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On the S^1-Segal Conjecture

By

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§ 1. Introduction

For a compact Lie group G the Segal conjecture can be formulated similarly to that for finite groups as follows. Let $\pi_k^G(S^0)$ be the equivariant stable homotopy group [5]. Let EG be a free contractible G-CW complex and let $EG^{(r)}$ be the equivariant skeleton. The projection $EG^{(r)} \to *$ induces a homomorphism $\pi_k^G(S^0) \cong \pi_G^{-k}(S^0) \to \pi_G^{-k}(EG_+^{(r)})$. It is well known that $\pi_G^{-k}(EG_+^{(r)}) \cong \pi^{-k}((EG^{(r)}/G)_+)$ and we have a homomorphism $\alpha_r \colon \pi_k^G(S^0) \to \pi^{-k}((EG^{(r)}/G)_+)$. Since $\varinjlim EG^{(r)}/G = BG$, we write $\varprojlim \pi^{-k}((EG^{(r)}/G)_+)$ as $\mathscr{H}^{-k}(BG; \boldsymbol{S})$, then we have a homomorphism

$$\alpha \colon \pi_k^G(S^0) \longrightarrow \mathscr{H}^{-k}(BG; \boldsymbol{S}).$$

Note that if G is not finite then $\mathscr{H}^{-k}(BG; \boldsymbol{S})$ is not isomorphic to the actual stable cohomotopy group $\pi^{-k}(BG_+)$.

Let $A(G) \cong \pi_0^G(S^0)$ be the Burnside ring of G defined by tom Dieck [5]. It is clear that α is continuous with respect to the $I(G)$-adic topology on $\pi_k^G(S^0)$ and the inverse limit topology on $\mathscr{H}^{-k}(BG; \boldsymbol{S})$. Hence we have a continuous homomorphism

$$\hat{\alpha} \colon \pi_k^G(S^0)^{\wedge}_{I(G)} \longrightarrow \mathscr{H}^{-k}(BG; \boldsymbol{S}).$$

If G is finite then the solution of the Segal conjecture [4] asserts that $\hat{\alpha}$ is a topological isomorphism. But if G is not finite then $\hat{\alpha}$ is seen to be not an isomorphism by a trivial reason. Let $G = S^1$, then $I(S^1) = 0$ and the $I(S^1)$-adic completion is the identity. Let $k = 1$, then by the tom Dieck splitting [6], $\pi_1^{S^1}(S^0)$ is a countable direct sum of $\boldsymbol{Z}$. On the other hand $\mathscr{H}^{-1}(BS^1; \boldsymbol{S})$ is $\boldsymbol{Z} \oplus$profinite group. Therefore those groups have different cardinalities. If $G = S^1$ and $k \leq 0$, then J. F. Adams [12] has announced that $\hat{\alpha}$ is an isomorphism. But even when $k = 0$ the situation is still bad. For example let $G = O(2)$. Then $I(O(2))$ is a

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countable direct sum of $\boldsymbol{Z}$. N. Minami has pointed out that $I(O(2))$-adic topology on $I(O(2))$ is the 2-adic topology. Therefore $I(O(2))_2^{\wedge}$ is not compact but $\tilde{\mathscr{H}}^0(BO(2); \boldsymbol{S}) = \varprojlim \{BO(2)^{(r)}, S^0\}$ is compact. So the $I(G)$-adic topology is inadequate for compact Lie groups and the Segal conjecture for non finite groups should be stated as follows.

Conjecture. *α: $\pi_k^G(S^0) \to \mathscr{H}^{-k}(BG; \boldsymbol{S})$ has a dense image for $k \in \boldsymbol{Z}$.*

For $k \geq 0$, M. Feshbach [13] has shown that the conjecture holds for any compact Lie group. Now the purpose of this paper is to prove the following.

Theorem. *The Segal conjecture holds for $G = S^1$. Moreover α is an isomorphism if $k \leq 0$.*

Our method is an approximation of S^1 by finite cyclic groups. For this we use the S^1-transfer and in Section 2 we explain this in more general situation. In Section 3 we show approximation theorems for stable cohomotopy and stable homotopy of BS^1, and in Section 4 the proof of the theorem will be given.

