_j}{p_i-q_j}\right)\exp\sum_{i=1}^{r}(\xi(x,\,p_i)-\xi(x,\,q_i)) \\ \times\prod_{i=1}^{r} a(p_i,\,q_i)dp_1\cdots dp_r dq_1\cdots dq_r\end{aligned}$$

depending on an arbitrary function $a(p,\,q)$ of two variables.

§3. Reduction to $A_1^{(1)}$

In Section 2 we have seen that the group orbit of the highest weight vector in the Fock representation of A_∞ represents the totality of polynomial solutions to the KP hierarchy. In this section we show that this correspondence induces a similar one between the affine Lie algebra $A_1^{(1)}$ and the hierarchy of the *KdV equation.*

Before going into the subject, we prepare some terminologies of Lie algebras which are frequently used in this and the later sections.

An integral matrix $A=(a_{ij})_{i,j\in I}$ is called a Cartan matrix, if it satisfies the following conditions: $a_{ii}=2$ for all $i\in I$; $a_{ij}\leqq 0$ if $i\neq j$; $a_{ij}=0$ if and only if $a_{ji}=0$.

A set of generators $\{e_i, f_i, h_i\}$ of a Lie algebra $\mathscr{L}$ is called a Chevalley basis, if it satisfies

$$[e_i, f_j]=\delta_{ij}h_i, \quad [h_i, e_j]=a_{ij}e_j, \quad [h_i, f_j]=-a_{ij}f_j,$$
$$[h_i, h_j]=0, \quad (\mathrm{ad}\, e_i)^{1-a_{ij}}e_j=0, \quad (\mathrm{ad}\, f_i)^{1-a_{ij}}f_j=0.$$

For a given Cartan matrix A, we associate a diagram called the Dynkin diagram for A as follows: The set of vertices is I; if $a_{ij}\neq 0$ the vertices i and j are connected by $a_{ij}a_{ji}$ lines; if $|a_{ij}|>|a_{ji}|$, an arrow pointing the vertex i is attached to these lines.

We denote by $\mathfrak{h}$ the linear span of $(h_i)_{i\in I}$ and by $\mathfrak{h}^*$ its dual space. The elements $(\Lambda_i)_{i\in I}\in\mathfrak{h}^*$ are called fundamental weights if they satisfy $\Lambda_i(h_j)=\delta_{ij}$. An irreducible $\mathscr{L}$ module L is called a highest weight module if it is generated by a vector $v\in L$ satisfying $e_iv=0$ and $h_iv=\Lambda(h_i)v$ for $\Lambda\in\bigoplus_i \boldsymbol{Z}\Lambda_i$. The vector v is called the highest weight vector and Λ is called its highest weight.

Consider the subalgebra $A_\infty^{\iota_2}$ of A_∞ consisting of those elements whose adjoint representations on $\mathscr{W}$ commute with ι_2:

$$A_\infty^{\iota_2}=\{X\in A_\infty \mid [\mathrm{ad}\, X, \iota_2]|_{\mathscr{W}}=0\}.$$

It contains a Heisenberg subalgebra $\mathscr{H}_2$ spanned by H_n (n: even) and 1, and splits into the direct sum of $\bigoplus_{n\in 2\boldsymbol{Z}} \boldsymbol{C}H_n$ and an algebra isomorphic to $A_1^{(1)}$. In fact, we can choose the Chevalley basis for $A_1^{(1)}$ as follows.

$$(3.1)\qquad \begin{aligned} e_j &= \sum_{n\equiv j \bmod 2} \psi_{n-1}\psi_n^*, \\ f_j &= \sum_{n\equiv j \bmod 2} \psi_n\psi_{n-1}^*, \\ h_j &= \sum_{n\equiv j \bmod 2} (:\psi_{n-1}\psi_{n-1}^*: - :\psi_n\psi_n^*:)+\delta_{j0}, \qquad (j=0, 1). \end{aligned}$$

We consider $A_1^{(1)}$ as a subalgebra in A_∞ in this way.

The highest weight vectors $|l\rangle$ generate highest weight modules for $A_1^{(1)}$. If l is even (resp. odd), the weight of $|l\rangle$ with respect to $A_1^{(1)}$ is Λ_0 (resp. Λ_1).

Consider $\tau_l(x; g)$ with $X_1,\dots, X_k$ in (2.2) belonging to $A_1^{(1)}$. We abbreviate $\tau_l(x; g)$ to $\tau_l(x)$ when we consider a fixed g. Then the following additional constraints are imposed on $\tau_l(x)$:

(3.2) $$\tau_{l+2}(x)=\tau_l(x),$$

(3.3) $$\partial\tau_l(x)/\partial x_{2j}=0, \qquad (j=1, 2, 3,\dots).$$

Below we show the Dynkin diagrams of A_∞ and $A_1^{(1)}$, which fairly illustrate (3.2).

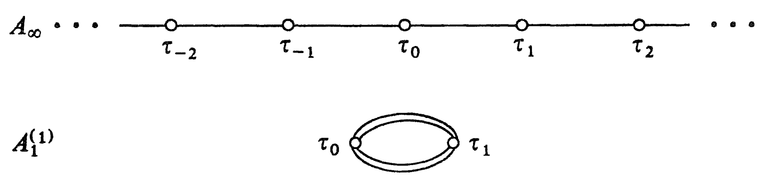


Fig. 7. Dynkin diagrams for A_∞ and $A_1^{(1)}$.

Under the conditions (3.2) and (3.3), the KP or the modified KP hierarchies reduce to a subfamily of equations, called the *KdV (or the modified KdV) hierarchies* ([35] [36]). These equations are obtained simply by omitting the derivatives $D_2, D_4,\dots$ in the KP hierarchy. Thus the n-th modified KP hierarchies for n even are reduced to give

(3.4) $$(D_1^4-4D_1D_3)\tau_l\cdot\tau_l=0,$$
$$(D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5)\tau_l\cdot\tau_l=0,\dots$$

whereas the n-th modified KP hierarchies for n odd yield

(3.5) $$D_1^2\tau_l\cdot\tau_{l+1}=0,$$
$$(D_1^3-4D_3)\tau_l\cdot\tau_{l+1}=0,\dots.$$

With the aid of the formula

(3.6) $$\frac{e^{\langle y,D\rangle}f\cdot f}{f\cdot f}=\frac{f(x+y)f(x-y)}{f(x)^2}=\exp\Big(2\sum_{|\nu|\,\mathrm{even}>0}\frac{y^\nu}{\nu!}\partial^\nu\log f\Big)$$
$$\langle y, D\rangle=\sum_{j=1}^{\infty}y_jD_j,\quad \nu=(\nu_1,\nu_2,\dots),\quad |\nu|=\nu_1+\nu_2+\cdots,$$

we may rewrite the bilinear equations (3.4) in terms of $u=2\partial^2\log\tau_l/\partial x_1^2$. The results are

$$4\frac{\partial u}{\partial x_3}=\frac{\partial^3 u}{\partial x_1^3}+6u\frac{\partial u}{\partial x_1} \qquad \text{(KdV equation)},$$

$$16\frac{\partial u}{\partial x_5}=\frac{\partial^5 u}{\partial x_1^5}+20\frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}+10u\frac{\partial^3 u}{\partial x_1^3}+30u^2\frac{\partial u}{\partial x_1}$$

(5-th order KdV equation)

and so on. Similarly, using

$$\frac{e^{\langle y,D\rangle}g\cdot f}{g\cdot f}=\frac{g(x+y)f(x-y)}{g(x)f(x)}=\frac{e^{\langle y,D\rangle}f\cdot f}{f\cdot f}\exp\Big(\sum_{|\nu|>0}\frac{y^\nu}{\nu!}\partial^\nu\log(g/f)\Big) \tag{3.7}$$

we obtain for u and $v=\partial\log(\tau_{l+1}/\tau_l)/\partial x_1$

$$\frac{\partial v}{\partial x_1}+v^2+u=0 \qquad \text{(Miura transformation [37])},$$

$$4\frac{\partial v}{\partial x_3}=-6v^2\frac{\partial v}{\partial x_1}+\frac{\partial^3 v}{\partial x_1^3} \qquad \text{(modified KdV equation)}$$

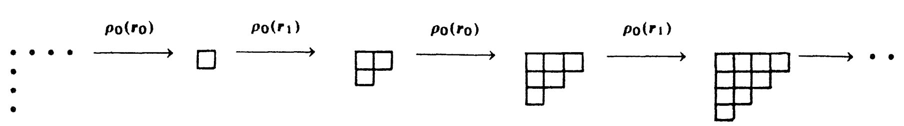
and the like. (Several useful formulas of the type (3.6), (3.7) are listed in the appendix of [38]).

Let us give some examples of solutions.

Example 1. By the definition, a KP–τ function $\tau(x)\in \boldsymbol{C}[x]$ solves the KdV hierarchy if and only if it is independent of $x_2, x_4,\dots$. In order to get homogeneous solutions, we put in parallel with (2.11)

$$-r_i=e^{e_i}e^{-f_i}e^{e_i} \qquad (i=0,\ 1),$$

where e_i, f_i are Chevalley basis (3.1). These are generators of the Weyl group of $A_1^{(1)}$. Successive application to 1 then produces all the Schur functions independent of $x_2, x_4,\dots$:



Example 2. In the definition of the vertex operator (1.17), set $p^2=q^2$. Then the variables $x_2, x_4,\dots$ drop out automatically, giving $x_1, x_3,\dots, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3},\dots$ (for $p=q$) and a vertex operator (for $p\neq q$) first employed in the realization of basic representations of $A_1^{(1)}$ ([22]). Correspondingly we get the N-soliton solutions to the KdV hierarchy in the form (2.12), where

$$\eta_i=2\sum_{n\ odd}p_i^n x_n+\eta_i^0$$

$$c_{ij}=(p_i-p_j)^2/(p_i+p_j)^2$$

with η_i^0, p_i being arbitrary parameters.

§4. Fermions with 2 Components

In this section we consider an alternative realization of the Fock representation by exploiting free fermions with 2 components.

Consider free fermions $\psi_n^{(j)}$, $\psi_n^{(j)*}$ indexed by $n \in \boldsymbol{Z}$ and $j=1, 2$, satisfying

$$[\psi_m^{(j)}, \psi_n^{(k)}]_+ = 0, \quad [\psi_m^{(j)*}, \psi_n^{(k)*}]_+ = 0,$$
$$[\psi_m^{(j)}, \psi_n^{(k)*}]_+ = \delta_{jk}\delta_{mn}.$$

Such fermions are obtainable by renumbering the fermions of a single component. For example, the simplest choice is

$$\psi_n^{(1)} = \psi_{2n}, \quad \psi_n^{(2)} = \psi_{2n+1}, \qquad (4.1)$$
$$\psi_n^{(1)*} = \psi_{2n}^*, \quad \psi_n^{(2)*} = \psi_{2n+1}^*.$$

Fixing the renumbering (4.1), we identify the Clifford algebra, the Fock space, the vacuum, etc. for the 2 component fermions with the previous ones.

A significant difference of two theories lies in the time flows. The natural time flows for the 2 component fermions are induced by the following Hamiltonian: We introduce time variables $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$ $(j=1, 2)$ and set

$$H(x^{(1)}, x^{(2)}) = \sum_{\substack{l=1,2,\ldots \\ n\in\boldsymbol{Z} \\ j=1,2}} x_l^{(j)} \psi_n^{(j)} \psi_{n+l}^{(j)*}.$$

We are going to construct an alternative realization of the Fock space by using $H(x^{(1)}, x^{(2)})$ instead of $H(x)$. In the previous case we had to consider the inner products with the vectors $\langle l | e^{H(x)} (l \in \boldsymbol{Z})$ in order to recover $a|\text{vac}\rangle$. This is because the flows induced by $e^{H(x)}$ preserves the charge l. In the present case the charge is preserved componentwise. Therefore, we have to choose representative vectors, one from each sector of fixed charges l_1 and l_2. (Here we denote by l_j the charge with respect to the j-th component of fermions.) Our choice is

$$|l_2, l_1\rangle = \Psi_{l_2}^{(2)} \Psi_{l_1}^{(1)} |\text{vac}\rangle, \qquad (4.2)$$
$$\langle l_1, l_2| = \langle \text{vac}| \Psi_{l_1}^{(1)*} \Psi_{l_2}^{(2)*},$$

where $\Psi_l^{(j)}$ and $\Psi_l^{(j)*}$ stand for Ψ_l and Ψ_l^* of (1.10) with ψ_n, ψ_n^* replaced by $\psi_n^{(j)}$, $\psi_n^{(j)*}$. The following are immediate consequences of (1.21).

$$\langle l_1, l_2 | \psi^{(1)}(k) e^{H(x^{(1)}, x^{(2)})}$$
$$= (-)^{l_2} k^{l_1 - 1} \langle l_1 - 1, l_2 | e^{H(x^{(1)} - \varepsilon(k^{-1}), x^{(2)})}$$
$$\langle l_1, l_2 | \psi^{(2)}(k) e^{H(x^{(1)}, x^{(2)})}$$
$$= k^{l_2 - 1} \langle l_1, l_2 - 1 | e^{H(x^{(1)}, x^{(2)} - \varepsilon(k^{-1}))},$$
$$\langle l_1, l_2 | \psi^{(1)*}(k) e^{H(x^{(1)}, x^{(2)})}$$
$$= (-)^{l_2} k^{-l_1} \langle l_1 + 1, l_2 | e^{H(x^{(1)} + \varepsilon(k^{-1}), x^{(2)})},$$

$$\langle l_1, l_2 | \psi^{(2)*}(k) e^{H(x^{(1)},x^{(2)})}$$
$$= k^{-l_2} \langle l_1, l_2+1 | e^{H(x^{(1)},x^{(2)}+\epsilon(k^{-1}))} .$$

Now we give the 2-component realization of $\mathscr{F}$. Let V_{l_1,l_2} $(l_1, l_2 \in \boldsymbol{Z})$ denote copies of the polynomial ring $\boldsymbol{C}[x^{(1)}, x^{(2)}]$. Then we have an isomorphism

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \bigoplus_{l_1,l_2\in\boldsymbol{Z}} V_{l_1,l_2} \\ \Psi & & \Psi \\ a|\mathrm{vac}\rangle & & \bigoplus_{l_1,l_2\in\boldsymbol{Z}} \langle l_1, l_2 | e^{H(x^{(1)},x^{(2)})} a | \mathrm{vac}\rangle \end{array}$$

The action of $\psi_n^{(j)}$ (resp. $\psi_n^{(j)*}$) is realized by $\hat{X}_n^{(j)}$ (resp. $\hat{X}_n^{(j)*}$) given below (See §1, Theorem 1.2.):

$$\hat{X}_n^{(1)}: V_{l_1,l_2} \longrightarrow V_{l_1+1,l_2}, \quad f_{l_1l_2}(x) \longrightarrow (-)^{l_2} X_{n-l_1}\Big(x^{(1)}, \frac{\partial}{\partial x^{(1)}}\Big) f_{l_1l_2}(x),$$

$$\hat{X}_n^{(1)*}: V_{l_1,l_2} \longrightarrow V_{l_1-1,l_2}, \quad f_{l_1l_2}(x) \longrightarrow (-)^{l_2} X^*_{n-l_1+1}\Big(x,^{(1)} \frac{\partial}{\partial x^{(1)}}\Big) f_{l_1l_2}(x),$$

$$\hat{X}_n^{(2)}: V_{l_1,l_2} \longrightarrow V_{l_1,l_2+1}, \quad f_{l_1l_2}(x) \longrightarrow X_{n-l_2}\Big(x^{(2)}, \frac{\partial}{\partial x^{(2)}}\Big) f_{l_1l_2}(x),$$

$$\hat{X}_n^{(2)*}: V_{l_1,l_2} \longrightarrow V_{l_1,l_2-1}, \quad f_{l_1l_2}(x) \longrightarrow X^*_{n-l_2+1}\Big(x^{(2)}, \frac{\partial}{\partial x^{(2)}}\Big) f_{l_1l_2}(x).$$

If we adopt (3.1) with the interpretation (4.1) as the Chevalley basis, the highest weight vectors are $|l-1, l\rangle$ and $|l, l\rangle$ $(l \in \boldsymbol{Z})$. This is dependent on the particular choice of the renumbering. Therefore, it is natural to consider general vectors $|l_2, l_1\rangle$ when we define the τ functions. Noting that g preserves the total charge l_1+l_2, we define

$$\tau_{l_1,l_2;l}(x^{(1)}, x^{(2)}) = \langle l_1, l_2 | e^{H(x^{(1)},x^{(2)})} g | l_2-l, l_1+l\rangle . \tag{4.3}$$

The bilinear identity is valid in the following form. For $l_1-l_1' \geqq l'-l \geqq l_2'-l_2+2$, we have

$$\sum_{j=1,2} \oint \frac{dk}{2\pi ik} \langle l_1, l_2 | e^{H(x^{(1)},x^{(2)})} \psi^{(j)}(k) g | l_2-l-1, l_1+l\rangle$$
$$\times \langle l_1', l_2' | e^{H(x^{(1)\prime},x^{(2)\prime})} \psi^{(j)*}(k) g | l_2'-l'+1, l_1'+l'\rangle = 0 .$$

Rewriting this we obtain

$$\oint \frac{dk}{2\pi ik} (-)^{l_2+l_2'} k^{l_1-1-l_1'} e^{\xi(x^{(1)}-x^{(1)\prime}, k)} \tag{4.4}$$
$$\times \tau_{l_1-1,l_2;l+1}(x^{(1)} - \epsilon(k^{-1}), x^{(2)})$$
$$\times \tau_{l_1'+1,l_2';l'-1}(x^{(1)\prime} + \epsilon(k^{-1}), x^{(2)\prime})$$

$$+\oint\frac{dk}{2\pi ik}k^{l_2-1-l_2'}e^{\xi(x^{(2)}-x^{(2)\prime},k)}$$
$$\times\tau_{l_1,l_2-1;l}(x^{(1)},x^{(2)}-\epsilon(k^{-1}))\tau_{l_1',l_2'+1;l'}(x^{(1)\prime},x^{(2)\prime}+\epsilon(k^{-1}))=0\,.$$

In particular, setting $l_1'=l_1-2$, $l_2'=l_2$, $l'=l+2$ and $x^{(2)}=x^{(2)\prime}$ we obtain (2.4) for $\tau(x)=\tau_{l_1-1,l_2;l+1}(x,x^{(2)})$. In other words the τ function (4.3) for the 2 component theory also solves the single component KP hierarchy.

