Manifestly Covariant Canonical Formalism of Quantum Gravity

——Systematic Presentation of the Theory——

By

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Contents

§ 1. Introduction

The manifestly covariant canonical formalism of quantum gravity [1]–[19] is an outstandingly beautiful unification of general relativity and quantum field theory. The present article is its review written primarily for mathematicians. Physicists are advised to read other reviews [20]–[24] (especially, [23]) before

reading the present article.

Since various formulae have been presented in many papers [1]–[19], it is perhaps worthwhile to collect them into a single article. In this review, the theory is described in a systematic way, without mentioning how this research has been developed. It is also omitted to reproduce the proofs or derivations of most formulae, which are presented in the original papers.*)

The author would like to celebrate the twentieth anniversary of the foundation of the Research Institute for Mathematical Sciences, Kyoto University.

§2. Quantum Field Theory and General Relativity

Quantum field theory is the standard theory of elementary-particle physics. A *quantum field* (or simply, a *field*) is a finite set of operator-valued generalized functions of spacetime coordinates x^μ. They form a linear representation of *Poincaré group*, which consists of all translations and all Lorentz transformations. Though, of course, the spacetime of reality is of four dimensions, we present the expressions valid in the n-dimensional spacetime, which has one time x^0 and $(n-1)$-dimensional space $(x^1, ..., x^{n-1})$.

The operand of quantum fields is called a *state*. The totality of states is an infinite-dimensional complex linear space equipped with an indefinite inner product, which is called an *indefinite-metric Hilbert space*. Except for the positive-definite case, no one yet knows how to introduce appropriate topology into it. Properties concerning topology are suitably assumed, whenever necessary, on the basis of physical intuition, as long as we do not explicitly encounter internal inconsistency.

Canonical formalism based on the Lagrangian is the most elegant and transparent method of formulating physical theories; every fundamental theory seems to have this framework. Unfortunately, however, the canonical formalism of quantum field theory is devoid of mathematical rigor, but what is importnat here is to formulate a physical theory but not to construct a theory of mathematics. Mathematical rigor is important at the final stage of accomplishing the theory, but it cannot be a guiding principle for constructing a physical theory.

In field theories, the most fundamental quantity is the *action*, which is an

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*) Some notational changes are made; e.g., $\kappa \Rightarrow 1$, $\bar{c}_\rho \Rightarrow -\bar{c}_\rho$, $\Gamma_\mu{}^{ab} \Rightarrow -\omega_\mu{}^{ab}$, $\Gamma_\mu \Rightarrow -\omega_\mu$, $Q_c \Rightarrow -Q_c$, etc.

n-dimensional integral of a local*⁾ function of fields, called the *Lagrangian density*. We can always rearrange the Lagrangian density in such a way that it contains no second or higher derivatives of fields and is at most quadratic with respect to first derivatives. In the Lagrangian density, the constant term is meaningless and the part linear with respect to fields can be eliminated by redefining fields. The part quadratic with respect to fields is called the *free Lagrangian density*, and the remainder is the *interaction Lagrangian density*, in which coefficients are called *coupling constants*. All field equations are derived from the action on the basis of the variational principle.

In contrast with the classical theory, quantum fields are not subject to any controllable conditions such as initial conditions. This is because quantum fields are the most fundamental objects representing natural laws themselves. In quantum field theory, what supplement field equations are *canonical commutation relations*. To set them up is called *canonical quantization*. The rules of canonical quantization are the straightforward extension of those in quantum mechanics (see Section 8). Since the number of spatial points is continuously infinite, we use the $(n-1)$-dimensional delta function in place of a Kronecker delta. Since there are, in general, fields obeying Fermi statistics in addition to those obeying Bose one, an anticommutator is used for two operators obeying Fermi statistics in place of a commutator.

Unless the Lagrangian density is a free one, field equations contain non-linear terms. Furthermore, the *canonical conjugate*, which is defined by differentiating the Lagrangian density with respect to the time derivative of a *canonical field* (a continuously infinite set of canonical variables), is, in general, non-linear with respect to fields. Since fields are operator-valued generalized functions, it is not mathematically sensible to consider their product at the same spacetime point. This is the origin of the well-known *divergence difficulty* of quantum field theory. In the four-dimensional world, the divergence difficulty can be made harmless by a subtraction procedure, called *renormalization*, in perturbation theory (power series expansion with respect to coupling constants), provided that the theory is renormalizable.**⁾ Even for a renormalizable theory, however, no one knows how to deal with the divergence problem in a non-perturbative way. Instead of pursuing this mathematical problem, we naively

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*⁾ In general, "local" means depending on a single spacetime point.
**⁾ Roughly speaking, a theory is renormalizable if all coupling constants appearing in its Lagrangian density have non-positive dimension of length in natural units.

assume that *any product of fields at the same spacetime point exists*, though its definition is unknown, and that it is unique, that is, *operator ordering in any local product of fields has no meaning.*[*]   Indeed, without this understanding, it would be impossible to start with the Lagrangian density in quantum field theory.

If the action is invariant under a group of transformations of fields, we say that the theory has a *symmetry* defined by those transformations.   If they transform spacetime coordinates, too, it is a *spacetime symmetry*, and otherwise it is an *internal symmetry*.   As is well known, the symmetry defined by continuous transformations can well be described by infinitesimal ones.   Given a continuous symmetry, one obtains the corresponding *conserved current* through the Noether theorem.   The spatial $(n-1)$-dimensional integral of its zeroth component formally defines a *conserved charge*, which may or may not be a well-defined operator, but is sensible as the *generator* of the original symmetry (if the theory is consistent).

Though spacetime coordinates x^μ are not transformed in an internal symmetry, its transformation law may depend on x^μ.   The transformations containing arbitrary (C^∞ class) functions of x^μ are called *local gauge transformations*.   The theory invariant under local gauge transformations is called a *gauge theory*, which usually contains a specially transforming vector field, called a *gauge field*.   A gauge theory is called *abelian* or *non-abelian* according as the corresponding spacetime-independent internal symmetry group is abelian or not. The Maxwell theory of electromagnetism is an abelian gauge theory, while the *Yang-Mills theory* is used as a synonym of the non-abelian gauge theory.

The gauge theory of the above definition in its strict sense can exist only as the classical theory.   It is impossible to construct quantum fields if the theory is invariant under local gauge transformations.   In order to quantize a gauge theory, therefore, one must introduce a *gauge-fixing* Lagrangian density and the corresponding *FP-ghost*[**] one.   Thus the quantum gauge theory is no longer invariant under local gauge transformations.   But, instead, it becomes invariant under a new spacetime-independent symmetry, called the *BRS symmetry,*[***] which obeys Fermi statistics.   The existence of the BRS symmetry is extremely

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[*] Of course, we must take account of a signature factor owing to the ordering of fields obeying Fermi statistics.

[**] Faddeev-Popov ghost [25].

[***] Becchi-Rouet-Stora symmetry [26].

important because it guarantees the probabilistic interpretability of the quantum gauge theory (see Section 16).

Quantum field theory is quite a successful fundamental theory, which describes strong, electromagnetic and weak interactions of elementary particles. On the other hand, the correct theory describing gravitational interaction is undoubtedly Einstein's general relativity. In contrast with special relativity, which is purely kinematic, general relativity contains the dynamics of the gravitational field. It is, therefore, impossible to extend quantum field theory by simply requiring invariance under general coordinate transformations instead of Poincaré invariance, that is, the gravitational field should be regarded as another quantum field.

It is, however, quite unsatisfactory to deal with the gravitational field in a rigid framework of Minkowski space, because general relativity is an extension of special relativity to non-inertial frames. Indeed, at the presence of the gravitational field, it is unreasonable to presuppose the existence of the undeformable light-cone. Since the gravitational field plays the role of spacetime metric in general relativity, when it is quantized, spacetime can no longer be a manifold nor an object of geometry. Planck length ($\sim 10^{-33}$ cm) is a scale in which spacetime loses its geometrical structure. Spacetime coordinates x^μ are now n parameters, which can be identified with the geometrical spacetime only asymptotically.

General relativity is similar to a gauge theory in the sense that general coordinate transformations contain arbitrary functions of x^μ. Hence it is natural to apply the method of quantizing a gauge field to gravity, though general relativity is qualitatively different from a gauge theory in the following respects.

1° The Einstein-Hilbert Lagrangian density is a highly non-polynomial, unrenormalizable one.

2° Poincaré invariance is not an appropriate framework.

3° General covariance is a spacetime symmetry.

4° No gauge field is present as a fundamental field.

In the following sections, taking account of the above observations, we present a very satisfactory quantum field theory of general relativity.

§3. Notation

Throughout the present article, the following notation is used.

We employ natural units, that is, $c = \hbar = 1$, where c and $2\pi\hbar$ denote the light velocity and the Planck constant, respectively. Also, the Einstein gravitational constant $\kappa = 8\pi c^{-4} G$ is set equal to unity, where G denotes the Newton gravitational constant.

The spacetime dimension is denoted by n. General relativity does not exist for $n = 2$, whence we assume $n \geq 3$.

The conventional notation of tensor calculus is employed. In particular, summation over repeated indices should always be understood. We use μ, ν, λ, ρ, σ, τ, etc. for spacetime (world) indices, k, l, etc. for spatial indices, and a, b, c, d, etc. for indices of the internal Minkowski space, whose metric is denoted by η_{ab} ($\eta_{00} = -\eta_{11} = \cdots = -\eta_{n-1,n-1} = 1$, $\eta_{ab} = 0$ for $a \neq b$).

A middle dot ($\cdot$) indicates that differentiation does not act beyond it; for instance, $\partial A \cdot B = (\partial A)B$. Differentiation with respect to x^μ is written as ∂_μ. If more than one spacetime points are relevant, $\partial/\partial x^\mu$ and $\partial/\partial y^\mu$ are distinguished by writing ∂_μ^x and ∂_μ^y, respectively. Differentiation with respect to x^0 is sometimes denoted by an upper dot; for example, $\dot{\Phi} = \partial_0 \Phi$.

The integral is always written as $\int dx(\cdots)$ rather than $\int(\cdots)dx$. Furthermore, we use the following abbreviations:

$$(3.1) \qquad \int d^{n-1}x(\cdots) = \int_{-\infty}^{+\infty} dx^1 \cdots \int_{-\infty}^{+\infty} dx^{n-1}(\cdots),$$

$$(3.2) \qquad \int d^n x(\cdots) = \int_{-\infty}^{+\infty} dx^0 \int d^{n-1}x(\cdots).$$

The expression which is obtained from its preceding one only by interchanging some indices is abbreviated as ($\leftrightarrow$); for instance,

$$(3.3) \qquad A_\mu B_\nu - C_{\mu\nu} - (\mu \leftrightarrow \nu) = A_\mu B_\nu - C_{\mu\nu} - A_\nu B_\mu + C_{\nu\mu}.$$

The Dirac notation is used for states and their inner products. A state having a name f is denoted by $|f\rangle$ instead of writing Ψ_f. The inner product (Ψ_f, Ψ_h) in the mathematical notation is written as $\langle h|f \rangle$, and if A is an operator, $(A\Psi_f, \Psi_h)$ is written as $\langle h|A|f \rangle$. We do not take care of the domain of any operator. Hence $\langle h|A|f \rangle$ represents $(\Psi_f, A^\dagger \Psi_h)$, too, where a dagger ($\dagger$) indicates hermitian conjugation. In particular, "hermitian" ($A = A^\dagger$) is used as

a synonym of self-adjoint.

It should always be understood that differentiation with respect to a quantity obeying Fermi statistics is made from the *left*.

Some special symbols are introduced for simplifying the description. We often use a signature factor, $\epsilon(\Phi_1, \Phi_2)$, defined by

$$(3.4) \quad \begin{cases} \epsilon(\Phi_1, \Phi_2) = -1 & \text{if both} \quad \Phi_1 \text{ and } \Phi_2 \text{ obey Fermi statistics}, \\ \epsilon(\Phi_1, \Phi_2) = +1 & \text{otherwise}. \end{cases}$$

Square root of a signature factor is defined by $\sqrt{+1} = +1$ and $\sqrt{-1} = +i$. Since we must use a commutator $[\Phi_1, \Phi_2]$ or an anticommutator $\{\Phi_1, \Phi_2\}$ appropriately according to the statistics of Φ_1 and Φ_2, it is convenient to introduce a $*$-commutator defined by

$$(3.5) \qquad [\Phi_1, \Phi_2]_* = \Phi_1\Phi_2 - \epsilon(\Phi_1, \Phi_2)\Phi_2\Phi_1.$$

We use the following abbreviations:

$$(3.6) \qquad [\Phi_1, \Phi_2']_* = [\Phi_1(x), \Phi_2(x')]_*|_{x^0 = x'^0},$$

$$(3.7) \qquad \delta^{n-1} = \prod_{k=1}^{n-1} \delta(x^k - x'^k).$$

§4. Fundamental Fields

In order to describe gravity, we need the following seven fundamental fields:

$$(4.1) \qquad h_{\mu a}(x); \ b_\rho(x), \ c^\sigma(x), \ \bar{c}_\tau(x); \ s_{ab}(x), \ t_{cd}(x), \ \bar{t}_{ef}(x).$$

They are all hermitian.

In the four-dimensional world, $h_{\mu a}(x)$ is called *vierbein* or *tetrad*, but, in the n-dimensional world, it is called *vielbein*. In the classical theory, it means a set of n basis vectors in the tangent space of the spacetime manifold, but, in the quantum theory, it is merely a set of n vector fields.

The symmetric gravitational field $g_{\mu\nu}(x)$, which represents the world metric in the classical theory, is defined by

$$(4.2) \qquad g_{\mu\nu}(x) = \eta^{ab}h_{\mu a}(x)h_{\nu b}(x) = g_{\nu\mu}(x).$$

Its contravariant component $g^{\mu\nu}(x)$ is defined by

$$(4.3) \qquad g_{\mu\lambda}(x)g^{\lambda\nu}(x) = \delta_u^\nu$$

as usual.   Raising and lowering a world index is made by means of $g^{\mu\nu}(x)$ and $g_{\mu\nu}(x)$, respectively.   Though general covariance is broken in quantum gravity, the distinction between "covariant" and "contravariant" is still very important because general linear invariance is retained.