§2. Compact Lie Group and Higher Transfer

Let G be a compact Lie group and V a real G-module. We say that a closed G-manifold M has a stable V-framing if there is a G-bundle monomorphism

$$\varphi: \boldsymbol{V} \oplus \boldsymbol{W} \longrightarrow \tau_G(M) \oplus \boldsymbol{W}$$

onto a G-subbundle of $\tau_G(M) \oplus \boldsymbol{W}$ for some G-module W where $\boldsymbol{V} = M \times V$ and $\tau_G(M)$ is the tangent G-bundle. Choose a G-invariant metric on M, then a V-framing determines a G-bundle α and a G-bundle isomorphism $\tau_G(M) \oplus \boldsymbol{W} \cong \boldsymbol{V} \oplus \boldsymbol{W} \oplus \alpha$. Choose a G-embedding $M \to U$ into a G-module U and let ν be the normal bundle. If U is large enough then the above isomorphism induces a G-bundle isomorphism $\nu \oplus \alpha \cong \boldsymbol{U} - \boldsymbol{V}$. Let f be the composite

$$U^c \xrightarrow{\gamma} \nu^c \subset (\nu \oplus \alpha)^c \xrightarrow{\pi} (U - V)^c$$

where γ is the Pontrjagin-Thom map and π is the projection. We denote the stable class of f by $\chi_V(M) \in \pi_V^G(S^0)$, the equivariant V-stem. $\chi_V(M)$ depends only on the stable class of a V-framing. For $V=0$, $\pi_0^G(S^0)$ is identified with the Burnside ring $A(G)$ and clearly $\chi_0(M) = [M] \in A(G)$ in the sense of tom Dieck [5].

Let H be a closed subgroup of G and let $W_H = N_G(H)/H$. In [6] tom Dieck has shown that there is a homomorphism $\lambda_H\colon \pi_n^{W_H}(EW_{H+})\to\pi_n^G(S^0)$ such that

$$\lambda=\bigoplus_{(H)}\lambda_H\colon \bigoplus_{(H)}\pi_n^{W_H}(EW_{H+})\longrightarrow \pi_n^G(S^0)$$

is an isomorphism where (H) runs through the conjugacy classes of subgroups of G. Let M be an n-dim. free W_H-manifold with an R^n-framing. Then the Pontrjagin-Thom construction of the classifying W_H-map $M\to EW_H$ determines a class $[M]\in\pi_n^{W_H}(EW_{H+})$. It is clear that the G-manifold $G\times_{N(H)}M$ has an R^n-framing induced from that of M. Then from the construction we easily see the following.

Lemma 2.1. $\lambda_H([M])=\chi_{R^n}(G\times_{N(H)}M)\in\pi_n^G(S^0)$.

Let now $F\xrightarrow{i}E\xrightarrow{\pi}B$ be a fibre bundle associated with a principal G-bundle $\tilde{E}\to B$. We suppose that F is a closed G-manifold and B is compact. Let $\tilde{\tau}$ be the tangent bundle along the fibre, i.e., $\tilde{\tau}=\tilde{E}\times_G\tau_G(F)$. Let ξ be a vector bundle over B. Then a stable map called a bundle transfer (Boardman [1])

$$t\colon SB^{\xi}\longrightarrow SE^{\pi^*\xi-\tilde{\tau}}$$

is defined by a similar way to the Becker-Gottlieb transfer [2]. Let now suppose that the fibre F is V-framed so that $\tau_G(F)\underset{s}{\simeq}V\oplus\alpha$. Let $\tilde{\alpha}=\tilde{E}\times_G\alpha$ and $\tilde{V}=(\tilde{E}\times F\times V)/G$, then $\tilde{\alpha}-\tilde{\tau}\underset{s}{\simeq}-\tilde{V}$. Let $\xi=0$, then composing t with the canonical inclusion $E^{-\tilde{\tau}}\xrightarrow{j}E^{-\tilde{\tau}+\tilde{\alpha}}$ we obtain a stable map

$$t=t_V\colon SB^0\longrightarrow SE^{-\tilde{V}}$$

which is called a V-transfer. If $V=0$, then it is clearly the Becker-Gottlieb transfer.