As an example of (4.4), we have the following bilinear equations for $f=\tau_{l_1l_2;l}$, $g=\tau_{l_1-1,l_2+1;l+1}$ and $g^*=\tau_{l_1+1,l_2-1;l-1}$:

$$(D_{x_2^{(1)}}-D^2_{x_1^{(1)}})f\cdot g=0,\qquad (D_{x_2^{(1)}}-D^2_{x_1^{(1)}})g^*\cdot f=0,$$
$$(D_{x_2^{(2)}}+D^2_{x_1^{(2)}})f\cdot g=0,\qquad (D_{x_2^{(2)}}+D^2_{x_1^{(2)}})g^*\cdot f=0,$$
$$D_{x_1^{(1)}}D_{x_1^{(2)}}f\cdot f-2gg^*=0\,.$$

Setting $x_2^{(1)}=-x_2^{(2)}=t$, $x_1^{(1)}=x$ and $x_1^{(2)}=y$, we obtain the *2-dimensional non-linear Schrödinger equation* (see [39]).

Now we are interested in the reduction to $A_1^{(1)}$. In terms of $\psi_n^{(j)}$, $\psi_n^{(j)*}$ the automorphism ι_2 reads as

$$\iota_2(\psi_n^{(j)})=\psi_{n+1}^{(j)},\quad \iota_2(\psi_n^{(j)*})=\psi_{n+1}^{(j)*}. \tag{4.5}$$

The corresponding τ functions satisfy

$$\tau_{l_1+n,l_2+n;l}(x^{(1)}+a,x^{(2)}+a)=(-)^{ln}\tau_{l_1,l_2;l}(x^{(1)},x^{(2)})\,.$$

In Appendix 2, the lower order Hirota equations are given for this reduced 2 component KP hierarchy.

The following soliton equations are contained in this hierarchy:

The non-linear Schrödinger equation [40]

$$\frac{\partial q}{\partial x_2}+\frac{\partial^2 q}{\partial x_1^2}-2q^*q^2=0$$
$$-\frac{\partial q}{\partial x_2}+\frac{\partial^2 q}{\partial x_1^2}-2q^{*2}q=0\,,$$

where $q=G/F$, $q^*=G^*/F$.

The non-linear Schrödinger equation with a derivative coupling [41]

$$\frac{\partial q}{\partial x_2}+\frac{\partial^2 q}{\partial x_1^2}-2q^*q\frac{\partial q}{\partial x_1}=0$$
$$-\frac{\partial q^*}{\partial x_2}+\frac{\partial^2 q^*}{\partial x_1^2}+2q^*q\frac{\partial q^*}{\partial x_1}=0$$

where $q=f/F$, $q^*=G^*/g^*$.

The Heisenberg ferromagnet equation [42]

$$\frac{\partial S}{\partial x_2}=iS\times\frac{\partial^2 S}{\partial x_1^2},\ S^2=1$$

where $S=\left(\frac{f^*g+g^*f}{f^*f+g^*g},\ -i\frac{f^*g-g^*f}{f^*f+g^*g},\ \frac{f^*f-g^*g}{f^*f+g^*g}\right)$.

We remark that all the results given here have straightforward generalizations to the N-component case.

§5. Algebras B_∞ and C_∞

In this section we introduce Lie algebras B_∞ and C_∞, which are the infinite dimensional analogues of the classical Lie algebras B_l and C_l, respectively.

Consider the automorphisms σ_l ($l\in \boldsymbol{Z}$) of the Clifford algebra A given by

$$\sigma_l(\psi_n)=(-)^{l-n}\psi^*_{l-n},\qquad (5.1)$$
$$\sigma_l(\psi^*_n)=(-)^{l-n}\psi_{l-n}.$$

We define B_∞ and C_∞ as subalgebras in A_∞ consisting of those elements which are fixed by σ_0 and σ_{-1}, respectively.

$$B_\infty=\{X\in A_\infty\mid\sigma_0(X)=X\},$$
$$C_\infty=\{X\in A_\infty\mid\sigma_{-1}(X)=X\}.$$

Since $\iota_j^{-1}\sigma_l\iota_j=\sigma_{l+2j}$ on A_∞, it is general enough to consider σ_0 and σ_{-1}. We note also that

$$\sigma_l(H_n)=\begin{cases} H_n & n\text{: odd}. \\ -H_n & n\text{: even}\end{cases}$$

Here we give the Dynkin diagrams and the Chevalley basis for B_∞ and C_∞.

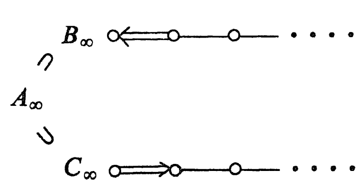


Fig. 8. Dynkin diagrams for B_∞ and C_∞.

The Chevalley basis for B_∞:

$$e_0=\sqrt{2}(\psi_{-1}\psi_0^*+\psi_0\psi_1^*),\qquad (5.2)$$
$$e_j=\psi_{-j-1}\psi^*_{-j}+\psi_j\psi^*_{j+1},\qquad (j\geqq 1),$$

$$f_0=\sqrt{2}(\psi_0\psi^*_{-1}+\psi_1\psi^*_0),$$
$$f_j=\psi_{-j}\psi^*_{-j-1}+\psi_{j+1}\psi^*_j, \qquad (j\geqq 1),$$
$$h_0=2(\psi_{-1}\psi^*_{-1}-\psi_1\psi^*_1),$$
$$h_j=\psi_{-j-1}\psi^*_{-j-1}-\psi_{-j}\psi^*_{-j}+\psi_j\psi^*_j-\psi_{j+1}\psi^*_{j+1}, \qquad (j\geqq 1).$$

The Chevalley basis for C_∞:

(5.3)
$$e_0=\psi_{-1}\psi^*_0,$$
$$e_j=\psi_{j-1}\psi^*_j+\psi_{-j-1}\psi^*_{-j}, \qquad (j\geqq 1),$$
$$f_0=\psi_0\psi^*_{-1},$$
$$f_j=\psi_j\psi^*_{j-1}+\psi_{-j}\psi^*_{-j-1}, \qquad (j\geqq 1),$$
$$h_0=\psi_{-1}\psi^*_{-1}-\psi_0\psi^*_0,$$
$$h_j=\psi_{j-1}\psi^*_{j-1}-\psi_j\psi^*_j+\psi_{-j-1}\psi^*_{-j-1}-\psi_{-j}\psi^*_{-j}, \qquad (j\geqq 1).$$

The highest weight vectors $|l\rangle$ generate highest weight modules for B_∞ and C_∞. Here we give the table of the correspondence between $|l\rangle$ and its weight as the highest weight vector of the B_∞ module or the C_∞ module:

	B_l		C_l	
$\|l\rangle$	Λ_{l-1}	$(l\geqq 2)$	Λ_l	$(l\geqq 0)$
	$2\Lambda_0$	$(l=0, 1)$	Λ_{-l}	$(l<0)$
	Λ_{-l}	$(l\leqq -1)$		

As a B_∞ module, $\mathscr{F}_l$ is irreducible. On the other hand, as a C_∞ module, $\mathscr{F}_l$ splits into irreducible components. Denoting by $\mathscr{F}'_l$ the C_∞ module generated by $|l\rangle$ we have ([13])

$$\mathscr{F}_l\cong\mathscr{F}_{-l}\cong\mathscr{F}'_l\oplus\mathscr{F}'_{l+2}\oplus\mathscr{F}'_{l+4}\oplus\cdots.$$

Consider $\tau_l(x)=\tau_l(x;g)$ of (2.2) with $X_1,\ldots,X_k$ belonging to B_∞ or C_∞. We use the notation $\tilde{x}=(x_1,-x_2,x_3,-x_4,\ldots)$. The σ_j-invariance $(j=0,-1)$ of X_i $(i=1,\ldots,k)$ implies the following invariance of the respective τ functions:

(5.4)
$$\tau_l(x)=\tau_{1-l}(\tilde{x}), \qquad \text{for } B_\infty,$$
$$\tau_l(x)=\tau_{-l}(\tilde{x}), \qquad \text{for } C_\infty.$$

Consider the case of B_∞, and let $l=0$. Substituting the Taylor expansions with respect to $x_2, x_4,\ldots$ (cf. (5.4))

$$\tau_0(x)=f_0(x_{odd})+x_2f_1(x_{odd})+\frac{1}{2}x_2^2f_2(x_{odd})+x_4f_3(x_{odd})+\cdots$$
$$\tau_1(x)=f_0(x_{odd})-x_2f_1(x_{odd})+\frac{1}{2}x_2^2f_2(x_{odd})-x_4f_3(x_{odd})+\cdots$$

into the modified KP hierarchy, we obtain

$$D_1^2 f_0\cdot f_0+2f_0\cdot f_1=0$$
$$(D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5)f_0\cdot f_0+30(4D_1D_3-D_1^4)f_1\cdot f_0=0.$$

(Using the KP and the modified KP hierarchies, higher order terms $f_2, f_3,\dots$ are solved in terms of f_0 and f_1.) We thus get the BKP equation

(5.5) $\quad l=0$
$$9\frac{\partial^2 u}{\partial x_1\partial x_5}-5\frac{\partial^2 u}{\partial x_3^2}+\frac{\partial}{\partial x_1}\Big(-5\frac{\partial^3 u}{\partial x_1^2\partial x_3}-15\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}+\frac{\partial^5 u}{\partial x_1^5}+15\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3}+15\Big(\frac{\partial u}{\partial x_1}\Big)^3\Big)=0,$$
$$u=\frac{\partial}{\partial x_1}\log\tau_0(x)|_{x_2=x_4=\cdots=0}.$$

Equations corresponding to other vertices are obtained in a similar manner. For instance, the case $l=-1$ reads

$l=-1$
$$9\frac{\partial^2 u}{\partial x_1\partial x_5}-5\frac{\partial^2 u}{\partial x_3^2}+\frac{\partial}{\partial x_1}\Big(-5\frac{\partial^3 u}{\partial x_1^2\partial x_3}-15\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}+\frac{\partial^5 u}{\partial x_1^5}+15\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3}+15\Big(\frac{\partial u}{\partial x_1}\Big)^3+\frac{45}{4}\Big(\frac{\partial^2 u}{\partial x_1^2}\Big)^2\Big)=\frac{45}{2}\frac{\partial v}{\partial x_1}\frac{\partial^2 v}{\partial x_1^2},$$
$$\frac{\partial v}{\partial x_3}+3v\frac{\partial v}{\partial x_1}+2\frac{\partial^3 v}{\partial x_1^3}+6\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+3\frac{\partial^2 u}{\partial x_1^2}v+3\frac{\partial^2 u}{\partial x_1\partial x_3}=0,$$
$$u=\frac{\partial}{\partial x_1}\log\tau_{-1}(x)|_{x_2=x_4=\cdots=0},\quad v=\frac{\partial}{\partial x_2}\log\tau_{-1}(x)|_{x_2=x_4=\cdots0}.$$

In the case of C_∞ we have likewise:

$l=0$
$$9\frac{\partial^2 u}{\partial x_1\partial x_5}-5\frac{\partial^2 u}{\partial x_3^2}+\frac{\partial}{\partial x_1}\Big(-5\frac{\partial^3 u}{\partial x_1^2\partial x_3}-15\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}+\frac{\partial^5 u}{\partial x_1^5}+15\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3}+15\Big(\frac{\partial u}{\partial x_1}\Big)^3+\frac{45}{4}\Big(\frac{\partial^2 u}{\partial x_1^2}\Big)^2\Big)=0,$$
$$u=\frac{\partial}{\partial x_1}\log\tau_0(x)|_{x_2=x_4=\cdots0},$$

$l=1$
$$9\frac{\partial^2 u}{\partial x_1\partial x_5}-5\frac{\partial^2 u}{\partial x_3^2}+\frac{\partial}{\partial x_1}\Big(-5\frac{\partial^3 u}{\partial x_1^2\partial x_3}-15\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}+\frac{\partial^5 u}{\partial x_1^5}+15\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3}+15\Big(\frac{\partial u}{\partial x_1}\Big)^3+\frac{45}{4}\Big(\frac{\partial^2 u}{\partial x_1^2}\Big)^2\Big)=\frac{45}{2}\frac{\partial v}{\partial x_1}\frac{\partial^2 v}{\partial x_1^2},$$
$$2\frac{\partial v}{\partial x_3}-6v\frac{\partial v}{\partial x_1}+\frac{\partial^3 v}{\partial x_1^3}+6\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+\frac{\partial}{\partial x_1}\Big(-4\frac{\partial u}{\partial x_3}+6\Big(\frac{\partial u}{\partial x_1}\Big)^2+\frac{\partial^3 u}{\partial x_1^3}\Big)=0,$$
$$u=\frac{\partial}{\partial x_1}\log\tau_1(x)|_{x_2=x_4=\cdots=0},\quad v=\frac{\partial}{\partial x_2}\log\tau_1(x)|_{x_2=x_4=\cdots=0}.$$

§ 6. Spin Representation of B_∞

In this section we construct the spin representation of B_∞ by exploiting neutral free fermions ϕ_n ($n \in \boldsymbol{Z}$) satisfying

$$[\phi_m, \phi_n]_+ = (-)^m \delta_{m,-n}.$$

By the spin representation we mean the representation with the highest weight Λ_0. Note that the construction in Section 5 affords us the representation of B_∞ with the highest weight $2\Lambda_0$, but not Λ_0.

The charged free fermions introduced in Section 1 split into two sets of neutral free fermions. Namely, if we set

$$(6.1)\qquad \phi_m = \frac{\psi_m + (-)^m \psi^*_{-m}}{\sqrt{2}}, \quad \hat{\phi}_m = i\frac{\psi_m - (-)^m \psi^*_{-m}}{\sqrt{2}}, \quad (m \in \boldsymbol{Z}),$$

we have $[\phi_m, \phi_n]_+ = (-)^m \delta_{m,-n}$, $[\hat{\phi}_m, \hat{\phi}_n]_+ = (-)^m \delta_{m,-n}$ and $[\phi_m, \hat{\phi}_n]_+ = 0$.

We denote by $\mathbf{A}'$ (resp. $\hat{\mathbf{A}}'$) the subalgebra of $\mathbf{A}$ generated by ϕ_m (resp. $\hat{\phi}_m$) ($m \in \boldsymbol{Z}$), and by $\mathscr{F}'$ (resp. $\hat{\mathscr{F}}'$) the $\mathbf{A}'$ (resp. $\hat{\mathbf{A}}'$) submodule of $\mathscr{F}$ generated by $|0\rangle$. Note that

$$\phi_n|0\rangle = \hat{\phi}_n|0\rangle = 0 \ (n<0), \ \langle 0|\phi_n = \langle 0|\hat{\phi}_n = 0 \ (n>0).$$

We also remark that

$$\langle 0|\phi_m\hat{\phi}_n|0\rangle = -\langle 0|\hat{\phi}_n\phi_m|0\rangle = \delta_{m0}\delta_{n0}\frac{i}{2},$$

$$\langle 1|\phi_m\hat{\phi}_n|1\rangle = -\langle 1|\hat{\phi}_n\phi_m|1\rangle = -\delta_{m0}\delta_{n0}\frac{i}{2}.$$

An even element in $\mathbf{A}'$ (resp. $\hat{\mathbf{A}}'$) can be written as $a+\phi_0 b$ (resp. $\hat{a}+\hat{\phi}_0\hat{b}$) with a and b (resp. $\hat{a}$ and $\hat{b}$) not containing ϕ_0 (resp. $\hat{\phi}_0$). Then we have

$$(6.2)\qquad \begin{aligned}\langle 0|(a+\phi_0 b)(\hat{a}+\hat{\phi}_0\hat{b})|0\rangle &= \langle 1|(a+\phi_0 b)(\hat{a}+\hat{\phi}_0\hat{b})|1\rangle \\ &= \langle 0|a|0\rangle \cdot \langle 0|\hat{a}|0\rangle.\end{aligned}$$

Consider the Lie algebra

$$B'_\infty = \{\textstyle\sum a_{ij} {:}\, \phi_i\phi_j {:} \mid {}^\exists N,\ a_{ij}=0 \quad \text{if} \quad |i+j|>N\}.$$

This is isomorphic to B_∞. We define an automorphism κ of $\mathbf{A}$ by $\kappa(\psi_m) = i\psi_m$, $\kappa(\psi^*_m) = -i\psi^*_m$, or equivalently, $\kappa(\phi_m) = \hat{\phi}_m$, $\kappa(\hat{\phi}_m) = -\phi_m$. Then

$$(6.3)\qquad \begin{array}{ccc} B'_\infty & \xrightarrow{\cong} & B_\infty \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ X & \longmapsto & X+\kappa(X), \end{array}$$

is an isomorphism.

The Chevalley basis of B'_∞ translated from (5.2) is as follows.

$$\begin{aligned} &e_0=\sqrt{2}\phi_{-1}\phi_0, \quad e_j=\phi_{-j-1}\phi_j, && (j\geqq 1),\\ &f_0=\sqrt{2}\phi_1\phi_0, \quad f_j=\phi_{j+1}\phi_{-j}, && (j\geqq 1),\\ &h_0=2\phi_1\phi_{-1}+1, \quad h_j=(-)^j(\phi_j\phi_{-j}+\phi_{j+1}\phi_{-j-1}), && (j\geqq 1). \end{aligned}$$

The Lie algebra B'_∞ does not belong to A_∞, but its action on $\mathscr{F}$ is well-defined by (1.7). In particular, $\mathscr{F}'$ is a B'_∞ module. It splits into two irreducible B'_∞ modules. Namely $\mathscr{F}'=\mathscr{F}'_{even}\oplus\mathscr{F}'_{odd}$, where $\mathscr{F}'_{even}$ (resp. $\mathscr{F}'_{odd}$) is generated by the highest weight vector $|0\rangle$ (resp. $|1\rangle$). Its highest weight is Λ_0 (resp. Λ_1). (Note that $|1\rangle=\sqrt{2}\phi_0|0\rangle$.) Thus we have constructed the spin representation of $B'_\infty\cong B_\infty$.