We set

(4.4) $h(x) = \det h_\mu{}^a(x).$

Then (4.2) implies that

(4.5) $[h(x)]^2 = -\det g_{\mu\nu}(x).$

We also set

(4.6) $\tilde{g}^{\mu\nu}(x) = h(x)g^{\mu\nu}(x).$

We call $b_\rho(x)$ the *gravitational B field* and $c^\sigma(x)$ and $\bar{c}_\tau(x)$ the *gravitational FP ghosts*.   They are world vectors.   Likewise, we call $s_{ab}(x)$ the *internal-Lorentz B field* and $t_{cd}(x)$ and $\bar{t}_{ef}(x)$ the *internal-Lorentz FP ghosts*.   They are world scalars but *antisymmetric* tensors under local internal Lorentz transformations.[*]

The three fields $h_{\mu a}(x)$, $b_\rho(x)$, and $s_{ab}(x)$ obey Bose statistics, while the four FP ghosts $c^\sigma(x)$, $\bar{c}_\tau(x)$, $t_{cd}(x)$, and $\bar{t}_{ef}(x)$ obey Fermi statistics.

Of course, gravity couples with matter fields, but no one knows the correct set of fundamental matter fields.   Fortunately, the construction of the quantum-gravity theory is essentially independent of the information about matter fields, as long as their Lagrangian density is a scalar density under general coordinate transformations and invariant under local internal Lorentz transformations. But whenever a concrete example is necessary, we consider quantum electro-dynamics, in which we have the following fundamental fields:

(4.7) $\psi(x); A_\mu(x), B(x), C(x), \bar{C}(x).$

Here, $\psi(x)$ is non-hermitian, while all others are hermitian.

We call $\psi(x)$ the *Dirac field*, which is *not* a world spinor but a world scalar and is a spinor under local internal Lorentz transformations.   We call $A_\mu(x)$ the *electromagnetic field*, $B(x)$ the *electromagnetic B field*, and $C(x)$ and $\bar{C}(x)$ the *electromagnetic FP ghosts*. $A_\mu(x)$ is a world vector and others are world scalars.   The two fields $A_\mu(x)$ and $B(x)$ obey Bose statistics, while the three fields

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[*] Local internal Lorentz transformations are local gauge transformations, but there is no corresponding fundamental gauge field.

$\psi(x)$, $C(x)$, and $\bar{C}(x)$ obey Fermi statistics.

§5. Classical Results

For convenience of later uses, we here summarize some classical results of general relativity, which remain valid also in quantum gravity.

Hereafter we omit to write the argument x^μ of a local quantity explicitly unless necessary.

Let $\partial_\mu \varepsilon^\nu \cdot [\varphi_\Sigma]^\mu{}_\nu$ be the infinitesimal change of a local quantity φ_Σ under an infinitesimal general coordinate transformation $x'^\nu = x^\nu + \varepsilon^\nu$. For a tensor $\varphi_\Sigma = \varphi^{\sigma_1\cdots\sigma_r}{}_{\tau_1\cdots\tau_s}$, we have

$$(5.1) \qquad [\varphi^{\sigma_1\cdots\sigma_r}{}_{\tau_1\cdots\tau_s}]^\mu{}_\nu = \sum_{p=1}^{r} \delta^{\sigma_p}_\nu \varphi^{\sigma_1\cdots\sigma_{p-1}\mu\sigma_{p+1}\cdots\sigma_r}{}_{\tau_1\cdots\tau_s} - \sum_{q=1}^{s} \delta^\mu_{\tau_q} \varphi^{\sigma_1\cdots\sigma_r}{}_{\tau_1\cdots\tau_{q-1}\nu\tau_{q+1}\cdots\tau_s}.$$

If $\varphi^{\sigma_1\cdots\sigma_r}{}_{\tau_1\cdots\tau_s}$ is a tensor density,

$$(5.2) \qquad\qquad\qquad -\delta^\mu_\nu \varphi^{\sigma_1\cdots\sigma_r}{}_{\tau_1\cdots\tau_s}$$

should be added to the right-hand side of (5.1).

Then the *covariant derivative* is expressed as

$$(5.3) \qquad\qquad\qquad \nabla_\mu \varphi_\Sigma = \partial_\mu \varphi_\Sigma + \Gamma_{\mu\lambda}{}^\nu [\varphi_\Sigma]^\lambda{}_\nu,$$

where the *affine connection* $\Gamma_{\mu\lambda}{}^\nu$ is defined by

$$(5.4) \qquad\qquad \Gamma_{\mu\lambda}{}^\nu = \frac{1}{2} g^{\nu\sigma} (\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\mu\sigma} - \partial_\sigma g_{\mu\lambda}) = \Gamma_{\lambda\mu}{}^\nu,$$

so as to have

$$(5.5) \qquad\qquad\qquad \nabla_\mu g_{\sigma\tau} = 0.$$

For vielbein, its covariant derivative including internal Lorentz part vanishes:

$$(5.6) \qquad\qquad \partial_\mu h_\nu{}^a - \Gamma_{\mu\nu}{}^\lambda h_\lambda{}^a + \omega_\mu{}^{ab} h_{\nu b} = 0,$$

where the *spin connection* $\omega_\mu{}^{ab}$ is defined by

$$(5.7) \qquad\qquad \omega_\mu{}^{ab} = h^{\lambda a}(\partial_\mu h_\lambda{}^b - \Gamma_{\mu\lambda}{}^\nu h_\nu{}^b) = -\omega_\mu{}^{ba}.$$

In contrast with $\Gamma_{\mu\lambda}{}^\nu$, $\omega_\mu{}^{ab}$ is a world vector, but it is not covariant under local internal Lorentz transformations.

The *Riemann tensor* $R^\lambda{}_{\sigma\mu\nu}$ is defined by

$$(5.8) \qquad\qquad\qquad [\nabla_\mu,\ \nabla_\nu]\varphi^\lambda = R^\lambda{}_{\sigma\mu\nu}\varphi^\sigma$$

for an arbitrary vector φ^λ. The *Ricci tensor* $R_{\mu\nu}$ is given by

(5.9) $R_{\mu\nu} = R^\lambda{}_{\nu\mu\lambda} = R_{\nu\mu} = \partial_\mu \Gamma_{\nu\sigma}{}^\sigma - \partial_\sigma \Gamma_{\mu\nu}{}^\sigma + \Gamma_{\mu\sigma}{}^\lambda \Gamma_{\nu\lambda}{}^\sigma - \Gamma_{\mu\nu}{}^\lambda \Gamma_{\lambda\sigma}{}^\sigma,$

and we set $R = R^\mu{}_\mu$. Then we have an identity

(5.10) $$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0.$$

The following identities are derived by the theory of invariant variations (second Noether theorem). Let $\mathscr{F}$ be an arbitrary local function of fields Φ_A, containing no second or higher derivatives of them. If $\mathscr{F}$ is a scalar density under general coordinate transformations, then we have three identities

(5.11) $$\sum_A \partial_\mu \left([\Phi_A]^\mu{}_\nu \frac{\delta \mathscr{F}}{\delta \Phi_A} \right) + \sum_A \partial_\nu \Phi_A \cdot \frac{\delta \mathscr{F}}{\delta \Phi_A} = 0,$$

(5.12) $$\sum_A \partial_\nu \Phi_A \cdot \frac{\partial \mathscr{F}}{\partial (\partial_\mu \Phi_A)} - \delta^\mu_\nu \mathscr{F} - \sum_A [\Phi_A]^\mu{}_\nu \frac{\delta \mathscr{F}}{\delta \Phi_A} - \partial_\lambda \mathscr{K}^{\mu\lambda}{}_\nu = 0,$$

(5.13) $$\mathscr{K}^{\mu\lambda}{}_\nu = - \mathscr{K}^{\lambda\mu}{}_\nu,$$

where $\delta/\delta \Phi_A$ denotes the Euler derivative, namely,

(5.14) $$\frac{\delta \mathscr{F}}{\delta \Phi_A} = \frac{\partial \mathscr{F}}{\partial \Phi_A} - \partial_\mu \frac{\partial \mathscr{F}}{\partial (\partial_\mu \Phi_A)},$$

and

(5.15) $$\mathscr{K}^{\mu\lambda}{}_\nu = - \sum_A [\Phi_A]^\lambda{}_\nu \frac{\partial \mathscr{F}}{\partial (\partial_\mu \Phi_A)}.$$

If $\mathscr{F}$ is invariant under local internal Lorentz transformations, we have two identities

(5.16) $$\sum_A (\Phi_A)^a{}_b \frac{\delta \mathscr{F}}{\delta \Phi_A} = 0,$$

(5.17) $$\sum_A (\Phi_A)^a{}_b \frac{\partial \mathscr{F}}{\partial (\partial_\mu \Phi_A)} = 0,$$

where $\varepsilon_a{}^b (\Phi)^a{}_b$ denotes the infinitesimal change of Φ under a local internal Lorentz transformation described by an infinitesimal function ε_{ab} antisymmetric in a and b.

§6. Lagrangian Density

The classical theory of general relativity formulated in terms of vielbein is

invariant both under general coordinate transformations (having n degrees of freedom) and under local internal Lorentz transformations (having $n(n-1)/2$ degrees of freedom). In the quantum-gravity theory, however, those local symmetries must be explicitly broken.

The Lagrangian density of our theory of quantum gravity is given by

$$(6.1) \qquad \mathscr{L} = \mathscr{L}_{E} + \mathscr{L}_{GF} + \mathscr{L}_{FP} + \mathscr{L}_{LGF} + \mathscr{L}_{LFP} + \mathscr{L}_{MF}.$$

Here $\mathscr{L}_{E}$ denotes the *Einstein-Hilbert Lagrangian density*

$$(6.2) \qquad \mathscr{L}_{E} = (1/2)hR.$$

$\mathscr{L}_{GF}$ and $\mathscr{L}_{FP}$ are called the *gravitational gauge-fixing Lagrangian density* and the *gravitational FP-ghost Lagrangian density*, respectively, which are defined by (cf. [1])

$$(6.3) \qquad \mathscr{L}_{GF} = -\tilde{g}^{\mu\nu}\partial_{\mu}b_{\nu},$$

$$(6.4) \qquad \mathscr{L}_{FP} = -i\tilde{g}^{\mu\nu}\partial_{\mu}\bar{c}_{\rho}\cdot\partial_{\nu}c^{\rho}.$$

$\mathscr{L}_{LGF}$ and $\mathscr{L}_{LFP}$ are called the *internal-Lorentz gauge-fixing Lagrangian density* and the *internal-Lorentz FP-ghost Lagrangian density*, respectively, which are defined by (cf. [5])

$$(6.5) \qquad \mathscr{L}_{LGF} = -\tilde{g}^{\mu\nu}\omega_{\nu}{}^{ab}\partial_{\mu}s_{ab},$$

$$(6.6) \qquad \mathscr{L}_{LFP} = -i\tilde{g}^{\mu\nu}\partial_{\mu}\bar{t}_{ab}\cdot(D_{\nu}t)^{ab},$$

where

$$(6.7) \qquad (D_{\nu}\varphi)^{ab} = \partial_{\nu}\varphi^{ab} + \omega_{\nu}{}^{ac}\varphi_{c}{}^{b} - \omega_{\nu}{}^{bc}\varphi_{c}{}^{a}.$$

$\mathscr{L}_{MF}$ denotes the *matter-field Lagrangian density*.

It is important to note that $\mathscr{L}_{E}$, $\mathscr{L}_{LGF}$, $\mathscr{L}_{LFP}$ and $\mathscr{L}_{MF}$ are scalar densities under general coordinate transformations, but $\mathscr{L}_{GF}$ and $\mathscr{L}_{FP}$ break this property; they are scalar densities only under *general linear transformations*. On the other hand, $\mathscr{L}_{E}$, $\mathscr{L}_{GF}$, $\mathscr{L}_{FP}$ and $\mathscr{L}_{MF}$ are invariant under local internal Lorentz transformations, but $\mathscr{L}_{LGF}$ and $\mathscr{L}_{LFP}$ are non-invariant; they are invariant only under *global*[*)] internal Lorentz transformations.

The difference in form between $\mathscr{L}_{FP}$ and $\mathscr{L}_{LFP}$ reflects the fact that the translation group is abelian, whereas the Lorentz group is non-abelian.

As a concrete example of $\mathscr{L}_{MF}$, we consider the case of quantum electrodynamics, in which we have

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[*)] "Global" means not involving an arbitrary spacetime function.

(6.8) $$\mathscr{L}_{\mathrm{MF}} = \mathscr{L}_{\mathrm{EM}} + \mathscr{L}_{\mathrm{EMGF}} + \mathscr{L}_{\mathrm{EMFP}} + \mathscr{L}_{\mathrm{D}}.$$

Here $\mathscr{L}_{\mathrm{EM}}$ is the *Maxwell Lagrangian density*

(6.9) $$\mathscr{L}_{\mathrm{EM}} = -\frac{1}{4} h g^{\mu\sigma} g^{\nu\tau} F_{\mu\nu} F_{\sigma\tau},$$

where

(6.10) $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}.$$

$\mathscr{L}_{\mathrm{EMGF}}$ and $\mathscr{L}_{\mathrm{EMFP}}$ are called the *electromagnetic gauge-fixing Lagrangian density* and the *electromagnetic FP-ghost Lagrangian density*, respectively, which are defined by

(6.11) $$\mathscr{L}_{\mathrm{EMGF}} = -\tilde{g}^{\mu\nu} A_\nu \partial_\mu B,$$

(6.12) $$\mathscr{L}_{\mathrm{EMFP}} = -i\tilde{g}^{\mu\nu} \partial_\mu \bar{C} \cdot \partial_\nu C.$$

The similarity between $\mathscr{L}_{\mathrm{LGF}}$ and $\mathscr{L}_{\mathrm{EMGF}}$ and that between $\mathscr{L}_{\mathrm{FP}}$ and $\mathscr{L}_{\mathrm{EMFP}}$ are noteworthy. $\mathscr{L}_{\mathrm{D}}$ is the *Dirac Lagrangian density*

(6.13) $$\mathscr{L}_{\mathrm{D}} = ih\psi^\dagger \mathring{\gamma}_0 [\gamma^\mu (\partial_\mu + \omega_\mu - ieA_\mu) + im]\psi,$$

where

(6.14) $$\gamma^\mu = h^{\mu a} \mathring{\gamma}_a,$$

(6.15) $$\omega_\mu = \frac{1}{2} \mathring{\sigma}_{ab} \omega_\mu{}^{ab},$$

(6.16) $$\mathring{\sigma}_{ab} = \frac{1}{4} [\mathring{\gamma}_a, \mathring{\gamma}_b],$$

(6.17) $$\{\mathring{\gamma}_a, \mathring{\gamma}_b\} = 2\eta_{ab},$$

and e and m are real constants (bare charge and bare mass). Though the above expression for $\mathscr{L}_{\mathrm{D}}$ is non-hermitian, it is possible to replace it by a hermitian quantity without changing the corresponding action.