Let h^* be a multiplicative cohomology theory. Suppose that vector bundles ξ, $\tilde{\tau}$ and $\tilde{V}$ are h^*-oriented. Then all stable bundles in the above construction are canonically h^*-oriented. Then *via* Thom isomorphisms t and t_V induce homomorphisms

$$\pi_!\colon h^i(E)\longrightarrow h^{i-n}(B)$$

and

$$\pi_{V!}\colon h^i(E)\longrightarrow h^{i-d}(B)$$

where $n=\dim F$ and $d=\dim V$. Note that F is then h^*-oriented and let $[F]\in h^n(F)$ be the cohomology fundamental class.

Proposition 2.2. i) *$\pi_!$ is independent of ξ.*

ii) $\pi_!(x\cdot\pi^*(y))=\pi_!(x)\cdot y$ *for* $x\in h^*(E)$ *and* $y\in h^*(B)$.

iii) *Suppose that there is an element* $u\in h^n(E)$ *such that* $i^*(u)=[F]$, i^*: $h^*(E)\to h^*(F)$, *then* $\pi_!(u)\in h^0(B)$ *is a unit.*

iv) $\pi_{V!}(x)=\pi_!(\chi(\tilde{\alpha})\cdot x)$, $\chi(\tilde{\alpha})\in h^{n-d}(E)$ *is the Euler class of* $\hat{\alpha}$.

Proof. i), ii) and iii) are obvious from [1] and iv) is clear from the fact that the composition $h^*(E)\xrightarrow{\cong}\tilde{h}^*(E^{-\tilde{\tau}+\tilde{\alpha}})\xrightarrow{j^*}\tilde{h}^*(E^{-\tilde{\tau}})\xrightarrow{\cong}h^*(E)$ is just the multiplication with $\chi(\tilde{\alpha})$.

Let H be a subgroup of G. Then we have fibre bundle $G/H\xrightarrow{i}\tilde{E}/H\xrightarrow{\pi}\tilde{E}/G=B$. Let ad$(G)$ be the adjoint representation of G on the tangent space $T(G)_e$ and let $\xi=\tilde{E}\times_G\mathrm{ad}(G)$. Note that $\tau_G(G/H)\cong G\times_H(\mathrm{ad}(G)/\mathrm{ad}(H))$ as G-vector bundles. Then we see that $\pi^*\xi-\tilde{\tau}\cong\tilde{E}\times_H\mathrm{ad}(H)$ and we have a bundle transfer t: $\boldsymbol{S}(\tilde{E}_+\wedge_G\mathrm{ad}(G)^c)\to\boldsymbol{S}(\tilde{F}_+\wedge_H\mathrm{ad}(H)^c)$ which is just the transfer of Becker-Schultz [3]. On the other hand let $d=d(G,H)=\dim W_H$. Then it is well known that $\dim(\mathrm{ad}(G)/\mathrm{ad}(H))^H=d$ and the inclusion $W_H\to G/H$ determines a canonical R^d-framing on G/H. Hence we have a transfer t: $\boldsymbol{S}\Sigma^d(\tilde{E}/G_+)\to\boldsymbol{S}(\tilde{E}/H_+)$. Let $\tilde{E}=EG^{(r)}$ and using the naturality of the transfer we can take a limit and obtain stable maps

$$t=t_{\mathrm{ad}}\colon \boldsymbol{S}(EG_+\wedge_G\mathrm{ad}(G)^c)\longrightarrow\boldsymbol{S}(EH_+\wedge_H\mathrm{ad}(H)^c)$$

and

$$t=t_{G/H}\colon \boldsymbol{S}\Sigma^d(BG_+)\longrightarrow\boldsymbol{S}(BH_+)$$

which will be called a G/H-transfer. Let K be a subgroup of H. Then in general $t_{H/K}\circ\Sigma^{d(H,K)}t_{G/H}\neq t_{G/K}$, but if G is abelian the equality clearly holds.

