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Chronoprojective Cartan Structures on Four-Dimensional Manifolds

By

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§ 0. Introduction

As indicated by its denomination Cartan structures have been derived from Cartan's works [1] initiating the projective and conformal geometries. In the fifties a precise description of this notion in a modern mathematical language has been given by using the fibre bundle of second order frames [2, 3].

The starting point can be viewed as a generalization of the Klein's Erlangen program. Indeed Cartan considered various spaces at each point of which an homogeneous space of the same dimension is tangentially associated, with the possibility of connecting these tangent spaces at different neighbouring points of the base space. Moreover these spaces were endowed with a "normal" connection which allows to develop the base space on the tangent homogeneous space along a curve.

In a geometrical language the above depicted situation is described by using the notions of Cartan connection and Cartan structure. The classical geometries i.e. the projective [4] and conformal [5] geometries are the standard examples of Cartan structures; they correspond to the case where the bigger concerned Lie group is semisimple and its Lie algebra is |1|-graded. A general study of this case can be found in the literature [6]. On the contrary the geometrical structures considered in this paper, do not enter this scheme. They deal with a group which is not semi-simple and whose Lie algebra is |2|-graded, the so-called chronopro-

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jective group, which is a generalization with respect to a degenerate quadratic form of the orthogonal group.

It will be shown that the geometry which derives from the chronoprojective group, the so-called chronoprojective geometry, is a kind of Weyl's geometry [7] in the sense that it reconciles the notions of conformal equivalence over a Galilean manifold and of projective equivalence between Newtonian connections. It appears that the chronoprojective geometry is the very geometry of the Newtonian cosmology. Moreover it also explains various "accidental symmetries" in classical mechanics, e.g. the Kepler similitudes, the new kinematical symmetries of the system of a charged particle in a Dirac magnetic monopole field etc.… [8, 9]

This paper is organized as follows:

—In Section 1 the chronoprojective group is defined and a particular four-dimensional homogeneous space is introduced.

—Chronoprojective Cartan connections are described in Section 2 and conditions under which there exists a uniquely defined chronoprojective connection are exhibited.

—Section 3 deals with the chronoprojective Cartan structures. A notion of admissibility for a linear connection to belong to a chronoprojective structure is given. The notion of chronoprojective equivalence is presented as the condition that two admissible connections belong to the same chronoprojective structure. The chronoprojective Weyl's curvature tensor is defined and chronoprojectively flat structures are introduced.

—In Section 4 chronoprojective structures are constructed over Galilean manifolds and the similitudes with Weyl's structures over Riemannian manifolds is established.

§ 1. A Homogeneous Space of the Chronoprojective Group

First let us define a generalization of the orthogonal and conformal groups with respect to degenerate quadratic forms. For $n \geq p \in \boldsymbol{N}$ we set

$$\mathrm{O}^{n-p}(p) = \left\{ g \in \mathrm{Gl}(n, \boldsymbol{R}) \mid g S^p(n)\, {}^t g = S^p(n),\ S^p(n) = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & \mathrm{O}_{n-p} \end{pmatrix} \in M_n \right\}$$

$$\mathrm{O}_{n-p}(p) = \left\{ g \in \mathrm{Gl}(n, \boldsymbol{R}) \mid {}^t g S_p(n)\, g = S_p(n),\ S_p(n) = \begin{pmatrix} \mathrm{O}_{n-p} & 0 \\ 0 & \mathbf{1}_p \end{pmatrix} \in M_n \right\}$$

M_n denoting the $n\times n$ square matrices

$$\mathrm{CO}^{n-p}(p) = \{g\in \mathrm{Gl}(n, \boldsymbol{R}) \mid gS^p(n)\,{}^tg = \lambda^2 S^p(n),\ \lambda\in\dot{\boldsymbol{R}} := \boldsymbol{R}-\{O\}\},$$

$$\mathrm{CO}_{n-p}(p) = \{g\in \mathrm{Gl}(n, \boldsymbol{R}) \mid {}^tgS_p(n)\,g = \lambda^2 S_p(n),\ \lambda\in\dot{\boldsymbol{R}}\}.$$

Note that $\mathrm{O}^0(p) = \mathrm{O}_0(p)$ is the usual orthogonal group denoted by $\mathrm{O}(n)$.

Now, let us consider the group $\mathrm{O}^2(3)$ which will be called the chronoprojective group (cf. § 4). Its canonical representation is given in terms of 5×5 matrices by

$$\begin{pmatrix} A & \overline{B} & \overline{C} \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \tag{1.1}$$

where A is a 3×3 matrix of $\mathrm{O}(3)$, $\overline{B}$ and $\overline{C}$ are column 3×1 matrices $\in\boldsymbol{R}^3$ and $\begin{pmatrix} d & e \\ f & g\end{pmatrix}\in \mathrm{Gl}(2, \boldsymbol{R})$. $\mathrm{O}^2(3)$ is a 13-dimensional Lie group which can be decomposed as $(\boldsymbol{R}^3\otimes\boldsymbol{R}^3)\rtimes(\mathrm{O}(3)\otimes \mathrm{Gl}(2, \boldsymbol{R}))$ and contains the orthochronous Galilei group $(d=g=1, f=0)$ isomorphic to $(\boldsymbol{R}^3\otimes\boldsymbol{R}^3)\rtimes(\mathrm{O}(3)\otimes\boldsymbol{R})$. Let us introduce a particular subgroup L^0 of $\mathrm{O}^2(3)$ generated by the elements of $\mathrm{O}^2(3)$ which admit ${}^t(00001)$ as eigenvector. L^0 is the group of matrices of the form:

$$h=\begin{pmatrix} A & \overline{B} & 0 \\ 0 & d & 0 \\ 0 & f & g \end{pmatrix} \tag{1.2}$$

with $A\in\mathrm{O}(3)$, $\overline{B}\in\boldsymbol{R}^3$, $d,f,g\in\boldsymbol{R}$ such that $dg\neq 0$, and can be written as $\boldsymbol{R}^3\rtimes(\mathrm{O}(3)\otimes\boldsymbol{R}\otimes \mathrm{S}_2)$, where S_2 denotes the 2-dimensional solvable group. Let $o^2(3)$ (resp. l^0) denote the Lie algebra of $\mathrm{O}^2(3)$ (resp. L^0). As a vector space $o^2(3)$ can be decomposed as

$$o^2(3) = a + l^0 \tag{1.3}$$

where a is a 4-dimensional Abelian Lie algebra ($[l^0, a]$ being not contained into a, this decomposition is not reductive).

It is easy to show that $o^2(3)$ is a $|2|$-graded Lie algebra i.e. it can be decomposed as

$$o^2(3) = g_{-2}+g_{-1}+g_0+g_1+g_2 \tag{1.4}$$

such that $[g_p, g_q]\subset g_{p+q}$ with $g_p=0$ for $|p|>2$, since there exists a unique (up to a conjugation) element D in g_0 such that $[D, g_p] = p\, g_p$. In fact $g_{-2} = g_2 = \boldsymbol{R}$ and $[g_2, g_{-2}]$ is proportional to D. Moreover $g_{-1} = g_1 = \boldsymbol{R}^3$, $[g_1, g_{-1}] = 0$ and $g_0 = o(3) + \boldsymbol{R}^2 \cdot g_{-2} + g_{-1}$ and $g_2 + g_1$ are both 4-dimensional Abelian subalgebras. We have

$$l^0 = g_0 + g_1 + g_2$$

so

$$a = g_{-2} + g_{-1}\,.$$

(A description including commutation relations of the Lie algebra $o^2(3)$ is given in [8] § 7. a)

Now, let us describe the homogeneous space $\mathrm{M} = O^2(3)/\mathrm{L}^0$. It is easy to see that $\mathrm{M} = (\boldsymbol{R}^3 \times (\boldsymbol{R}^2 - \{0\}))/\dot{\boldsymbol{R}}$. Taking into account that $\boldsymbol{R}^2 - \{0\}$ can be considered as a non-trivial principal $\dot{\boldsymbol{R}}$-bundle over the 1-dimensional projective space over $\boldsymbol{R}$, i.e. the circle S^1, M can be described as a vector bundle of standard fibre $\boldsymbol{R}^3$ over S^1 associated to $\boldsymbol{R}^2 - \{0\}$. Otherwise $\boldsymbol{R}^2 - \{0\}$ is also a trivial $\boldsymbol{R}^+$-principal bundle, so M can be equivalently written as $(\boldsymbol{R}^3 \times S^1)/\boldsymbol{Z}^2$ and appears as a generalized Möbius space. In Section 4 M will be called the chronoprojective space-time. The class of the identity $e \in \mathrm{O}^2(3)$ in M will be called the origin of M and denoted by o.