§7. Field Equations

Since $\mathscr{L}_{\mathrm{E}}$ contains second derivatives, it is convenient to replace it by

(7.1) $$\tilde{\mathscr{L}}_{\mathrm{E}} = \frac{1}{2} \tilde{g}^{\mu\nu} (\Gamma_{\mu\nu}{}^\lambda \Gamma_{\lambda\sigma}{}^\sigma - \Gamma_{\mu\sigma}{}^\lambda \Gamma_{\nu\lambda}{}^\sigma).$$

Since the difference $\mathscr{L}_{\mathrm{E}} - \tilde{\mathscr{L}}_{\mathrm{E}}$ is a total divergence, the action remains unchanged by this replacement.

In order to eliminate the derivative of b_ρ, we also replace $\mathscr{L}_{GF}$ by

(7.2)
$$\tilde{\mathscr{L}}_{GF} = \partial_\mu \tilde{g}^{\mu\nu} \cdot b_\nu .$$

On the other hand, we do not eliminate the derivative of s_{ab}, because $\omega_\mu{}^{ab}$ already contains the derivatives of vielbeins.

Thus the starting Lagrangian density of canonical formalism is*)

(7.3)
$$\tilde{\mathscr{L}} = \tilde{\mathscr{L}}_E + \tilde{\mathscr{L}}_{GF} + \mathscr{L}_{FP} + \mathscr{L}_{LGF} + \mathscr{L}_{LFP} + \mathscr{L}_{MF} ,$$

where

(7.4)
$$\mathscr{L} - \tilde{\mathscr{L}} = \partial_\mu \mathscr{D}^\mu ,$$

(7.5)
$$\mathscr{D}^\mu = \frac{1}{2}(\tilde{g}^{\mu\lambda}g^{\sigma\tau} - \tilde{g}^{\mu\sigma}g^{\lambda\tau})\partial_\lambda g_{\sigma\tau} - \tilde{g}^{\mu\nu}b_\nu .$$

Field equations are

(7.6)
$$\frac{\delta\tilde{\mathscr{L}}}{\delta\Phi_A} = 0 ,$$

where Φ_A goes over all fields involved in $\tilde{\mathscr{L}}$.

From the Euler equation for $h_{\mu a}$, we obtain the *quantum Einstein equation* [1], [5]:

(7.7)
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - E_{\mu\nu} + \frac{1}{2}g_{\mu\nu}E = -T_{\mu\nu} ,$$

where

(7.8)
$$E_{\mu\nu} = \partial_\mu b_\nu + i\partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho + (\mu \leftrightarrow \nu) = E_{\nu\mu} ,$$

(7.9)
$$E = E^\mu{}_\mu ,$$

(7.10)
$$T_{\mu\nu} = T_{L\mu\nu} + T_{M\mu\nu} ,$$

(7.11)
$$T_{L\mu\nu} = -h^{-1}h^\nu{}_a \frac{\delta(\mathscr{L}_{LGF} + \mathscr{L}_{LFP})}{\delta h_{\mu a}} ,$$

(7.12)
$$T_{M\mu\nu} = -h^{-1}h^\nu{}_a \frac{\delta\mathscr{L}_{MF}}{\delta h_{\mu a}} .$$

From the Euler equations for b_ν, $\bar{c}_\sigma$, and c^τ, we obtain

(7.13)
$$\partial_\mu \tilde{g}^{\mu\nu} = 0 ,$$

(7.14)
$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu c^\sigma) = 0 ,$$

(7.15)
$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu \bar{c}_\tau) = 0 ,$$

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*) In $\mathscr{L}_{MF}$, $\mathscr{L}_{EMGF}$ should also be replaced by $\tilde{\mathscr{L}}_{EMGF} = \partial_\mu(\tilde{g}^{\mu\nu}A_\nu) \cdot B$.

respectively.   The equation (7.13) has the same form as the *de Donder condition* or the *harmonic coordinate condition*, but in quantum gravity it is a field equation at the same level as the quantum Einstein equation.

From the Euler equations for s_{ab}, $\bar{t}_{ab}$, and t_{ab}, we obtain

$$(7.16) \qquad \qquad \partial_\mu(\tilde{g}^{\mu\nu}\omega_\nu{}^{ab}) = 0,$$

$$(7.17) \qquad \qquad \partial_\mu[\tilde{g}^{\mu\nu}(D_\nu t)^{ab}] = 0,$$

$$(7.18) \qquad \qquad \partial_\mu[\tilde{g}^{\mu\nu}(D_\nu \bar{t})^{ab}] = 0,$$

respectively.   Here use has been made of (7.16) in deriving (7.18).   In (7.14)–(7.18), we may interchange the order of ∂_μ and $\tilde{g}^{\mu\nu}$ because of (7.13).

Now, since $\mathscr{L}_{\mathrm{LGF}}$, $\mathscr{L}_{\mathrm{LFP}}$, and $\mathscr{L}_{\mathrm{MF}}$ are scalar densities under general coordinate transformation, it follows from (5.11) and the field equations (7.6) for $\Phi_A \neq h_{\mu a}$ that

$$(7.19) \qquad \qquad \nabla_\mu T_{\mathrm{L}}^{\mu\nu} = \nabla_\mu T_{\mathrm{M}}^{\mu\nu} = 0.$$

Hence the quantum Einstein equation (7.7) implies that

$$(7.20) \qquad \qquad \nabla_\mu\left(E^{\mu\nu} - \frac{1}{2}g^{\mu\nu}E\right) = 0.$$

With the aid of (7.13)–(7.15), (7.20) reduces to a remarkably simple equation [1], [5]:

$$(7.21) \qquad \qquad \partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu b_\rho) = 0.$$

As is discussed in Section 11, it is very important to note that (7.13), (7.21), (7.14), and (7.15) can be put together into

$$(7.22) \qquad \qquad \partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu X) = 0,$$

where $X = (x^\lambda, b_\rho, c^\sigma, \bar{c}_\tau)$.

Since $\mathscr{L}_{\mathrm{MF}}$ is invariant under local internal Lorentz transformations, it follows from (5.16) and the field equations (7.6) for $\Phi_A \neq h_{\mu a}$ that

$$(7.23) \qquad \qquad T_{\mathrm{M}\mu\nu} = T_{\mathrm{M}\nu\mu}.$$

But one must note that the same does not hold for $T_{\mathrm{L}\mu\nu}$.   Hence the antisymmetric part of the quantum Einstein equation (7.7) is

$$(7.24) \qquad \qquad T_{\mathrm{L}\mu\nu} - T_{\mathrm{L}\nu\mu} = 0,$$

or equivalently [5],

(7.25) $\partial_\mu(\tilde{g}^{\mu\nu}\{(D_\nu s)^{ab} - i[\bar{t}^a{}_c(D_\nu t)^{cb} - (a \leftrightarrow b)]\}) = 0$.

It is remarkable that all field equations, except for the symmetric part of the quantum Einstein equation, have the form of the conservation law.

Finally, if we adopt quantum electrodynamics as an example of matter-field part [8], we obtain

(7.26) $\partial_\nu(hF^{\mu\nu}) + \tilde{g}^{\mu\nu}\partial_\nu B = ej^\mu$,

(7.27) $\partial_\mu(\tilde{g}^{\mu\nu}A_\nu) = 0$,

(7.28) $\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu C) = 0$,

(7.29) $\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu \bar{C}) = 0$,

(7.30) $[\gamma^\mu(\partial_\mu + \omega_\mu - ieA_\mu) + im]\psi = 0$.

Here j^μ is the *electric current* defined by

(7.31) $j^\mu = h\psi^\dagger \bar{\gamma}_0 \gamma^\mu \psi$,

which is conserved, namely,

(7.32) $\partial_\mu j^\mu = 0$

because of the electromagnetic $U(1)$ symmetry. Note that (7.32) is equivalent to $\nabla_\mu j^\mu = 0$ in contrast with the case of $hT^{\mu\nu}$. From the *quantum Maxwell equation* (7.26) and the current conservation (7.32), we obtain

(7.33) $\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu B) = 0$.

The equations (7.33), (7.28), and (7.29) have the form of (7.22).

§8. Canonical Quantization

Given a Lagrangian density $\tilde{\mathscr{L}}$ and a canonical field Φ_A, the canonical conjugate, Π^A, of Φ_A is defined by

(8.1) $$\Pi^A = \frac{\partial \tilde{\mathscr{L}}}{\partial \dot{\Phi}_A}.$$

Then canonical quantization is carried out by setting up the canonical *-commutation relations:

(8.2) $[\Phi_A, \Phi_B'']_* = 0$,

(8.3) $[\Pi^A, \Phi_B'']_* = -i\delta^A_B \delta^{n-1}$,

(8.4) $[\Pi^A, \Pi^{B'}]_* = 0$,

where the abbreviated notation given in (3.6) and (3.7) is used.

Among the seven fundamental fields of quantum gravity listed in (4.1), the six fields other than b_ρ are canonical fields. Likewise, among the five fundamental fields (4.7) of quantum electrodynamics, the four fields other than B are canonical fields. The B fields are generally non-canonical, but there is an important exception of s_{ab}. The correct general rule is that the total *canonical* degrees of freedom of a gauge-like field and its B field *minus* that of its FP ghosts is equal to the degrees of physical (i.e., observable) freedom of the system. In the four-dimensional world ($n=4$), this arithmetic for quantum gravity goes like $16+6-4\times2-6\times2=2$, that is, gravitons (i.e., quanta of gravitational wave) have two degrees of physical freedom, just as photons (i.e., quanta of electromagnetic wave) do ($4-1\times2=2$). The basis of this rule is the quartet mechanism explained in Section 16.

It is of fundamental importance that canonical quantization can be carried out *consistently* owing to the introduction of $\mathscr{L}_{\mathrm{GF}}$ and $\mathscr{L}_{\mathrm{LGF}}$. Our theory is *not* of a constrained system in sharp contrast with the classical general relativity. We therefore need not apply the Dirac canonical method for constrained systems [27], which is known to be unsatisfactory in the quantum treatment of constraints.

§9. Equal-Time Commutators

From the canonical $*$-commutation relations (8.2)–(8.4) together with (7.13) and n components $(\cdots)_\mu{}^0$ of (7.7),[*] we can calculate all of the equal-time $*$-commutators $[\Phi_A, \Phi_B']_*$ and $[\Phi_A, \dot{\Phi}_B']_*$ for the seven fundamental fields (4.1) of gravity. It is quite remarkable that they are explicitly obtained *in closed form*, though the calculation is very elaborate and much involved [2], [6], [8]. Their expressions are independent of $\mathscr{L}_{\mathrm{MF}}$, but there is no general proof of this proposition, because the canonical conjugate of $h_{\mu a}$ receives the contribution from $\mathscr{L}_{\mathrm{MF}}$ if it contains $\dot{h}_{vb}$. The following results are derived under the assumption that $\mathscr{L}_{\mathrm{MF}}$ may contain $\mathscr{L}_{\mathrm{D}}$ but has no other terms containing $\dot{h}_{vb}$.

From (8.2), $[\Phi_A, \Phi_B']_*$ vanishes if neither Φ_A nor Φ_B are b_ρ. By explicit computation, we find that

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[*] Necessity of field equations is due to the non-canonical nature of b_ρ.

(9.1) $$[h_{\mu a}, b_\rho{}'] = -i\delta^0_\mu h_{\rho a}(\tilde{g}^{00})^{-1}\delta^{n-1}$$

and

(9.2) $$[\Phi, b_\rho{}'] = 0$$

for the fields Φ other than $h_{\mu a}$.

The $*$-commutators $[\Phi_A, \dot{\Phi}_B{}']_*$ are as follows, where we omit to present $[\Phi_B, \dot{\Phi}_A{}']_*$ if $[\Phi_A, \dot{\Phi}_B{}']_*$ is given, because

(9.3) $$[\dot{\Phi}_A, \Phi_B{}']_* = \partial_0[\Phi_A, \Phi_B{}']_* - [\Phi_A, \dot{\Phi}_B{}']_*.$$

$1°$ Commutators involving $h_{\mu a}$ and b_ρ only

(9.4) $$[h_{\mu a}, h_{\nu b}{}'] = -\frac{1}{2}i\left[\left(\frac{n}{2}-1\right)^{-1}h_{\mu a}h_{\nu b} - h_{\mu b}h_{\nu a} - g_{\mu\nu}\eta_{ab}\right.$$
$$\left. + (g^{00})^{-1}(\delta^0_\mu\delta^0_\nu\eta_{ab} + \delta^0_\mu h^0{}_b h_{\nu a} + \delta^0_\nu h^0{}_a h_{\mu b} + h^0{}_a h^0{}_b g_{\mu\nu})\right](\tilde{g}^{00})^{-1}\delta^{n-1};$$

(9.5) $$[h_{\mu a}, b_\rho{}'] = i[\partial_\rho h_{\mu a} \cdot (\tilde{g}^{00})^{-1} - \delta^0_\mu h_{\rho a}\partial_0(\tilde{g}^{00})^{-1}]\delta^{n-1}$$
$$- i[2(\tilde{g}^{00})^{-1}\tilde{g}^{0k}\delta^0_\mu - \delta^k_\mu]h_{\rho a}(\delta^{n-1})_k,$$

where

(9.6) $$(\delta^{n-1})_k = \partial_k[(\tilde{g}^{00})^{-1}\delta^{n-1}];$$

(9.7) $$[b_\lambda, b_\rho{}'] = i(\partial_\lambda b_\rho + \partial_\rho b_\lambda)(\tilde{g}^{00})^{-1}\delta^{n-1}.$$

$2°$ Independent *non-vanishing* $*$-commutators involving either c^σ or $\bar{c}_\tau$

(9.8) $$[c^\sigma, b_\rho{}'] = i\partial_\rho c^\sigma \cdot (\tilde{g}^{00})^{-1}\delta^{n-1},$$

(9.9) $$[\bar{c}_\tau, b_\rho{}'] = i\partial_\rho \bar{c}_\tau \cdot (\tilde{g}^{00})^{-1}\delta^{n-1};$$

(9.10) $$\{c^\sigma, \dot{\bar{c}}_\tau{}'\} = \delta^\sigma_\tau(\tilde{g}^{00})^{-1}\delta^{n-1}.$$

$3°$ *Non-vanishing* $*$-commutators between either $h_{\mu a}$ or b_ρ and one of s_{cd}, t_{cd}, and $\bar{t}_{cd}$

(9.11) $$[h_{\mu a}, \dot{s}_{cd}{}'] = \frac{1}{2}i(\eta_{ac}h_{\mu d} - \eta_{ad}h_{\mu c})(\tilde{g}^{00})^{-1}\delta^{n-1};$$

(9.12) $$[\varphi_{cd}, b_\rho{}'] = i\partial_\rho \varphi_{cd} \cdot (\tilde{g}^{00})^{-1}\delta^{n-1} \quad \text{for} \quad \varphi_{cd} = s_{cd}, t_{cd}, \text{ and } \bar{t}_{cd}.$$

$4°$ Independent *non-vanishing* $*$-commutators involving s_{ab}, t_{cd}, and $\bar{t}_{ef}$ only

(9.13) $$[s_{ab}, t_{cd}{}'] = \frac{1}{2}i[\eta_{bd}t_{ac} - \eta_{ad}t_{bc} - (c\leftrightarrow d)](\tilde{g}^{00})^{-1}\delta^{n-1};$$

(9.14) $$\{t_{ab}, \bar{t}_{cd}{}'\} = -\frac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})(\tilde{g}^{00})^{-1}\delta^{n-1}.$$

Finally, we mention the $*$-commutators related to the fields (4.7) of quantum electrodynamics. Since the number of independent $[\Phi_A, \dot{\Phi}_B{}']_*$'s is large, we here present the non-vanishing $[\Phi_A, \Phi_B{}']_*$'s only.