In [7] Hauschild has shown that there is an isomorphism

$$\mu=\mu_G\colon \pi_k(EG_+\wedge_G\mathrm{ad}(G)^c)\longrightarrow\pi_k^G(EG_+).$$

Then from the construction we easily see the following

Lemma 2.3. *The following diagram is commutative*

$$\begin{array}{ccc}\pi_k(EG_+\wedge_G\mathrm{ad}(G)^c) & \xrightarrow{\mu} & \pi_K^G(EG_+)\\ {\scriptstyle t_*}\downarrow & & \downarrow{\scriptstyle r}\\ \pi_k(EH_+\wedge_H\mathrm{ad}(H^c) & \xrightarrow[\mu]{} & \pi_k^H(EH_+)\end{array}$$

where r *is the homomorphism given by restricting the* G*-action.*

§3. Approximation by Cyclic Groups

Let X be a connected CW-complex and let p be a prime number. Let $X_p^\wedge$ denote the p-adic completion of X of Sullivan [10]. Let F be a connected H-space such that $\pi_i(F)$ is a finite p-group for any i. Then by the obstruction theory we easily see that the natural map

$$[X_p^\wedge, F] \longrightarrow [X, F]$$

is an isomorphism. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a direct system of finite CW-complexes with a countable index set Λ. Let $X=\text{hocolim } X_\lambda$. Let $\boldsymbol{E}$ be a connected locally finite spectrum. We put

$$\tilde{\mathscr{H}}^i(X; \boldsymbol{E}_p^\wedge) = \varprojlim (\tilde{h}^i(X_\lambda; \boldsymbol{E}) \otimes \hat{\boldsymbol{Z}}_p)$$

where $h^i(X_\lambda; \boldsymbol{E})$ is the generalized cohomology theory defined by $\boldsymbol{E}$. Let $\{X_\lambda\}\to\{Y_\mu\}$ be a morphism of direct systems and let $f\colon X\to Y$ be the induced map. Then we obtain an induced homomorphism

$$f^*;\ \tilde{\mathscr{H}}^i(Y; \boldsymbol{E}_p^\wedge) \longrightarrow \tilde{\mathscr{H}}^i(X; \boldsymbol{E}_p^\wedge).$$

Let $\boldsymbol{E}\to\boldsymbol{F}\to\boldsymbol{G}$ be a cofibration of spectra. Note that $\tilde{h}^i(X_\lambda; \boldsymbol{E})\otimes \boldsymbol{Z}_p^\wedge$ is a compact topological group. Hence there is no $\varprojlim^1$ and we obtain an exact sequence

$$\longrightarrow \mathscr{H}^i(X; \boldsymbol{E}_p^\wedge) \longrightarrow \mathscr{H}^i(X; \boldsymbol{F}_p^\wedge) \longrightarrow \mathscr{H}^i(X; \boldsymbol{G}_p^\wedge) \longrightarrow .$$

Let $\boldsymbol{Z}_{p^r}\subset S^1$ be the standard inclusion. Let $\boldsymbol{Z}_{p^\infty}=\varinjlim \boldsymbol{Z}_{p^r}$, then we have an inclusion $\boldsymbol{Z}_{p^\infty}\subset S^1$. Note that it is factored as $\boldsymbol{Z}_{p^\infty}\subset \boldsymbol{Q}/\boldsymbol{Z}\subset S^1$. Those inclusions induce maps $B\boldsymbol{Z}_{p^\infty}\to B(\boldsymbol{Q}/\boldsymbol{Z})\to BS^1$ which are all denoted by j. It is well known [10] that

$$j_p^\wedge\colon (B\boldsymbol{Z}_{p^\infty})_p^\wedge \longrightarrow (BS^1)_p^\wedge$$

and

$$j^\wedge\colon (B(\boldsymbol{Q}/\boldsymbol{Z}))^\wedge \longrightarrow (BS^1)^\wedge$$

are homotopy equivalences, where $(\ )^\wedge$ is the profinite completion. We have $BS^1=\varinjlim (BS^1)^{(n)}$ and $B\boldsymbol{Z}_{p^\infty}=\varinjlim (B\boldsymbol{Z}_{p^r})^{(n)}$ and the map $j\colon B\boldsymbol{Z}_{p^\infty}\to BS^1$ is clearly filtrated. Let $\boldsymbol{S}$ be the sphere spectrum.