Now we construct the realization of $\mathscr{F}'$. Set

$$x_{odd}=(x_1, x_3, x_5,\dots)$$

and

$$H'(x_{odd})=\frac{1}{2}\sum_{\substack{l=1,3,\dots\\ n\in\mathbf{Z}}}(-)^{n+1}x_l\phi_n\phi_{-n-l}.$$

Then we have

$$H(x)|_{x_2=x_4=\cdots=0}=H'(x_{odd})+\kappa(H'(x_{odd})). \tag{6.4}$$

Setting $\phi(k)=\sum_{n\in\mathbf{Z}}\phi_n k^n$, we have

$$\begin{aligned} &\langle 0|\phi(k)e^{H'(x_{odd})}=\frac{1}{\sqrt{2}}\langle 1|e^{H'(x_{odd}-\epsilon'(k^{-1}))},\\ &\langle 1|\phi(k)e^{H'(x_{odd})}=\frac{1}{\sqrt{2}}\langle 0|e^{H'(x_{odd}-\epsilon'(k^{-1}))}, \end{aligned} \tag{6.5}$$

where $\epsilon'(k^{-1})=\left(\frac{2}{k}, \frac{2}{3k^3}, \frac{2}{5k^5},\dots\right)$.

Let V'_i $(i=0, 1)$ be copies of the polynomial ring $\mathbf{C}[x_{odd}]$. By using formulas (6.5), we obtain an isomorphism:

$$\begin{array}{ccc} \mathscr{F}' & \xrightarrow{\ \cong\ } & V'_0\oplus V'_1\\ \cup & & \cup\\ a|0\rangle & \longmapsto & \langle 0|e^{H'(x_{odd})}a|0\rangle\oplus\langle 1|e^{H'(x_{odd})}a|0\rangle. \end{array}$$

We introduce the following vertex operator.

$$X'(k)=\frac{1}{\sqrt{2}}\exp\Big(\sum_{n=1,3,\dots}x_nk^n\Big)\exp\Big(-2\sum_{n=1,3,\dots}\frac{1}{n}\frac{\partial}{\partial x_n}k^{-n}\Big).$$

Then the action of $\phi(k)$ on $\mathscr{F}'$ is realized as follows.

$$V'_0 \longrightarrow V'_1,\quad f_0(x_{odd}) \longmapsto X'(k)f_0(x_{odd}),$$
$$V'_1 \longrightarrow V'_0,\quad f_1(x_{odd}) \longmapsto X'(k)f_1(x_{odd}).$$

Now consider the τ function

$$\tau(x_{odd}) = \langle 0 | e^{H'(x_{odd})} g | 0 \rangle,$$
$$= \langle 1 | e^{H'(x_{odd})} g | 1 \rangle,$$

where $g = e^{X_1} \cdots e^{X_k}$ with locally nilpotent $X_1, \ldots, X_k \in B'_\infty$. We have

$$\oint \frac{dk}{2\pi ik} \langle 1 | e^{H'(x_{odd})} \phi(k) g | 0 \rangle \langle 1 | e^{H'(x'_{odd})} \phi(-k) g | 0 \rangle$$
$$= \frac{1}{2} \tau(x_{odd}) \tau(x'_{odd}).$$

By using (6.4) we rewrite this to obtain

(6.6) $$\sum_{j=1}^{\infty} \tilde{p}_j(-2y_{odd}) \tilde{p}_j(2\tilde{D}_{odd}) \exp\Big(\sum_{l:odd} y_l D_l\Big) \tau(x_{odd}) \cdot \tau(x_{odd}) = 0,$$

where $\exp \sum_{l:odd} k^l x_l = \sum_{j=0}^{\infty} \tilde{p}_j(x_{odd}) k^j$, $y_{odd} = (y_1, y_3, \ldots, y_{2n+1}, \ldots)$ and $\tilde{D}_{odd} = (D_1, D_3/3, \ldots, D_{2n+1}/(2n+1), \ldots)$. The lower order equations are explicitly given in Appendix 3.

In Section 5 we defined the τ function $\tau_0(x)$, which corresponds to the highest weight $2\Lambda_0$, and in this section we obtained $\tau(x_{odd})$, which corresponds to Λ_0. Choose the group element g for $\tau_0(x)$ and g' for $\tau(x_{odd})$ so that they correspond to each other by (6.3). Then, they are actually related to each other by

(6.7) $$\tau(x_{odd})^2 = \tau_0(x) |_{x_2 = x_4 = \cdots = 0}.$$

This is a consequence of (6.2), (6.3) and (6.4), and implies that the non linear equations for the variable $u(x_{odd}) = \partial^2 \log \tau(x_{odd}) / \partial x_1^2$ are the same as (5.5).

§7. Algebras D_∞ and D'_∞

In this section we introduce the algebras D_∞ and D'_∞, which are infinite dimensional versions of even demensional orthogonal Lie algebras. Actually, D_∞ and D'_∞ are isomorphic. The difference is that D'_∞ is appropriate for the spin representations with the highest weights Λ_0 and Λ_1, and D_∞ is appropriate for the representations with the highest weights $2\Lambda_0$, $2\Lambda_1$, $\Lambda_0 + \Lambda_1$ and $\Lambda_j\ (j \geq 2)$.

Denote by σ an automorphism of the Clifford algebra of the 2 component charged free fermion (see Section 4) given by

$$\sigma(\psi_n^{(j)})=(-)^n\psi_{-n}^{(j)*},\quad \sigma(\psi_n^{(j)*})=(-)^n\psi_{-n}^{(j)}.$$

Then we define

$$D_\infty=\{X\in A_\infty \mid \sigma(X)=X\}. \tag{7.1}$$

We can take the following Chevalley basis.

$$\left.\begin{matrix} e_0 \\ e_1 \end{matrix}\right\} =\frac{1}{\sqrt{2}}(\psi_{-1}^{(1)}(\psi_0^{(1)*}\pm i\psi_0^{(2)*})+(\psi_0^{(1)}\pm i\psi_0^{(2)})\psi_1^{(1)*}),$$

$$e_{2j}=\psi_j^{(1)}\psi_j^{(2)*}-\psi_{-j}^{(2)}\psi_{-j}^{(1)*}, \qquad (j\geq 1),$$

$$e_{2j+1}=\psi_j^{(2)}\psi_{j+1}^{(1)*}+\psi_{-j-1}^{(1)}\psi_{-j}^{(2)*},$$

$$\left.\begin{matrix} f_0 \\ f_1 \end{matrix}\right\} =\frac{1}{\sqrt{2}}((\psi_0^{(1)}\mp i\psi_0^{(2)})\psi_{-1}^{(1)*}+\psi_1^{(1)}(\psi_0^{(1)*}\mp i\psi_0^{(2)*})),$$

$$f_{2j}=\psi_j^{(2)}\psi_j^{(1)*}-\psi_{-j}^{(1)}\psi_{-j}^{(2)*}, \qquad (j\geq 1),$$

$$f_{2j+1}=\psi_{j+1}^{(1)}\psi_j^{(2)*}+\psi_{-j}^{(2)}\psi_{-j-1}^{(1)*},$$

$$\left.\begin{matrix} h_0 \\ h_1 \end{matrix}\right\} =\psi_{-1}^{(1)}\psi_{-1}^{(1)*}-\psi_1^{(1)}\psi_1^{(1)*}\pm i(\psi_0^{(2)}\psi_0^{(1)*}-\psi_0^{(1)}\psi_0^{(2)*})$$

$$h_{2j}=\psi_j^{(1)}\psi_j^{(1)*}-\psi_j^{(2)}\psi_j^{(2)*}+\psi_{-j}^{(2)}\psi_{-j}^{(2)*}-\psi_{-j}^{(1)}\psi_{-j}^{(1)*}, \qquad (j\geq 1).$$

$$h_{2j+1}=\psi_j^{(2)}\psi_j^{(2)*}-\psi_{j+1}^{(1)}\psi_{j+1}^{(1)*}+\psi_{-j-1}^{(1)}\psi_{-j-1}^{(1)*}-\psi_{-j}^{(2)}\psi_{-j}^{(2)*},$$

Then the Dynkin diagram for D_∞ is as follows.

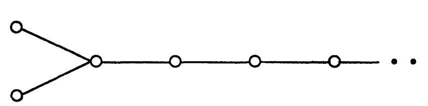


Fig. 9. Dynkin diagram for D_∞.

Because of (7.1) the A_∞ module $\mathscr{F}$ can be considered as a D_∞ module. It splits into irreducible components with the following highest weight vectors and the highest weights:

(7.2)

highest weight vectors	highest weights
$\lvert 0, 0\rangle$, $\lvert 1, 1\rangle$	$\Lambda_0+\Lambda_1$
$\lvert 0, 1\rangle+i\lvert 1, 0\rangle$	$2\Lambda_0$
$\lvert 0, 1\rangle-i\lvert 1, 0\rangle$	$2\Lambda_1$
$\left\lvert -\left[\frac{j-1}{2}\right], -\left[\frac{j}{2}\right]\right\rangle$, $\left\lvert \left[\frac{j+1}{2}\right], \left[\frac{j}{2}\right]+1\right\rangle$	$\Lambda_j \quad (j\geq 2)$

For $g=e^{X_1}\cdots e^{X_k}$ with locally nilpotent $X_1,\dots, X_k\in D_\infty$ the τ functions $\tau_{l_1,l_2;l}(x^{(1)}, x^{(2)})$ defined in (4.3) satisfy the following symmetry.

$$\tau_{l_1,l_2;l}(x^{(1)}, x^{(2)})=(-)^{l(l_1+l_2)}\tau_{1-l_1,1-l_2;-l}(\tilde{x}^{(1)}, \tilde{x}^{(2)}),$$

where $\tilde{x}^{(j)}=(x_1^{(j)}, -x_2^{(j)}, x_3^{(j)}, -x_4^{(j)},\dots)$.

Next, we define D'_∞ in terms of the charged free fermions as follows

$$\begin{aligned} D'_\infty=\{\textstyle\sum a_{jk}:\psi_j\psi_k^*: +\sum b_{jk}\psi_j\psi_k \\ +c_{jk}\textstyle\sum \psi_j^*\psi_k^*+d\,|\,{}^\exists N,\ a_{jk}=b_{j-k}=c_{-jk}=0 \quad\text{if}\quad |j-k|>N\}. \end{aligned}$$

In Section 6 we introduced the neutral free fermions ϕ_n and $\hat{\phi}_n$, which are related to ψ_n and ψ_{-n}^* by (6.1).

The algebra D'_∞ is equivalently defined as

$$\begin{aligned} D'_\infty=\{\textstyle\sum a_{jk}:\phi_j\phi_k: +\sum b_{jk}:\hat{\phi}_j\hat{\phi}_k: \\ +\textstyle\sum c_{jk}\phi_j\hat{\phi}_k+d\,|\,{}^\exists N,\ a_{jk}=b_{jk}=c_{jk}=0 \quad\text{if}\quad |j+k|>N\}. \end{aligned}$$

Note that

$$\begin{aligned} &(\phi_0+i\hat{\phi}_0)|0\rangle=0, \quad (\phi_0-i\hat{\phi}_0)|0\rangle=\sqrt{2}\,|1\rangle, \\ &(\phi_0+i\hat{\phi}_0)|1\rangle=\sqrt{2}\,|0\rangle, \quad (\phi_0-i\hat{\phi}_0)|1\rangle=0. \end{aligned}$$

We choose the Chevalley basis for D'_∞ so that the vacuums $|0>$ and $|1>$ are annihilated by e_j's. For notational simplicity we set

$$\begin{aligned} &\varphi_0=(\phi_0-i\hat{\phi}_0)/\sqrt{2}, \quad \varphi_{2j-1}=\phi_j, \quad \varphi_{2j}=\hat{\phi}_j, \qquad (j\geq 1), \\ &\varphi_0^*=(\phi_0+i\hat{\phi}_0)/\sqrt{2}, \quad \varphi_{2j-1}^*=(-)^j\phi_{-j}, \quad \varphi_{2j}^*=(-)^j\hat{\phi}_{-j}, \qquad (j\geq 1). \end{aligned}$$

Then we have

$$[\varphi_j, \varphi_k]_+=[\varphi_j^*, \varphi_k^*]_+=0, \quad [\varphi_j, \varphi_k^*]_+=\delta_{jk},$$

and φ_j^* annihilates $|0\rangle$. Our choice of the Chevalley basis is as follows.

$$\begin{aligned} &e_0=\varphi_1^*\varphi_0^*, \quad e_j=\varphi_j^*\varphi_{j-1}, \qquad (j\geq 1), \\ &f_0=\varphi_0\varphi_1, \quad f_j=\varphi_{j-1}^*\varphi_j, \qquad (j\geq 1), \\ &h_0=-\varphi_1\varphi_1^*-\varphi_0\varphi_0^*+1, \quad h_j=-\varphi_j\varphi_j^*+\varphi_{j-1}\varphi_{j-1}^*, \qquad (j\geq 1). \end{aligned}$$

This choice gives the Dynkin diagram of D'_∞ which is the same as that for D_∞. (See Fig. 9) In fact, by using a similar argument as in Section 6 we can show that D'_∞ is isomorphic to D_∞.

As a D'_∞ module, $\mathscr{F}$ splits into two irreducible highest weight modules. They are generated by $|0\rangle$ and $|1\rangle$, respectively, and their highest weights are Λ_0 and Λ_1, respectively. In this sense we call those representations the spin representations.

Let us consider the τ functions for the spin representations. First, we consider the time flows $x_{odd}=(x_1, x_3,\dots)$ and $\hat{x}_{odd}=(\hat{x}_1, \hat{x}_3,\dots)$ induced by the Hamiltonian

$$H'(x_{odd}, \hat{x}_{odd})=\frac{1}{2}\sum_{l:odd}(-)^{n+1}x_l\phi_n\phi_{-n-l}+\frac{1}{2}\sum_{\substack{l:odd\\ n\in\mathbf{Z}}}(-)^{n+1}\hat{x}_l\hat{\phi}_n\hat{\phi}_{-n-l}.$$

We set

$$\text{(7.3)}\qquad \begin{aligned}\tau_0(x_{odd}, \hat{x}_{odd})&=\langle 0|e^{H'(x_{odd},\hat{x}_{odd})}g|0\rangle,\\ \tau_1(x_{odd}, \hat{x}_{odd})&=\langle 1|e^{H'(x_{odd},\hat{x}_{odd})}g|1\rangle,\end{aligned}$$

where $g=e^{X_1}\cdots e^{X_k}$ with locally nilpotent $X_1,\dots, X_k\in D'_\infty$. As for $\hat{\phi}(k) = \sum_{n\in\mathbf{Z}}\hat{\phi}_n k^n$ we have the following formulas. (See (6.5).)

$$\text{(7.4)}\qquad \begin{aligned}\langle 0|\hat{\phi}(k)e^{H'(x_{odd},\hat{x}_{odd})}&=\frac{i}{\sqrt{2}}\langle 1|e^{H'(x_{odd},\hat{x}_{odd}-\epsilon'(k^{-1}))},\\ \langle 1|\hat{\phi}(k)e^{H'(x_{odd},\hat{x}_{odd})}&=\frac{-i}{\sqrt{2}}\langle 0|e^{H'(x_{odd},\hat{x}_{odd}-\epsilon'(k^{-1}))}.\end{aligned}$$

By using (7.3) and (7.4) we obtain the bilinear identities. (For notational simplicity we set $\phi^{(1)}(k)=\phi(k)$, $\phi^{(2)}(k)=\hat{\phi}(k)$ and $0^*=1$, $1^*=0$.)

$$\text{(7.5)}\qquad \begin{aligned}&\sum_{k=1,2}\oint\frac{dk}{2\pi ik}\langle l|e^{H'(x_{odd},\hat{x}_{odd})}\phi^{(j)}(k)g|l^*\rangle\\ &\quad\times\langle l'|e^{H'(x'_{odd},\hat{x}'_{odd})}\phi^{(j)}(-k)g|l'^*\rangle\\ &\quad=(1-\delta_{ll'})\tau_l(x_{odd}, \hat{x}_{odd})\tau_{l'}(x'_{odd}, \hat{x}'_{odd}),\qquad (l, l'=0, 1),\\ &(\sum_{j\geq 1}\tilde{p}_j(-2y_{odd})\tilde{p}_j(2\tilde{D}_{odd})+(-)^{\delta_{ll'}}\sum_{j\geq 1}\tilde{p}_j(-2\hat{y}_{odd})\tilde{p}_j(2\tilde{\hat{D}}_{odd}))\\ &\quad\times\exp(\sum_{l:odd}y_lD_l+\sum_{l:odd}\hat{y}_l\hat{D}_l)\tau_l\cdot\tau_{l'}\\ &\quad=(1-\delta_{ll'})2\exp(\sum_{l:odd}y_lD_l+\sum_{l:odd}\hat{y}_l\hat{D}_l)\tau_{l'}\cdot\tau_l,\qquad (l, l'=0, 1).\end{aligned}$$

The τ functions corresponding to $\Lambda_0+\Lambda_1$, $2\Lambda_0$, $2\Lambda_1$ and those corresponding to Λ_0, Λ_1 are related as follows (cf. (6.7)):

$$\text{(7.6)}\qquad \begin{aligned}&\langle 0, 0|e^{H(x,\hat{x})}g|0, 0\rangle|_{x_{even}=\hat{x}_{even}=0}\\ &\quad=\langle 1, 1|e^{H(x,\hat{x})}g|1, 1\rangle|_{x_{even}=\hat{x}_{even}=0}\\ &\quad=\langle 0|e^{H'(x_{odd},\hat{x}_{odd})}g'|0\rangle\cdot\langle 1|e^{H'(x_{odd},\hat{x}_{odd})}g'|1\rangle,\\ &\langle 0, 0|(\psi_0^{(1)*}-i\psi_0^{(2)*})e^{H(x,\hat{x})}g(\psi_0^{(1)}+i\psi_0^{(2)})|0, 0\rangle|_{x_{even}=\hat{x}_{even}=0}\\ &\quad=\langle 0|e^{H'(x_{odd},\hat{x}_{odd})}g'|0\rangle^2,\\ &\langle 0, 0|(\psi_0^{(1)*}+i\psi_0^{(2)*})e^{H(x,\hat{x})}g(\psi_0^{(1)}-i\psi_0^{(2)})|0, 0\rangle|_{x_{even}=\hat{x}_{even}=0}\\ &\quad=\langle 1|e^{H'(x_{odd},\hat{x}_{odd})}g'|1\rangle^2.\end{aligned}$$

Here g and g' should correspond to each other by the isomorphism $D_\infty\cong D'_\infty$.