(9.15) $$[A_\mu, b_\rho{}'] = -i\delta_\mu^0 A_\rho(\tilde{g}^{00})^{-1}\delta^{n-1},$$

(9.16) $$[A_\mu, B'] = i\delta_\mu^0(\tilde{g}^{00})^{-1}\delta^{n-1} ;$$

(9.17) $$\{\psi_\alpha, (\psi'^\dagger\tilde{\gamma}^0)_\beta\} = (\gamma^0)_{\alpha\beta}(\tilde{g}^{00})^{-1}\delta^{n-1}.$$

§10. Tensorlike Commutation Relations

There is a remarkable regularity in the equal-time commutators involving b_ρ. From a large number of examples, we can deduce the following general formula [8]:[*]

(10.1) $$[\varphi_\Sigma, b_\rho{}'] = i[\varphi_\Sigma]^0{}_\rho(\tilde{g}^{00})^{-1}\delta^{n-1},$$

where φ_Σ is a local operator which is a tensor or a tensor density under general coordinate transformations, and $[\varphi_\Sigma]^\mu{}_\nu$ is defined by (5.1). Since $\partial_\mu\varepsilon^\nu\cdot[\varphi_\Sigma]^\mu{}_\nu$ is the infinitesimal change of φ_Σ under an infinitesimal general coordinate transformation $x'^\nu = x^\nu + \varepsilon^\nu$, (10.1) is consistent with the rules of tensor calculus, that is, the form of (10.1) is preserved for linear combination, for tensor product, and for contraction of indices, and therefore for raising or lowering any tensor index by means of $g^{\mu\nu}$ or $g_{\mu\nu}$. Hence we call (10.1) a *tensorlike commutation relation*. Though general covariance is lost through quantization, it is revived at least partially in the form of the tensorlike commutation relation.

The validity of (10.1) is confirmed for the fundamental fields $h_{\mu a}$, s_{ab}, t_{ab}, A_ν, B, C, $\bar{C}$, their first covariant derivatives, $T_{L\mu\nu}$, and $R_{\mu\nu}$, and therefore all tensors and tensor densities composed from them according to the rules of tensor calculus [8].

Since the affine connection $\Gamma_{\mu\lambda}{}^\nu$ is not a tensor in contrast with $\omega_\mu{}^{ab}$, the commutator between it and b_ρ contains a non-tensorlike term:

(10.2) $$[\Gamma_{\mu\lambda}{}^\nu, b_\rho{}'] = i(\delta_\rho^\nu\Gamma_{\mu\lambda}{}^0 - \delta_\mu^0\Gamma_{\rho\lambda}{}^\nu - \delta_\lambda^0\Gamma_{\mu\rho}{}^\nu)(\tilde{g}^{00})^{-1}\delta^{n-1}$$
$$+ i\delta_\rho^\nu[2\delta_\mu^0\delta_\lambda^0(\tilde{g}^{00})^{-1}\tilde{g}^{0k} - \delta_\mu^0\delta_\lambda^k - \delta_\mu^k\delta_\lambda^0](\delta^{n-1})_k,$$

where $(\delta^{n-1})_k$ is defined by (9.6).

By using (10.2), we see that if (10.1) is valid for φ_Σ, then it is also valid for

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[*] The non-canonical nature of b_ρ is crucial for the validity of (10.1).

its spatial covariant derivative $\nabla_k \varphi_\Sigma$. Since (10.1) is an equal-time commutation relation, we cannot discuss its validity for the time covariant derivative $\nabla_0 \varphi_\Sigma$. But, if we *assume* the validity of (10.1) for $\nabla_0 \varphi_\Sigma$, then we can calculate $[\dot{\varphi}_\Sigma, b_\rho']$ and hence $[\varphi_\Sigma, \dot{b}_\rho']$ by means of (9.3). In this way, we find that [15]

$$(10.3) \qquad [\varphi_\Sigma, \dot{b}_\rho'] = i\partial_\rho \varphi_\Sigma \cdot (\tilde{g}^{00})^{-1}\delta^{n-1} + i[\varphi_\Sigma]^0{}_\rho \{\partial_0(\tilde{g}^{00})^{-1}\delta^{n-1}$$
$$+ 2(\tilde{g}^{00})^{-1}\tilde{g}^{0k}(\delta^{n-1})_k\} - i[\varphi_\Sigma]^k{}_\rho (\delta^{n-1})_k .$$

This formula is again tensorlike in the sense stated above.

Though (10.3) holds for several examples [see (9.5), (9.8), (9.9),*) and (9.12)], a counterexample to it is found. Since we can rewrite (7.7) as

$$(10.4) \qquad\qquad E_{\mu\nu} = R_{\mu\nu} + T_{\mu\nu} - (n-2)^{-1}g_{\mu\nu}T,$$

$E_{\mu\nu}$ is a tensor, though it has no classical counterpart. The commutator $[E_{\mu\nu}, b_\rho']$, however, contains a non-tensorlike term [15]

$$(10.5) \qquad 2i\partial_k\partial_l b_\rho \cdot [\delta^k_\mu \delta^l_\nu \tilde{g}^{00} - \delta^k_\mu \delta^0_\nu \tilde{g}^{0l} - \delta^0_\mu \delta^l_\nu \tilde{g}^{0k} + \delta^0_\mu \delta^0_\nu \tilde{g}^{kl}](\tilde{g}^{00})^{-2}\delta^{n-1}$$

in addition to the tensorlike terms expected from the right-hand side of (10.3). Note that the coefficient of $\ddot{g}_{kl}$ in $R_{\mu\nu}$ is precisely proportional to the quantity in the square brackets of (10.5).

The commutator between $E_{\mu\nu}$ and $\dot{c}^\sigma$ (or $\dot{\bar{c}}_\tau$) is precisely equal to (10.5) with replacement of b_ρ by c^σ (or $\bar{c}_\tau$). In general, $[\varphi_\Sigma, b_\rho']$ and $[\varphi_\Sigma, \dot{b}_\rho']$ are totally tensorlike if and only if $[\varphi_\Sigma, c^{\sigma'}] = 0$ and if and only if $[\varphi_\Sigma, \dot{c}^{\sigma'}] = 0$, respectively (see Section 15).

§11. Choral Symmetry

As is suggested in Section 7, it is a very attractive idea to introduce the concept of the *4n-dimensional supercoordinate*

$$(11.1) \qquad\qquad X = (x^\lambda, b_\rho, c^\sigma, \bar{c}_\tau).$$

We use X, Y, Z, U, etc. to express 4n-dimensional supercoordinates. Note that all fundamental fields other than b_ρ, c^σ and $\bar{c}_\tau$ are quantities transforming as tensors under general coordinate transformations. The remarkably beautiful structure of the theory becomes clearer by rewriting it into the 4n-dimensional form.

First, we introduce the *4n-dimensional supermetric* $\eta(X, Y)$ by

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*) In this respect, c^σ and $\bar{c}_\tau$ should be regarded as scalars.

(11.2)
$$\begin{cases} \eta(x^\lambda,\, b_\rho)=\eta(b_\rho,\, x^\lambda)=\eta(c^\lambda,\, \bar{c}_\rho)= -\eta(\bar{c}_\rho,\, c^\lambda)=\delta^\lambda_\rho, \\ \eta(X,\, Y)=0 \qquad \text{otherwise}. \end{cases}$$

Here, supercoordinates are used as indices, that is, X and Y play the role of μ and ν in $\eta_{\mu\nu}$. Corresponding to $\eta^{\mu\nu}$, the contravariant component, $\tilde{\eta}(X,\, Y)$, of $\eta(X,\, Y)$ is defined by

(11.3)
$$\begin{cases} \tilde{\eta}(x^\lambda,\, b_\rho)=\tilde{\eta}(b_\rho,\, x^\lambda)= -\tilde{\eta}(c^\lambda,\, \bar{c}_\rho)=\tilde{\eta}(\bar{c}_\rho,\, c^\lambda)=\delta^\rho_\lambda, \\ \tilde{\eta}(X,\, Y)=0 \qquad \text{otherwise}, \end{cases}$$

in such a way that

(11.4)
$$\sum_Z \tilde{\eta}(X,\, Z)\eta(Z,\, Y)=\delta(X,\, Y),$$

where $\delta(X,\, Y)$ denotes the $4n$-dimensional Kronecker delta.

Next, we introduce two kinds of *$4n$-dimensional supertransformations* defined by the following (Z_2-graded) derivations [14]:

(11.5) $\delta^*{}_X(\varPhi)=\delta_X(\varPhi)-\delta_X(x^\nu)\partial_\nu\varPhi,$

(11.6) $\delta^*{}_{X,Y}(\varPhi)=\delta_{X,Y}(\varPhi)-\delta_{X,Y}(x^\nu)\partial_\nu\varPhi$

with the Leibniz rules

(11.7) $\delta_X(\varPhi_1\varPhi_2)=\delta_X(\varPhi_1)\varPhi_2+\epsilon\,(X,\, \varPhi_1)\varPhi_1\delta_X(\varPhi_2),$

(11.8) $\delta_{X,Y}(\varPhi_1\varPhi_2)=\delta_{X,Y}(\varPhi_1)\varPhi_2+\epsilon\,(XY,\, \varPhi_1)\varPhi_1\delta_{X,Y}(\varPhi_2),$

where

(11.9) $\delta_X(U)=\mathrm{i}\sqrt{-\epsilon\,(X,\, U)}\,\eta(X,\, U),$

(11.10) $\delta_{X,Y}(U)=\mathrm{i}\sqrt{-\epsilon\,(XY,\, U)}\,[\eta(Y,\, U)X-\epsilon\,(X,\, Y)\eta(X,\, U)Y]$

for a $4n$-dimensional supercoordinate U, and

(11.11) $\delta_X(\varphi_\Sigma)=0,$

(11.12) $\delta_{X,Y}(\varphi_\Sigma)= -\{\eta(Y,\, x^\nu)\partial_\mu X\cdot[\varphi_\Sigma]^\mu{}_\nu-(X\leftrightarrow Y)\}$

for a tensor field φ_Σ.

The first terms and the second terms in the right-hand sides of (11.5) and (11.6) are the *intrinsic parts* and the *orbital parts*, respectively. It is very important to recognize that the *intrinsic supertransformations* δ_X and $\delta_{X,Y}$ are more fundamental concepts than $\delta^*{}_X$ and $\delta^*{}_{X,Y}$.

According to (11.9), the orbital term of (11.5) is non-vanishing if and only if X is b_ρ, and likewise that of (11.6) is non-vanishing if and only if at least either

X or Y is b_ρ. In those cases, the supertransformations are spacetime ones; otherwise δ^*_X and $\delta^*_{X,Y}$ are supertransformations for internal symmetries. The spacetime intrinsic supertransformations do not commute with ∂_μ, but δ^*_X and $\delta^*_{X,Y}$ always commute with ∂_μ, corresponding to the fact that $\delta^*_X(x^\nu)$ $=\delta^*_{X,Y}(x^\nu)=0$.

For any local operator Φ which has its classical counterpart, (11.5), (11.11), and (11.9) imply that

$$(11.13) \qquad \delta^*_X(\Phi)=\eta(X,\ x^\nu)\partial_\nu\Phi\,.$$

As for $\delta^*_{X,Y}$, from (11.6), (11.12), and (11.10), we have

$$(11.14) \qquad \delta^*_{X,Y}(\varphi_\Sigma)=\{-\eta(Y,\ x^\nu)(\partial_\mu X\cdot[\varphi_\Sigma]^\mu{}_\nu-X\partial_\nu\varphi_\Sigma)\}-(X\leftrightarrow Y)\,.$$

This relation holds for any local operator φ_Σ which is a tensor or a tensor density at the classical level, because (11.14) is tensorlike in the sense stated in Section 10 and its form is preserved under covariant differentiation, as is confirmed by using

$$(11.15)\quad \delta^*_{X,Y}(\Gamma_{\sigma\tau}{}^\lambda)=\{-\eta(Y,\ x^\nu)(\partial_\mu X\cdot[\Gamma_{\sigma\tau}{}^\lambda]^\mu{}_\nu-X\partial_\nu\Gamma_{\sigma\tau}{}^\lambda)$$
$$+\eta(Y,\ x^\lambda)\partial_\sigma\partial_\tau X\}-(X\leftrightarrow Y)\,,$$

where $[\Gamma_{\sigma\tau}{}^\lambda]^\mu{}_\nu$ is defined directly by (5.1) though $\Gamma_{\sigma\tau}{}^\lambda$ is not a tensor.

In particular, if $\mathscr{F}$ is a scalar density ($[\mathscr{F}]^\mu{}_\nu=-\delta^\mu_\nu\mathscr{F}$), (11.13) and (11.14) show that both $\delta^*_X(\mathscr{F})$ and $\delta^*_{X,Y}(\mathscr{F})$ are total divergences. *Without using field equations*, we thus see that $\int d^n x \mathscr{L}_E$, $\int d^n x \mathscr{L}_{LGF}$, $\int d^n x \mathscr{L}_{LFP}$, and $\int d^n x \mathscr{L}_{MF}$ are invariant under the supertransformations δ^*_X and $\delta^*_{X,Y}$.