The linear isotropy representation ρ of L^0 (cf. Appendix A, Rel. (A. 2)) is not faithful. Its kernel N is a subgroup of L^0 isomorphic to $\boldsymbol{R}$, the Lie algebra of which is g_2. Explicitly one gets

$$\rho\left(\begin{pmatrix} A & \overline{B} & \overline{C} \\ 0 & d & e \\ 0 & f & g \end{pmatrix}\right) = \frac{1}{g}\begin{pmatrix} A & \overline{B} \\ 0 & d \end{pmatrix}. \tag{1.5}$$

It is easy to verify that $\mathrm{L}^I = \rho(\mathrm{L}^0) = \mathrm{CO}^1(3) \cap \mathrm{CO}_3(1) = \boldsymbol{R}^3 \rtimes (\mathrm{CO}(3) \otimes \boldsymbol{R})$. Let H denote the full homogeneous Galilei group which is a double covering of $\boldsymbol{R}^3 \rtimes \mathrm{O}(3)$. Then L^I can be written as the semi-direct product $\mathrm{H} \rtimes (\dot{\boldsymbol{R}}_s \otimes \dot{\boldsymbol{R}}_t)$ where $\dot{\boldsymbol{R}}_s$ and $\dot{\boldsymbol{R}}_t$ denote two distinct dilation subgroups defined as follows: let $S = \begin{pmatrix} \tilde{A} & \tilde{\overline{B}} \\ \mathrm{O} & \tilde{d} \end{pmatrix} \in \mathrm{L}^I$ where $\tilde{A} \in \mathrm{CO}(3)$, $\tilde{\overline{B}} \in \boldsymbol{R}^3$, $\tilde{d} \in \dot{\boldsymbol{R}}$, then $\dot{\boldsymbol{R}}_s$ is parametrized by $(\det \tilde{A})^{-1/3}$ and $\dot{\boldsymbol{R}}_t$ is parametrized by $\tilde{d}$. L^I can be called the conformal homogeneous Galilei group (cf. § 2 of [8]). It should also be noticed that $\mathrm{L}^0 = \boldsymbol{R}_n \otimes \mathrm{L}^I$ with the group law

(1.6) $$(S, f)(S', f') = \left(SS', \frac{\tilde{d}'}{|\det \tilde{A}'|^{1/3}} f + f'\right)$$

where $S \in \mathrm{L}^I$, and $f \in \boldsymbol{R}_n$ the kernel of the linear isotropy representation of L^0. This group law corresponds to the following choice of the injective homomorphism k: $\mathrm{L}^I \to \mathrm{L}^0$

(1.7) $$k\left(\begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{d} \end{pmatrix}\right) = \frac{1}{|\det A|^{1/3}} \begin{pmatrix} \tilde{A} & \bar{\tilde{B}} & 0 \\ 0 & \tilde{d} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally as a vector space $o^2(3)$ can be decomposed as

(1.8) $$o^2(3) = a + l^I + \boldsymbol{R}_n.$$

§ 2. Chronoprojective Connections

2. A) Definition and Structure Equations

The Cartan connection notion is described in Appendix A. According to the notations used therein, let L be the chronoprojective group $\mathrm{O}^2(3)$ and L^0 its subgroup considered in Section 1.

Definition 2.1. *Let* $\mathrm{L}^0(V_4)$ *be a principal* L^0*-bundle over a* 4*-dimensional manifold* V_4. *A chronoprojective connection is a Cartan connection in* $\mathrm{L}^0(V_4)$ *with respect to the chronoprojective group.*

Hence a chronoprojective connection form ω is $o^2(3)$-valued and can be written under the form

(2.1) $$\omega = \begin{pmatrix} w & \bar{w}_0 & \bar{w}_{0'} \\ 0 & w_0^0 & w_{0'}^0 \\ 0 & w_0^{0'} & w_{0'}^{0'} \end{pmatrix}$$

where $w = \{w_j^k, j, k = 1, 2, 3\}$ is $o(3)$-valued and $\{w, w_0^0, w_{0'}^{0'}\}$ is g_0-valued,

$\bar{w}_0 = \{w_0^j, j = 1, 2, 3\}$ is g_1-valued, $w_0^{0'}$ is g_2-valued

$\bar{w}_{0'} = \{w_{0'}^j, j = 1, 2, 3\}$ is g_{-1}-valued, $w_{0'}^0$ is g_{-2}-valued

so that $\{\bar{w}_0, w_{0'}^0\} = w_a$ is a-valued according to the decomposition (1.3).

Let us set $\tilde{w} = w - w_{0'}^{0'} \mathbf{1}_3$ and $w_D = w_0^0 - w_{0'}^{0'}$, then $w_I = \{\tilde{w}, \bar{w}_0, w_D\}$

is a l^I-valued 1-form which can be written as

$$w_I = \begin{pmatrix} \tilde{w} & \bar{w}_0 \\ 0 & w_D \end{pmatrix} \tag{2.2}$$

and $\{w_I, w_0^{0'}\}$ is l^0-valued.

Therefore a chronoprojective connection ω can be written as a set $\{w_\alpha, w_I, w_0^{0'}\}$ according to the vector space decomposition (1.8).

Proposition 2.2. *Let ω be a chronoprojective connection form for $\mathrm{L}^0(V_4)$ whose components are gathered under the form $\{w_\alpha, w_I, w_0^{0'}\}$ then*

***P*'1.** *Restricted to each fibre of $\mathrm{L}^0(V_4)$, $\{w_I, w_0^{0'}\}$ is the Maurer-Cartan form on L^0.*

***P*'2.** *The subspaces spanned by $\{w_\alpha\}$ and by $\{w_\alpha, w_I\}$ are stable under the right action of L^0, indeed one has*

(2.3) (a) $R_h^*(w_\alpha) = \rho(h^{-1})\, w_\alpha$,

(b) $R_h^*(w_I) = \rho(h^{-1}) \left(w_I + \frac{f}{d}\boldsymbol{B} \right) \rho(h)$,

(c) $R_h^*(w_0^{0'}) = \frac{d}{g} w_0^{0'} - \frac{f^2}{dg} w_{0'}^0 - \frac{f}{g} w_D$,

where $h \in \mathrm{L}^0$ is parametrized according to (1.2) *and $\boldsymbol{B}$ denotes the following 1-form matrix*

$$\boldsymbol{B} = \begin{pmatrix} w_{0'}^0 \mathbf{1}_3 & \bar{w}_{0'} \\ 0 & 2w_{0'}^0 \end{pmatrix}. \tag{2.4}$$

***P*'3.** *The subspace of $T_u(\mathrm{L}^0(V_4))$ defined by the four equations $w_{0'}^\mu(X_u) = 0$, $\mu = 0, 1, 2, 3$ is the subspace of vertical tangent vectors to $\mathrm{L}^0(V_4)$ at $u \in \mathrm{L}^0(V_4)$.*

The above properties are just the expression of the properties ***P*1**, 2, 3 of Appendix A in this particular situation. □

According to (A.1) the components of the 2-form Ω of a chronoprojective connection are given by

(2.5) (a) $\Omega_a = dw_a + w_I \wedge w_a$

(b) $\Omega_I = dw_I + w_I \wedge w_I - w_0^{0'} \wedge \boldsymbol{B}$

(c) $\Omega_0^{0'} = dw_0^{0'} + w_0^{0'} \wedge w_D$

Ω_a is called the torsion form and $\{\Omega_I, \Omega_0^{0'}\}$ the curvature form of the chronoprojective connection.

There is no conservation property concerning all the components of Ω. More precisely, if $\boldsymbol{D}$ denotes the exterior covariant differentiation, the second set of structure equations is the following

(2.6) (a) $\boldsymbol{D}\Omega_a = \Omega_I \wedge w_a$

(b) $\boldsymbol{D}\Omega_I = -\Omega_0^{0'} \wedge \boldsymbol{B}$

(c) $\boldsymbol{D}\Omega_0^{0'} = 0$.

2. B) Existence of a Uniquely Defined Chronoprojective Connection

Being given a set $\mathfrak{T} = \{w_a, w_I\}$ of twelve differential 1-forms whose values in each point are linearly independent and which satisfy Properties $\boldsymbol{P}'1, 2, 3$ of Proposition 2.2, there is at least one 1-form $w_0^{0'}$ such that $\{\mathfrak{T}, w_0^{0'}\}$ is a chronoprojective connection for $\mathrm{L}^0(V_4)$, (See Appendix A). Here we want to show that $\mathfrak{T}$ can be completed in such a way that the so-obtained chronoprojective connection is uniquely defined owing to specific properties of its curvature.