Proposition 3.1. *For any prime p and any integer i, the homomorphism $j^*\colon \tilde{\mathscr{H}}^i(BS^1; \boldsymbol{S}_p^\wedge)\to\tilde{\mathscr{H}}^i(B\boldsymbol{Z}_{p^\infty}; \boldsymbol{S}_p^\wedge)$ is an isomorphism.*

Proof. Let $\tilde{S}\to S\to K(Z)$ be the 0-connective fibration of the sphere spectrum. Then we have a commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow \tilde{\mathscr{H}}^i(BS^1;\tilde{S}_p^\wedge) & \longrightarrow & \tilde{\mathscr{H}}^i(BS^1;S_p^\wedge) & \longrightarrow & \tilde{\mathscr{H}}^i(BS^1;K(Z)_p^\wedge) & \longrightarrow \\
\downarrow j^* & & \downarrow j^* & & \downarrow j^* & \\
\longrightarrow \tilde{\mathscr{H}}^i(BZ_{p^\infty};\tilde{S}_p^\wedge) & \longrightarrow & \tilde{\mathscr{H}}^i(BZ_{p^\infty};S_p^\wedge) & \longrightarrow & \tilde{\mathscr{H}}^i(BZ_{p^\infty};K(Z)_p^\wedge) & \longrightarrow .
\end{array}$$

First note that

$$\begin{aligned} j^*;\ \tilde{\mathscr{H}}^i(BS^1;K(Z)_p^\wedge) \cong H^i(BS^1;\hat{Z}_p) &\to \tilde{\mathscr{H}}^i(BZ_{p^\infty};K(Z)_p^\wedge) \\ &\cong \varprojlim (H^i(B)Z_{p^r};Z_p^\wedge)) \end{aligned}$$

is an isomorphism. Next for given i choose $l\geq 0$ such that $l+i\geq 0$. Then one easily see that

$$\tilde{\mathscr{H}}^i(X;\tilde{S}_p^\wedge)\cong[\Sigma^l(X_+),(\widetilde{QS^{l+i}})_p^\wedge]$$

where $\widetilde{QS^{l+i}}$ is the $l+i$ connective fibre space of QS^{l+i}. Note that $\pi_j((\widetilde{QS^{l+i}})_p^\wedge)$ is a finite p-group for any j. Then since $j_p^\wedge\colon (BZ_{p^\infty})_p^\wedge\to(BS^1)_p^\wedge$ is a homotopy equivalence we see that

$$j^*:\tilde{\mathscr{H}}^i(BS^1:\tilde{S}_p^\wedge)\longrightarrow\tilde{\mathscr{H}}^i(BZ_{p^\infty}:\tilde{S}_p^\wedge)$$

is an isomorphism. Hence the proposition follows from the five lemma.

Now consider the commutative diagram

$$\begin{array}{ccc}
S\Sigma^i(BS^1_+) & \xrightarrow{\ t\ } & S(BZ_{p^r+}) \\
 & \searrow^{t} & \downarrow t \\
 & & S(BZ_{p^{r-1}+})
\end{array}$$

of the transfer maps associated with $Z_{p^{r-1}}\subset Z_{p^r}\subset S^1$. Then we have a homomorphism

$$\varprojlim t^*:\pi_i(S\Sigma^1(BS^1_+))\longrightarrow\varprojlim\pi_i(S(BZ_{p^r+})).$$

Proposition 3.2. $\varprojlim t^*$ *is a p-adic complection.*

Proof. From the cofibration $S^0\to BZ_{p^r+}\to BZ_{p^r}$ we obtain an inverse system of cofibrations $\{S^0\}_r\to\{S(BZ_{p^r+})\}_r\to\{S(BZ_{p^r})\}_r$. Note that $\{S^0\}_r$ is $S^0\xleftarrow{p}S^0\xleftarrow{p}\cdots$. Then we easily see that $\varprojlim\pi_i(S(BZ_{p^r+}))\cong\varprojlim\pi_i(SBZ_{p^r})$. Let $(S\Sigma^1(BS^1_+))_p^\wedge$ be the p-adic completion of the spectrum $X=S\Sigma^1(BS^1_+)$. Let $f\colon X\to F$ be a spectra map where F is a connective CW-spectrum such that $\pi_i(F)$ is a finite p-group for any i. Then $X_p^\wedge$ can be given as a functorial inverse limit $\varprojlim F$. We are then enough to show that $\{X \xrightarrow{t} SBZ_{p^r}\}_r$ is cofinal in $\{X \to F\}$. Let $\pi\colon BZ_{p^r} \to BS^1$ be the projection. Then S^1/Z_{p^r}-transfer t induces a homomorphism