The quantity $E_{\mu\nu}$ defined by (7.8) can be rewritten as

$$(11.16) \qquad E_{\mu\nu}=\sum_{U,V}\sqrt{\epsilon(U,\ V)}\bar{\eta}(U,\ V)\partial_\mu U\cdot\partial_\nu V\,,$$

that is, $E_{\mu\nu}$ is essentially a "$4n$-dimensional scalar product" of two supercoordinates apart from the ordinary tensor indices μ and ν. Indeed, from (11.5)–(11.10), we can show that $\delta^*_X(E_{\mu\nu})$ and $\delta^*_{X,Y}(E_{\mu\nu})$ behave as if $E_{\mu\nu}$ were a covariant tensor at the classical level. Since

$$(11.17) \qquad \mathscr{L}_{GF}+\mathscr{L}_{FP}=-\frac{1}{2}\tilde{g}^{\mu\nu}E_{\mu\nu}\,,$$

we see that $\int d^n x(\mathscr{L}_{GF}+\mathscr{L}_{FP})$ is also invariant. Thus the action $\int d^n x \mathscr{L}$ is invariant under our $4n$-dimensional supertransformation.

This symmetry is called the *choral symmetry*[*].

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[*] The name "choral" is owing to the fact that it was proposed in the *nineth* paper [9] of the series.

§12.   4n-Dimensional Poincaré-like Superalgebra

Since the action is invariant under the $4n$-dimensional supertransformations (11.5) and (11.6), the Noether theorem implies that the *Noether supercurrents*

$$(12.1) \qquad \hat{\mathscr{P}}^\mu(X) = \sum_A \delta^*_X(\Phi_A)\frac{\partial\tilde{\mathscr{L}}}{\partial(\partial_\mu\Phi_A)} + \delta_X(x^\mu)\mathscr{L} + \delta^*_X(\mathscr{D}^\mu),$$

$$(12.2) \qquad \hat{\mathscr{M}}^\mu(X,\,Y) = \sum_A \delta^*_{X,Y}(\Phi_A)\frac{\partial\tilde{\mathscr{L}}}{\partial(\partial_\mu\Phi_A)} + \delta_{X,Y}(x^\mu)\mathscr{L} + \delta^*_{X,Y}(\mathscr{D}^\mu)$$

are conserved, where Φ_A goes over all fundamental fields.

By using the identity (5.12) (we set $\mathscr{F} = \mathscr{L}_{\mathrm{LGF}} + \mathscr{L}_{\mathrm{LFP}} + \mathscr{L}_{\mathrm{MF}}$) together with the field equations other than

$$(12.3) \qquad \partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu X) = 0,$$

we can rewrite (12.1) and (12.2) as [14]

$$(12.4) \qquad \hat{\mathscr{P}}^\mu(X) = \mathscr{P}^\mu(X) - \partial_\lambda\mathscr{J}^{\mu\lambda}(X),$$

$$(12.5) \qquad \hat{\mathscr{M}}^\mu(X,\,Y) = \mathscr{M}^\mu(X,\,Y) - \partial_\lambda\mathscr{J}^{\mu\lambda}(X,\,Y),$$

respectively.   Here we set

$$(12.6) \qquad \mathscr{P}^\mu(X) = \tilde{g}^{\mu\nu}\partial_\nu X,$$

$$(12.7) \qquad \mathscr{M}^\mu(X,\,Y) = \sqrt{\epsilon(X,\,Y)}\,\tilde{g}^{\mu\nu}(X\partial_\nu Y - \partial_\nu X\cdot Y),$$

and

$$(12.8) \quad \mathscr{J}^{\mu\lambda}(X,\,Y) = \delta_{X,Y}(x^\nu)\Big[\frac{1}{2}h(g^{\mu\sigma}g^{\lambda\tau} - g^{\mu\tau}g^{\lambda\sigma})\partial_\sigma g_{\nu\tau}$$

$$- \sum_A[\Phi_A]^\lambda{}_\nu\frac{\partial(\mathscr{L}_{\mathrm{LGF}} + \mathscr{L}_{\mathrm{LFP}} + \mathscr{L}_{\mathrm{MF}})}{\partial(\partial_\mu\Phi_A)}\Big] + \frac{1}{2}[\tilde{g}^{\mu\nu}\partial_\nu\delta_{X,Y}(x^\lambda) - \tilde{g}^{\lambda\nu}\partial_\nu\delta_{X,Y}(x^\mu)],$$

$\mathscr{J}^{\mu\lambda}(X)$ being obtained from (12.8) by replacing $\delta_{X,Y}(x^\nu)$ by $\delta_X(x^\nu) = -\eta(X,\,x^\nu)$.

Since both $\mathscr{J}^{\mu\lambda}(X)$ and $\mathscr{J}^{\mu\lambda}(X,\,Y)$ are antisymmetric under the interchange $\mu\leftrightarrow\lambda$ [cf. (5.13)], both $\mathscr{P}^\mu(X)$ and $\mathscr{M}^\mu(X,\,Y)$ satisfy the conservation law $\partial_\mu\mathscr{P}^\mu(X) = \partial_\mu\mathscr{M}^\mu(X,\,Y) = 0$, as is manifestly seen from (12.6) and (12.7).   Thus the conserved *supercharges* are given by

$$(12.9) \qquad P(X) = \int d^{n-1}x\,\mathscr{P}^0(X),$$

$$(12.10) \qquad M(X,\,Y) = \int d^{n-1}x\,\mathscr{M}^0(X,\,Y)$$

apart from possible surface terms. Here, in the left-hand sides, supercoordinates play the role of indices only.

From (12.10), we see that

$$(12.11) \qquad M(Y, X) = -\epsilon(X, Y)M(X, Y).$$

Hence the number of independent $M(X, Y)$'s is $8n^2$. Since that of $P(X)$ is, of course, $4n$, we have $4n(2n+1)$ independent supercharges.

The spatial integrations in (12.9) and (12.10) may not be convergent, that is, $P(X)$ and $M(X, Y)$ may be ill-defined operators. Given an ill-defined charge, it is sometimes possible to make it well-defined by adding a certain surface integral, but sometimes it is impossible to construct a well-defined one.

In any case, however, all charges are sensible as the generators of the corresponding transformations for any local operator Φ, as noted in Section 2, in the following sense. Since a charge is (at least formally) independent of x^0, we may set the x^0 of its integrand equal to the x'^0 of $\Phi(x')$. Then the *-commutator between the integrand and Φ can be calculated by means of equal-time *-commutators. Since it then has a support only at $x^k = x'^k$, the spatial integration is obviously convergent. Thus a charge is always sensible as a generator if it is understood that the spatial integration is carried out *after* taking the *-commutator.

In the way explained above, we can calculate $[P(X), \Phi]_*$ and $[M(X, Y), \Phi]_*$ explicitly. We find (cf. [9], [10])

$$(12.12) \qquad [P(X), \Phi]_* = -i\delta^*_X(\Phi),$$

$$(12.13) \qquad [M(X, Y), \Phi]_* = -i\delta^*_{X,Y}(\Phi)$$

for any fundamental field Φ. That is, $P(X)$ and $M(X, Y)$ are correctly the generators of the choral symmetry. Note that non-infinitesimal transformations are not necessarily sensible.

From (12.12) and (12.13), we can calculate the *-commutators between the supercharges. We find (cf. [10])

$$(12.14) \quad [P(X), P(U)]_* = 0,$$

$$(12.15) \quad [M(X, Y), P(U)]_*$$
$$= \sqrt{-\epsilon(XY, U)}\,[\eta(Y, U)P(X) - \epsilon(X, Y)\eta(X, U)P(Y)],$$

$$(12.16) \quad [M(X, Y), M(U, V)]_*$$
$$= \sqrt{-\epsilon(XY, UV)}\,[\eta(Y, U)M(X, V) - \epsilon(X, Y)\eta(X, U)M(Y, V)$$

$$+\epsilon(X, Y)\epsilon(U, V)\eta(X, V)M(Y, U)-\epsilon(U, V)\eta(Y, V)M(X, U)]\,.$$

Thus they form a Z_2-graded Lie algebra. Since it is remarkably similar to the Poincaré algebra, we call it the *4n-dimensional Poincaré-like superalgebra*. This superalgebra is quite remarkable in the sense that *it is a natural extension of the ordinary n-dimensional spacetime symmetry.*

Some special cases of the generators are of particular interest.

1° Translation generators [3]

$$(12.17) \qquad P_v = P(b_v)\,;$$

$$(12.18) \qquad [P_v, \Phi] = -i\partial_v\Phi\,.$$

2° Generators of general linear transformations [3]

$$(12.19) \qquad \hat{M}^\mu{}_v = -M(b_v, x^\mu)-M(\bar{c}_v, c^\mu)\,;$$

$$(12.20) \qquad [\hat{M}^\mu{}_v, \Phi] = i[\Phi]^\mu{}_v - ix^\mu\partial_v\Phi\,,$$

where $[\Phi]^\mu{}_v$ is defined by (5.1) even for $\Phi = b_\rho$, c^σ, and $\bar{c}_\tau$.

3° Gravitational BRS charge [2]

$$(12.21) \qquad Q_b = M(b_\lambda, c^\lambda)\,;$$

$$(12.22) \qquad [Q_b, \varphi_\Sigma]_* = -i\partial_\mu c^v \cdot [\varphi_\Sigma]^\mu{}_v + ic^v\partial_v\varphi_\Sigma\,,$$

$$(12.23) \qquad [Q_b, b_\rho] = ic^v\partial_v b_\rho\,,$$

$$(12.24) \qquad \{Q_b, c^\sigma\} = ic^v\partial_v c^\sigma\,,$$

$$(12.25) \qquad \{Q_b, \bar{c}_\tau\} = b_\tau + ic^v\partial_v\bar{c}_\tau\,;$$

$$(12.26) \qquad \{Q_b, Q_b\} = 0\,.$$

4° Gravitational FP-ghost number

$$(12.27) \qquad iQ_c = iM(\bar{c}_\lambda, c^\lambda)\,;$$

$$(12.28) \qquad [iQ_c, \Phi] = 0 \qquad \text{for} \quad \Phi \neq c^\sigma, \bar{c}_\tau\,,$$

$$(12.29) \qquad [iQ_c, c^\sigma] = c^\sigma\,,$$

$$(12.30) \qquad [iQ_c, \bar{c}_\tau] = -\bar{c}_\tau\,;$$

$$(12.31) \qquad [iQ_c, Q_b] = Q_b\,.$$

Finally, we consider the case in which $\mathscr{L}_{\mathrm{MF}}$ contains the Lagrangian density of quantum electrodynamics. In this case, the choral symmetry is extended by three dimensions, because, as is seen from (7.33), (7.28), and (7.29), B, C, and $\bar{C}$ satisfy an equation of the form (12.3). We therefore define the *extended super-*

coordinate $\hat{X}$ by

(12.32) $$\hat{X} = (x^\lambda, \, b_\rho, \, c^\sigma, \, \bar{c}_\tau; \, B, \, C, \, \bar{C}) \, .$$

The *extended supermetric* $\eta(\hat{X}, \, \hat{Y})$ is defined by

(12.33) $$\begin{cases} \eta(x^\lambda, \, b_\rho) = \eta(b_\rho, \, x^\lambda) = \eta(c^\lambda, \, \bar{c}_\rho) = -\eta(\bar{c}_\rho, \, c^\lambda) = \delta^\lambda_\rho \, , \\ \eta(C, \, \bar{C}) = -\eta(\bar{C}, \, C) = 1 \\ \eta(\hat{X}, \, \hat{Y}) = 0 \quad \text{otherwise.} \end{cases}$$

Then all the previous results concerning the generators are extended to this case [12]. The *electric BRS charge* Q_B and the *electric FP-ghost number* iQ_C are expressed as $M(B, \, C)$ and $iM(\bar{C}, \, C)$, respectively.

Note that $\eta(\hat{X}, \, B) = 0$ for any $\hat{X}$. Hence all generators considered above commute with $P(B)$. There is, however, owing to (7.27), another generator,

(12.34) $$P(A) = \int d^{n-1}x \, \tilde{g}^{0\nu} A_\nu \, ,$$

which does not commute with $P(B)$. The commutators between $M(\hat{X}, \, \hat{Y})$ and $P(A)$ can be described by (12.15) with

(12.35) $$\eta(\hat{X}, \, A) = \delta(\hat{X}, \, B) \, .$$

As is seen from the quantum Maxwell equation (7.26), $P(B)$ equals the electromagnetic charge operator apart from a surface integral, that is, $P(B)$ is nothing but the electromagnetic $U(1)$ generator. Thus we have an indecomposable Z_2-graded Lie algebra including both the spacetime symmetry and the electromagnetic $U(1)$ symmetry.

§13. Superalgebra in the Internal Lorentz Part

In this section, we describe the symmetry concerning the internal Lorentz part [19].

Corresponding to the conservation laws (7.16), (7.25), (7.17), and (7.18), we have the following $2n(n-1)$ generators:

(13.1) $$P(\omega^{ab}) = \int d^{n-1}x \, \tilde{g}^{0\nu}\omega_\nu{}^{ab} \, ,$$

(13.2) $$P(s^{ab}) = \int d^{n-1}x \, \tilde{g}^{0\nu}\{(D_\nu s)^{ab} - i[\bar{t}^a{}_c(D_\nu t)^{cb} - (a \leftrightarrow b)]\} \, ,$$

(13.3) $$P(t^{ab}) = \int d^{n-1}x \, \tilde{g}^{0\nu}(D_\nu t)^{ab} \, ,$$

(13.4) $P(\bar{t}^{ab}) = \displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}(D_\nu \bar{t})^{ab}\,.$

Because of the non-abelian nature of the Lorentz group, it is generally impossible to have the quadratic-type generators. We can construct the generators having no Lorentz indices only:

(13.5) $Q(s,\,t) = \displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}[s_{ab}(D_\nu t)^{ab} - \partial_\nu s_{ab}\cdot t^{ab} + i\partial_\nu \bar{t}_{ab}\cdot t^a{}_c t^{cb}]\,,$

(13.6) $Q(s,\,\bar{t}) = \displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}[s_{ab}(D_\nu \bar{t})^{ab} - \partial_\nu s_{ab}\cdot \bar{t}^{ab} + i\bar{t}_{ac}\bar{t}^c{}_b(D_\nu t)^{ab}]\,,$

(13.7) $Q(t,\,t) = i\displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}[t_{ab}(D_\nu t)^{ab} - \partial_\nu t_{ab}\cdot t^{ab}]\,,$

(13.8) $Q(\bar{t},\,t) = i\displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}[\bar{t}_{ab}(D_\nu t)^{ab} - \partial_\nu \bar{t}_{ab}\cdot t^{ab}]\,,$

(13.9) $Q(\bar{t},\,\bar{t}) = i\displaystyle\int d^{n-1}x\, \tilde{g}^{0\nu}[\bar{t}_{ab}(D_\nu \bar{t})^{ab} - \partial_\nu \bar{t}_{ab}\cdot \bar{t}^{ab}]\,.$

The independent non-vanishing ∗-commutators between the above $2n(n-1) + 5$ generators and the fields of gravity together with $\omega_\mu{}^{ab}$ are as follows.