Let ω and $\hat{\omega}$ be two chronoprojective connections built by supplementing the same set $\mathfrak{T}$ with two 1-forms namely $w_0^{0'}$ and $\hat{w}_0^{0'}$. From $\boldsymbol{P}'3$ one deduces that $w_0^{0'}$ and $\hat{w}_0^{0'}$ are related by

$$w_0^{0'} - \hat{w}_0^{0'} = \sum_{\mu=0}^{3} A_\mu w_{0'}^{\mu} \tag{2.7}$$

where the coefficients $\{A_\mu\}$ are a set of four C^∞-functions on $\mathrm{L}^0(V_4)$. On the other hand, according to (A.2), any component of Ω_I can be written as

$$\Omega_\alpha^\beta = \frac{1}{2} \sum_{j,k=1}^{3} (K_\alpha^\beta)_{jk} w_{0'}^j \wedge w_{0'}^k + \sum_{j=1}^{3} (K_\alpha^\beta)_{j0} w_{0'}^j \wedge w_{0'}^0 \tag{2.8}$$

where the $(K_\alpha^\beta)_{\mu\nu}$'s are functions on $\mathrm{L}^0(V_4)$.

Then the components $\{\Omega_\alpha^\beta\}$ and $\{\hat{\Omega}_\alpha^\beta\}$ of the 2-forms corresponding to ω and $\hat{\omega}$ can be written either from their definition (2.5b) by taking (2.7) into account, or by means of their development (2.8). By identifying these two expressions one gets

$$(2.9)\qquad (a)\quad \frac{1}{3}\sum_{j=1}^{3}(K_0^j)_{j0}-(\hat{K}_0^j)_{j0}=A_0\,,$$

$$(b)\quad \frac{1}{2}\sum_{j=1}^{3}(K_0^j)_{jk}-(\hat{K}_0^j)_{jk}=-(K_D)_{k0}+(\hat{K}_D)_{k0}$$

$$=\sum_{j=1}^{3}-(K_j^j)_{k0}+(\hat{K}_j^j)_{k0}=A_k\,.$$

Therefore all the A_μ will vanish if the coefficients K and $\hat{K}$ both satisfy

$$(2.10)\qquad \begin{array}{llll} (a) & (K_D)_{j0}=0 & (c) & \sum_{j=1}^{3}(K_0^j)_{jk}=0 \\ (b) & \sum_{j=1}^{3}(K_j^j)_{k0}=0 & (d) & \sum_{j=1}^{3}(K_0^j)_{j0}=0 \end{array}$$

One has:

Proposition 2.3. *There is a unique chronoprojective connection associated to a given set* $\mathfrak{T}$*, such that its curvature possesses the following properties*

$$(2.11)\qquad a)\quad \forall j=1,2,3,\ \sum_{k,l=1}^{3}\varepsilon_{jkl}\Omega_D\wedge w_{0'}^k\wedge w_{0'}^l=0\,,$$

$$b)\quad \forall j=1,2,3,\ \sum_{k,l=1}^{3}\varepsilon_{jkl}\Omega_{0'}^{0'}\wedge w_{0'}^k\wedge w_{0'}^l=0\,,$$

$$c)\quad \sum_{j,k,l=1}^{3}\varepsilon_{jkl}\Omega_0^j\wedge w_{0'}^k\wedge w_{0'}^l=0\,,$$

$$d)\quad \forall l=1,2,3,\ \sum_{j,k=1}^{3}\varepsilon_{jkl}\Omega_0^j\wedge w_{0'}^k\wedge w_{0'}^0=0\,,$$

where ε_{jkl} *denotes the three-index permutation symbol.*

Notice that if $\mathfrak{T}$ defines a torsionless connection, the condition (2.11a) together with the second structure equation (which in particular gives $\Omega_D\wedge w_0^{0'}=0$) lead to the condition $\Omega_D=0$.

§ 3. Chronoprojective Cartan Structure on a 4-Dimensional Manifold

3. A) Definition and Properties

The general definition of a Cartan structure is given in Appendix B. Let us introduce here what is called a chronoprojective Cartan structure.

Definition 3.1. *Let* $\mathrm{O}^2(3)$ *be the chronoprojective group with subgroup* L^0 *as in Section* 1. *A* $\mathrm{O}^2(3)/\mathrm{L}^0$ *Cartan structure over a* 4-*dimensional manifold* V_4 *will be called a chronoprojective Cartan structure on* V_4.

To show the existence of such a Cartan structure we have first to prove the following

Proposition 3.2. *There is an isomorphic embedding of* L^0 *into* $\mathrm{G}^2(4)$ *which can be described by the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{N}(4) & \longrightarrow & \mathrm{G}^2(4) & \longrightarrow & \mathrm{G}^1(4) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \boldsymbol{R} & \xrightarrow{i} & \mathrm{L}^0 & \underset{\rho}{\overset{k}{\rightleftarrows}} & \mathrm{L}^I & \longrightarrow & 1. \end{array}$$

With respect to the local coordinate system in $P^2(V_4)$ *introduced in Appendix B each element* $h\in\mathrm{L}^0$, *given by* (1.2), *is represented by*

$$S^\lambda_\mu=(\rho(h))^\lambda_\mu \tag{3.1}$$

$$S^\lambda_{\mu\nu}=-\frac{1}{2}\frac{f}{g}[\delta^0_\mu(\rho(h))^\lambda_\nu+\delta^0_\nu(\rho(h))^\lambda_\mu]$$

where ρ *is the linear isotropy representation of* L^0 *defined in Section* 1 *and* f *and* g *the parameters of* L^0 *defined in Section* 1 (*f parametrizes the kernel of* ρ).

By a direct computation one verifies that the group law (1.6) of L^0 is recovered by using (B.1) from the above expressions of S^λ_μ and $S^\lambda_{\mu\nu}$. □

Let us then consider a subbundle P of $P^2(V_4)$ with structure group

L^0, and let us denote by (ϑ', Θ') the restriction to P of the canonical form of $P^2(V_4)$.

We have seen (cf. Section 2. B) that starting from a set $\mathfrak{T}$ of twelve $(a+l^I)$-valued 1-forms whose values at each point are linearly independent it is possible to construct a unique torsionless Cartan connection with respect to the chronoprojective group $O^2(3)$ (Proposition 2. 3). By comparing over any open set of V_4 the right action of the structure group L^0, on one hand on the Cartan connection 1-forms given in (2. 3), and on the other hand on the restriction to P of the canonical form of $P^2(V_4)$, one can verify that the set $\mathfrak{T}$ can be canonically realized from $\theta' = (\vartheta', \Theta')$ in the following way:

Definition 3. 3. *The unique chronoprojective connection obtained by supplementing the set $\mathfrak{T} = \{w_a, w_I\}$ of differential 1-forms given by*

$$w_a = \{w_{0'}^{\mu} = \frac{1}{2}\theta'^{\mu}, \mu = 0, 1, 2, 3\} \tag{3.2}$$

$$w_I = \left\{\tilde{w}_j^k = \frac{1}{2}(\theta'^k_j - \theta'^j_k) + \frac{1}{3}\delta_j^k \sum_l \theta'^l_l, \quad w_D = \theta'^0_0, w_0^j = \theta'^j_0 \quad j, k, l = 1, 2, 3\right\} \tag{3.3}$$

as described in Section 2. B, is called the natural chronoprojective connection.

Considering then P, subbundle of $P^2(V_4)$ equipped with the natural chronoprojective Cartan connection provides us with a chronoprojective Cartan structure on V_4.

Proposition 3. 4. *The 2-form Ω of the natural chronoprojective Cartan connection has the following properties*

(3. 4)
i) *Ω has a vanishing torsion*

ii) $\Omega_D = 0$

iii) $\sum_{j,k} \varepsilon_{jkl} \Omega_{0'}^{0'} \wedge \theta'^j \wedge \theta'^k = 0 \quad j, k, l = 1, 2, 3$

iv) $\sum_{jkl} \varepsilon_{jkl} \Omega_0^j \wedge \theta'^k \wedge \theta'^l = 0 \quad \forall l$

v) $\sum_{j,k} \varepsilon_{jkl} \Omega_0^j \wedge \theta'^k \wedge \theta'^0 = 0 \quad \forall l$

Proof. i) is an immediate consequence of (B. 5).

ii) see the remark after Proposition 2. 3.

iii), iv) and v) are the analogous of Rel. (2. b, c, d) respectively by taking into account the canonical realization of $\{w_0^{\mu}\}$ given in (3. 2).