Consider the time flows induced by the Hamiltonian $H(x)=\sum_{\substack{l=1,2,\dots\\ n\in\mathbf{Z}}} x_l\psi_n\psi^*_{n+l}$. We set

$$\tau_n(x)=\begin{cases}\langle n|e^{H(x)}g|0\rangle, & n:\text{ even}.\\ \langle n|e^{H(x)}g|1\rangle, & n:\text{ odd}\end{cases}$$

The following identities are valid.

$$\begin{aligned}(7.7)\qquad &\oint\frac{dk}{2\pi ik}\langle n|e^{H(x)}\psi(k)g|l\rangle\langle n'|e^{H(x')}\psi^*(k)g|l'\rangle\\ &\quad+\oint\frac{dk}{2\pi ik}\langle n|e^{H(x)}\psi^*(k)g|l\rangle\langle n'|e^{H(x')}\psi(k)g|l'\rangle\\ &=(1-\delta_{ll'})\tau_n(x)\tau_{n'}(x'),\qquad (n\equiv l+1,\ n'\equiv l'+1 \bmod 2),\\ &\sum p_j(-2y)p_{j+n-n'-1}(\tilde{D})\exp(\sum y_iD_i)\tau_{n'+1}(x)\cdot\tau_{n-1}(x)\\ &\quad+\sum p_j(2y)p_{j+n'-n-1}(\tilde{D})\exp(-\sum y_iD_i)\tau_{n+1}(x)\tau_{n'-1}(x)\\ &=(1-\delta_{ll'})\exp(\sum y_iD_i)\tau_{n'}(x)\tau_n(x).\end{aligned}$$

For example, (7.7) contains the equations

$$\begin{aligned}&(2D_3-3D_1D_2+D_1^3)\tau_{n+1}\cdot\tau_{n-1}=0\\ &(D_2-D_1^2)\tau_{n+1}\cdot\tau_n+2\tau_{n+2}\cdot\tau_{n-1}=0\\ &(4D_3-3D_1D_2-2D_1^3)\tau_{n+1}\cdot\tau_n-6D_1\tau_{n+2}\cdot\tau_{n-1}=0.\end{aligned}$$

Setting $u=\log(\tau_{n+1}/\tau_n)$, $v=\tau_{n+2}/\tau_n$ and $v^*=\tau_{n-1}/\tau_{n+1}$ we obtain

$$\begin{aligned}&4\frac{\partial^2u}{\partial x_1\partial x_3}-3\frac{\partial^2u}{\partial x_2^2}+3\frac{\partial}{\partial x_2}\left(\left(\frac{\partial u}{\partial x_1}\right)^2-2vv^*\right)+\frac{\partial}{\partial x_1}\left(-9\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right.\\ &\quad\left.-2\frac{\partial^3u}{\partial x_1^3}+4\left(\frac{\partial u}{\partial x_1}\right)^3-6vv^*\frac{\partial u}{\partial x_1}-6v^*\frac{\partial v}{\partial x_1}+6v\frac{\partial v^*}{\partial x_1}\right)=0,\\ &2\frac{\partial v}{\partial x_3}-3\frac{\partial^2v}{\partial x_1\partial x_2}+\frac{\partial^3v}{\partial x_1^3}+3\left(\frac{\partial u}{\partial x_2}-\frac{\partial^2u}{\partial x_1^2}-\left(\frac{\partial u}{\partial x_1}\right)^2+4v^*v\right)\frac{\partial v}{\partial x_1}-\left(4\frac{\partial u}{\partial x_3}\right.\\ &\quad\left.-3\frac{\partial^2u}{\partial x_1\partial x_2}-9\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}+4\left(\frac{\partial u}{\partial x_1}\right)^3-2\frac{\partial^3u}{\partial x_1^3}-6v^*v\frac{\partial u}{\partial x_1}+6v\frac{\partial v^*}{\partial x_1}\right)v=0\end{aligned}$$

and the equation obtained by the replacement $u\to -u$, $v\leftrightarrow v^*$, $x_2\to -x_2$.

§8. Reduction to Kac-Moody Lie Algebras

In Section 3 we have seen that the Kac-Moody Lie algebra $A_1^{(1)}$ is contained in A_∞ as a subalgebra, and that, correspondingly, the KP hierarchy reduces to the KdV hierarchy. In this section we list up such reductions for $A_l^{(1)}$, $C_l^{(1)}$, $D_l^{(1)}$, $A_{2l}^{(2)}$, $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$.

We call $X=\sum_{i,j\in\mathbf{Z}} a_{ij}:\psi_i\psi_j^*:+c\in A_\infty$ l-reduced if and only if the following conditions (i) and (ii) are satisfied:

(i) $$a_{i+l,j+l}=a_{i,j}, \qquad (i, j\in \boldsymbol{Z}),$$

(ii) $$\sum_{i=0}^{l-1} a_{i,i+jl}=0, \qquad (j\in \boldsymbol{Z}).$$

The condition (i) is equivalent to the commutativity $[\mathrm{ad}\,X, \iota_l]=0$. Note that $H_{jl}=\sum_{i\in \boldsymbol{Z}} :\psi_i\psi^*_{i+jl}:$ satisfies (i), but not (ii). We call $X=\sum_{\mu,\nu=1,2}\sum_{i,j\in \boldsymbol{Z}} a^{(\mu\nu)}_{ij}:\psi^{(\mu)}_i\psi^{(\nu)*}_j: +c\in A_\infty$ (l_1, l_2)-reduced if and only if the following conditions (i)′ and (ii)′ are satisfied.

(i)′ $$a^{(\mu\nu)}_{i+l_\mu,j+l_\nu}=a^{(\mu\nu)}_{i,j}, \qquad (\mu, \nu=1, 2, \quad i, j\in \boldsymbol{Z}),$$

(ii)′ $$\sum_{\mu=1,2}\sum_{i=0}^{l_\mu-1} a^{(\mu\mu)}_{i,i+jl_\mu}=0, \qquad (j\in \boldsymbol{Z}).$$

The condition (ii)′ is equivalent to the commutativity $[\mathrm{ad}\,X, \iota_{l_1,l_2}]=0$, where $\iota_{l_1,l_2}(\psi^{(\mu)}_i)=\psi^{(\mu)}_{i+l_\mu}$ and $\iota_{l_1,l_2}(\psi^{(\mu)*}_i)=\psi^{(\mu)*}_{i+l_\mu}$.

The algebras $A^{(1)}_l$, $D^{(2)}_{l+1}$, $A^{(2)}_{2l}$, $C^{(1)}_l$, $D^{(1)}_l$ and $A^{(2)}_{2l-1}$ are obtained as follows:

$$\begin{aligned}
A^{(1)}_l&=\{X\in A_\infty \mid X\colon (l+1)\text{-reduced}\},\\
D^{(2)}_{l+1}&=\{X\in B_\infty \mid X\colon 2(l+1)\text{-reduced}\}=A^{(1)}_{2l+1}\cap B_\infty,\\
A^{(2)}_{2l}&=\{X\in B_\infty \mid X\colon (2l+1)\text{-reduced}\}=A^{(1)}_{2l}\cap B_\infty,\\
&\cong\{X\in C_\infty \mid X\colon (2l+1)\text{-reduced}\}=A^{(1)}_{2l}\cap C_\infty,\\
C^{(1)}_l&=\{X\in C_\infty \mid X\colon 2l\text{-reduced}\}=A^{(1)}_{2l-1}\cap C_\infty,\\
D^{(1)}_l&=\{X\in D_\infty \mid X\colon (2l-2, 2)\text{-reduced}\},\\
A^{(2)}_{2l-1}&=\{X\in D_\infty \mid X\colon (2l-1, 1)\text{-reduced}\}.
\end{aligned}$$

Remark. The algebra $D^{(1)}_l$ (resp. $A^{(2)}_{2l-1}$) is also obtained as $(2(l-s), 2s)$-reduction (resp. $(2(l-s)-1, 2s+1)$-reduction) (see [7]).

To be explicit we give the list of the Chevalley basis in Table 1. For notational simplicity we omit $h_n=[e_n, f_n]$.

In the previous sections, we constructed highest weight modules of A_∞, B_∞, C_∞ and D_∞ by using the Fock representations and the vectors $|n\rangle$ or $|n_2, n_1\rangle$. Those vectors also serve as highest weight vectors of subalgebras. The following Table 2 gives such highest weight vectors and their weights.

By exploiting the isomorphism $B_\infty\cong B'_\infty$ and $D_\infty\cong D'_\infty$, we can construct highest weight modules corresponding to the spin representations of $D^{(2)}_{l+1}$, $A^{(2)}_{2l}$, $D^{(1)}_l$ and $A^{(2)}_{2l-1}$. We leave it to the reader to make tables of the Chevalley basis and the highest weight vectors.

In Table 3 we give the extra identities satisfied by the τ functions of reduced hierarchies.

Some examples of the corresponding soliton equations for algebras of lower rank are tabulated in Table 4. We also refer to the paper [19].

Table 1. The Chevalley basis

$A_l^{(1)}$

$$e_n=\sum_{\nu\in\mathbf{Z}}\psi_{n-1+(l+1)\nu}\psi^*_{n+(l+1)\nu},\qquad f_n=\sum_{\nu\in\mathbf{Z}}\psi_{n+(l+1)\nu}\psi^*_{n-1+(l+1)\nu},\qquad (n=0,\dots,l)$$

$D_{l+1}^{(2)}\subset A_{2l+1}^{(1)}$

$$e_n=\begin{cases}\sqrt{2}(\tilde e_0+\tilde e_1) & (n=0)\\ \tilde e_{n+1}+\tilde e_{2l+2-n} & (1\leqq n\leqq l-1)\\ \sqrt{2}(\tilde e_{l+1}+\tilde e_{l+2}) & (n=l)\end{cases}\qquad f_n=\begin{cases}\sqrt{2}(\tilde f_0+\tilde f_1) & (n=0)\\ \tilde f_{n+1}+\tilde f_{2l+2-n} & (1\leqq n\leqq l-1)\\ \sqrt{2}(\tilde f_{l+1}+\tilde f_{l+2}) & (n=l)\end{cases}$$

$A_{2l}^{(2)}\subset A_{2l}^{(1)}$

$$e_n=\begin{cases}\sqrt{2}(\tilde e_0+\tilde e_1) & (n=0)\\ \tilde e_{n+1}+\tilde e_{2l+1-n} & (1\leqq n\leqq l-1)\\ \tilde e_{l+1} & (n=l)\end{cases}\qquad f_n=\begin{cases}\sqrt{2}(\tilde f_0+\tilde f_1) & (n=0)\\ \tilde f_{n+1}+\tilde f_{2l+1-n} & (1\leqq n\leqq l-1)\\ \tilde f_{l+1} & (n=l)\end{cases}$$

$C_l^{(1)}\subset A_{2l-1}^{(1)}$

$$e_n=\begin{cases}\tilde e_n & (n=0,\ l)\\ \tilde e_n+\tilde e_{2l-n} & (1\leqq n\leqq l-1)\end{cases}\qquad f_n=\begin{cases}\tilde f_n & (n=0,\ l)\\ \tilde f_n+\tilde f_{2l-n} & (1\leqq n\leqq l-1)\end{cases}$$

$D_l^{(1)}$

$$\left.\begin{matrix}e_0\\ e_1\end{matrix}\right\}=\sum_{\nu\in\mathbf{Z}}\frac{1}{\sqrt{2}}\left(\psi^{(1)}_{-1+2(l-1)\nu}(\psi^{(1)*}_{2(l-1)\nu}\pm i\psi^{(2)*}_{2\nu})+(\psi^{(1)}_{2(l-1)\nu}\pm i\psi^{(2)}_{2\nu})\psi^{(1)*}_{1+2(l-1)\nu}\right),$$

$$e_n=\sum_{\nu\in\mathbf{Z}}\left(\psi^{(1)}_{-n+2(l-1)\nu}\psi^{(1)*}_{-n+1+2(l-1)\nu}+\psi^{(1)}_{n-1+2(l-1)\nu}\psi^{(1)*}_{n+2(l-1)\nu}\right),\qquad (2\leqq n\leqq l-2),$$

$$\left.\begin{matrix}e_{l-1}\\ e_l\end{matrix}\right\}=\sum_{\nu\in\mathbf{Z}}\frac{1}{\sqrt{2}}\Big((\psi^{(1)}_{-l+1+2(l-1)\nu}\pm\varepsilon^{-1}\psi^{(2)}_{-1+2\nu})\psi^{(1)*}_{-l+2+2(l-1)\nu}$$
$$+\psi^{(1)}_{l-2+2(l-1)\nu}(\psi^{(1)*}_{l-1+2(l-1)\nu}\mp\varepsilon\psi^{(2)*}_{1+2\nu})\Big),$$

$$\left.\begin{matrix}f_0\\ f_1\end{matrix}\right\}=\sum_{\nu\in\mathbf{Z}}\frac{1}{\sqrt{2}}\left((\psi^{(1)}_{2(l-1)\nu}\mp i\psi^{(2)}_{2\nu})\psi^{(1)*}_{-1+2(l-1)\nu}+\psi^{(1)*}_{1+2(l-1)\nu}(\psi^{(1)}_{2(l-1)\nu}\mp i\psi^{(2)}_{2\nu})\right),$$

$$f_n=\sum_{\nu\in\mathbf{Z}}\left(\psi^{(1)}_{-n+1+2(l-1)\nu}\psi^{(1)*}_{-n+2(l-1)\nu}+\psi^{(1)}_{n+2(l-1)\nu}\psi^{(1)*}_{n-1+2(l-1)\nu}\right),\qquad (2\leqq n\leqq l-2),$$

$$\left.\begin{matrix}f_{l-1}\\ f_l\end{matrix}\right\}=\sum_{\nu\in\mathbf{Z}}\frac{1}{\sqrt{2}}\Big(\psi^{(1)}_{-l+2+2(l-1)\nu}(\psi^{(1)*}_{-l+1+2(l-1)\nu}\pm\varepsilon\psi^{(2)*}_{-1+2\nu})$$
$$+(\psi^{(1)}_{l-1+2(l-1)\nu}\mp\varepsilon^{-1}\psi^{(2)}_{1+2\nu})\psi^{(1)*}_{l-2+2(l-1)\nu}\Big),$$

where $\varepsilon=\begin{cases}1 & l\text{: odd}\\ i & l\text{: even}\end{cases}$.

$A_{2l-1}^{(2)}$

$$\left.\begin{matrix}e_0\\ e_1\end{matrix}\right\}=\sum_{\nu\in\mathbf{Z}}\frac{1}{\sqrt{2}}\left(\psi^{(1)}_{-1+(2l-1)\nu}(\psi^{(1)*}_{(2l-1)\nu}\pm i\psi^{(2)*}_{\nu})+(\psi^{(1)}_{(2l-1)\nu}\pm i\psi^{(2)}_{\nu})\psi^{(1)*}_{1+(2l-1)\nu}\right),$$

$$e_n=\sum_{\nu\in Z}\psi^{(1)}_{-n+(2l-1)\nu}\psi^{(1)*}_{-n+1+(2l-1)\nu}+\psi^{(1)}_{n-1+(2l-1)\nu}\psi^{(1)*}_{n+(2l-1)\nu}, \qquad (2\leqq n\leqq l-1)$$

$$e_l=\sum_{\nu\in Z}\psi^{(1)}_{l-1+(2l-1)\nu}\psi^{(1)*}_{l+(2l-1)\nu},$$

$$\left.\begin{matrix} f_0 \\ f_1 \end{matrix}\right\}=\sum_{\nu\in Z}\frac{1}{\sqrt{2}}((\psi^{(1)}_{(2l-1)\nu}\mp i\psi^{(2)}_{\nu})\psi^{(1)*}_{-1+(2l-1)\nu}+\psi^{(1)}_{1+(2l-1)\nu}(\psi^{(1)*}_{(2l-1)\nu}\mp i\psi^{(2)*}_{\nu})),$$

$$f_n=\sum_{\nu\in Z}\psi^{(1)}_{-n+1+(2l-1)\nu}\psi^{(1)*}_{-n+(2l-1)\nu}+\psi^{(1)}_{n+(2l-1)\nu}\psi^{(1)*}_{n-1+(2l-1)\nu}, \qquad (2\leqq n\leqq l-1)$$

$$f_l=\sum_{\nu\in Z}\psi^{(1)}_{l+(2l-1)\nu}\psi^{(1)*}_{l-1+(2l-1)\nu}.$$

Remark. In the above list $\tilde{e}_n$ and $\tilde{f}_n$ denote the Chevalley basis for $A^{(1)}_{2l+1}$, $A^{(1)}_{2l}$, $A^{(1)}_{2l-1}$ and $D^{(1)}_{l+1}$, respectively.