(13.10) $[P(\omega^{ab}),\, s_{cd}] = \dfrac{1}{2}i(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)\,;$

(13.11) $[P(s^{ab}),\, h_{\mu c}] = -\dfrac{1}{2}i(\delta^a_c h_\mu{}^b - \delta^b_c h_\mu{}^a)\,,$

(13.12) $[P(s^{ab}),\, \varphi^{cd}] = -\dfrac{1}{2}i[\eta^{ac}\varphi^{bd} - \eta^{ad}\varphi^{bc} - (a\leftrightarrow b)]$

                          for $\varphi^{cd} = \omega_\mu{}^{cd},\ s^{cd},\ t^{cd},$ and $\bar{t}^{cd}\,;$

(13.13) $\{P(t^{ab}),\, \bar{t}_{cd}\} = \dfrac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)\,;$

(13.14) $[P(\bar{t}^{ab}),\, s_{cd}] = \dfrac{1}{2}i[\delta^b_c \bar{t}^a{}_d - \delta^b_d \bar{t}^a{}_c - (a\leftrightarrow b)]\,,$

(13.15) $\{P(\bar{t}^{ab}),\, t_{cd}\} = -\dfrac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)\,;$

(13.16) $[Q(s,\,t),\, h_{\mu a}] = it_a{}^b h_{\mu b}\,,$

(13.17) $[Q(s,\,t),\, \omega_\mu{}^{ab}] = -i(D_\mu t)^{ab}\,,$

(13.18) $\{Q(s,\,t),\, t_{ab}\} = it_{ac}t^c{}_b\,,$

(13.19) $\{Q(s,\,t),\, \bar{t}_{ab}\} = s_{ab}\,;$

$$(13.20) \qquad [Q(s, \bar{\imath}), h_{\mu a}] = i \bar{\imath}_a{}^b h_{\mu b},$$

$$(13.21) \qquad [Q(s, \bar{\imath}), \omega_\mu{}^{ab}] = -i(D_\mu \bar{\imath})^{ab},$$

$$(13.22) \qquad [Q(s, \bar{\imath}), s_{ab}] = i(\bar{\imath}_a{}^c s_{cb} - \bar{\imath}_b{}^c s_{ca}),$$

$$(13.23) \qquad \{Q(s, \bar{\imath}), t_{ab}\} = i(\bar{\imath}_a{}^c t_{cb} - \bar{\imath}_b{}^c t_{ca}) - i s_{ab},$$

$$(13.24) \qquad \{Q(s, \bar{\imath}), \bar{\imath}_{ab}\} = i \bar{\imath}_{ac} \bar{\imath}{}^c{}_b \; ;$$

$$(13.25) \qquad [Q(t, t), s_{ab}] = -2 t_{ac} t^c{}_b,$$

$$(13.26) \qquad [Q(t, t), \bar{\imath}_{ab}] = 2 i t_{ab} \; ;$$

$$(13.27) \qquad [Q(\bar{\imath}, t), t_{ab}] = -i t_{ab},$$

$$(13.28) \qquad [Q(\bar{\imath}, t), \bar{\imath}_{ab}] = i \bar{\imath}_{ab} \; ;$$

$$(13.29) \qquad [Q(\bar{\imath}, t), s_{ab}] = 2 \bar{\imath}_{ac} \bar{\imath}{}^c{}_b,$$

$$(13.30) \qquad [Q(\bar{\imath}, t), t_{ab}] = -2 i \bar{\imath}_{ab}.$$

Each of those generators *-commutes with any generator of the $4n$-dimensional Poincaré-like superalgebra, that is, they separately form a Z_2-graded Lie algebra. In the following, we present the independent *non-vanishing* *-commutators.

1° $[P, P]_*$-type

$$(13.31) \qquad [(P(s^{ab}), P(\varphi^{cd})] = \frac{1}{2} i [\eta^{ad} P(\varphi^{bc}) - \eta^{ac} P(\varphi^{bd}) - (a \leftrightarrow b)]$$

$$\text{for} \quad \varphi^{cd} = \omega^{cd}, \, s^{cd}, \, t^{cd}, \quad \text{and} \quad \bar{\imath}^{cd};$$

$$(13.32) \qquad \{P(t^{ab}), P(\bar{\imath}^{cd})\} = \frac{1}{2} [\eta^{ad} P(\omega^{bc}) - \eta^{ac} P(\omega^{bd}) - (a \leftrightarrow b)].$$

2° $[Q, P]_*$-type

$$(13.33) \qquad [Q(s, t), P(\omega^{ab})] = -i P(t^{ab}),$$

$$(13.34) \qquad [Q(s, \bar{\imath}), P(\omega^{ab})] = -i P(\bar{\imath}^{ab}) \; ;$$

$$(13.35) \qquad \{Q(s, \bar{\imath}), P(t^{ab})\} = -P(s^{ab}),$$

$$(13.36) \qquad [Q(\bar{\imath}, t), P(t^{ab})] = -i P(t^{ab}),$$

$$(13.37) \qquad [Q(\bar{\imath}, t), P(t^{ab})] = -2 i P(\bar{\imath}^{ab}) \; ;$$

$$(13.38) \qquad \{Q(s, t), P(\bar{\imath}^{ab})\} = P(s^{ab}),$$

$$(13.39) \qquad [Q(t, t), P(\bar{\imath}^{ab})] = 2 i P(t^{ab}),$$

$$(13.40) \qquad [Q(\bar{\imath}, t), P(\bar{\imath}^{ab})] = i P(\bar{\imath}^{ab}).$$

$3°$ $[Q, Q]_*$-type

(13.41) $[Q(s, t), Q(\bar{t}, t)] = iQ(s, t),$

(13.42) $[Q(s, t), Q(\bar{t}, \bar{t})] = 2iQ(s, \bar{t})$;

(13.43) $[Q(s, \bar{t}), Q(t, t)] = -2iQ(s, t),$

(13.44) $[Q(s, \bar{t}), Q(\bar{t}, t)] = -iQ(s, \bar{t})$;

(13.45) $[Q(t, t), Q(\bar{t}, t)] = 2iQ(t, t),$

(13.46) $[Q(t, t), Q(\bar{t}, \bar{t})] = 4iQ(\bar{t}, t),$

(13.47) $[Q(\bar{t}, t), Q(\bar{t}, \bar{t})] = 2iQ(\bar{t}, \bar{t}).$

There is a complete parallelism between the above symmetry properties in the internal Lorentz part of quantum gravity and those in the quantum theory of the Yang-Mills field (in the Landau gauge).

Some generators of the above symmetry are of particular interest. The *global internal Lorentz generators*[*] are given by

(13.48) $M_L{}^{ab} = 2P(s^{ab}),$

for which

(13.49) $[M_L{}^{ab}, \psi] = -i\hat{\sigma}^{ab}\psi.$

We call $Q_s = Q(s, t)$ the *internal-Lorentz BRS charge* and $iQ_t = iQ(\bar{t}, t)$ the *internal-Lorentz FP-ghost number*.

§14.   Spontaneous Breakdown of Symmetries

In quantum field theory, it is usually postulated that there is a unique state, called the *vacuum* and denoted by $|0\rangle$, which is Poincaré invariant and of the lowest energy $(=0)$ and for which $\langle 0|0\rangle = 1$. The vacuum, however, may not be an eigenstate of a conserved charge. Given a generator of symmetry, which we generically denote by Q, if there is a local operator Φ such that

(14.1) $\langle 0|[Q, \Phi]_*|0\rangle \neq 0,$

then it is said that the symmetry generated by Q is *spontaneously broken*. Note that the left-hand side of (14.1) must vanish if $|0\rangle$ is an eigenstate of Q. *If* (14.1) *holds, then* Φ *contains a massless discrete spectrum, which implies*

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[*]   Their Noether currents can be calculated by using (5.17).

the existence of massless particles having the quantum numbers of Φ (Goldstone theorem).

From (12.12) and (12.13), we have

(14.2) $\langle 0 | [P(x^{\mu}),\, b_{\rho}] | 0 \rangle = i\delta_{\rho}^{\mu},$

(14.3) $\langle 0 | \{P(c^{\mu}),\, \bar{c}_{\rho}\} | 0 \rangle = \delta_{\rho}^{\mu},$

(14.4) $\langle 0 | \{P(\bar{c}_{\mu}),\, c^{\rho}\} | 0 \rangle = -\delta_{\mu}^{\rho};$

(14.5) $\langle 0 | [M(x^{\mu},\, x^{\nu}),\, \partial_{\lambda} b_{\rho} - \partial_{\rho} b_{\lambda}] | 0 \rangle = 2i(\delta_{\lambda}^{\mu}\delta_{\rho}^{\nu} - \delta_{\lambda}^{\nu}\delta_{\rho}^{\mu}),$

(14.6) $\langle 0 | \{M(x^{\mu},\, c^{\nu}),\, \partial_{\lambda} \bar{c}_{\rho}\} | 0 \rangle = \delta_{\lambda}^{\mu}\delta_{\rho}^{\nu},$

(14.7) $\langle 0 | \{M(x^{\mu},\, \bar{c}_{\nu}),\, \partial_{\lambda} c^{\rho}\} | 0 \rangle = -\delta_{\lambda}^{\mu}\delta_{\nu}^{\rho}.$

Thus the symmetries generated by $P(x^{\mu})$, $P(c^{\mu})$, $P(\bar{c}_{\mu})$, $M(x^{\mu},\, x^{\nu})$, $M(x^{\mu},\, c^{\nu})$, and $M(x^{\mu},\, \bar{c}_{\nu})$ are necessarily spontaneously broken, and b_{ρ}, c^{σ}, and $\bar{c}_{\tau}$ contain a massless discrete spectrum [7].

Likewise, from (13.10)–(13.30), we have

(14.8) $\langle 0 | [P(\omega^{ab}),\, s_{cd}] | 0 \rangle = \frac{1}{2}i(\delta_{c}^{a}\delta_{d}^{b} - \delta_{c}^{b}\delta_{d}^{a}),$

(14.9) $\langle 0 | \{P(t^{ab}),\, \bar{t}_{cd}\} | 0 \rangle = \frac{1}{2}(\delta_{c}^{a}\delta_{d}^{b} - \delta_{c}^{b}\delta_{d}^{a}),$

(14.10) $\langle 0 | \{P(\bar{t}^{ab}),\, t_{cd}\} | 0 \rangle = -\frac{1}{2}(\delta_{c}^{a}\delta_{d}^{b} - \delta_{c}^{b}\delta_{d}^{a}).$

Thus the symmetries generated by $P(\omega^{ab})$, $P(t^{ab})$, and $P(\bar{t}^{ab})$ are necessarily spontaneously broken, and s_{ab}, t_{cd}, and $\bar{t}_{ef}$ contain a massless discrete spectrum [7].

For the other gravitational symmetries, we can say nothing definite about their spontaneous breakdown without additional information. We must specify the representation of field operators, which is usually characterized by giving the *vacuum expectation values* $\langle 0 | \Phi | 0 \rangle$ of local operators Φ. Since we postulate that translational invariance is not spontaneously broken, $\langle 0 | \Phi | 0 \rangle$ must be a constant.

Given $h_{\mu a}$, we have tacitly assumed the existence of $h^{\nu b}$, which is understandable only when the $n \times n$ matrix given by

(14.11) $\langle 0 | h_{\mu a} | 0 \rangle = \overset{\circ}{h}_{\mu a}$

is non-singular. We then consider

$$(14.12) \qquad \mathring{g}_{\mu\nu} = \eta_{ab}\mathring{h}_\mu{}^a\mathring{h}_\nu{}^b \,.$$

Since $\det \mathring{h}_\mu{}^a \neq 0$, we can transform $\mathring{g}_{\mu\nu}$ into $\eta_{\mu\nu}$ by a general linear transformation according to Sylvester's law of inertia. Denoting it by $u_\lambda{}^\sigma$, we have

$$(14.13) \qquad \eta_{ab}(u_\mu{}^\sigma \mathring{h}_\sigma{}^a)(u_\nu{}^\tau \mathring{h}_\tau{}^b) = \eta_{\mu\nu} \,.$$

This relation shows that $u_\mu{}^\sigma \mathring{h}_\sigma{}^a$ $(\mu = 0, 1, \ldots, n-1)$ are the basis vectors in the internal Minkowski space. Hence, there exists a Lorentz transformation $v_c{}^a$ such that

$$(14.14) \qquad u_\mu{}^\sigma \mathring{h}_\sigma{}^c v_c{}^a = \delta_\mu^a \,.$$

Now, we consider the generators $\hat{M}^\mu{}_\nu$ of general linear transformations. From (12.20), we have

$$(14.15) \qquad [\hat{M}^\mu{}_\nu, h_{\lambda c}] = -i\delta_\lambda^\mu h_{\nu c} - ix^\mu \partial_\nu h_{\lambda c} \,.$$

Hence (14.11) implies that

$$(14.16) \qquad \langle 0|[\hat{M}^\mu{}_\nu, h_{\lambda c}]|0\rangle = -i\delta_\lambda^\mu \mathring{h}_{\nu c} \,.$$

Thus the general linear invariance is spontaneously broken, and $h_{\lambda c}$ has a massless discrete spectrum, that is, *gravitons must be exactly massless* [17].