3. B) Admissible Linear Connections

P being a chronoprojective Cartan structure on V_4, then $P/\ker(\rho)$ is a subbundle Q of $P^1(V_4)$, considered as a subbundle of $P^2(V_4)$, with $\mathrm{L}^I \subset G^1(4)$ as structure group. Conversely let $\mathrm{L}^I(V_4)$ be a L^I-structure of degree one and k the chosen homomorphism (1. 7) $\mathrm{L}^I \to \mathrm{L}^0$. L^I acts on the left on L^0 as follows: $(a, m) \mapsto k(a)m$, $a \in L^I$, $m \in L^0$. So we can introduce the associated fibre bundle $(\mathrm{L}^I(V_4) \times \mathrm{L}^0)/\mathrm{L}^I = Q_k$ which is a principal fibre bundle, the $\mathfrak{K}$-extension of $\mathrm{L}^I(V_4)$, with respect to the right action of L^0 over Q_k given by

$$((q^{\cdot}m), m') = q^{\cdot}(mm') \quad q \in Q, \ m \text{ and } m' \in L^0,$$

$q^{\cdot}m$ denotes the class of (q, m) into Q_k. With k chosen as above then the $\mathfrak{K}$-extension of $\mathrm{L}^I(V_4)$ will be a L^0-structure of degree two. Every connection in $\mathrm{L}^I(V_4)$ determines a linear connection of V_4 in the bundle of linear frames. According to a general result every torsionfree connection in $\mathrm{L}^I(V_4)$ corresponds precisely to a section Γ: $V_4 \to P^2(V_4)/\mathrm{L}^I$. Composed with the natural mapping: μ: $P^2(V_4)/\mathrm{L}^I \to P^2(V_4)/\mathrm{L}^0$ these connections give rise to sections $\mu \circ \Gamma$: $V_4 \to P^2(V_4)/\mathrm{L}^0$ i.e. to reductions of $P^2(V_4)$. In other words every torsionfree connection Γ in $\mathrm{L}^I(V_4)$ defines a reduction of the structure group of $P^2(V_4)$ to L^0 and induces an isomorphism $\mathfrak{J}$ of $\mathrm{L}^I(V_4)$ into $P^2(V_4)$, such that $\mathfrak{J}(\mathrm{L}^I(V_4))$ can be identified with Q. Then $\mathfrak{J}^*(\theta)$ is the canonical 1-form of $P^1(V_4)$ restricted to $\mathrm{L}^I(V_4)$, which can be decomposed into the a-valued 1-form ϑ and the l^I-valued component φ the 1-form of Γ, $\varphi = \mathfrak{J}^*(w_I)$ with w_I given by (3. 3).

Definition 3. 5. *Γ will be said admissible if it belongs to a chronoprojective structure P, that is to say if it induces P in the above described manner i.e. if the corresponding subbundle $\mathfrak{J}(\mathrm{L}^I(V_4))$ of $P^2(V_4)$ is contained into P identified with the $\mathfrak{K}$-extension Q_k of*

$\mathrm{L}^I(V_4)$.

The admissibility conditions have a double origin. Firstly they come from the pull-back to the bundle $\mathrm{L}^I(V_4)$ of Rel. (2.5b) defining the unique Cartan connection which leads to the following

Proposition 3.6. *Let us denote by Φ the l^I-valued curvature form of a torsionfree connection Γ in $\mathrm{L}^I(V_4)$. If Γ belongs to a chronoprojective structure its curvature form is related to the corresponding chronoprojective connection curvature as follows*

$$\mathfrak{K}^*(\Omega_I) = \Phi - \mathfrak{K}^*(w_0^{0'})\wedge\mathfrak{K}^*(\boldsymbol{B}). \tag{3.5}$$

Obviously $\mathfrak{K}^*(\boldsymbol{B})$ is expressed in terms of φ.

This is evident from (2.5b). □

Proposition 3.7. *Γ will belong to a chronoprojective structure provided that its curvature fulfills the following necessary conditions*

$$\text{(3.6)}\qquad \text{a)}\quad \sum_{j,k} \varepsilon_{jkl}\Big(\sum_{i=1}^{3} \Phi_i^i\Big)\wedge\theta^j\wedge\theta^k = \frac{1}{2}\mathfrak{K}^*(w_0^{0'})\wedge\Big(\sum_{j,k}\varepsilon_{jkl}\theta^j\wedge\theta^k\wedge\theta^0\Big)$$

$$\text{b)}\quad \Phi_0^0 = \mathfrak{K}^*(w_0^{0'})\wedge\theta^0$$

$$\text{c)}\quad \sum_{j,k,l}\varepsilon_{jkl}\Phi_0^j\wedge\theta^k\wedge\theta^l = \frac{1}{2}\mathfrak{K}^*(w_0^{0'})\wedge\Big(\sum_{jkl}\varepsilon_{jkl}\theta^j\wedge\theta^k\wedge\theta^l\Big)$$

$$\text{d)}\quad \sum_{j,k}\varepsilon_{jkl}\Phi_0^j\wedge\theta^k\wedge\theta^0 = \frac{1}{2}\mathfrak{K}^*(w_0^{0'})\wedge\Big(\sum_{j,k}\varepsilon_{jkl}\theta^j\wedge\theta^k\wedge\theta^0\Big).$$

These conditions are nothing but the pull-back to $\mathrm{L}^I(V_4)$ of the unicity conditions (2.11). Here θ^μ's denote the components of the canonical form ϑ on $\mathrm{L}^I(V_4)$.

Secondly, one can show that the pull-back to $\mathrm{L}^I(V_4)$ of the right action of L^0 on the chronoprojective connection and on its curvature gives rise to constraints which can be expressed by the following condition on the Ricci curvature tensor of Γ

$$\mathrm{Ric}\,(e_j, e_k) = 0 \quad \text{for} \quad j, k = 1, 2, 3, \tag{3.7}$$

where $\{e_\mu, \mu = 0, 1, 2, 3\}$ denote the basis of the comoving frame i.e.

$\theta^\mu(e_\nu)=\delta^\mu_\nu$.

In particular from (3.6) and (3.7) we deduce that the Ricci tensor takes the particular following form

$$\mathrm{Ric}=\chi\otimes\theta^0-4\theta^0\otimes\chi \tag{3.8}$$

where χ is an arbitrary covariant tensor field of degree one. So we have the following

Proposition 3.8. *A connection Γ is admissible if its curvature form satisfies* (3.6) *and* (3.7) *which imply* (3.8).

Another consequence of (3.6) and (3.5) is the following

Proposition 3.9. *The pull-back to $\mathrm{L}^I(V_4)$ of the g_2-valued component of the natural chronoprojective connection which completes the canonical set $\mathfrak{T}$ is given by*

$$\mathfrak{R}^*(w_0^{0\prime})=2\,\mathrm{Ric}\,(e_\mu, e_0)\,\theta^\mu\,. \tag{3.9}$$

3.C) Chronoprojective Equivalence and Weyl's Curvature Tensor

Definition 3.10. *Two admissible torsionfree connections are said to be chronoprojectively equivalent if they belong to the same chronoprojective structure P.*

Proposition 3.11. *Two admissible torsionfree connections defined by the l^I-valued 1-forms φ and φ' are chronoprojectively equivalent if and only if there exists a g_2-valued function ξ on V_4 such that*

$$\varphi'-\varphi=\mathfrak{R}^*([\vartheta', \xi\circ\pi]) \tag{3.10}$$

where π is the projection $\pi\colon P\to V_4$.