Table 2. Highest weight vectors

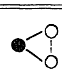
subalgebra	highest weight vector	weight
$A^{(1)}_l$	$\|n\rangle$	Λ_j $(n\equiv j \bmod l+1)$
$D^{(2)}_{l+1}$	$\|n\rangle$	$2\Lambda_0$ $(n\equiv 0,\ 1 \bmod 2(l+1))$
		Λ_j $(n\equiv -j, j+1 \bmod 2(l+1))$
		$2\Lambda_l$ $(n\equiv l+1,\ l+2 \bmod 2(l+1))$
$A^{(2)}_{2l}$	$\|n\rangle$	$2\Lambda_0$ $(n\equiv 0,\ 1 \bmod 2l+1)$
		Λ_j $(n\equiv -j, j+1 \bmod 2l+1)$
		Λ_l $(n\equiv l+1 \bmod 2l+1)$
$C^{(1)}_l$	$\|n\rangle$	Λ_0 $(n\equiv 0 \bmod 2l)$
		Λ_j $(n\equiv \pm j \bmod 2l)$
		Λ_l $(n\equiv l \bmod 2l)$
$D^{(1)}_l$	$\|2\nu, 2(l-1)\nu\rangle$, $\|2\nu+1, 2(l-1)\nu+1\rangle$	$\Lambda_0+\Lambda_1$
	$\|2\nu, 2(l-1)\nu+1\rangle$	$\left\{\begin{matrix}2\Lambda_0\\2\Lambda_1\end{matrix}\right.$
	$\pm i\|2\nu+1, 2(l-1)\nu\rangle$	
	$\|1+2\nu, j+2(l-1)\nu\rangle$,	Λ_j $(2\leq j\leq l-2)$
	$\|2+2\nu, j+l-1+2(l-1)\nu\rangle$	
	$\|1+2\nu, l-1+2(l-1)\nu\rangle$,	$\Lambda_{l-1}+\Lambda_l$
	$\|2+2\nu, l+2(l-1)\nu\rangle$	
	$\|1+2\nu, l+2(l-1)\nu\rangle$	$\left\{\begin{matrix}2\Lambda_{l-1}\\2\Lambda_l\end{matrix}\right.$
	$\pm\varepsilon^{-1}\|2+2\nu, l-1+2(l-1)\nu\rangle$	
$A^{(2)}_{2l-1}$	$\|\nu, (2l-1)\nu\rangle$, $\|1+\nu, 1+(2l-1)\nu\rangle$	$\Lambda_0+\Lambda_1$
	$\|\nu, 1+(2l-1)\nu\rangle\pm i\|1+\nu, (2l-1)\nu\rangle$	$\left\{\begin{matrix}2\Lambda_0\\2\Lambda_1\end{matrix}\right.$
	$\|1+\nu, j+(2l-1)\nu\rangle$,	Λ_j $(2\leq j\leq l-1)$
	$\|1+\nu, l+j-1+(2l-1)\nu\rangle$	Λ_l
	$\|1+\nu, l+(2l-1)\nu\rangle$	

Table 3. Identities for the reduced hierarchy

$A_l^{(1)}$	$\tau_{n+l+1}(x)=\tau_n(x)$, $\partial\tau_n(x)/\partial x_{(l+1)\nu}=0$.
$D_{l+1}^{(2)}$	$\tau_{n+2(l+1)}(x)=\tau_{1-n}(\tilde{x})=\tau_n(x)$, $\partial\tau_n(x)/\partial x_{2(l+1)\nu}=0$.
$A_{2l}^{(2)}$	$\tau_{n+2l+1}(x)=\tau_{1-n}(\tilde{x})=\tau_n(x)$, $\partial\tau_n(x)/\partial x_{(2l+1)\nu}=0$.
$C_l^{(1)}$	$\tau_{n+2l}(x)=\tau_{-n}(\tilde{x})=\tau_n(x)$, $\partial\tau_n(x)/\partial x_{2l\nu}=0$.
$D_l^{(1)}$	$\tau_{l_1+2(l-1)\nu,l_2+2\nu;l}(x^{(1)},x^{(2)})$ $=(-)^{l(l_1+l_2)}\tau_{1-l_1,1-l_2;-l}(\tilde{x}^{(1)},\tilde{x}^{(2)})$ $=\tau_{l_1,l_2;l}(x^{(1)},x^{(2)})$, $\partial\tau_{l_1,l_2;l}(x)/\partial x^{(1)}_{2(l-1)\nu}+\partial\tau_{l_1,l_2;l}(x)/\partial x^{(2)}_{2\nu}=0$.
$A_{2l-1}^{(2)}$	$\tau_{l_1+(2l-1)\nu,l_2+\nu;l}(x^{(1)},x^{(2)})$ $=(-)^{l(l_1+l_2)}\tau_{1-l_1,1-l_2;-l}(\tilde{x}^{(1)},\tilde{x}^{(2)})$ $=\tau_{l_1,l_2;l}(x^{(1)},x^{(2)})$, $\partial\tau_{l_1,l_2;l}(x)/\partial x^{(1)}_{(2l-1)\nu}+\partial\tau_{l_1,l_2;l}(x)/\partial x^{(2)}_{\nu}=0$.
$A_{2l}^{(2)}$	(the spin representation) $\partial\tau(x_{odd})/\partial x_{(2l+1)(2\nu+1)}=0$
$A_{2l-1}^{(2)}$	(the spin representation) $\partial\tau_i(x_{odd})/\partial x^{(1)}_{(2l-1)(2\nu+1)}+\partial\tau_i(x_{odd})/\partial x^{(2)}_{2\nu+1}=0$, $(i=0,1)$

Remark. As for the extra identities for the τ functions of the spin representations of $D_{l+1}^{(2)}$ and $D_l^{(1)}$, we refer the reader to [7].

Table 4. Example of soliton equations

$A_2^{(1)}$ 

$$(D_1^4+3D_2^2)f\cdot f=0,\quad f=\tau_0,\tau_1,\tau_2.$$

$$3\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^4 u}{\partial x_1^4}+6u\frac{\partial^2 u}{\partial x_1^2}+6\Big(\frac{\partial u}{\partial x_1}\Big)^2=0,\quad u=2\frac{\partial^2}{\partial x_1^2}\log f.$$

(Boussinesq equation (see [31])†))

$A_2^{(2)}$ 

$$\begin{cases}(D_1^6+144D_1D_5)f\cdot f-90D_1^2f\cdot g=0, & f=\tau_0|_{x_2=x_4=\cdots=0},\\ D_1^4f\cdot f+6f\cdot g=0, & g=\dfrac{\partial^2\tau_0}{\partial x_2^2}\Big|_{x_2=x_4=\cdots=0}.\end{cases}$$

$$9\frac{\partial u}{\partial x_5}+\frac{\partial}{\partial x_1}\Big(\frac{\partial^4 u}{\partial x_1^4}+15u\frac{\partial^2 u}{\partial x_1^2}+15u^3+\frac{45}{4}\Big(\frac{\partial u}{\partial x_1}\Big)^2\Big)=0,$$

$$u=\frac{\partial^2}{\partial x_1^2}\log f.$$

†) The definition of u here differs from the one in [31] by an additive constant.

(Kaup equation [43])

○⇛● $$\begin{cases} (D_1^6+144D_1D_5)f\cdot f-30D_1^4f\cdot g=0, & f=\tau_1|_{x_2=x_4=\cdots=0}, \\ D_1^2f\cdot f+2f\cdot g=0, & g=\dfrac{\partial\tau_1}{\partial x_2}|_{x_2=x_4=\cdots=0}. \end{cases}$$

$$9\frac{\partial u}{\partial x_5}+\frac{\partial}{\partial x_1}\left(\frac{\partial^4 u}{\partial x_1^4}+15u\frac{\partial^2 u}{\partial x_1^2}+15u^3\right)=0, \quad u=\frac{\partial^2}{\partial x_1^2}\log f.$$

(Sawada-Kotera equation [44])

○⇛○ $$\begin{aligned} &(D_1^6+144D_1D_5)\hat{f}\cdot f+15D_1^4\hat{f}\cdot g=0, && \hat{f}=\tau_0|_{x_2=x_4=\cdots=0}, \\ &D_1^2\hat{f}\cdot f-\hat{f}g=0, && f=\tau_1|_{x_2=x_4=\cdots=0}, \\ &D_1^2f\cdot f+2gf=0, && g=\frac{\partial\tau_1}{\partial x_2}|_{x_2=x_4=\cdots=0}. \end{aligned}$$

$$9\frac{\partial u}{\partial x_5}+\frac{\partial}{\partial x_1}\left(\frac{\partial^4 u}{\partial x_1^4}-5u^2\frac{\partial^2 u}{\partial x_1^2}-5\frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\right.$$
$$\left.-5u\left(\frac{\partial u}{\partial x_1}\right)^2+u^2\right)=0, \quad u=\frac{\partial}{\partial x_1}\log(f/\hat{f}).$$

$C_2^{(1)}$ ●⇒○⇐○ $$\begin{cases} (D_1^4-4D_1D_3)f\cdot f+6f\cdot g=0, & f=\tau_0|_{x_2=x_4=\cdots=0}, \\ (D_1^3+2D_3)f\cdot g=0, & g=\dfrac{\partial^2\tau_0}{\partial x_2^2}|_{x_2=x_4=\cdots=0}. \end{cases}$$

$$\begin{cases} -4\dfrac{\partial u}{\partial x_3}+12u\dfrac{\partial u}{\partial x_1}+\dfrac{\partial^3 u}{\partial x_1^3}+3\dfrac{\partial v}{\partial x_1}=0, & u=\dfrac{\partial^2}{\partial x_1^2}\log f, \\ 2\dfrac{\partial v}{\partial x_3}+6u\dfrac{\partial v}{\partial x_1}+\dfrac{\partial^3 v}{\partial x_1^3}=0, & v=g/f. \end{cases}$$

○⇒●⇐○ $$\begin{cases} (D_1^4-4D_1D_3)f\cdot f-6g^2=0, & f=\tau_1|_{x_2=x_4=\cdots=0}, \\ (D_1^3+2D_3)f\cdot g=0, & g=\dfrac{\partial\tau_1}{\partial x_2}|_{x_2=x_4=\cdots=0}. \end{cases}$$

$$\begin{cases} -4\dfrac{\partial u}{\partial x_3}+12u\dfrac{\partial u}{\partial x_1}+\dfrac{\partial^3 u}{\partial x_1^3}-6v\dfrac{\partial v}{\partial x_1}=0, & u=\dfrac{\partial^2}{\partial x_1^2}\log f, \\ 2\dfrac{\partial v}{\partial x_3}+6u\dfrac{\partial v}{\partial x_1}+\dfrac{\partial^3 v}{\partial x_1^3}=0, & v=g/f. \end{cases}$$

(coupled KdV equation of Hirota-Satsuma [45]~[47])

○⇒○⇐○ $$\begin{cases} D_1^2\hat{f}\cdot f-\hat{f}\cdot g=0, & \hat{f}=\tau_0|_{x_2=x_4=\cdots=0}, \\ (D_1^3-4D_3)\hat{f}\cdot f+3D_1\hat{f}\cdot g=0, & f=\tau_1|_{x_2=x_4=\cdots=0}, \\ (D_1^3+2D_3)\hat{f}\cdot g=0, & g=\dfrac{\partial\tau_1}{\partial x_2}|_{x_2=x_4=\cdots=0}. \end{cases}$$

$$\begin{cases} -4\dfrac{\partial u}{\partial x_3}-4u\dfrac{\partial u}{\partial x_1}+\dfrac{\partial^3 u}{\partial x_1^3}+3\dfrac{\partial^2 v}{\partial x_1^2}=0, & u=\dfrac{\partial}{\partial x_1}\log(f/\hat{f}), \\ 2\dfrac{\partial v}{\partial x_3}+3v\dfrac{\partial v}{\partial x_1}+3\dfrac{\partial u}{\partial x_1}\dfrac{\partial v}{\partial x_1}-3u^2\dfrac{\partial v}{\partial x_1}+\dfrac{\partial^3 v}{\partial x_1^3}=0, & v=g/f. \end{cases}$$

$D_3^{(2)}$ ●⇐○⇒○ $$\begin{cases} D_1^2f\cdot f+2f\cdot g=0, & f=\tau_0|_{x_2=x_4=\cdots=0}, \\ (D_1^3D_3+2D_3^2)f\cdot f-6D_1D_3f\cdot g=0, & g=\dfrac{\partial\tau_0}{\partial x_2}|_{x_2=x_4=\cdots=0}. \end{cases}$$

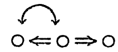
$$\frac{\partial^3 u}{\partial x_3^2}+2\frac{\partial}{\partial x_1}\left(\frac{\partial^3 u}{\partial x_1^2\partial x_3}+3\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\right)=0, \quad u=\frac{\partial}{\partial x_1}\log f.$$

(Ito equation [48])

$\circ\Leftarrow\bullet\Rightarrow\circ$

$$\begin{cases} D_1D_3 f\cdot f+g^2=0, & f=\tau_2|_{x_2=x_4=\cdots=0},\\ (D_1^3D_3+2D_3^2)f\cdot f-3D_1^2 g\cdot g=0, & g=\frac{\partial \tau_2}{\partial x_2}|_{x_2=x_4=\cdots=0}.\end{cases}$$

$$\begin{cases} 2\frac{\partial u}{\partial x_3}+v^2=0, & u=\frac{\partial}{\partial x_1}\log f,\\ v\frac{\partial v}{\partial x_3}+\frac{\partial}{\partial x_1}\left(2v\frac{\partial^2 v}{\partial x_1^2}-\left(\frac{\partial v}{\partial x_1}\right)^2+3v^2\frac{\partial u}{\partial x_1}\right)=0, & v=g/f.\end{cases}$$



$$\begin{cases} D_1^2\hat f\cdot\hat f+2\hat f\cdot\hat g=0, & \hat f=\tau_0|_{x_2=x_4=\cdots=0},\\ D_1D_3 f\cdot f+g^2=0, & \hat g=\frac{\partial\tau_0}{\partial x_2}|_{x_2=x_4=\cdots=0},\\ D_1^2 f\cdot\hat f+g\cdot\hat f-\hat g\cdot f=0, & f=\tau_2|_{x_2=x_4=\cdots=0},\\ D_3 f\cdot\hat f+D_1 g\cdot\hat f=0, & g=\frac{\partial\tau_2}{\partial x_2}|_{x_2=x_4=\cdots=0}.\end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial x_3}+v\frac{\partial u}{\partial x_1}+u\frac{\partial v}{\partial x_1}+\frac{\partial^2 v}{\partial x_1^2}=0, \quad u=\frac{\partial}{\partial x_1}\log(f/\hat f),\\ \frac{\partial v}{\partial x_3}-3v\frac{\partial v}{\partial x_1}+2\frac{\partial^3 v}{\partial x_1^3}+2\left(\frac{\partial u}{\partial x_1}-u^2\right)\frac{\partial v}{\partial x_1}\\ \qquad +2\left(\frac{\partial^2 u}{\partial x_1^2}-u\frac{\partial u}{\partial x_1}\right)v=0, \quad v=g/f.\end{cases}$$

§9. Time Evolutions with Singularities Other than $k=\infty$ — The 2 Dimensional Toda Lattice —

As we have seen, there are two ways for describing free fermions. One is to deal with discrete indices by considering ψ_n, ψ_n^*, and the other is to deal with continuum parameters by considering $\psi(k)$, $\psi^*(k)$. The advantage of the former is that the creation part and the annihilation part are separated as (1.1), while the advantage of the latter is that the time evolutions are diagonalized as (1.20). So far, we mainly adopted the former in order to emphasize the aspect of the representation theory. In this section, we adopt the latter in order to treat more general time evolutions other than (1.20). As an example we treat the 2 dimensional Toda lattice.

In terms of $\psi(k)$ and $\psi^*(k)$, the vacuum expectation value is given by

$$\langle 0|\psi(p)\psi^*(q)|0\rangle=-\langle 0|\psi^*(q)\psi(p)|0\rangle=\frac{q}{p-q}, \qquad (p\neq q).$$

The following formulas are also available:

$$\langle l|\psi(p_1)\cdots\psi(p_{m+l'})\psi^*(q_1)\cdots\psi^*(q_m)|0\rangle$$
$$=\delta_{ll'}\frac{\prod_{i<i'}(p_i-p_{i'})\sum_{j<j'}(q_j-q_{j'})\prod_j q_j}{\prod_{i,j}(p_i-q_j)}, \qquad (l\geqq 0),$$

$$\langle -l|\psi^*(p_1)\cdots\psi^*(p_{m+l'})\psi(q_1)\cdots\psi(q_m)|0\rangle$$
$$=\frac{\delta_{ll'}\prod_{i<i'}(p_i-p_{i'})\prod_{j<j'}(q_j-q_{j'})\prod_i p_i}{\prod_{i,j}(p_i-q_j)}, \qquad (l\geqq 0).$$

The time evolution (1.20) is singular at $k=\infty$ in the sense that $\exp\xi(x, k)$ has an essential singularity there. In general, we introduce the following time evolution which is singular at $k=k_0$:

$$\psi(k)\longmapsto\left\{(k-k_0)^n\exp\xi\left(x, \frac{1}{k-k_0}\right)\right\}\psi(k), \tag{9.1}$$
$$\psi^*(k)\longmapsto\left\{(k-k_0)^{-n}\exp\xi\left(x, \frac{1}{k-k_0}\right)\right\}\psi^*(k).$$

For an element $a\in A$, we denote by $a(n, x)$ the image of a by the automorphism (9.1). We call $a(n, x)$ the time evolution of a.