It is convenient to set

$$(14.17) \qquad \hat{M}^{*\mu}{}_\nu = u^\mu{}_\sigma u_\nu{}^\tau \hat{M}^\sigma{}_\tau \,,$$

$$(14.18) \qquad h^*{}_\lambda{}^c = u_\lambda{}^\rho h_\rho{}^d v_d{}^c \,,$$

where

$$(14.19) \qquad u^\mu{}_\sigma u_\lambda{}^\sigma = \delta_\lambda^\mu \,.$$

Then (14.16) is rewritten as

$$(14.20) \qquad \langle 0|[M^{*\mu}{}_\nu, h^*{}_\lambda{}^c]|0\rangle = -i\delta_\lambda^\mu \delta_\nu^c \,.$$

Next, we consider the generators, $M_L{}^{ab} = 2P(s^{ab})$, of global internal Lorentz transformations. From (13.11), we have

$$(14.21) \qquad \langle 0|[M_L{}^{ab}, h_{\lambda c}]|0\rangle = -i(\delta_c^a \mathring{h}_\lambda{}^b - \delta_c^b \mathring{h}_\lambda{}^a) \,,$$

or equivalently,

$$(14.22) \qquad \langle 0|[M_L{}^{*ab}, h^*{}_\lambda{}^c]|0\rangle = -i(\eta^{ac}\delta_\lambda^b - \eta^{bc}\delta_\lambda^a) \,,$$

where

$$(14.23) \qquad M_L{}^{*ab} = M_L{}^{cd} v_c{}^a v_d{}^b \,.$$

Hence, the global internal Lorentz invariance is spontaneously broken.

It should be noted, however, that combining (14.20) and (14.22), we have

$$(14.24) \qquad \langle 0|[M_{\alpha\beta}, h^{*}{}_{\lambda}{}^{c}]|0\rangle = 0,$$

where

$$(14.25) \qquad M_{\alpha\beta} = \eta_{\alpha\mu}\hat{M}^{*\mu}{}_{\beta} - \eta_{\beta\mu}\hat{M}^{*\mu}{}_{\alpha} + \eta_{\alpha a}\eta_{\beta b}M_{\mathrm{L}}{}^{*ab}.$$

Thus the spacetime symmetry generated by $M_{\alpha\beta}$ is *not* spontaneously broken [7], [14].

The generators $M_{\alpha\beta}$ are nothing but the *Lorentz generators of elementary-particle physics*. Indeed, it is easy to show, for instance, that

$$(14.26) \qquad [M_{\alpha\beta}, \psi] = -\mathrm{i}(\dot{\sigma}^{*}{}_{\alpha\beta} + x^{*}{}_{\alpha}\partial^{*}{}_{\beta} - x^{*}{}_{\beta}\partial^{*}{}_{\alpha})\psi,$$

where

$$(14.27) \qquad \dot{\sigma}^{*}{}_{\alpha\beta} = \frac{1}{4}[\hat{\gamma}^{*}{}_{\alpha}, \hat{\gamma}^{*}{}_{\beta}], \quad \hat{\gamma}^{*}{}_{\alpha} = \eta_{\alpha a}\hat{\gamma}^{c}v_{c}{}^{a},$$

$$(14.28) \qquad x^{*}{}_{\alpha} = \eta_{\alpha\mu}u^{\mu}{}_{\sigma}x^{\sigma}, \quad \partial^{*}{}_{\beta} = u_{\beta}{}^{\tau}\partial_{\tau}.$$

That is, the Dirac field ψ behaves as a world spinor under $M_{\alpha\beta}$. It should be emphasized that there is no reason why we regard ψ as a world spinor in the classical generally-covariant Dirac theory. *Only in the manifestly covariant canonical formalism of quantum gravity, we can uniquely characterize the genuine Lorentz generators $M_{\alpha\beta}$ without making any ad hoc assumption.*

The fact that the Lorentz generators are characterized at the level of the representation of field operators has a very fundamentally important implication in trying to construct a unified field theory including gravity: *It is not the right way to extend the Lorentz invariance of elementary-particle physics at the level of determining the fundamental Lagrangian density.*

The six fields of gravity other than $h_{\mu a}$ cannot have any non-vanishing vacuum expectation value without breaking the Poincaré invariance spontaneously. Hence it is natural to postulate that their vacuum expectation values vanish. Then we may claim that in addition to the Poincaré invariance, the $n(9n+1)/2$ symmetries generated by $M(X, Y)$ $(X \neq x^{\mu}, Y \neq x^{\nu})$ and the five ones generated by $Q(\varphi, \varphi')$ are *not* spontaneously broken.

The vacuum expectation value of the *-commutator between a generator of unbroken symmetry and any time-ordered product of local operators vanish. This relation is called a *Ward-Takahashi-type identity*. For the unbroken

$M(X, Y)$'s, the validity of their Ward-Takahashi-type identities are explicitly confirmed in perturbation theory (at tree and one-loop levels) [13]. For spontaneously broken $M(\hat{X}, \hat{Y})$'s, it is analyzed how to modify Ward-Takahashi-type identities [11], [12].

§15. Quantum-Gravity Invariant D-Function

The tensorlike commutation relations discussed in Section 10 can be extended to the *unequal-time* one, to which one can apply covariant differentiation.

In the ordinary quantum field theory, there is an important generalized function, $D(x-y)$, called the *invariant D-function*. It is defined by the three properties $\eta^{\mu\nu}\partial_\mu^x\partial_\nu^x D = 0$, $D|_0 = 0$, and $\partial_0^x D|_0 = -\delta^{n-1}$, where the symbol $|_0$ means setting $y^0 = x^0$, and we always identify y^k with x'^k at $y^0 = x^0$ for convenience of notation.   The explicit expression for $D(x-y)$ is known; it is odd and Poincaré invariant (and vanishes for $x^\mu - y^\mu$ spacelike). $D(x-y)$ is encountered in the n-dimensional (i.e., unequal-time) *-commutator involving a free massless field.

We extend $D(x-y)$ to quantum gravity [15].   Let $\mathscr{D}(x, y)$ be the *quantum-gravity extension of the invariant D-function*.   Since the metric $g_{\mu\nu}(x)$ is an operator, $\mathscr{D}(x, y)$ must be a bilocal operator.   In analogy with the definition of $D(x-y)$, we define $\mathscr{D}(x, y)$ by the following four properties:

(15.1) $$\mathscr{D}(x, y) = -\mathscr{D}(y, x),$$

(15.2) $$\partial_\mu^x[\tilde{g}^{\mu\nu}(x)\partial_\nu^x\mathscr{D}(x, y)] = 0,$$

(15.3) $$\mathscr{D}(x, y)|_0 = 0,$$

(15.4) $$\partial_0^x\mathscr{D}(x, y)|_0 = -(\tilde{g}^{00})^{-1}\delta^{n-1}.$$

We could also define $\mathscr{D}(x, y)$ by (15.1), (15.3), (15.4) and[*]

(15.5) $$\partial_\mu^x[\partial_\nu^x\mathscr{D}(x, y)\cdot\tilde{g}^{\mu\nu}(x)] = 0.$$

It can be proved, however, that both definitions give one and the same $\mathscr{D}(x, y)$.

As is expected, $\mathscr{D}(x, y)$ can be shown to be affine invariant in the sense that (cf. [15])

(15.6) $$[P_\nu, \mathscr{D}(x, y)] = -i(\partial_\nu^x + \partial_\nu^y)\mathscr{D}(x, y),$$

(15.7) $$[\hat{M}^\mu{}_\nu, \mathscr{D}(x, y)] = -i(x^\mu\partial_\nu^x + y^\mu\partial_\nu^y)\mathscr{D}(x, y).$$

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[*] Since $\mathscr{D}(x, y)$ depends on y^0, operator ordering is non-trivial.

Just as in the case of $D(x - y)$, we encounter $\mathscr{D}(x, y)$ in the n-dimensional *-commutator involving a field $\hat{X}$ satisfying

$$(15.8) \qquad \partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\hat{X})=0.$$

From (15.8) and (15.2)–(15.4), we have an integral representation of $\hat{X}$:

$$(15.9) \qquad \hat{X}(y)=\int \mathrm{d}^{n-1}z[\partial_\nu\hat{X}(z)\cdot\tilde{g}^{0\nu}(z)-\hat{X}(z)\tilde{g}^{0\nu}(z)\partial_\nu^z]\mathscr{D}(z, y).$$

Since the right-hand side of (15.9) is *independent* of z^0, the n-dimensional *-commutator $[\varPhi(x), \hat{X}(y)]_*$ can be expressed in terms of equal-time *-commutators by setting $z^0=x^0$.

The tensorlike commutation relations presented in Section 10 can be extended into the following n-dimensional form. Let φ_Σ be a local operator which is a tensor or a tensor density at the classical level and X be a supercoordinate. Then, by using (15.9), we can show that [15]

$$(15.10) \quad [\varphi_\Sigma(x), X(y)]_* = -i\eta(X, x^\nu)\{[\varphi_\Sigma(x)]^\mu{}_\nu\partial_\mu^x$$
$$-\partial_\nu\varphi_\Sigma(x)\}\mathscr{D}(x, y)+\mathscr{C}(x, y; \varphi_\Sigma, X).$$

Here the first term is nothing but the n-dimensional form of the tensorlike part; it vanishes except for $X=b_\rho$. The second term is non-tensorlike but its form is *common* for any X; more precisely, it is a linear functional of X (at the leftmost position). Furthermore, the fact that it vanishes for $X=x^\lambda$ implies that X stands as the form of a second derivative, because $\partial_\mu\partial_\nu X=0$ for $X=x^\lambda$.

§16. Unitarity of the Physical S-Matrix

The *S-matrix* in quantum field theory is a generalization of the scattering matrix in quantum mechanics. From the S-matrix elements, one can immediately calculate the probabilities in which various reactions of elementary particles take place. In the Heisenberg picture, the S-matrix is defined as follows.

Let $\varPhi(x)$ be a local operator having a discrete spectrum. Essentially owing to the Riemann-Lebesgue lemma, $\varPhi(x)$ approaches $\varPhi^{\mathrm{in}}(x)$ as $x^0\to -\infty$ and $\varPhi^{\mathrm{out}}(x)$ as $x^0\to +\infty$ in the sense of weak topology, where $\varPhi^{\mathrm{in}}(x)$ and $\varPhi^{\mathrm{out}}(x)$, which are called an *in-field* and an *out-field*, respectively, are quantum fields having common free-field properties.

We can construct a *Fock space* by applying in-fields on the vacuum $|0\rangle$.

Its standard basis vectors are called *in-states*. Likewise, we define *out-states*. According to the postulate of *asymptotic completeness*, both Fock spaces coincide with the whole space of states, which we denote by $\mathscr{V}$.

The S-matrix S is defined as the transformation matrix from in-states to out-states. If $\mathscr{V}$ has positive-definite metric, then S is unitary, and this fact allows the usual probabilistic interpretation of the quantum theory. But if the indefinite inner product is used as in gauge theories and in quantum gravity, then S is *not* unitary though $SS^{\dagger} = S^{\dagger}S = 1$. Hence we encounter serious difficulty in the probabilistic interpretation of the S-matrix.

The most reasonable and almost unique method of restoring the probabilistic interpretability is to show the existence of a positive-semidefinite subspace of $\mathscr{V}$ invariant under S and $S^{\dagger}$. If it exists at all, it is called the *physical subspace* and denoted by $\mathscr{V}_{\mathrm{phys}}$. The totality of the states of $\mathscr{V}_{\mathrm{phys}}$ which are orthogonal to any state of $\mathscr{V}_{\mathrm{phys}}$ is a subspace of $\mathscr{V}_{\mathrm{phys}}$, which is denoted by $\mathscr{V}_{0}$. The *physical S-matrix* S_{phys} is defined by restricting S to $\mathscr{V}_{\mathrm{phys}}$. Then S_{phys} can be shown to be unitary in the quotient space $\mathscr{V}_{\mathrm{phys}}/\mathscr{V}_{0}$, which is the space of "observable" states.

The physical subspace $\mathscr{V}_{\mathrm{phys}}$ is usually defined by *subsidiary conditions*, that is, $|f\rangle \in \mathscr{V}_{\mathrm{phys}}$ if and only if $|f\rangle$ satisfies all subsidiary conditions. If subsidiary conditions are linear and time-independent and if they altogether exclude any state $|g\rangle \in \mathscr{V}$ such that $\langle g|g\rangle < 0$, then $\mathscr{V}_{\mathrm{phys}}$ has the properties stated above. In the Kugo-Ojima formulation [28]–[30] of a BRS-invariant theory, we can always successfully construct the subsidiary conditions having the required properties. In the following, we describe how they are realized.

For the time being, we consider the case in which there is only one type of the BRS charge, which we generically denote by Q_{BRS}. There is the corresponding FP-ghost number, which we generically denote by iQ_{FP}. Both Q_{BRS} and Q_{FP} are hermitian, and supposed to be well-defined. Then their $*$-commutation relations imply

(16.1) $$Q_{\mathrm{BRS}}^{2} = 0,$$

(16.2) $$(iQ_{\mathrm{FP}})Q_{\mathrm{BRS}} = Q_{\mathrm{BRS}}(iQ_{\mathrm{FP}} + 1).$$

Because of (16.1), the irreducible representations of Q_{BRS} are a *BRS singlet* $\{|a\rangle\}$ $(Q_{\mathrm{BRS}}|a\rangle = 0$ but there is no $|b\rangle$ such that $Q_{\mathrm{BRS}}|b\rangle = |a\rangle)$ and a *BRS doublet* $\{|a\rangle, Q_{\mathrm{BRS}}|a\rangle\}$ $(Q_{\mathrm{BRS}}|a\rangle \neq 0)$. On the other hand, the anti-hermitian operator iQ_{FP} has integer eigenvalues. Since it is a conserved number, two

eigenstates of iQ_{FP} are orthogonal unless the sum of their eigenvalues is zero. We may assume, without loss of generality, that for any state in $\mathscr{V}$ there exists at least one state in $\mathscr{V}$ which is not orthogonal to it. Then two eigenstates of iQ_{FP} having a non-zero eigenvalue form a pair so as to be mutually non-orthogonal. Such a pair of BRS singlets is called a *singlet pair*. Correspondingly, a non-paired BRS singlet is called a *pure singlet*. Since (16.2) implies that Q_{BRS} increases the FP-ghost number by one, at least one member of BRS doublet has a non-zero eigenvalue if they are eigenstates of iQ_{FP}. Hence, BRS doublets necessarily form a pair; this pair is called a *quartet*: $\{|a\rangle, Q_{BRS}|a\rangle, |b\rangle, Q_{BRS}|b\rangle\}$ $(\langle b|Q_{BRS}|a\rangle \neq 0)$.