Proof. Let us consider the bundle $P^2(V_4)/\mathrm{L}^I$ with fibre $\mathrm{G}^2(4)/\mathrm{L}^I$ associated to the principal L^I-bundle. We introduce a local coordinate system $(b^\tau, b^\tau_\rho, b^\tau_{\rho\sigma})$ in $P^2(V_4)/\mathrm{L}^I$ in such a way that the projection $P^2(V_4)\to P^2(V_4)/\mathrm{L}^I$ is given by the equations

(3.11) $b^\tau = x^\tau$, b^τ_ρ undetermined, $b^\tau_{\rho\sigma} = \sum e^\tau_{\lambda\eta}(e^{-1})^\lambda_\rho(e^{-1})^\eta_\sigma$

Then a cross-section $\Gamma: V_4 \to P^2(V_4)/\mathrm{L}^I$ is locally given by $b^\tau = x^\tau$ and a set of functions $b^\tau_{\rho\sigma} = -\Gamma^\tau_{\rho\sigma}(x)$ with $\Gamma^\tau_{\rho\sigma} = \Gamma^\tau_{\sigma\rho}$. Let Γ and Γ' be two cross-sections: $V_4 \to P^2(V_4)/\mathrm{L}^I$ given in the above defined coordinate system by $(b^\tau, b^\tau_\rho, b^\tau_{\rho\sigma})$ and $(b^\tau, b^\tau_\rho, b'^\tau_{\rho\sigma})$ respectively and let φ and φ' be the corresponding connections forms on Q. It is easy to check that $b^\tau_{\rho\sigma}$ given by (3.11) is invariant under the action induced by an element $(S^\tau_\rho, 0)$ of $\mathrm{L}^I \subset \mathrm{G}^2(4)$. But under the action of $(S^\tau_\rho, S^\tau_{\rho\sigma})$ belonging to L^0 one gets $b^\tau_{\rho\sigma} \mapsto \hat{b}^\tau_{\rho\sigma} = b^\tau_{\rho\sigma} - 2/1\,(\delta^\tau_\sigma h_\rho + \delta^\tau_\rho h_\sigma)$ with $h_\rho = f(S^{-1})^0_0(e^{-1})^0_\rho$. So Γ and Γ' give rise to the same section $\hat{\Gamma}: V_4 \to P^2(V_4)/\mathrm{L}^0$ if and only if there is a $\boldsymbol{R}$-valued function ξ on V_4 such that

$$\Gamma'^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} = \xi[(e^{-1})^0_\beta \delta^\alpha_\gamma + (e^{-1})^0_\gamma \delta^\alpha_\beta]. \tag{3.12}$$

In terms of local coordinates this relation is equivalent to $\varphi' - \varphi = 2\xi \mathfrak{R}^*(\boldsymbol{B})$ i.e. (3.10). □

Now let us introduce a particular curvature tensor, the so-called chronoprojective Weyl's curvature tensor. The component Ω_I of the natural chronoprojective connection of P is a 2-form with values in the Lie algebra l^I which can be lifted to the L^I-structure Q identified with the quotient $P/\ker(\rho)$. From Ω_I can be constructed the following endomorphism $W(X, Y)$ of $T_x(V_4)$

$$W(X, Y)Z = u(2\mathfrak{R}^*(\Omega_I(\overline{X}, \overline{Y})) \cdot (u_*^{-1}Z)) \in T_x(V_4), \tag{3.13}$$

for $X, Y, Z \in T_x(V_4)$, $u \in \mathrm{L}^I(V_4)$ such that $\pi(u) = x$, $\overline{X}, \overline{Y} \in T_u(\mathrm{L}^I(V_4))$ being such that $\pi_* \overline{X} = X$, $\pi_* \overline{Y} = Y$, and where:

i) u is considered as a linear invertible mapping: $\boldsymbol{R}^4 \to T_x(V_4)$, ($T(V_4)$ being the tangent fibre bundle to V_4 considered as an associated fibre bundle to $\mathrm{L}^I(V_4)$).

ii) $(2\mathfrak{R}^*\Omega_I(\overline{X}, \overline{Y}))$ $(u^{-1}Z)$ denotes the image of $u^{-1}Z \in \boldsymbol{R}^4$ by the linear endomorphism $2\mathfrak{R}^*\Omega_I(\overline{X}, \overline{Y}) \in l^I \subset \mathrm{gl}(4, \boldsymbol{R})$.

W is then a trilinear mapping from $\mathfrak{H} \times \mathfrak{H} \times \mathfrak{H}$ into $\mathfrak{H}$ where $\mathfrak{H}$ denotes the space vector fields on V_4. From a classical theorem such a mapping can be considered as a tensor field of type (1.3) on V_4 which will be the Weyl's curvature tensor of the chronoprojective geometry.

Definition 3.12. *The chronoprojective Weyl's curvature tensor is a tensor field of type* (1.3) *on* V_4 *constructed from the endomorphism* $W(X, Y)$ *of* $T_x(V_4)$ *defined in* (3.14).

By construction W is chronoprojectively invariant.

Proposition 3.13. *In terms of the curvature R and of the Ricci tensor Ric of an admissible* L^I*-connection the Weyl's curvature tensor is expressed by*

$$(3.14)\qquad W(X,Y)=R(X,Y)+(X\top Y)$$
$$-\frac{1}{15}\{[\mathrm{Ric}(X,e_0)+4\,\mathrm{Ric}(e_0,X)]\vartheta^0(Y)$$
$$-[\mathrm{Ric}(Y,e_0)+4\,\mathrm{Ric}(e_0,Y)]\vartheta^0(X)\}$$

where $(X\top Y)$ *denotes the following antisymmetric endomorphism of* $T_x(V_4)$ *defined for any* Z *by*

$$(X\top Y)=-\frac{1}{15}(\mathrm{Ric}(Z,e_0)+4\,\mathrm{Ric}(e_0,Z))\,(\vartheta(Y)X-\vartheta(X)Y).$$

Proposition 3.14. *The Weyl's curvature tensor vanishes if and only if the natural chronoprojective connection has a vanishing curvature.*

This is a direct consequence of the definition of the Weyl's curvature tensor by taking into account the exterior derivative of (2.5b). This property is used to characterize a chronoprojectively flat manifold in what follows. Let P and P' be chronoprojective structures on four-dimensional manifolds V_4 and V_4'.

Definition 3.15. *A diffeomorphism* f: $V_4 \to V_4$ *is called chronoprojective (with respect to* P *and* P'*) if prolonged to a mapping of* $P^2(V_4)$ *onto* $P^2(V_4')$, f *maps* P *onto* P'.

Hence f is a bundle isomorphism which can be called chronoprojective with respect to the admissible connections Γ and Γ' which induce P and

P' respectively.

Definition 3.16. *A chronoprojective structure P is called flat if, for each point of V_4, there exists a neighbourhood $\mathfrak{U}$ and a chronoprojective diffeomorphism of $\mathfrak{U}$ onto an open subset of $M=\mathrm{O}^2(3)/\mathrm{L}^0$.*

This is the usual definition of a flat manifold in a Cartan geometry. Let us consider $\mathrm{O}^2(3)$ as a principal L^0-bundle over M. $\mathrm{O}^2(3)$ can be identified with a chronoprojective structure in the following manner: on one hand each $f\in\mathrm{O}^2(3)$ is a transformation of M, on the other hand any neighbourhood of the origin o of M can be identified with a neighbourhood of 0 in $\boldsymbol{R}^4$ in a natural way. Then any 2-jet of f can be considered as a 2-frame of M and the set $f(o)$ of all 2-frames thus obtained defines a chronoprojective structure which can be identified with $\mathrm{O}^2(3)$. The Maurer-Cartan form of $\mathrm{O}^2(3)$ becomes the natural Cartan connection of this chronoprojective structure, so it has no curvature and no torsion.

Proposition 3.17. *A chronoprojective structure P on a four-dimensional manifold V_4 is flat if and only if the natural chronoprojective connection has vanishing curvature.*

Proof. Since the natural chronoprojective connection of the chronoprojective structure on the chronoprojective space has vanishing curvature the natural connection of a flat chronoprojective structure has also vanishing curvature. The proof of the converse is similar to the proof of the corresponding property in the projective and conformal cases, see for instance [4b, 5b]. □

Proposition 3.18. *A chronoprojective structure on a four-dimensional manifold is flat if and only if the chronoprojective Weyl's curvature tensor vanishes.*

This proposition is deduced from Propositions (3.14) and (3.17). □

§ 4. The Chronoprojective Geometry over a Galilean Manifold

4. A) Conformal Galilean Equivalence and Chronoprojective Equivalence

In the previous sections the basis manifold V_4 have not been supposed to be endowed with some particular geometrical structure. Now we want to show that the chronoprojective geometry is naturally associated to the notion of conformal equivalence over a Galilean manifold. First let us recall the following definition (see [10]).

Definition 4.1. *A Galilean manifold is a triple* (V_4, ψ, γ) where V_4 *is a four-dimensional* C^∞*-manifold,* ψ *is a differential 1-form of class one and* γ *is a positive semi-definite symmetric contravariant tensor field of degree two such that* $\operatorname{Ker}\gamma$ *is generated by* ψ.

In fact a Galilean manifold can also be described as a fibre bundle over a one-dimensional manifold (the "time axis" $V_4/\operatorname{Ker}\psi$), the projection being known as the "universal time". Let H denote the neutral component of the full homogeneous Galilei group, i.e. the group of matrices

$$\begin{pmatrix} A & \bar{B} \\ 0 & 1 \end{pmatrix} \text{ with } A \in \mathrm{SO}(3) \text{ and } \bar{B} \in \boldsymbol{R}^3 .$$

Definition 4.2. *The bundle of Galilean frames* $\mathrm{H}(V_4)$ *over a Galilean manifold* (V_4, ψ, γ) *is a* H*-structure of degree one, subbundle of* $P^1(V_4)$ *corresponding to the reduction of* $\mathrm{Gl}(4, \boldsymbol{R})$ *to* H.