The basic formulas (1.21) are valid in the form

$$\langle l|\psi(k)a(n, x)|0\rangle=(-)^{l-1}\langle l-1|a(n+1, x-\epsilon(k-k_0))|0\rangle, \tag{9.2}$$
$$\langle l|\psi^*(k)a(n, x)|0\rangle=(-)^{l+1}k\langle l+1|a(n-1, x+\epsilon(k-k_0))|0\rangle.$$

Now we consider the τ functions. First we consider the single component theory with the time evolutions singular at $k=\infty$ and $k=0$:

$$\psi(k)\longmapsto k^n e^{\xi(x,k)+\xi(y,k^{-1})}\psi(k), \tag{9.3}$$
$$\psi^*(k)\longmapsto k^{-n}e^{-\xi(x,k)-\xi(y,k^{-1})}\psi^*(k),$$

and denote by $g(n, x, y)$ the time evolution of $g=\exp(\sum_i a_i\psi(p_i)\psi^*(q_i))$. Then the τ function

$$\tau_n(x, y)=\langle 0|g(n, x, y)|0\rangle$$

satisfies the bilinear identity in the following form: Choosing a contour C as in Figure 10, we have

$$\oint_C\frac{dk}{2\pi ik}\langle 1|(\psi(k)g)(n, x, y)|0\rangle\langle -1|(\psi^*(k)g)(n', x', y')|0\rangle=0. \tag{9.4}$$

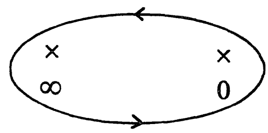


Fig. 10. Contour for the integration in (9.4).

Note that if $y=0$, then $\tau_n(x, y)$ coincides with $\langle n|g(x)|n\rangle$ of (2.3). Formula (9.4) generalizes $(2.4)_{ll'}$ where the restriction $l \geqq l'$ is removed by taking into account the contribution from $k=0$.

The simplest bilinear equation contained in (9.4) is

$$\frac{1}{2} D_{x_1} D_{y_1} \tau_n \cdot \tau_n = \tau_n^2 - \tau_{n+1}\tau_{n-1}.$$

Setting

$$u_n = \log(\tau_n^2/\tau_{n+1}\tau_{n-1}),$$

we have the equation of the 2 dimensional Toda lattice ([49], [50]):

$$\frac{\partial^2 u_m}{\partial\xi\partial\eta} = -\sum_n a_{nm} e^{-u_n}, \tag{9.5}$$

where $\xi = x_1$, $\eta = y_1$ and (a_{mn}) is the Cartan matrix for A_∞:

$$a_{mn} = \begin{cases} 2 & m=n \\ -1 & m=n\pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

In general the 2 dimensional Toda lattice of type $\mathscr{L}$ ($\mathscr{L} = B_\infty, C_\infty, A_l^{(1)}$, etc.) is (9.5) with (a_{mn}) corresponding to the Dynkin diagram of $\mathscr{L}$ (See Fig.). If we choose g corresponding to B_∞, e.g.

$$g = \exp\left(\sum_{i=1}^{N} c_i(\psi(p_i)\psi^*(q_i) - \psi(-q_i)\psi^*(-p_i))\right),$$

then we have the following solution to the 2 dimensional Toda lattice of type B_∞ (cf. (5.4)):

$$u_0 = \log(\tau_1/\tau_2) + \log 2,$$
$$u_n = \log(\tau_{n+1}^2/\tau_{n+2}\tau_n), \qquad (n \geqq 1).$$

Similarly, if we choose g corresponding to C_∞, e.g.

$$g = \exp\left(\sum_{i=1}^{N} c_i(p_i\psi(p_i)\psi^*(q_i) - q_i\psi(-q_i)\psi^*(-p_i))\right),$$

then we have the following solution to the 2 dimensional Toda lattice of type C_∞ (cf. (5.4)):

$$u_0 = 2\log(\tau_0/\tau_1),$$
$$u_n = \log(\tau_n^2/\tau_{n+1}\tau_{n-1}), \qquad (n \geqq 1).$$

Likewise, the symmetry relations for τ functions listed in Table 3 afford us the following solutions to the 2 dimensional Toda lattice of type $A_l^{(1)}$, $D_{l+1}^{(2)}$, $A_{2l}^{(2)}$ and $C_l^{(1)}$, respectively:

$A_l^{(1)}$
$$u_0 = \log(\tau_0^2/\tau_1\tau_l),$$
$$u_n = \log(\tau_n^2/\tau_{n+1}\tau_{n-1}), \qquad (n=1,\dots,l-1),$$
$$u_l = \log(\tau_l^2/\tau_{l-1}\tau_0),$$
$$g = \exp\left(\sum_{i=1}^N c_i\psi(p_i)\psi^*(\omega_i p_i)\right), \quad \omega_i^{l+1}=1.$$

$D_{l+1}^{(2)}$
$$u_0 = \log(\tau_1/\tau_2) + \log 2,$$
$$u_n = \log(\tau_{n+1}^2/\tau_{n+2}\tau_n), \qquad (n=1,\dots,l-1),$$
$$u_l = \log(\tau_{l+1}/\tau_l) + \log 2,$$
$$g = \exp\left(\sum_{i=1}^N c_i(\psi(p_i)\psi^*(\omega_i p_i) - \psi(\omega_i p_i)\psi^*(p_i))\right), \quad \omega_i^{2(l+1)}=1.$$

$A_{2l}^{(2)}$
$$u_0 = \log(\tau_1/\tau_2) + \log 2,$$
$$u_n = \log(\tau_{n+1}^2/\tau_{n+2}\tau_n), \qquad (n=1,\dots,l-1),$$
$$u_l = 2\log(\tau_{l+1}/\tau_l),$$
$$g = \exp\left(\sum_{i=1}^N c_i(\psi(p_i)\psi^*(\omega_i p_i) - \psi(\omega_i p_i)\psi^*(p_i))\right), \quad \omega_i^{2l+1}=1.$$

$A_{2l}^{(2)}$
$$u_0 = 2\log(\tau_0/\tau_1),$$
$$u_n = \log(\tau_n^2/\tau_{n+1}\tau_{n-1}), \qquad (n=1,\dots,l-1),$$
$$u_l = \log(\tau_l/\tau_{l-1}) + \log 2,$$
$$g = \exp\left(\sum_{i=1}^N c_i(\psi(p_i)\psi^*(\omega_i p_i) - \omega_i\psi(-\omega_i p_i)\psi^*(-p_i))\right), \quad \omega_i^{2l+1}=1.$$

$C_l^{(1)}$
$$u_0 = 2\log(\tau_0/\tau_1),$$
$$u_n = \log(\tau_n^2/\tau_{n+1}\tau_{n-1}), \qquad (n=1,\dots,l-1),$$
$$u_l = 2\log(\tau_l/\tau_{l-1}),$$
$$g = \exp\left(\sum_{i=1}^N c_i(\psi(p_i)\psi^*(\omega_i p_i) - \omega_i\psi(-\omega_i p_i)\psi^*(-p_i))\right), \quad \omega_i^{2l}=1.$$

Next, we consider the 2 component theory with the time evolutions singular at $k=\infty$ and $k=0$:

(9.6)
$$\psi^{(j)}(k) \longmapsto k^{n^{(j)}} e^{\xi(x^{(j)},k)+\xi(y^{(j)},k^{-1})}\psi^{(j)}(k),$$
$$\psi^{(j)*}(k) \longmapsto k^{-n^{(j)}} e^{-\xi(x^{(j)},k)-\xi(y^{(j)},k^{-1})}\psi^{(j)*}(k).$$

The τ functions are

$$\tau_{l,n^{(1)},n^{(2)}}(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$$
$$=\langle l, -l| g(n^{(1)}, x^{(1)}, y^{(1)}, n^{(2)}, x^{(2)}, y^{(2)})|0, 0\rangle,$$

where $g(n^{(1)}, x^{(1)}, y^{(1)}, n^{(2)}, x^{(2)}, y^{(2)})$ is the time evolution of $g=\exp(\sum_k a_k\cdot \psi^{(i_k)}(p_k)\psi^{(j_k)*}(q_k))$ ($i_k, j_k=1$ or 2). They satisfy the following bilinear identity:

$$0=\sum_{j=1,2}\oint_C \frac{dk}{2\pi ik}\langle l+1, -l|(\psi^{(j)}(k)g)(n^{(1)}, x^{(1)}, y^{(1)}, n^{(2)}, x^{(2)}, y^{(2)})|0, 0\rangle$$
$$\times\langle l'-1, -l'|(\psi^{(j)*}(k)g)(n^{(1)\prime}, x^{(1)\prime}, y^{(1)\prime}, n^{(2)\prime}, x^{(2)\prime}, y^{(2)\prime})|0, 0\rangle.$$

In particular, we have

$$\frac{1}{2}D_{x_1^{(1)}}D_{y_1^{(1)}}\tau_{l,n^{(1)},n^{(2)}}\cdot\tau_{l,n^{(1)},n^{(2)}} = \tau^2_{l,n^{(1)},n^{(2)}}-\tau_{l,n^{(1)}+1,n^{(2)}}\cdot\tau_{l,n^{(1)}-1,n^{(2)}}. \quad (9.7)$$

Setting

$$u_n=\log(\tau^2_{l,n,n^{(2)}}/\tau_{l,n+1,n^{(2)}}\tau_{l,n-1,n^{(2)}}),$$

we have (9.5) again with $\xi=x_1^{(1)}$ and $\eta=y_1^{(1)}$.

We shall show that the 2 dimensional Toda lattice of type D_∞ is obtained from (9.7) by the reduction to D_∞.

We set

$$\phi^{(i)}(k)=\frac{1}{\sqrt{2}}(\psi^{(i)}(k)+\psi^{(i)*}(-k)),$$
$$\hat{\phi}^{(i)}(k)=\frac{1}{\sqrt{2}}(\psi^{(i)}(k)-\psi^{(i)*}(-k)),$$

and denote by κ the automorphism of the Clifford algebra generated by $\psi_j^{(i)}$, $\psi_j^{(i)*}$ ($j\in \boldsymbol{Z}$, $i=1, 2$) satisfying

$$\kappa(\phi^{(i)}(k))=\hat{\phi}^{(i)}(k), \quad \kappa(\hat{\phi}^{(i)}(k))=-\phi^{(i)}(k).$$

Then the group element g satisfying $(\iota(g)=g)$ is written as

$$g=g_0\cdot\kappa(g_0)$$

where g_0 belongs to the Clifford algebra generated by $\phi^{(i)}(k)$ $(i=1, 2)$. We denote by $g_0(x^{(1)}_{odd}, y^{(1)}_{odd}, x^{(2)}_{odd}, y^{(2)}_{odd})$ the time evolution of g_0 caused by the time evolution of free fermions,

$$\phi^{(i)}(k)\longmapsto e^{\xi'(x^{(i)}_{odd},k)+\xi'(y^{(i)}_{odd},k^{-1})}\phi^{(i)}(k).$$

We denote by π an isomorphism such that $\pi(\phi^{(1)}(k))=\phi(k)$ and $\pi(\phi^{(2)}(k))=\hat{\phi}(k)$, and define

$$\tau_i=\langle i\,|\,\pi(g_0(x^{(1)}_{odd},\ y^{(1)}_{odd},\ x^{(2)}_{odd},\ y^{(2)}_{odd}))\,|\,i\rangle,\qquad i=0,\ 1\,.$$

We also define

$$f=\langle 0,\ 0\,|\,g_0(x^{(1)}_{odd},\ y^{(1)}_{odd},\ x^{(2)}_{odd},\ y^{(2)}_{odd})\,|\,0,\ 0\rangle\,,$$
$$f^*=2\langle 0,\ 0\,|\,g_0(x^{(1)}_{odd},\ y^{(1)}_{odd},\ x^{(2)}_{odd},\ y^{(2)}_{odd})\phi^{(1)}_0\phi^{(2)}_0\,|\,0,\ 0\rangle\,.$$

Then we have $\tau_0=f-if^*$ and $\tau_1=f+if^*$. The correct choice of u_n is as follows.

$$u_0=\log(\tau_0^2/\tau_{2,1}),\quad u_1=\log(\tau_1^2/\tau_{2,1})$$
$$u_n=\log(\tau^2_{n,1}/\tau_{n+1,1}\tau_{n-1,1}),\qquad (n\geqq 2)\,,$$

where $\tau_{m,n}=\tau_{0,m,n}(x^{(1)},\ y^{(1)},\ x^{(2)},\ y^{(2)})$ with $x^{(i)}_2=x^{(i)}_4=\cdots=y^{(i)}_2=y^{(i)}_4=\cdots=0$.

In fact, we have

$$\tau_{0,1}=\frac{1}{2}(\tau_0^2+\tau_1^2)\,,$$
$$\tau_{1,1}=\tau_0\tau_1\,.$$

Hence the equation (9.7) implies

$$\frac{1}{2}D_\xi D_\eta(\tau_0\tau_1)\cdot(\tau_0\tau_1)=(\tau_0\tau_1)^2-\frac{1}{2}\tau_{2,1}(\tau_0^2+\tau_1^2)\,,\tag{9.8}$$

where $\xi=x^{(1)}_1$ and $\eta=x^{(2)}_1$. On the other hand, we can show that (see (39) in [6] and (2.4) in [11]V)

$$(D_\xi D_\eta-1)f\cdot f^*=0\,,$$

which is rewritten as

$$(D_\xi D_\eta-1)(\tau_0\cdot\tau_0-\tau_1\cdot\tau_1)=0\,.\tag{9.9}$$

From (9.8) and (9.9) we have

$$(D_\xi D_\eta-1)\tau_i\cdot\tau_i=-\tau_{2,1},\qquad i=0,\ 1\,.\tag{9.10}$$

The equations (9.7) and (9.10) imply (9.5) with (a_{ij}) corresponding to D_∞.

Reductions to $A^{(2)}_{2l-1}$, $D^{(1)}_l$ (see Section 8) afford us the following solutions to the 2 dimensional Toda lattice of $A^{(2)}_{2l-1}$, $D^{(1)}_l$ type, respectively:

$A^{(2)}_{2l-1}$

$$u_0=\log(\tau_0^2/\tau_{2,1})+\log 2$$
$$u_1=\log(\tau_1^2/\tau_{2,1})+\log 2$$
$$u_n=\log(\tau^2_{n,1}/\tau_{n+1,1}\tau_{n-1,1}),\qquad (n=2,\dots,\ l-1)$$
$$u_l=2\log(\tau_{l,1}/\tau_{l-1,1})\,,$$

where $\tau_{0,1}=(\tau_0^2+\tau_1^2)/2$ and $\tau_{1,1}=\tau_0\tau_1$,

$$g=\exp\left(\sum_{i=1}^{N}(c_i(\psi^{(1)}(p_i)\psi^{(1)*}(\omega_i p_i)-\psi^{(1)}(-\omega_i p_i)\psi^{(1)*}(-p_i))\right.$$
$$\left.+c_i'(\psi^{(1)}(p_i^{2l-1})\psi^{(2)*}(\omega_i p_i)-\psi^{(2)}(-\omega_i p_i)\psi^{(1)*}(-p_i^{2l-1})))\right),\quad \omega_i^{2l-1}=1.$$

$D_l^{(1)}$
$$u_0=\log(\tau_0^2/\tau_{2,1})+\log 2,$$
$$u_1=\log(\tau_1^2/\tau_{2,1})+\log 2,$$
$$u_n=\log(\tau_{n,1}^2/\tau_{n+1,1}\tau_{n-1,1}),\qquad (n=2,\dots,l-2)$$
$$u_{l-1}=\log(\tau_{l-1}^2/\tau_{l-2,1})+\log 2,$$
$$u_l=\log(\tau_l^2/\tau_{l-2,1})+\log 2,$$

where $\tau_{0,1}=(\tau_0^2+\tau_1^2)/2$, $\tau_{1,1}=\tau_0\tau_1$,

$$\tau_{l-1,1}=\tau_{l-1}\tau_l \quad\text{and}\quad \tau_{l,1}=(\tau_{l-1}^2+\tau_l^2)/2,$$

$$g=\exp\left(\sum_{i=1}^{N}(c_i(\psi^{(1)}(p_i)\psi^{(1)*}(\omega_i p_i)-\psi^{(1)}(-\omega_i p_i)\psi^{(1)*}(-p_i))\right.$$
$$\left.+c_i'(\psi^{(1)}(p_i^{2l-2})\psi^{(2)*}(\omega_i p_i)-\psi^{(2)}(-\omega_i p_i)\psi^{(1)*}(-p_i^{2l-1})))\right),\quad \omega_i^{2l-2}=1.$$

The bilinear equations of low degree corresponding to the reduction (4.5) and the time evolution (9.6) are listed in Appendix 2. A typical example contained in this class is the Pohlmeyer-Lund-Regge equation (see [51]):

$$D_1(f^*\cdot f+g^*\cdot g)=0,\quad \hat{D}_1(f^*\cdot f-g^*\cdot g)=0,$$
$$(D_1\hat{D}_1-1)f^*\cdot g=0,\quad (D_1\hat{D}_1-1)g^*\cdot f=0,$$
$$D_1\hat{D}_1(f^*\cdot f-g^*\cdot g)+2g^*g=0.$$

The considerations in this Section applies also to the case of neutral free fermions. For example the BKP hierarchy with the time evolution $\phi(k)\to\phi(k)\exp(xk+yk^3+\cdots+tk^{-1})$ contains ([11])

$$(D_yD_t-D_x^3D_t+3D_x^2)\tau\cdot\tau=0.$$

When specialized to $y=x$, this reduces to the model equation for shallow water waves [52].

§10. Difference Equations
— The Principal Chiral Field —

So far we have discussed various non-linear partial differential equations arising from representations of infinite dimensional Lie algebras. In this section we explain a method for generating their difference analogues by introducing discrete time evolutions.