$$\pi_{1!}\colon H^i(BZ_{p^r};\, Z_p) \longrightarrow H^{i-1}(BS^1;\, Z_p).$$

By Proposition 2.2, iii) we see that $\pi_{1!}$ is an isomorphism if i is odd. Then one easily see that

$$\varinjlim t^*\colon \varinjlim \tilde{H}^i(BZ_{p^r};\, Z_p) \longrightarrow \tilde{H}^i(\Sigma^1(BS^1_+);\, Z_p)$$

is an isomorphism for any i. Then by the obstruction theory (Postnikov system) similar to Sullivan [10], we see that $\{S\Sigma^1(BS^1_+) \xrightarrow{t} SBZ_{p^r}\}$ is cofinal.

§ 4. Proof of the Theorem

Let $Z_{p^{r-1}} \subset Z_{p^r} \subset S^1$ be the standard inclusions. Consider the commutative diagram of the restriction homomorphisms

$$\begin{array}{ccccc}
\pi_k^{S^1}(S^0) & \xrightarrow{r} & \pi_k^{Z_{p^r}}(S^0) & \xrightarrow{c} & \pi_k^{Z_{p^r}}(S^0)^\wedge_I \\
& \searrow^{r} & \downarrow r & & \downarrow r \\
& & \pi_k^{Z_{p^{r-1}}}(S^0) & \xrightarrow[c]{} & \pi_k^{Z_{p^{r-1}}}(S^0)^\wedge_I
\end{array}$$

where $(\ )^\wedge_I$ is the $I(G)$-adic completion. Then we have a homomorphism

$$\varprojlim r\colon \pi_k^{S^1}(S^0) \longrightarrow \varprojlim (\pi_k^{Z_{p^r}}(S^0)^\wedge_I).$$

By the Milnor exact sequence for $BZ_{p^\infty} = \varinjlim BZ_{p^r}$ we see that the canonical map

$$\omega\colon h^{-k}(BZ_{p^\infty};\, S) \longrightarrow \varprojlim h^{-k}(BZ_{p^r}\colon S)$$

is an isomorphism for any k. For the reduced groups we see that $\tilde{h}^{-k}(BZ_{p^\infty};\, S) = \tilde{\mathscr{H}}^{-k}(BZ_{p^\infty};\, S) \to \tilde{\mathscr{H}}^{-k}(BZ_{p^\infty};\, S^\wedge_p)$ is an isomorphism. Then we have a commutative diagram

(D)

$$\begin{array}{ccccc}
\pi_k^{S^1}(S^0) & \xrightarrow{\alpha} & \mathscr{H}^{-k}(BS^1;\, S) = \tilde{\mathscr{H}}^{-k}(BS^1;\, S) \oplus \pi_k(S^0) & \to & \tilde{\mathscr{H}}^{-k}(BS^1;\, S^\wedge_p) \oplus \pi_k(S^0) \\
\downarrow \varprojlim r & & \downarrow \varprojlim j^* & & \downarrow j^* \\
\varprojlim \pi_k^{Z_{p^r}}(S^0)^\wedge_I & \xrightarrow[\varprojlim \alpha]{} & \varprojlim \mathscr{H}^{-k}(BZ_{p^r};\, S) & \xleftarrow[\omega]{} & \tilde{\mathscr{H}}^{-k}(BZ_{p^\infty};\, S^\wedge_p) \oplus \pi_k(S^0).
\end{array}$$

By the above argument ω is an isomorphism. By Proposition 3.1, j^* is an isomorphism. By the solution of the Segal conjecture for cyclic groups [9], we see that $\varprojlim \alpha$ is an isomorphism.

Now according to k, the proof is devided into two cases. First suppose that $k<0$. Then by the tom Dieck splitting, $\pi_k^{S^1}(S^0)\cong\pi_k^{Z_{p^r}}(S^0)\cong 0$. Hence we see that $\tilde{\mathscr{H}}^{-k}(BS^1; S_p^\wedge)=0$ for any p. Then we easily see that $\mathscr{H}^{-k}(BS^1; \boldsymbol{S})=0$ and hence $\tilde{\mathscr{H}}^{-k}(BS^1; \boldsymbol{S})=0$. This shows that α is an isomorphism.