Definition 4.3. *A Galilean connection is a linear connection reducible to a connection in* $\mathrm{H}(V_4)$, *with respect to which* ψ *and* γ *are parallel* $\nabla\psi = 0$, $\nabla\gamma = 0$.

It is worth noticing that this definition is not sufficiently compelling to imply the existence of any privileged (torsionfree) Galilean connection over a Galilean manifold.

Let us denote by Φ the curvature form of a Galilean connection and by $\vartheta=\{\theta^0, \bar{\theta}\}$ the $\boldsymbol{R}^4$-valued canonical form of $P^1(V_4)$ restricted to $\mathrm{H}(V_4)$.

Definition 4. 4. *A Newtonian connection is a torsionless Galilean connection which is such that* ${}^t\bar{\theta}_\wedge\bar{\Phi}_0=0$, *and a Newtonian space-time is a connected Galilean manifold equipped with a Newtonian connection* ($\bar{\Phi}_0$ *denotes the* $\boldsymbol{R}^3$*-component of* Φ).

Definition 4. 5. *Two Galilean manifolds* (V_4, ψ, γ) *and* (V_4, ψ', γ') *are said conformally equivalent iff* $\psi'=\rho_t\psi$ *and* $\gamma'=\rho_s\gamma$, *where* ρ_t *and* ρ_s *are positive suitably differentiable functions on* V_4.

Corollary 4. 6. *The functions* ρ_t *is the pull-back of a function on the time axis.*

Proof. From Definition 4. 1 ψ and ψ' are closed 1-forms, so $d\psi'=0 = d\rho_{t\wedge}\psi$ implies that $\mathrm{Ker}(\psi)\subset\mathrm{Ker}(d\rho_t)$. □

Let $\{\Gamma^\gamma_{\alpha\beta}\}$ denote the components (or Christoffel's symbols) of the Galilean connection with respect to the local coordinate system $\{x^\tau, \tau =0, 1, 2, 3\}$. We want to derive the most general expression relating two torsionless Galilean connections Γ and Γ', respectively associated to two conformally equivalent manifolds (V_4, ψ, γ) and (V_4, ψ', γ'). Let us set $\Gamma'^\gamma_{\alpha\beta}-\Gamma^\gamma_{\alpha\beta}=\Delta^\gamma_{\alpha\beta}$.

Proposition 4. 7. *The connection* Γ' *is a torsionless Galilean connection for the Galilean manifold* (V_4, ψ', γ') *conformally equivalent to* (V_4, ψ, γ) *if and only if the following holds*

$$\Delta^\gamma_{[\alpha\beta]}=0\,, \tag{4.1}$$

$$\Delta^\mu_{\alpha\beta}\psi_\mu=\partial_\alpha(\mathrm{Log}\,\rho_t)\psi_\beta\,, \tag{4.2}$$

$$\Delta^{(\alpha}_{\mu\nu}\gamma^{\beta)\mu}=-\frac{1}{2}\partial_\nu(\mathrm{Log}\,\rho_s)\gamma^{\alpha\beta}\,. \tag{4.3}$$

Proof. (4.1) expresses the torsionfree condition. By definition $\nabla'_\alpha\psi'_\beta=(\rho_t\nabla_\alpha+\partial_\alpha\rho_t)\psi_\beta-\Delta^\mu_{\alpha\beta}\psi'_\mu=0$, since $\nabla_\alpha\psi_\beta=0$ one gets (4.2).

In the same manner $\nabla'_\nu\gamma'^{\alpha\beta}=(\rho_s\nabla_\nu+\partial_\nu\rho_s)\gamma^{\alpha\beta}+\Delta^\alpha_{\nu\mu}\gamma'^{\mu\beta}+\Delta^\beta_{\nu\mu}\gamma'^{\mu\alpha}=0$, then (4.3) comes from the fact that γ is parallel.

The general solution of the set of linear equations (4.1, 2, 3) is obtained by a direct calculation in a Galilean frame, from which one establishes the following:

Proposition 4.8. *The most general equivalence relation between two torsionless Galilean connections, compatible with the conformal equivalence of their respective Galilean manifolds is given by*

$$\Delta^\gamma_{\alpha\beta}=(\partial_\alpha(\mathrm{Log}\,\rho_t)\psi_\beta+\psi_{(\alpha}\partial_{\beta)}\,\mathrm{Log}\,\rho_s)U-\delta^\gamma_{(\alpha}\partial_{\beta)}\,\mathrm{Log}\,\rho_s+\frac{1}{2}\gamma^{\nu\gamma}\partial_\nu(\mathrm{Log}\,\rho_s)\overset{U}{\gamma}_{\alpha\beta}+\Xi^\gamma_{(\alpha}\psi_{\beta)} \tag{4.4}$$

where U is an arbitrary timelike unit vector field i.e. such that $\psi(U)=1$, $\overset{U}{\gamma}$ is the twice covariant symmetric tensor associated to U which is uniquely determined by the conditions $\overset{U}{\gamma}(U)=0$ and $\overset{U}{\gamma}_{\alpha\nu}\gamma^{\nu\beta}=\delta^\beta_\alpha-U^\beta\psi_\alpha$ and Ξ is an arbitrary skew-symmetric tensor $\Xi^{(\alpha\beta)}=0$ such that $\psi\cdot\Xi=0$.

Hence, due to the absence of a privileged torsionless Galilean connection, eleven functions are necessary to define the notion of conformal equivalence between Galilean manifolds endowed with connections. Let us recall that only one function is necessary in the Riemannian case owing to the presence of the Levi-Cività connection. It is clear that the tensor Ξ may be discarded. If we want to make disappear the arbitrary vector U one have to set $\rho_t\circ\rho_s=$ constant function on V_4. In this situation (4.4) reduces to

$$\Delta^\gamma_{\alpha\beta}=-\delta^\gamma_{(\alpha}\partial_{\beta)}\,\mathrm{Log}\,\rho_s \tag{4.5}$$

and this relation is formally equivalent to the one which expresses the projective equivalence between linear connections [4]. But, by taking Corollary 4.6 into account, one can set $d(\mathrm{Log}\,\rho_s)=-2\zeta\psi$ where ζ is a

$\boldsymbol{R}$-valued function on V_4. Thus one has

$$\Delta^{\gamma}_{\alpha\beta} = -2\zeta\delta^{\gamma}_{(\alpha}\psi_{\beta)} . \tag{4.6}$$

If one remembers that in a Galilean frame $\psi_\beta = (e^{-1})^0_\beta$, it has been shown that (4.6) is nothing else than (3.12) which expresses the chronoprojective equivalence.

Finally what has been proved? Firstly, the chronoprojective geometry is perfectly adapted to Galilean manifolds since the chronoprojective equivalence is a subcase of the conformal Galilean equivalence (described in Proposition 4.8) obtained by reducing at most the number of arbitrary functions. Secondly, the chronoprojective geometry is the Newtonian counterpart of the Weyl geometry since it appears also as a restriction of the projective geometry compatible with the conformal one. We can pursue the parallel with the conformal geometry by noticing the following property which has been proved in [8].

Proposition 4.9. *Two conformally equivalent Galilean structures (V_4, ψ, γ) and (V_4, ψ', γ') are associated to two distinct embeddings h and h' of* $\mathrm{H}(V_4)$ *into* $\mathrm{L}^I(V_4)$. *On each fibre of the conformal Galilean bundle* $\mathrm{L}^I(V_4)$ *these two embeddings define two* H*-orbits and $h'(u) = h(u)\lambda(x)$ for any $u \in \mathrm{L}^I(V_4)$ which projects onto $x \in V_4$, where*

$$\lambda(x) = \begin{pmatrix} \mathbf{1}_3 \times \rho_s(x)^{1/2} & 0 \\ 0 & \rho_t(x)^{-1} \end{pmatrix}$$

is identified with an element of $\dot{\boldsymbol{R}}_s \otimes \dot{\boldsymbol{R}}_t$.