To illustrate the idea, let us take the KP hierarchy. Introducing small parameters a, b, c, we put $x=x_0+l\in(a)+m\in(b)+n\in(c)$ (l, m, $n\in\boldsymbol{Z}$). In view of the formula

$$e^{\xi(\in(a),k)}=(1-ak)^{-1},$$

this amounts to considering the time evolution with respect to discrete variables l, m, n:

$$\psi(k)\longmapsto(1-ak)^{-l}(1-bk)^{-m}(1-ck)^{-n}e^{\xi(x_0,k)}\psi(k)$$
$$\psi^*(k)\longmapsto(1-ak)^{l}(1-bk)^{m}(1-ck)^{n}e^{-\xi(x_0,k)}\psi^*(k).$$

If we set $x-x'=\in(a)+\in(b)+\in(c)$, then the bilinear identity (2.4) becomes

$$\oint\frac{dk}{2\pi i}\frac{1}{(1-ak)(1-bk)(1-ck)}\tau(x+\in(a)+\in(b)+\in(c)-\in(k^{-1}))\times\tau(x+\in(k^{-1}))=0.$$

Evaluating the residues, we thus get the following difference analogue of the bilinear KP equation for

(10.1) $\tau(lmn)=\tau(x_0+l\in(a)+m\in(b)+n\in(c))$:

$$a(b-c)\tau(l+1\ m\ n)\tau(l\ m+1\ n+1)+b(c-a)\tau(l\ m+1\ n)\tau(l+1\ m\ n+1)$$
$$+c(a-b)\tau(l\ m\ n+1)\tau(l+1\ m+1\ n)=0.$$

By construction, it is evident that the N-soliton solution (2.12) still solves (10.1) provided the exponential factors e^{η_i} are read as

$$e^{\eta_i}=\text{const.}\left(\frac{1-ap_i}{1-aq_i}\right)^{-l}\left(\frac{1-bp_i}{1-bq_i}\right)^{-m}\left(\frac{1-cp_i}{1-cq_i}\right)^{-n}.$$

In a similar manner we may introduce an arbitrary number of discrete variables and write down a hierarchy of discrete KP equations, which is equivalent to the continuous KP hierarchy.

The same procedure is applicable to all the equations discussed in Sections 1–9. Several examples are worked out in [11], including the BKP, KdV, sine-Gordon, non-linear Schrödinger, Heisenberg ferromagnet and other equations. Here we shall describe the method by using another example, the principal chiral field equation [53]:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial R}{\partial y_1}R^{-1}\right)+\frac{\partial}{\partial y_1}\left(\frac{\partial R}{\partial x_1}R^{-1}\right)=0 \tag{10.2}$$
$$\det R=1$$

where $R=R(x_1, y_1)$ is an N by N matrix function.

First let us give some preliminary remarks on the wave functions for the N-component KP theory. As in Section 4, we introduce N copies of free fermions $\psi^{(j)}(k)$, $\psi^{*(j)}(k)$ $(j=1,\dots,N)$. As the time evolution

$$\psi^{(j)}(k) \longmapsto e^{\xi^{(j)}(k)}\psi^{(j)}(k), \quad \psi^{*(j)}(k) \longmapsto e^{-\xi^{(j)}(k)}\psi^{*(j)}(k)$$

we take

$$\xi^{(j)}(k)=(k-k_0)^{m^{(j)}}\exp\left(\xi\left(x^{(j)}, \frac{1}{k-k_0}\right)\right), \tag{10.3}$$

or more generally the sum of (10.3) with different k_0's. Define the matrix of wave functions $W(x, m; k)$, $W^*(x, m; k)$ by

$$W(x, m; k)_{ij}=\langle 0\cdots\overset{i}{\check{1}}\cdots 0|(\psi^{(j)}(k)g)(x, m)|0\cdots 0\rangle/\tau(x, m) \tag{10.4}$$
$$W^*(x, m; k)_{ij}=\langle 0\cdots -\overset{i}{\check{1}}\cdots 0|(\psi^{*(j)}(k)g)(x, m)|0\cdots 0\rangle/k\tau(x, m)$$

with $\tau(x, m)=\tau_{0\cdots 0}(x, m)$, where we set in general

$$\tau_{l_1\cdots l_N}(x, m)=\langle l_1,\dots, l_N|g(x, m)|0\cdots 0\rangle. \tag{10.5}$$

(The vector $\langle l_1,\dots, l_N|$ is defined in the same way as $\langle l_1 l_2|$; see §4.) Writing $W(x, m; k)=\hat{W}(x, m; k)E(x, m; k)$, $W^*(x, m; k)=\hat{W}^*(x, m; k)E(x, m; k)^{-1}$ with $E(x, m; k)_{ij}=\delta_{ij}(k-k_0)^{m^{(j)}}\exp\left(\xi\left(x^{(j)}, \frac{1}{k-k_0}\right)\right)$, we have

$$\hat{W}(x, m; k)=1+O(k^{-1}),\ \hat{W}^*(x, m; k)=1+O(k^{-1}) \quad \text{as} \quad k\longrightarrow\infty.$$

By virtue of the Wick's theorem, we have further that

$$\det\hat{W}(x, m; k)=\langle 1\cdots 1|\psi^{(N)}(k)\cdots\psi^{(1)}(k)g(x, m)|0\cdots 0\rangle/\tau(x, m). \tag{10.6}$$

Next we impose on g the condition of reduction $\iota(g)=g$, where $\iota(\psi^{(j)}(k))=k\psi^{(j)}(k)$ and $\iota(\psi^{*(j)}(k))=k^{-1}\psi^{*(j)}(k)$. The Lie algebra $\{X\in A_\infty|\iota(X)=X\}$ is isomorphic to $A_{N-1}^{(1)}$. For example:

$$g=\exp\left(\sum_{i=1}^{M}\sum_{j\neq k}c_i^{(jk)}\psi^{(j)}(p_i)\psi^{*(k)}(p_i)\right).$$

This implies the translational invariance $g(x^{(1)}+x_0,\dots, x^{(N)}+x_0, m^{(1)}+m,\dots, m^{(N)}+m)=g(x, m)$. From the formulas (1.21) and (10.6) it follows that

$$\det\hat{W}(x, m; k)=1.$$

In terms of W and W^*, the bilinear identity takes the form

$$\oint_C\frac{dk}{2\pi i}\alpha(k)W(x, m; k)^t W^*(x', m'; k)=0,$$

where the contour C encircles $k=k_0$ and ∞ as in Figure 10, and $\alpha(k)$ is any meromorphic function having poles only at $k=k_0$ and ∞. In particular, the choice $x^{(i)}=x^{(i)\prime}+\epsilon(p-k_0)$, $m^{(i)}=m^{(i)\prime}$ $(i=1,\dots,N)$ and $\alpha(k)=(k-k_0)^{-1}$ leads to the identity

$$\hat{W}(x', m; p)^t\hat{W}^*(x', m; p)=1. \tag{10.7}$$

Let a be a small parameter. For a function $f(x, m)$ of (x, m) we adopt the notation $f[x^{(i)}+\epsilon(a), m^{(i)}+1]$ to signify $f(x^{(1)},\dots, x^{(i)}+\epsilon(a),\dots, x^{(N)}, m^{(1)},\dots, m^{(i)}+1,\dots, m^{(N)})$. We shall show that the matrix W satisfies the following linear difference equation

$$W=\left(1-\frac{aA^{(i)}}{k-k_0}\right)W[x^{(i)}+\epsilon(a)], \tag{10.8}$$

where $A^{(i)}=u^{(i)t}u^{*(i)}$ is a matrix of rank 1 given by

$$\tau\cdot u_j^{(i)}=\begin{cases} \mp\tau_{0\cdots\underset{-1}{\overset{i}{}}\cdots\underset{+1}{\overset{j}{}}\cdots 0}[m^{(i)}+1] & (i\gtrless j)\\ \tau[m^{(i)}+1] & (i=j)\end{cases} \tag{10.9}$$

$$\tau[x^{(i)}+\epsilon(a)]u_j^{*(i)}=\begin{cases} \mp\tau_{0\cdots\underset{+1}{\overset{i}{}}\cdots\underset{-1}{\overset{j}{}}\cdots 0}[x^{(i)}+\epsilon(a), m^{(i)}-1] & (i\gtrless j)\\ \tau[x^{(i)}+\epsilon(a), x^{(i)}-1] & (i=j).\end{cases}$$

To see (10.8), we note first that

$$W[x^{(i)}+\epsilon(a)]=\hat{W}[x^{(i)}+\epsilon(a)]\left(1-E_i+\frac{k-k_0}{k-k_0-a}E_i\right)E(x, m; k)$$

where $E_i=(\delta_{i\alpha}\delta_{i\beta})_{\alpha,\beta=1,\dots,N}$. Let $A(k)$ be a matrix whose only poles are at $k=k_0$ and ∞. Set $\hat{V}=\hat{W}-A(k)\hat{W}[x^{(i)}+\epsilon(a)]\left(1-E_i+\dfrac{k-k_0}{k-k_0-a}E_i\right)$. The bilinear identity ensures that

$$\oint_C\frac{dk}{2\pi i}\alpha(k)\hat{V}(x, m; k)^t\hat{W}^*(x, m; k)=0. \tag{10.10}$$

Let us choose $A(k)$ so as to satisfy the conditions

$$\begin{aligned}\hat{V}(x, m; k)&=O(1) \quad \text{at} \quad k=k_0+a \quad \text{and} \quad k=\infty,\\ &=O(k-k_0) \quad \text{at} \quad k=k_0.\end{aligned}$$

These conditions determine $A(k)$ uniquely in the form $B+\dfrac{A}{k-k_0}$. Taking $\alpha(k)=(k-k_0)^{-\nu}$ $(\nu=2, 3,\dots)$ in (10.10) we then conclude that $\hat{V}\equiv 0$, or equivalently that $W=\left(B+\dfrac{A}{k-k_0}\right)W[x^{(i)}+\epsilon(a)]$. Comparing the behavior at $k=\infty$ we get $B=1$. Likewise, from the behavior at $k=k_0$ we obtain

$$A=-a\hat{W}E_i\hat{W}[x^{(i)}+\epsilon(a)]^{-1}|_{k=k_0}.$$

Using (1.21) and (10.7) we arrive at the formula (10.9).

We remark that repeated use of (10.8) gives, for example,

$$W=\left(1-\frac{A^{(ij)}}{k-k_0}\right)W[x^{(i)}+\in(a),\ x^{(j)}+\in(b)] \qquad (i\neq j)$$

where $A^{(ij)}=aA^{(i)}+bA^{(j)}[x^{(i)}+\in(a)]$ is of rank two. (Here we have used $A^{(i)}A^{(j)}[x^{(i)}+\in(a)]=0$.)

Now let us introduce two singular points $k_0=\pm 1$ and the time evolutions attached to them. The corresponding time variables are denoted by (x, m) and (y, n), respectively. In the sequel we fix them so as to satisfy $\det E(x, m, y, n; k)=1$. Fix i, j arbitrarily, and put $W(\mu, \nu)=W[x^{(i)}+\mu\in(a),\ y^{(j)}+\nu\in(b)]$ for $\mu, \nu\in \boldsymbol{Z}$.

From (10.8) we have a linear system of the form

$$\begin{aligned}W(\mu, \nu)&=\left(1-\frac{aA(\mu, \nu)}{k-1}\right)W(\mu+1, \nu)\\&=\left(1-\frac{bB(\mu, \nu)}{k+1}\right)W(\mu, \nu+1).\end{aligned} \tag{10.11}$$

The integrability condition leads to the non-linear difference equation

$$\begin{aligned}&\frac{2}{b}(A(\mu\ \nu+1)-A(\mu\ \nu))\\&=\frac{2}{a}(B(\mu+1\ \nu)-B(\mu\ \nu))\\&=B(\mu\nu)A(\mu\ \nu+1)-A(\mu\nu)B(\mu+1\ \nu).\end{aligned}$$

Setting $k=0$ in (10.11) and eliminating A, B in terms of

$$R(\mu, \nu)=W(\mu, \nu)\,|_{k=0}\cdot(1+a)^{\mu/N}(1-b)^{\nu/N}$$

we obtain the following difference analogue of the principal chiral field equation (10.2)

$$\begin{aligned}&(1+a)^{1/N}(R(\mu\nu)R(\mu+1\ \nu)^{-1}-R(\mu\ \nu+1)R(\mu+1\ \nu+1)^{-1})\\&\quad+(1-b)^{1/N}(R(\mu\nu)R(\mu\ \nu+1)^{-1}-R(\mu+1\ \nu)R(\mu+1\ \nu+1)^{-1})=0,\\&\det R(\mu\nu)=1.\end{aligned} \tag{10.12}$$

Note that in the limit a, $b\to 0$, (10.12) and (10.11) reduce respectively to the chiral field equation (10.2) and its linearization

$$\frac{\partial}{\partial x_1}W=\frac{A}{k-1}W,\quad \frac{\partial}{\partial y_1}W=\frac{B}{k+1}W$$

with $x_1=x_1^{(i)}$, $y_1=y_1^{(j)}$. With due choice of variables, the expectation values

(10.4), (10.5) thus provide solutions to both the discrete and the continuous chiral field equations.

Let us write down the formulas above in the case $N=2$ more explicitly. Put $x=x^{(1)}-x^{(2)}$, $y=y^{(1)}-y^{(2)}$, $m=m^{(1)}-m^{(2)}$ and $n=n^{(1)}-n^{(2)}$. The relevant τ functions are

$$\left.\begin{matrix} f^* \\ f \end{matrix}\right\}=\langle 00|\mathbf{g}(x, y, m\pm 1, n)|00\rangle$$

$$\left.\begin{matrix} g^* \\ g \end{matrix}\right\}=\langle \mp 1, \pm 1|\mathbf{g}(x, y, m\pm 1, n)|00\rangle$$

$$\left.\begin{matrix} \hat{f}^* \\ \hat{f} \end{matrix}\right\}=\langle 00|\mathbf{g}(x, y, m, n\pm 1)|00\rangle$$

$$\left.\begin{matrix} \hat{g}^* \\ \hat{g} \end{matrix}\right\}=\langle \mp 1, \pm 1|\mathbf{g}(x, y, m, n\pm 1)|00\rangle .$$

Redefining $W(x, y, m, n; k)$ by multiplying $(k-1)^{-(m^{(1)}+m^{(2)})/2}(k+1)^{-(n^{(1)}+n^{(2)})/2}\exp\left(-\frac{1}{2}\left(\xi\left(x^{(1)}+x^{(2)}, \frac{1}{k-1}\right)+\xi\left(y^{(1)}+y^{(2)}, \frac{1}{k+1}\right)\right)\right)$ so that $\det W\equiv 1$ holds, we have

$$\begin{aligned} W(\mu\ \nu)&=\sqrt{1+a}\left(1-\frac{a}{2(k-1)}(S(\mu, \nu)+1)\right)W(\mu+1\ \nu) \\ &=\sqrt{1-b}\left(1-\frac{b}{2(k+1)}(\hat{S}(\mu, \nu)+1)\right)W(\mu\ \nu+1). \end{aligned}$$

Here

$$S(\mu\nu)=\sum_{\alpha=1}^{3}\sigma_\alpha S_\alpha(\mu\nu), \quad \hat{S}(\mu\nu)=\sum_{\alpha=1}^{3}\sigma_\alpha \hat{S}_\alpha(\mu\nu)$$
$$S(\mu\nu)^2=1, \quad \hat{S}(\mu\nu)^2=1,$$

σ_α are Pauli matrices, and $S_\alpha(\mu\nu)$ are given by

$$S_1(\mu\nu)=\frac{f^*(\mu\nu)g(\mu+1\ \nu)+g^*(\mu\nu)f(\mu+1\ \nu)}{f^*(\mu\nu)f(\mu+1\ \nu)+g^*(\mu\nu)g(\mu+1\ \nu)}$$

$$S_2(\mu\nu)=-i\frac{f^*(\mu\nu)g(\mu+1\ \nu)-g^*(\mu\nu)f(\mu+1\ \nu)}{f^*(\mu\nu)f(\mu+1\ \nu)+g^*(\mu\nu)g(\mu+1\ \nu)}$$

$$S_3(\mu\nu)=\frac{f^*(\mu\nu)f(\mu+1\ \nu)-g^*(\mu\nu)g(\mu+1\ \nu)}{f^*(\mu\nu)f(\mu+1\ \nu)+g^*(\mu\nu)g(\mu+1\ \nu)}.$$

$\hat{S}_\alpha(\mu\nu)$ are given by replacing $f^*(\mu\nu)$, $g^*(\mu\nu)$, $f(\mu+1\ \nu)$ and $g(\mu+1\ \nu)$ by $\hat{f}^*(\mu\nu)$, $\hat{g}^*(\mu\nu)$, $\hat{f}(\mu\ \nu+1)$ and $\hat{g}(\mu\ \nu+1)$, respectively. From the bilinear identity we have the bilinearization of the discrete chiral field equation

$$
\begin{aligned}
&f^*f+g^*g=\hat{f}^*\hat{f}+\hat{g}^*\hat{g}\,,\\
&f^*(\mu\nu)f(\mu+1\ \nu)+\left(1+\frac{a}{2}\right)g^*(\mu\nu)g(\mu+1\ \nu)\\
&\qquad\qquad=\hat{f}^*(\mu+1\ \nu)\hat{f}(\mu\nu)+\left(1+\frac{a}{2}\right)\hat{g}^*(\mu\nu)\hat{g}(\mu+1\ \nu)\\
&f^*(\mu\ \nu+1)f(\mu\nu)+\left(1-\frac{b}{2}\right)g^*(\mu\nu)g(\mu\ \nu+1)\\
&\qquad\qquad=\hat{f}^*(\mu\nu)\hat{f}(\mu\ \nu+1)+\left(1-\frac{b}{2}\right)\hat{g}^*(\mu\nu)\hat{g}(\mu\ \nu+1)\\
&\hat{f}^*(\mu+1\ \nu)\hat{g}(\mu\nu)-\left(1+\frac{a}{2}\right)\hat{f}^*(\mu\nu)\hat{g}(\mu+1\ \nu)+\frac{a}{2}f^*(\mu\nu)g(\mu+1\ \nu)=0\\
&f^*(\mu\ \nu+1)g(\mu\nu)-\left(1-\frac{b}{2}\right)f^*(\mu\nu)g(\mu\ \nu+1)-\frac{b}{2}\hat{f}^*(\mu\nu)\hat{g}(\mu\ \nu+1)=0.
\end{aligned}
$$

The equations obtained by the simultaneous exchange $f\leftrightarrow g$, $f^*\leftrightarrow g^*$, $\hat{f}\leftrightarrow\hat{g}$ and $\hat{f}^*\leftrightarrow\hat{g}^*$ are also valid.