The space $\mathscr{V}$ is decomposable into a direct sum of pure-singlet subspaces, singlet-pair subspaces and quartet subspaces, each of which is orthogonal to each other. Furthermore, from the BRS transformation properties of FP-ghosts, we can show that there are no singlet pairs in $\mathscr{V}$ [31]. Though the proofs of those properties may not be rigorous in the part concerning the topology of $\mathscr{V}$ which we do not know, we regard them as reasonable from the physicist's standpoint (see Section 2).

Let $\mathscr{V}_1$ be the subspace spanned by all *one-particle* in-states. Since it is invariant under Q_{BRS} and iQ_{FP}, restricting the above decomposition of $\mathscr{V}$ to $\mathscr{V}_1$, we see that $\mathscr{V}_1$ is decomposable into a direct sum of pure-singlet subspaces and quartet subspaces, each of which is orthogonal to any other. From this decomposition, we can define *pure-singlet particles* and *quartet particles*.

Let $P^{(N)}$ be the orthogonal projection operator to the subspace spanned by the in-states containing exactly N quartet particles. Then, of course, we have

$$(16.3) \qquad \sum_{N=0}^{\infty} P^{(N)} = 1 .$$

By explicit computation [30], we can show that there is an operator $R^{(N-1)}$ such that

$$(16.4) \qquad P^{(N)} = \{Q_{BRS}, R^{(N-1)}\} \qquad \text{for} \quad N \geq 1 .$$

Now, we set up the *Kugo-Ojima subsidiary condition*

$$(16.5) \qquad Q_{BRS}|f\rangle = 0 ,$$

which is manifestly linear and time-independent. Furthermore, from (16.3)–(16.5) together with the hermiticity of Q_{BRS}, we obtain

$$(16.6) \qquad \langle f|f\rangle = \langle f|P^{(0)}|f\rangle .$$

This extremely important result is called the *quartet mechanism* [30].

In reasonable theories, there are good physical reasons for postulating that $\langle f_0 | f_0 \rangle > 0$ for any pure-singlet one-particle in-state $|f_0\rangle$. Then (16.6) implies that

$$(16.7) \qquad\qquad \langle f | f \rangle \geqq 0 .$$

This establishes the positive semi-definiteness of $\mathscr{V}_{\text{phys}}$.

If there exist another independent BRS charge Q_{BRS}' and the corresponding FP-ghost number iQ_{FP}', then we apply the above reasoning to $\{P^{(0)}\mathscr{V}, Q_{\text{BRS}}', iQ_{\text{FP}}'\}$ in place of $\{\mathscr{V}, Q_{\text{BRS}}, iQ_{\text{FP}}\}$. Repeating this procedure until exhausting all BRS charges, we finally arrive at (16.7).

In our theory of quantum gravity, we set up two subsidiary conditions

$$(16.8) \qquad\qquad Q_b | f \rangle = 0 ,$$

$$(16.9) \qquad\qquad Q_s | f \rangle = 0 ,$$

where $Q_b = M(b_\lambda, c^\lambda)$ and $Q_s = Q(s, t)$, and if $\mathscr{L}_{\text{MF}}$ includes a gauge field, for example, the electromagnetic field, we further set up

$$(16.10) \qquad\qquad Q_B | f \rangle = 0 .$$

Then the physical S-matrix is unitary.

We can also make a more concrete proof under the assumption that in-fields are governed by the free part of $\tilde{\mathscr{L}}$ [1], [32], [5].

§17. Possible Resolution of the Divergence Difficulty

Quantum gravity is known to be unrenormalizable in the four-dimensional world; the divergence difficulty in perturbation theory cannot be remedied by introducing a finite number of counter terms into the Lagrangian density. If one adds higher-derivative terms to the Einstein-Hilbert Lagrangian density, quantum gravity becomes renormalizable in perturbation theory, but necessarily violates the unitarity of the physical S-matrix.

It is quite likely that what is responsible for the divergence difficulty in quantum gravity is not the theory itself but the perturbative approach. Indeed, since the perturbation expansion of quantum gravity is not a power series of an adjustable parameter*), it is quite unreasonable to discuss the divergence problem

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*) The gravitational constant should be regarded as a *unit* ($\kappa = 1$) in Nature just like $c = \hbar = 1$.

in each order.

On the other hand, the renormalizability of the ordinary quantum field theory should not be regarded as the ultimate resolution of the divergence difficulty. Though the renormalized S-matrix is finite in each order of perturbation theory, we must introduce divergent counter terms into the Lagrangian density. The requirements of Lorentz invariance, unitarity, and macrocausality make the total removal of divergences (in the four-dimensional world) prohibitively difficult.

There is an old expectation that quantum gravity might ultimately resolve the divergence difficulty of the ordinary quantum field theory. The ground of this expectation was that the quantum fluctuation of spacetime geometry would make the rigid light-cone singularity obscure. But such a geometrical consideration is not convincing because the divergence difficulty really arises from the presence of the products of field operators at the *same* spacetime point; indeed, the Euclidean quantum field theory has the same divergence difficulty as the Minkowski one.

In the ordinary quantum field theory, there is the only one reason for supporting the belief that the divergence difficulty remains even in the nonperturbative approach. The divergence problem in perturbation theory is essentially governed by the high-energy asymptotic behavior of the *Feynman propagator*, namely, the *free-field two-point function*. Here, the use of the free field is, of course, owing to the perturbative approach. If one wishes to discuss the problem in a non-perturbative way, one must take account of the effect of interaction simultaneously. Accordingly, one encounters the *exact two-point function*. But there is an important theorem (Lehmann theorem) [33], which states that *the high-energy asymptotic behavior of the exact two-point function cannot be milder than that of the corresponding Feynman propagator*. One therefore conjectures that the divergence problem cannot be resolved even in the exact treatment.

For clarity, we explain the Lehmann theorem by taking, as an example, a hermitian scalar field $\phi(x)$. Its two-point function is defined by

$$(17.1) \qquad \tau(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle ,$$

where the symbol T indicates time ordering of local operators, that is,

$$(17.2) \qquad T\phi(x)\phi(y) = \phi(x)\phi(y) \quad \text{for} \quad x^0 > y^0 ,$$
$$= \phi(y)\phi(x) \quad \text{for} \quad x^0 < y^0 .$$

If $\phi(x)$ is a free field, $\tau(x-y)$ reduces to the Feynman propagator $\Delta_F(x-y)$, whose fourier transform is given by

$$(17.3) \qquad \frac{-i}{m^2-p^2-i0},$$

where p_α and m denote the energy-momentum (i.e., fourier conjugate of $x^\alpha - y^\alpha$) and the mass of the scalar particle, respectively, and $p^2 = \eta^{\alpha\beta}p_\alpha p_\beta$. When interaction is present, the fourier transform of the exact two-point function has a spectral representation

$$(17.4) \qquad -i \int_0^\infty ds \frac{\rho(s)}{s-p^2-i0},$$

where $\rho(s)$ is a real generalized function defined by

$$(17.5) \qquad \rho(p^2) = (2\pi)^{-1} \int d^n x \, e^{ip(x-y)} \langle 0|\phi(x)\phi(y)|0\rangle.$$

If the space $\mathscr{V}$ has positive-definite metric, there is a complete set of the eigenstates $|h\rangle$, having an eigenvalue $p^{(h)}_\alpha (p^{(h)}_0 \geqq 0)$, of the energy-momentum operator P_α, the completeness relation is written symbolically as[*]

$$(17.6) \qquad \sum_h |h\rangle\langle h| = 1.$$

We insert (17.6) into (17.5) and use

$$(17.7) \qquad \phi(x) = e^{iPx}\phi(0)e^{-iPx},$$

$$(17.8) \qquad P_\alpha|0\rangle = 0, \quad P_\alpha|h\rangle = p^{(h)}_\alpha|h\rangle.$$

Then we easily see that

$$(17.9) \qquad \rho(p^2) = (2\pi)^{n-1} \sum_h |\langle h|\phi(0)|0\rangle|^2 \delta^n(p-p^{(h)}) \geqq 0.$$

The Lehmann theorem follows from (17.4) and (17.9).

Of course, the Lehmann theorem does *not*, in general, hold if $\mathscr{V}$ has indefinite metric. But, unfortunately, in such a case, unitarity is usually violated. The successful exceptions are gauge theories, in which the unitarity of the physical S-matrix is guaranteed as shown in Section 16 despite the use of indefinite metric. In gauge theories, however, the Lehmann theorem still holds for any local observable, namely, any local operator $\Phi(x)$ satisfying[**]

$$(17.10) \qquad [Q_{\mathrm{BRS}}, \Phi(x)]_* = 0,$$

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[*] $\sum_h$ includes integration over $p^{(h)}_\alpha$.
[**] If there are several BRS charges, we consider all of them simultaneously.

as shown below.

Since the vacuum $|0\rangle$ is BRS-invariant ($Q_{BRS}|0\rangle = 0$), we have

(17.11) $$Q_{BRS}\Phi(x)|0\rangle = 0,$$

that is, $\Phi(x)|0\rangle$ is a physical state. The physical subspace $\mathscr{V}_{phys}$ is a direct sum of two subspaces $\mathscr{V}_0$ and $\mathscr{V}_+$, where $\mathscr{V}_0$ is the totality of the states in $\mathscr{V}_{phys}$ orthogonal to any state of $\mathscr{V}_{phys}$ and $\mathscr{V}_+$ has positive-definite metric. Let P_+ be the orthogonal projection operator to $\mathscr{V}_+$. Then for any $|f\rangle \in \mathscr{V}_{phys}$, we have $(1 - P_+)|f\rangle \in \mathscr{V}_0$, whence for any $|f'\rangle \in \mathscr{V}_{phys}$ we obtain

(17.12) $$\langle f'|(1 - P_+)|f\rangle = 0,$$

that is,

(17.13) $$\langle f'|f\rangle = \langle f'|P_+|f\rangle.$$

Thus, for any two physical states, we can insert

(17.14) $$\sum_{|h\rangle \in \mathscr{V}_+} |h\rangle\langle h| = P_+$$

into their inner product. Accordingly, the positivity of $\rho(s)$ still remains valid for the two-point function of local observables.*)

The situation drastically changes in quantum gravity, in which there are *no* (non-trivial) local observables [18]. This is because Q_b is a generator of a *spacetime* symmetry, that is, $[Q_b, \Phi]_*$ *always contains an orbital term* $ic^\mu \partial_\mu \Phi$. Thus *the evasion of the Lehmann theorem is achieved in quantum gravity.* The exact two-point function *may* have milder high-energy asymptotic behavior than that of the corresponding Feynman propagator. We can expect that quantum gravity may ultimately resolve the divergence difficulty — the old expectation is revived on a different ground.

The above observation also dissolves the necessity of the *ad hoc* introduction of the pathological "Schwinger term" [18].

§18. Related Work

In this section, we very briefly mention related work done mainly by other authors.

Delbourgo and Medrano [35], Stelle [36], and Townsend and Nieuwen-

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*) More generally, any local observable corresponds to a local operator in the Hilbert space $\mathscr{V}_{phys}/\mathscr{V}_0$ [30], [34].

huizen [37] constructed the gravitational BRS transformation in a way quite analogous to the Yang-Mills case.

Nishijima and Okawa [38] and Kugo and Ojima [39] proposed a modified formalism of our theory of quantum gravity so as to become more similar to the Yang-Mills theory. They adoped $\mathscr{L}_{\mathrm{GF}}$ as the gravitational gauge-fixing Lagrangian density, but added a general linear *non*-invariant term

(18.1) $$\frac{1}{2}\alpha\eta^{\mu\nu}b_{\mu}b_{\nu},$$

where α is a real parameter, and required the gravitational BRS invariance of Delbourgo and Medrano and others to have

(18.2) $$\mathscr{L}_{\mathrm{FP}}' = -\mathrm{i}\partial_{\mu}\bar{c}_{\nu}\cdot[\tilde{g}^{\mu\lambda}\partial_{\lambda}c^{\nu}+\tilde{g}^{\lambda\nu}\partial_{\lambda}c^{\mu}-\partial_{\lambda}(\tilde{g}^{\mu\nu}c^{\lambda})]$$

as the gravitational FP-ghost Lagrangian density. The difference between $\mathscr{L}_{\mathrm{FP}}$ and $\mathscr{L}_{\mathrm{FP}}'$ arises from the difference in the BRS transforms of b_{ρ} and $\bar{c}_{\tau}$. It can be shown, however, that their $\mathscr{L}_{\mathrm{GF}}+\mathscr{L}_{\mathrm{FP}}'$ for $\alpha=0$ is equivalent to our $\mathscr{L}_{\mathrm{GF}}+\mathscr{L}_{\mathrm{FP}}$ by transforming b_{ρ} into $b_{\rho}+\mathrm{i}c^{\lambda}\partial_{\lambda}\bar{c}_{\rho}$ [16].

The covariant formalism of the quantum Yang-Mills theory is invariant also under the *dual* (or *anti-*) *BRS transformation* [40], [41], which is another BRS transformation in which the roles of the two FP ghosts are interchanged. The existence of the corresponding symmetry is evident in our theory of quantum gravity; indeed, its generator is given by $\eta^{\alpha\beta}M(b_{\alpha},\bar{c}_{\beta})$, though it is not invariant under general linear transformations. Without knowing this pointing-out [9], Delbourgo and Thompson [42] claimed the non-existence of the dual BRS symmetry on the basis of a Yang-Mills-like FP-ghost Lagrangian density as $\mathscr{L}_{\mathrm{FP}}'$. Delbourgo, Jarvis, and Thompson [43], [44], however, found a class of Lagrangian densities invariant under both BRS and dual BRS transformations by means of the "superfield" method. Similar analysis was made also by Pasti and Tonin [45], who extended our treatment of the gravitational BRS transformation [1], and by Hirayama and Hirai [46]. Pasti and Tonin [45] redescribed the choral symmetry of our theory from their standpoint. A further comment was made on the basis of $\mathscr{L}_{\mathrm{GF}}+\mathscr{L}_{\mathrm{FP}}$ by Hamamoto [47].

Kawasaki, Kimura, and Kitago [48], [49] made an extension of our theory to the case in which the Lagrangian density effectively contains the terms proportional to $hR^{\mu\nu}R_{\mu\nu}$ and to hR^{2} (but without $\mathscr{L}_{\mathrm{LGF}}+\mathscr{L}_{\mathrm{LFP}}+\mathscr{L}_{\mathrm{MF}}$). After extremely elaborate calculation of equal-time *-commutators, they confirmed that the choral symmetry remains valid at the operator level. The tensorlike com-

mutation relation (10.1) was also confirmed in this case.

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