4.B) The Chronoprojective Space-Time

In Section 1 the homogeneous space $M = \mathrm{O}^2(3)/\mathrm{L}^0$ has been described. Let us now introduce a ten-dimensional subgroup G of $\mathrm{O}^2(3)$, isomorphic to $(\boldsymbol{R}^3 \otimes \boldsymbol{R}^3) \otimes (\mathrm{SO}(3) \otimes \mathrm{SO}(2))$ and being the group of matrices

$$\begin{pmatrix} A & \overline{B} & \overline{C} \\ 0 & J\overline{V} & \overline{V} \end{pmatrix}$$

where $A \in \mathrm{SO}(3)$, $\overline{B}$ and $\overline{C} \in \boldsymbol{R}^3$, $\overline{V} \in \boldsymbol{R}^2$ such that $V^2 = 1$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is then easy to show that M is also a homogeneous coset space of G, $M=\mathrm{G/H}$, and G can be considered as the fibre bundle of Galilean frames over M. In fact M is a Galilean manifold such that ψ is given by the O(2)-component of the Maurer-Cartan form of G: $\psi={}^{t}(J\overline{V})\ d\overline{V}$ and the positive Riemannian metric induced on the standard fibres $\boldsymbol{R}^3$ is written as $\delta^{jk}\frac{\partial}{\partial c^j}\otimes\frac{\partial}{\partial c^k}$, $j, k=1, 2, 3$.

The Lie algebra g of G can be decomposed as $g=h+m$ where h denotes the Lie algebra of H and m is a complementary subspace. h is a reductive subalgebra of g. Hence the h-component of the Maurer-Cartan form on G can be considered as a Galilean connection form and the complementary m-component defines the soldering form ϑ.

Moreover the so-defined Galilean connection Γ^0 is a Newtonian connection so $(M, \psi, \gamma, \Gamma^0)$ is a Newtonian space-time and it is easy to verify that it is an exact solution of the Newton vacuum field equation with a cosmological constant $\Lambda=3$ (we recall that the field equations of the Newtonian cosmology can be written in terms of the Ricci tensor as

$$\mathrm{Ric}=(4\pi\rho G+\Lambda)\,\psi\otimes\psi \tag{4.7}$$

where G denotes the gravitational constant, ρ the matter density and Λ the cosmological constant).

Due to the above properties it is now clear why M has been called the chronoprojective space-time.

4. C) Chronoprojective Galilean Structures

In Section 3 a notion of admissible linear connection has been introduced to ensure the inclusion into a chronoprojective Cartan structure. Here we want to examine the restriction of these conditions to a torsionless Galilean connection.

Since, by definition the curvature of a Galilean connection is h-valued one has $\Phi_0^0=\Phi_j^j=0$. So (3.6b) implies $\mathfrak{R}^*(w_0^{0'})\wedge\theta^0=0$ and (3.6d) reduces to

$$\varepsilon_{jkl}\Phi_0^j\wedge\theta^k\wedge\theta^0=0$$

which implies

$$\mathrm{Ric}\,(e_0, e_j)=0\,.$$

From (3.8) one gets

$$0=\mathrm{Ric}\,(e_0, e_j)=-4\chi(e_j)\,.$$

Therefore

$$\mathrm{Ric}\,(e_j, e_0)=\chi(e_j)=0\,.$$

By taking (3.7) into account one deduces that

$$\mathrm{Ric}=\eta\psi\otimes\psi \tag{4.8}$$

where η is an arbitrary function on V_4, and (3.9) reduces to

$$\mathfrak{R}^*(w_0^{0'})=2\,\mathrm{Ric}\,(e_0, e_0)\,\theta^0\,. \tag{4.9}$$

The following has then been shown:

Proposition 4.10. *A torsionless Galilean connection is admissible i.e. it induces a chronoprojective Galilean structure if its Ricci tensor is given by* $\mathrm{Ric}=\eta\psi\otimes\psi$ *where* η *is an arbitrary function on* V_4.

Proposition 4.11. *Any solution of the Newton field equations can be embedded into a chronoprojective Galilean structure.*

Proof. It is sufficient to remark that (4.7) is identical to (4.8). □

So the chronoprojective geometry is perfectly adapted to the Newtonian cosmology.

Proposition 4.12. *The chronoprojective Weyl's curvature tensor of a torsionless admissible Galilean connection is given by*

$$W(X,Y)=R(X,Y)-\frac{\eta}{3}(\psi(Y)X-\psi(X)Y)\,.$$

This is an immediate consequence of Proposition 4.10 together with Proposition 3.13. □

It is then worth noticing that the vanishing of W given by $R(X,Y)$

$=\frac{\eta}{3}(\psi(Y)X-\psi(X)Y)$, is just the condition which is known as the cosmological isotropy hypothesis [8]. Therefore from Proposition 3.18 we deduce that the chronoprojective structure over an isotropic Newtonian space-time is flat. This is the "non-relativistic" version of the conformal flatness of the Friedmann model and once more the chronoprojective geometry appears as the very geometry of the Newtonian cosmology.

Appendix A: Cartan Connections

Let $L^0(V)$ denote a principal fibre bundle with structure group any Lie group L^0 over a manifold V. Let us suppose that L^0 is a connected subgroup of a Lie group L and that $\dim L/L^0=\dim V$. The group L^0 acts on the left on L by $(h,g)\mapsto hg$, $h\in L^0$, $g\in L$. Hence one can introduce the extension $L^0(V)^L$ of $L^0(V)$ i.e. the associated fibre bundle $L^0(V)^L=L^0(V)\otimes_{L^0}L$. Moreover let L act on the right on $L^0(V)$ as follows: $R_{g'}(u.g)=u.(gg')$ for $u\in L^0(V)$, g and $g'\in L$; therefore $L^0(V)^L$ is a principal fibre bundle over V with structure group L and the mapping $u\mapsto(u,e)$, e being the identity of L, defines an embedding of $L^0(V)$ into $L^0(V)^L$.

By definition a connection Γ in $L^0(V)^L$ is said to be a *Cartan connection* for $L^0(V)$ with respect to L if $H_u\cap T_u(L^0(V))=0$, where $T_u(L^0(V))$ is the tangent space to $L^0(V)$ at u and H_u is the horizontal subspace of $T_u(L^0(V)^L)$ with respect to Γ.

Let w denote the restriction to $L^0(V)$ of the connection form of Γ; then w is a differential 1-form on $L^0(V)$ with values in the Lie algebra l of L, the so-called *Cartan connection form* of $L^0(V)$. At any point u of $L^0(V)$, w_u defines a linear isomorphism of $T_u(L^0(V))$ into l.

Conversely let w be a differential 1-form on $L^0(V)$ with values into l and satisfying the following properties:

P1. $w(X^*)=X$ for every X belonging to the Lie algebra l^0 of L^0, X^* being the fundamental vector field corresponding to X.

P2. $(R_h)^*w=ad(h^{-1})w$ for every $h\in L^0$, ad denoting the adjoint representation of L^0 on l.

P3. $w(X)\neq 0$ for every non-zero vector field X on $L^0(V)$. Then

w can be uniquely extended to a usual connection form on $L^0(V)^L$ and the set of properties **P1–P3** can be taken as the definition of a Cartan connection on $L^0(V)$.

If there exists a Cartan connection for $L^0(V)$, then $L^0(V)$ is parallelizable, in others words its tangent bundle $T(L^0(V))$ is trivializable.

The 2-form Ω associated to a Cartan connection Γ is given by the reduction to $L^0(V)$ of the structure equations for Γ considered as a usual connection on $L^0(V)^L$ i.e.

$$\Omega = dw + \frac{1}{2}[w, w] \tag{A.1}$$

Ω is a l-valued 2-form on $L^0(V)$.

Since $L^0(V)$ is parallelizable the algebra of differential forms on $L^0(V)$ is generated by w and functions on $L^0(V)$. Let us denote by a a complementary subspace such that l can be decomposed into the vector space sum $l = l^0 + a$. Let w_{l^0} (resp. w_a) be the l^0 (resp. a) component of w. Then, if the components of w_a with respect to a basis of a are denoted by $\{w^\mu\}$, Ω can be written as

$$\Omega = \frac{1}{2}\sum_{\mu\nu} K_{\mu\nu} w^\mu{}_\wedge w^\nu \tag{A.2}$$

where each $K_{\mu\nu}$ is a l-valued function on $L^0(V)$.

There is a natural representation ρ usually called the *linear isotropy representation* of L^0 on the tangent space to L/L^0 at the origin o, the class of the identity of L. $T_0(L/L^0) = a$ from the decomposition of l and the linear isotropy representation is defined by

$$\rho(h)X = Ad(h)X(\mathrm{mod}\, L^0) \quad \text{for} \quad h \in L^0 \quad \text{and} \quad X \in a. \tag{A.3}$$

This representation is not faithful in general, we denote by N its kernel and by L^I its image: $L^I = \rho(L^0) \subset Gl(n, \boldsymbol{R})$. Hence L^0 can be written as an extension of L^I by N and one has the exact sequence

$$1 \longrightarrow N \xrightarrow{\ i\ } L^0 \underset{\rho}{\overset{k}{\rightleftarrows}} L^I \longrightarrow 1$$

where k is the section which defines the extension.