Next suppose that $k\geq 0$. Let H be a subgroup of S^1 and let $H_r=H\cap \boldsymbol{Z}_{p^r}$. Then $\boldsymbol{Z}_{p^r}/H_r\subset S^1/H$ and we have the transfer t: $\boldsymbol{S}\Sigma^1(B(S^1/H)_+\to \boldsymbol{S}(B(\boldsymbol{Z}_{p^r}/H_r)_+)$. By Lemmas 2.1 and 2.3 we have a commutative diagram

$$\begin{array}{ccccc} \pi_k(\boldsymbol{S}\Sigma^1(B(S^1/H)_+)) & \xrightarrow[\cong]{\mu} & \pi_k^{S^1/H}(E(S^1/H_+) & \xrightarrow{\lambda_H} & \pi_k^{S^1}(S^0) \\ \downarrow{t_*} & & \downarrow{r} & & \downarrow{r} \\ \pi_k(\boldsymbol{S}(B(\boldsymbol{Z}_{p^r}/H_r)_+)) & \xrightarrow[\cong]{\mu} & \pi_k^{\boldsymbol{Z}_{p^r}/H_r}(E(\boldsymbol{Z}_{p^r}/H_r)_+) & \xrightarrow[\lambda_{H_r}]{} & \pi_k^{\boldsymbol{Z}_{p^r}}(S^0)\,. \end{array}$$

Let $\tilde{\pi}_k^{S^1}(S^0)=\text{Coker}\,[\lambda_{S^1}; \pi_k(S^0)\to\pi_k^{S^1}(S^0)]$ and similarly for $\tilde{\pi}_k^{\boldsymbol{Z}_{p^r}}(S^0)$. Then we have the restriction homomorphism r: $\tilde{\pi}_k^{S^1}(S^0)\to\tilde{\pi}_k^{\boldsymbol{Z}_{p^r}}(S^0)$. Consider the diagram

$$\begin{array}{ccc} \bigoplus_{s<r} \pi_k(\boldsymbol{S}(B(\boldsymbol{Z}_{p^r}/\boldsymbol{Z}_{p^s})_+)) & \xrightarrow{\cong} & \tilde{\pi}_k^{\boldsymbol{Z}_{p^r}}(S^0) \\ \downarrow{\oplus\varphi_s} & & \downarrow{r} \\ \bigoplus_{s<r-1} \pi_k(\boldsymbol{S}(B(\boldsymbol{Z}_{p^{r-1}}/\boldsymbol{Z}_{p^s})_+)) & \xrightarrow[\cong]{} & \pi_k^{\boldsymbol{Z}_{p^{r-1}}}(S^0) \end{array}$$

where $\varphi_{r-1}=0$ and $\varphi_s=t_*$ if $s<r-1$ and t: $\boldsymbol{S}(B\boldsymbol{Z}_{p^{r-s}+})\to \boldsymbol{S}(B\boldsymbol{Z}_{p^{r-s-1}+})$ is the transfer. Then clearly the above diagram is commutative. Then from the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{H\cong \boldsymbol{Z}_{p^a}} \pi_k(\boldsymbol{S}\Sigma^1(B(S^1/H)_+)) & \xrightarrow{\oplus(\lambda_H\circ\mu)} & \tilde{\pi}_k^{S^1}(S^0) \\ \downarrow{\oplus t_*} & & \downarrow{r} \\ \bigoplus_{H\cong \boldsymbol{Z}_{p^a}} \pi_k(\boldsymbol{S}(B(\boldsymbol{Z}_{p^r}/H_r)_+)) & \xrightarrow[\oplus(\lambda_{H_r}\circ\mu)]{} & \pi_k^{\boldsymbol{Z}_{p^r}}(S^0) \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \bigoplus_{H\cong \boldsymbol{Z}_{p^a}} \pi_k(S\Sigma^1(B(S^1/H)_+)) & \xrightarrow{\oplus(\lambda_H\circ\mu)} & \tilde{\pi}_k^{S^1}(S^0) \\ \downarrow{\Pi(\varprojlim t_*)} & & \downarrow{\varprojlim r} \\ \prod_{H\cong \boldsymbol{Z}_{p^a}} \varprojlim \pi_k(\boldsymbol{S}(B(\boldsymbol{Z}_{p^r}/H_r)_+)) & \xrightarrow[\cong]{} & \varprojlim \tilde{\pi}_k^{\boldsymbol{Z}_{p^r}}(S^0)\,. \end{array}$$