Finally we give an example of soliton solutions corresponding to the choice $\mathbf{g}=\exp(c_1\psi^{(1)}(p)\psi^{*(2)}(p)+c_2\psi^{(2)}(q)\psi^{*(1)}(q))$:

$$
\tau R=\begin{pmatrix} 1+\dfrac{p^2}{(p-q)^2}e^{\eta_1+\eta_2} & e^{\eta_1}\\ e^{\eta_2} & 1+\dfrac{q^2}{(p-q)^2}e^{\eta_1+\eta_2}\end{pmatrix}\begin{pmatrix} e^{\xi/2} & \\ & e^{-\xi/2}\end{pmatrix}
$$

$$
\tau=1+\frac{pq}{(p-q)^2}e^{\eta_1+\eta_2}
$$

where

$$
e^{\eta_1}=c_1\left(\frac{p-1}{p-1-a}\right)^{\mu}\left(\frac{p+1}{p+1-b}\right)^{\nu},\quad e^{\eta_2}=c_2\left(\frac{q-1}{q-1-a}\right)^{-\mu}\left(\frac{q+1}{q+1-b}\right)^{-\nu}
$$
$$
e^{\xi/2}=(1+a)^{-\mu/2}(1-b)^{-\nu/2}
$$

in the discrete case,

$$
e^{\eta_1}=c_1e^{\xi\left(x,\frac{1}{p-1}\right)+\xi\left(y,\frac{1}{p+1}\right)},\quad e^{\eta_2}=c_2e^{-\xi\left(x,\frac{1}{q-1}\right)-\xi\left(y,\frac{1}{q+1}\right)}
$$
$$
e^{\xi/2}=e^{(\xi(x,-1)+\xi(y,1))/2}
$$

in the continuous case.

Appendix 1. Bilinear Equations for the (Modified) KP Hierarchies

We give below a list of bilinear differential equations of low degree for the KP and the modified KP hierarchies, where we count $\deg D_\nu=\nu$ $(\nu=1,2,\ldots)$. For the l-th modified KP hierarchy, the number of linearly independent bilinear equations of degree n is known to be $p(n-l-1)$, where

$p(\nu)=\#\{$partitions of ν into positive integral parts$\}$.

ν	0	1	2	3	4	5	6	7	8	9	10	11	12
$p(\nu)$	1	1	2	3	5	7	11	15	22	30	42	56	77

(Note that if $P(-D)=-P(D)$, then $P(D)f\cdot f=0$ holds for any function f. For $l=0$, these trivial equations are also included in the counting.) Formulas for the bilinear equations in terms of determinants are also available ([3]).

KP hierarchy $P(D)\tau_n\cdot\tau_n=0$ ([1])

degree 4	$D_1^4-4D_1D_3+3D_2^2$
degree 5	$(D_1^3+2D_3)D_2-3D_1D_4$
degree 6	$D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5-45D_1^2D_2^2$
	$D_1^6+4D_1^3D_3-32D_3^2-9D_1^2D_2^2+36D_2D_4$
degree 7	$(D_1^5+10D_1^2D_3+24D_5)D_2+5D_1^3D_4-40D_1D_6$
	$D_1D_2^3+(D_1^3+2D_3)D_4-4D_1D_6$
	$D_1^2D_3D_2+D_3D_4-2D_1D_6$
degree 8	$D_1^8+14D_1^5D_3+84D_1^3D_5-504D_3D_5-120D_1D_7-105D_1^2D_2D_4$ $+210D_4^2+420D_2D_6$
	$-2D_1^2D_3^2+4D_1^3D_5+4D_3D_5-12D_1D_7+D_1^4D_2^2-9D_4^2+14D_2D_6$
	$-6D_1^2D_3^2+4D_1^3D_5-4D_3D_5+12D_1D_7+D_2^4-6D_1^2D_2D_4-3D_4^2+2D_2D_6$
	$D_1^5D_3-16D_3D_5-5D_1D_3D_2^2+20D_2D_6$
	$2D_1^2D_3^2+4D_3D_5-12D_1D_7+2D_1D_3D_2^2+3D_1^2D_2D_4+3D_4^2-2D_2D_6$
degree 9	$(D_1^7+35D_1D_3^2-21D_1^2D_5+90D_7)D_2+105D_4D_5$ $+(35D_1^3-140D_3)D_6-105D_1D_8$
	$D_1^3D_2^3+9D_1D_2^2D_4+6D_1^2D_3D_4+(4D_1^3+16D_3)D_6-36D_1D_8$
	$(3D_1^4D_3+12D_1^2D_5+48D_7)D_2+(-21D_1^2D_3-36D_5)D_4-24D_1D_2^2D_4$ $+(2D_1^3-8D_3)D_6+24D_1D_8$
	$-3D_1^2D_3D_4+D_3D_2^3+2D_1^3D_6$
	$(3D_1D_3^2+3D_1^2D_5-6D_7)D_2+3D_1D_4D_2^2+(6D_1^2D_3+9D_5)D_4$ $+(-D_1^3+4D_3)D_6-21D_1D_8$
	$(D_1^5+20D_1^2D_3+22D_5)D_4+15D_1D_2^2D_4+2D_3D_6-60D_1D_8$

1st Modified KP hierarchy $P(D)\tau_n \cdot \tau_{n+1}=0$ (cf. [1])

degree 2	$D_1^2+D_2$
degree 3	$D_1^3-4D_3-3D_1D_2$
degree 4	$D_1^4+8D_1D_3+3D_1^2D_2-6D_2^2$
	$-D_1^2D_2+D_2^2+2D_4$
degree 5	$D_1^5-16D_5+5D_1D_2^2-10D_1D_4$
	$(D_1^3-4D_3)D_2+3D_1D_2^2+6D_1D_4$
	$D_1^5-4D_1^2D_3+3D_1^3D_2+6D_1D_2^2$
degree 6	$D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5+(-15D_1^4+60D_1D_3)D_2$
	$(-D_1^4-8D_1D_3)D_2-3D_1^2D_2^2+6D_1^2D_4$
	$-D_1^6+16D_1^3D_3+3D_1^4D_2+12D_2^3$
	$D_1^3D_3+2D_3^2-3D_1D_2D_3+6D_6$
	$3D_1^6+192D_1D_5+(-35D_1^4-160D_1D_3)D_2-90D_1^2D_2^2$ $+180D_1^2D_4-120D_2D_4$
degree 7	$11D_1^7-70D_1^4D_3-336D_1^2D_5+560D_1D_3^2-480D_7$ $+(-7D_1^5+490D_1^2D_3-168D_5)D_2+(210D_1^3+420D_3)D_2^2$
	$3D_1^7+112D_1^2D_5-640D_7+(14D_1^5-224D_5)D_2+35D_1^3D_2^2-70D_1^3D_4$ $-70D_1D_2^3+140D_1D_2D_4$
	$5D_1^4D_3-120D_1^2D_5+40D_1D_3^2+240D_7+(D_1^5+35D_1^2D_3+84D_5)D_2$ $+(-5D_1^3-10D_3)D_2^2+30D_1D_2^3$
	$(-3D_1^5+48D_5)D_2+(-5D_1^3+20D_3)D_2^2+(10D_1^3-40D_3)D_4$
	$D_1^7-280D_1D_3^2+294D_1^2D_5-120D_7+(7D_1^5-70D_1^2D_3-42D_5)D_2$ $+(-35D_1^3-70D_3)D_2^2-105D_1D_2^3$
	$3D_1^4D_3-56D_1^2D_5+24D_1D_3^2+80D_7+(17D_1^2D_3+28D_5)D_2+2D_3D_2^2$ $+2D_1^3D_4+14D_1D_2^3+8D_1D_2D_4$
	$-D_1^2D_5+4D_7+(D_1^2D_3-D_5)D_2+D_3D_2^2+D_1^3D_4+D_1D_2D_4+4D_1D_6$

2nd Modified KP hierarchy $P(D)\tau_n\cdot\tau_{n+2}=0$

degree 3	$D_1^3+2D_3+3D_1D_2$
degree 4	$D_1^4-4D_1D_3-3D_2^2-6D_4$
degree 5	$D_1^5-10D_1^2D_3+24D_5+(-5D_1^3+20D_3)D_2$
	$D_1^5+20D_1^2D_3+24D_5-15D_1D_2^2+30D_1D_4$
degree 6	$D_1^6+10D_1^3D_3-20D_3^2-36D_1D_5+15D_2^3-60D_6$
	$D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5+45D_1^2D_2^2+90D_1^2D_4$
	$D_1^6-40D_3^2+24D_1D_5+(5D_1^4-20D_1D_3)D_2+15D_1^2D_2^2$ $+30D_1^2D_4+15D_2^3+30D_2D_4$
degree 7	$(2D_1^7-70D_1^4D_3-252D_1^2D_5+140D_1D_3^2-1080D_7)$ $+(42D_1^5+210D_1^2D_3-252D_5)D_2$ $+(105D_1^3+210D_3)D_2^2-105D_1D_2^3-840D_1D_6$
	$(D_1^5+20D_1^2D_3+24D_5)D_2+(5D_1^3+10D_3)D_2^2-(10D_1^3+20D_3)D_4$
	$(D_1^7+84D_1^2D_5+140D_1D_3^2-120D_7)+(14D_1^5+70D_1^2D_3-84D_5)D_2$ $+(35D_1^3+70D_3)D_2^2-210D_1D_2D_4$
	$(3D_1^7-168D_1^2D_5+480D_7)+(14D_1^5+280D_1^2D_3+336D_5)D_2$ $+105D_1^3D_2^2-210D_1^3D_4$
	$(5D_1^7-70D_1^4D_3+560D_1D_3^2-1440D_7)+(63D_1^5-1008D_5)D_2$

3rd Modified KP hierarchy $P(D)\tau_n\cdot\tau_{n+3}=0$

degree 4	$D_1^4+8D_1D_3+6D_1^2D_2+3D_2^2+6D_4$
degree 5	$3D_1^5-48D_5+(10D_1^3-40D_3)D_2-15D_1D_2^2-30D_1D_4$
degree 6	$D_1^6-40D_1^3D_3-96D_1D_5+15D_1^2D_2^2-90D_1^2D_4+30D_2^3+60D_2D_4$
	$D_1^6-8D_1^3D_3+16D_3^2-9D_1^2D_2^2-18D_1^2D_4+6D_2^3+36D_2D_4+48D_6$
degree 7	$(5D_1^7-70D_1^4D_3+560D_1D_3^2-1440D_7)$ $+(-42D_1^5+420D_1^2D_3-1008D_5)D_2$ $+(-105D_1^3-210D_3)D_2^2+(-210D_1^3-420D_3)D_4$
	$(D_1^7+28D_1^4D_3-56D_1D_3^2-288D_7)+(-21D_1^3+84D_3)D_2^2$ $+(42D_1^3-168D_3)D_4+42D_1D_2^3-168D_1D_6$
	$(2D_1^7+35D_1^4D_3+168D_1^2D_5-280D_1D_3^2-240D_7)$ $+(7D_1^5-70D_1^2D_3+168D_5)D_2$ $+(140D_1^3-350D_3)D_4+105D_3D_2^2+105D_1D_2^3+210D_1D_2D_4$

Appendix 2. Bilinear Equations for the 2 Component Reduced KP Hierarchy

Here are the bilinear equations of low degree for the τ functions of the 2 component KP hierarchy with the reduction condition (4.5) and the time evolution (9.6). We set

$$\tau_{m,n}(x, y)=\tau_{l,n^{(1)},n^{(2)}}(x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)})$$

with $m=l$, $n=n^{(2)}-n^{(1)}-l$, $x=x^{(1)}-x^{(2)}$ and $y=y^{(1)}-y^{(2)}$. It is related to $\tau_{l_1,l_2:l}(x^{(1)}, x^{(2)})$ of (4.3) by

$$\tau_{m,n}(x, 0)=(-)^{l(l_2+1)}\tau_{l_1,l_2;l}(x^{(1)}, x^{(2)})$$

with $m=-l$, $n=l_2-l_1+l$ and $x=x^{(1)}-x^{(2)}$.

In the list the equations of low degree among the following τ functions are given:

$$F=\tau_{l_1.l_2},\quad G^*=\tau_{l_1-1,l_2+1},\quad G=\tau_{l_1+1,l_2-1},$$
$$f^*=\tau_{l_1,l_2-1},\quad f=\tau_{l_1,l_2+1},\quad g^*=\tau_{l_1+1,l_2},\quad g=\tau_{l_1-1,l_2}.$$

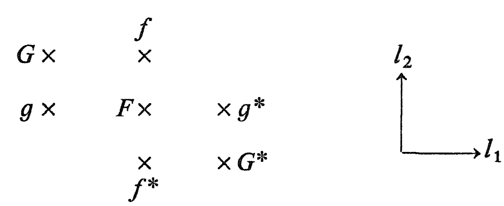


Fig. The τ functions of 2 component reduced KP hierarchy.

We omit those equations which are obtained from the ones in the list by the symmetry

$$\tau_{l_1,l_2}(x, y)\longrightarrow\tau_{l_1+l,l_2+l}(x, y)$$

or

$$\tau_{l_1,l_2}(x, y)\longrightarrow\tau_{l_1,l-l_2}(y, x).$$

$$
\begin{aligned}
& f^*f+g^*g-F^2=0, \quad f^*f-g^*g-F^2+D_1\hat{D}_1F\cdot F=0, \\
& (D_1\hat{D}_1-1)(f^*\cdot f-g^*\cdot g)+F^2=0, \quad D_1(f^*\cdot f+g^*\cdot g)=0, \\
& D_1^2(f^*\cdot f+g^*\cdot g+F\cdot F)=0, \quad D_1^2F\cdot F+2G^*G=0, \\
& (D_2+D_1^2)(f^*\cdot f+g^*\cdot g)=0, \quad (D_2-D_1^2)(f^*\cdot f-g^*\cdot g)=0, \\
& (D_2-D_1^2)F\cdot G=0, \quad (D_1\hat{D}_1-1)F\cdot G=0, \\
& D_1f\cdot g-FG=0, \quad D_2f\cdot g-D_1F\cdot G=0, \\
& D_1g^*\cdot F-G^*f=0, \quad D_1^2g^*\cdot F-D_1G^*\cdot f=0, \\
& D_2g^*\cdot F-D_1G^*\cdot f=0, \quad (D_1\hat{D}_1-2)g^*\cdot F+\hat{D}_1G^*\cdot f=0, \\
& D_1F\cdot f+g^*G=0, \quad D_1^2F\cdot f+D_1g^*\cdot G=0, \\
& D_2F\cdot f+D_1g^*\cdot G=0, \quad D_1\hat{D}_1F\cdot f-\hat{D}_1g^*\cdot G=0.
\end{aligned}
$$

Here $D_j=D_{x_j}$ and $\hat{D}_j=D_{y_j}$.

Appendix 3. Bilinear Equations Related to the Spin Representation of B_∞

Here we give a list of bilinear differential equations of low degree contained in (6.6). The number of equations of degree n is known to be

$$
\begin{aligned}
& \#\{(m_1,\dots,m_k)\mid m_i\in 2\boldsymbol{Z}+1,\ k\geqq 1,\ 1\leqq m_1\leqq\cdots\leqq m_k,\ \sum_{i=1}^{k} m_i=n\} \\
& -\#\{(m_1,\dots,m_k)\mid m_i\in 2\boldsymbol{Z},\ k\geqq 1,\ 1\leqq m_1\leqq\cdots\leqq m_k,\ \sum_{i=1}^{k} m_i=n\}.
\end{aligned}
$$

The equation of odd degree are all trivial in the sense mentioned in Appendix 1.

BKP hierarchy $P(D_{odd})\tau(x_{odd})\cdot\tau(x_{odd})=0$

degree 6	$D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5$
degree 8	$D_1^8+7D_1^5D_3-35D_1^2D_3^2-21D_1^3D_5-42D_3D_5+90D_1D_7$
degree 10	$D_1^{10}+63D_1^5D_5-225D_1^3D_7-175D_1D_3^3+525D_1D_9-189D_5^2$
	$6D_1^5D_5-5D_1^4D_3^2-15D_1^3D_7+15D_1^2D_3D_5-5D_1D_3^3+10D_1D_9-15D_3D_7+9D_5^2$
	$D_1^7D_3-21D_1^2D_3D_5+35D_1D_9-15D_3D_7$

Appendix 4. Bilinear Equations Related to the Spin Representation of D_∞

Here we give a list of bilinear differential equations of low degree contained in (7.5). We omit those equations which are obtained from the ones in the list by the symmetry $D_i \leftrightarrow \hat{D}_i$. We also omit equations in the BKP hierarchy (Appendix 3) from the list of the DKP hierarchy.

DKP hierarchy $P(D_{odd}, \hat{D}_{odd})\tau_l(x_{odd}, \hat{x}_{odd}) \cdot \tau_l(x_{odd}, \hat{x}_{odd})=0$

degree 4	$\hat{D}_1(D_1^3 - D_3)$
degree 6	$D_1^3\hat{D}_3 - \hat{D}_1^3 D_3$
	$\hat{D}_1(D_1^5 + 5D_1^2 D_3 - 6D_5)$
	$D_1^3\hat{D}_1^3 + D_1^3\hat{D}_3 - 2D_3\hat{D}_3$

modified DKP hierarchy $P(D_{odd}, \hat{D}_{odd})\tau_l(x_{odd}, \hat{x}_{odd})\tau_{l*}(x_{odd}, \hat{x}_{odd})=0$

degree 2	$D_1\hat{D}_1$
degree 3	$D_1^3 - D_3$
degree 4	$\hat{D}_1(D_1^3 + 2D_3)$
degree 5	$D_1^5 + 5D_1^2 D_3 - 6D_5$
	$\hat{D}_1^2(D_1^3 - D_3)$

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