Now let us consider a given set $\mathfrak{T}$ of $(k_*(l^I) + a)$-valued 1-forms over $L^0(V)$ where l^I is the Lie algebra of L^I, and let us denote by n a complementary subspace to $k_*(l^I)$ into l^0. Then there is always a n-valued 1-form $\{w_\mu\}=w_n$ such that $w=\mathfrak{X}+w_n$ is a Cartan connection in $L^0(V)$. This set of independent 1-forms is locally constructed as follows: fixing a cross-section $\sigma: V\to L^0(V)$, let us set $w_\mu(T)=0$ for every tangent vector T to $\sigma(V)$. If Z is an arbitrary tangent vector to $L^0(V)$ it can be uniquely written as $Z=R_{h*}(T)+Y_v$ where $h\in L^0$, Y_v being a vertical vector. Y_v extends to a unique fundamental vector field Y^* of $L^0(V)$ corresponding to $Y\in l^0$. By conditions **P1** and **P2**, $w(Z)=\mathrm{ad}(h^{-1})\cdot(w(T))+Y$ defines a Cartan connection and so the desired set $\{w_\mu\}$.

For reasons which will become clear in Appendix B we are faced with an interesting situation when a unique Cartan connection can be defined from a given set $\mathfrak{X}$ of independent 1-forms. A general description cannot be given; when such a unique Cartan connection exists it is characterized by some specific properties of its curvature and torsion.

Appendix B: Frames of Second Order Contact and Cartan Structures

Let V be a n-dimensional differentiable manifold. Let $P^r(V)$ be the set of r-frames of V i.e. the set of invertible r-jets $j_0^r(f)\in J_0^r(\mathbf{R}^n, V)$ of diffeomorphisms $f: \mathbf{R}^n\to V$ with source $o\in\mathbf{R}^n$. $P^r(V)$ is a principal fibre bundle over V with structure group $G^r(n)$ and a natural projection $j_0^r(f)\to f(o)$. $G^r(n)$ is the group of invertible r-jets $j_0^r(f)\in J_0^r(\mathbf{R}^n, \mathbf{R}^n)_0$ of diffeomorphisms $f: \mathbf{R}^n\to\mathbf{R}^n$ with source o and target $f(o)=o$.

By definition a reduced bundle P of $P^r(V)$ with structure group G, subgroup of $G^r(n)$, is called a G-*structure of degree* r on V. From now on we shall be concerned with $P^2(V)$ and $P^1(V)$ which is nothing else but the bundle of linear frames with structure group $G^1(n)=\mathrm{Gl}(n, \mathbf{R})$. Note that there is an exact sequence $0\to N(n)\to G^2(n)\to G^1(n)\to 1$ where $N(n)$ is the Abelian kernel of a natural homomorphism of $G^2(n)$ into $G^1(n)$, dim $N(n)=\frac{1}{2}n^2(n+1)$. A natural basis of $G^2(n)$ is provided by the set $\{S^\tau_\rho, S^\tau_{\rho\sigma}=S^\tau_{\sigma\rho}, \tau, \rho, \sigma\in[1, n]\}$ which satisfy the group law

$$(\mathrm{B.1})\qquad (S^\tau_\rho, S^\tau_{\rho\sigma})(S'^\tau_\rho, S'^\tau_{\rho\sigma})=\Big(\sum_\lambda S^\tau_\lambda S'^\lambda_\rho, \sum_{\lambda,\eta} S^\tau_\lambda S'^\lambda_{\rho\sigma}+S^\tau_{\lambda\eta}S'^\lambda_\rho S'^\eta_\sigma\Big).$$

A local coordinate system of $P^2(V)$ which arises in a natural way from a local coordinate system $\{x^\tau\}$ of V is given by the set $\{x^\tau, e^\tau_\rho, e^\tau_{\rho\sigma}, \tau, \rho, \sigma \in [1, n]\}$. Then the right action of $G^2(n)$ on $P^2(V)$ is locally given by

$$\text{(B. 2)} \qquad (x^\tau, e^\tau_\rho, e^\tau_{\rho\sigma})(S^\tau_\rho, S^\tau_{\rho\sigma}) = (x^\tau, \sum_\lambda e^\tau_\lambda S^\lambda_\rho, \sum_{\lambda,\eta} e^\tau_\lambda S^\lambda_{\rho\sigma} + e^\tau_{\lambda\eta} S^\lambda_\rho S^\eta_\sigma).$$

The natural homomorphism $P^2(V) \to P^1(V)$ and $G^2(n) \to G^1(n)$ are given by $\{x^\tau, e^\tau_\rho, e^\tau_{\rho\sigma}\} \to \{x^\tau, e^\tau_\rho\}$ and $\{S^\tau_\rho, S^\tau_{\rho\sigma}\} \to \{S^\tau_\rho\}$ respectively. It is possible to define a *canonical differential form* θ on $P^2(V)$ which takes its values in the Lie algebra $\boldsymbol{R}^n \oplus \mathrm{gl}(n, \boldsymbol{R})$ of the affine group (note that $P^r(\boldsymbol{R}^n), r > 1$ does not have a natural group structure, $P^1(\boldsymbol{R}^n)$ only is isomorphic to the affine group). Let us denote by e the 1-jet at o of the identity transformation of $\boldsymbol{R}^n$ which corresponds to the identity in the affine group under this isomorphism. Then $\{E_\tau = (\partial/\partial x^\tau)_e, E^\rho_\tau = (\partial/\partial e^\tau_\rho)_e\}$ defines a basis for the affine algebra and locally the canonical differential form can be written as

$$\text{(B. 3)} \qquad \theta = \sum_\tau \theta^\tau E_\tau + \sum_{\tau,\rho} \theta^\tau_\rho E^\rho_\tau$$

with

$$\theta^\rho = \sum_\sigma (e^{-1})^\rho_\sigma dx^\sigma$$

$$\theta^\rho_\tau = \sum_\sigma (e^{-1})^\rho_\sigma de^\sigma_\tau + \sum_{\lambda,\eta} (e^{-1})^\rho_\lambda e^\lambda_{\sigma\eta} (e^{-1})^\sigma_\tau dx^\eta .$$

It follows that θ can be decomposed as $\theta = \{\vartheta, \Theta\}$ where $\vartheta = \{\theta^\rho\}$ is $\boldsymbol{R}^n$-valued and $\Theta = \{\theta^\rho_\tau\}$ is $\mathrm{gl}(n, \boldsymbol{R})$-valued.

Under the action of $h \in G^2(n)$ the canonical form transforms as

$$\text{(B. 4)} \qquad R^*_h(\theta) = \mathrm{ad}(h^{-1})\theta$$

moreover we have to note the important property

$$\text{(B. 5)} \qquad d\vartheta = -1/2[\Theta, \vartheta], \text{ in components } d\theta^\rho = -\sum_\tau \theta^\rho_\tau \wedge \theta^\tau .$$

Now let us consider the situation described at the end of Appendix A, i.e. the existence of a unique Cartan connection on a fibre bundle $L^0(V)$ with respect to L; and let us suppose that

i) L^0 can be realized as a subgroup of $G^2(n)$ i.e. L^I is considered as a subgroup of $G^1(n) = \mathrm{Gl}(n, \boldsymbol{R})$ and N can be embedded into $N(n)$, so that $L^0(V)$ becomes a L^0-structure of degree 2 over V.

ii) the set $\mathfrak{T}$ of $(k_*(l^I) + a)$-valued 1-forms can be constructed from the restriction to P of the canonical form θ on $P^2(V)$. Due to i) there is no difficulty to obtain the set of $k_*(l^I)$-valued 1-forms from the restriction to P of Θ. But concerning the set of a-valued 1-forms, we have to distinguish the case where a is a subalgebra isomorphic to $\mathbb{R}^n$ for which it is clear from (B. 5) that the obtained Cartan connection is torsionless. Any other case must be studied specifically.

A *Cartan structure* is defined as follows:

Definition: *A* L/L^0 *Cartan structure over a manifold* V (dim V $=\dim L/L^0$) *is a* L^0*-structure* P *of degree* 2 *on* V, *with the unique Cartan connection obtained by supplementing the set* $\mathfrak{T}$ *of canonical* 1*-forms described in* ii) *above.*

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