Note that $H_r=\boldsymbol{Z}_{p^r}\cap \boldsymbol{Z}_{p^a}$ and $\varinjlim \boldsymbol{Z}_{p^r}/H_r=\boldsymbol{Z}_{p^\infty}/\boldsymbol{Z}_{p^a}\cong \boldsymbol{Z}_{p^\infty}$. Hence by Proposition 3.2, $\varprojlim t_*$ is a p-adic completion. This implies that Im ($\varprojlim r$) is dense in $\varprojlim \tilde{\pi}_k^{\boldsymbol{Z}_{p^r}}(S^0)$, and hence so is for $\varprojlim r$: $\pi_k^{S^1}(S^0)\to\varprojlim \pi_k^{\boldsymbol{Z}_{p^r}}(S^0)$. Let $k=0$, then $\tilde{\pi}_k^{S^1}(S^0)=0$ and hence $\varprojlim \pi_0^{\boldsymbol{Z}_{p^r}}(S^0)=\varprojlim A(\boldsymbol{Z}_{p^r})\cong \boldsymbol{Z}$. This clearly implies that

$\varprojlim A(Z_{p^r})^{\wedge}_I \cong Z$. Then $\alpha\colon Z\to Z$ is clearly an isomorphism. Finally let $k>0$. Then $\pi_k^{Z_{p^r}}(S^0)$ is a finite group and hence the canonical map $\pi_k^{Z_{p^r}}(S^0)\to\pi_k^{Z_{p^r}}(S^0)^{\wedge}_I$ is an epimorphism. Hence so is $\varprojlim \pi_k^{Z_{p^r}}(S^0)\to\varprojlim \pi_k^{Z_{p^r}}(S^0)^{\wedge}_I$. Then from the diagram (D) we see that $\operatorname{Im}\alpha$ is dense. This completes the proof.

As a remark we state the structure of the actual stable cohomotopy group $\tilde{h}^k(BS^1; S)=\{\Sigma^{-k}BS^1, S^0\}$ for $k\geq 0$.

Proposition 4.1. *Let $k\geq 0$, then $\tilde{h}^k(BS^1; S)\cong 0$ if k is even or $k=1$, and $\cong \hat{Z}/Z$ if $k=2i+1$, $i>0$.*

Proof. If k is even then $\varprojlim^1 \tilde{h}^{k-1}((BS^1)^{(r)}; S)=0$ and the result follows from the main theorem. Next consider the following commutative diagram

$$\begin{array}{ccccccccc}
\to\tilde{h}^{2i}(BS^1; S) & \to & \tilde{h}^{2i}(BS^1; K(Z)) & \to & \tilde{h}^{2i+1}(BS^1; \tilde{S}) & \to & \tilde{h}^{2i+1}(BS^1; S) & \to & 0\\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \\
\to\tilde{h}^{2i}(BQ/Z; S) & \to & \tilde{h}^{2i}(BQ/Z; K(Z)) & \to & \tilde{h}^{2i+1}(BQ/Z; \tilde{S}) & \to & \tilde{h}^{2i+1}(BQ/Z; S) & \to & 0.
\end{array}$$

By Proposition 3.1 we see that $j^*\colon \tilde{h}^*(BS^1; \tilde{S})\to\tilde{h}^*(BQ/Z; \tilde{S})$ is an isomorphism. Note that there is no $\varprojlim^1$ for BQ/Z. Then by the Segal conjecture for cyclic groups we see that $\tilde{h}^i(BQ/Z; S)=0$ for $i>0$. Then from the above diagram we immediately see that $\tilde{h}^{-1}(BS^1; S)=0$, and $\tilde{h}^{2i+1}(BS^1; S)\cong\operatorname{Coker}[j^*\colon \tilde{h}^{2i}(BS^1; K(Z))\to\tilde{h}^{2i}(BQ/Z; K(Z))]\cong Z/\tilde{Z}$ if $i>0$.

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