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Cohomology Vanishing Theorems on Weakly 1-Complete Manifolds

By

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§0. Introduction

The purpose of the present article is to give an expository account of the works by S. Nakano, A. Kazama, O. Suzuki, and others, on analytic cohomology groups of weakly 1-complete manifolds.

Let X be a paracompact complex manifold of dimension n, and let E be a holomorphic vector bundle over X. Then, studies on the cohomology groups $H^q(X, \Omega^p(E))$ have significant relationship with function-theoretic and geometric studies of X and E. Here $\Omega^p(E)$ denotes the sheaf of holomorphic p-forms with values in E. For example, the following theorem has fundamental importance in the theory of compact complex manifolds.

Theorem K.N. *If X is compact and E has a metric whose curvature form is Nakano-positive* (*cf. Section* 2), *then*

$$H^q(X, \Omega^n(E))=0, \qquad for \quad q \geqq 1 .$$

Originally Theorem K.N. was proved for line bundles by K. Kodaira [16], and it was generalized by Nakano [18] for vector bundles of arbitrary rank.

Since Theorem K.N. had so many applications, several mathematicians generalized it to non-compact complex manifolds (cf. Andreotti-Vesentini [4], Grauert-Riemenschneider [11]), and in [20] S. Nakano introduced the concept of weakly 1-complete manifold (cf. Section 1) to establish a vanishing theorem for relatively compact weakly 1-complete domains. Afterwards, A. Kazama [15] generalized Nakano's result and gave a vanishing theorem for weakly 1-complete manifolds, and O. Suzuki [28] gave a different proof in the spirit of Kodaira's origianl work.

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Another important example is Grauert's finiteness theorem on strongly pseudoconvex manifolds. Nakano conjectured that it has a relevant generalization to weakly 1-complete manifolds, which was the motivation of the author's works [23], [24], [26]. They shall be explained in the present article, too.

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§1. Preliminaries

1. Weakly 1-Complete Manifolds

Let X be a complex manifold of dimension n. X is said to be weakly 1-complete if there exists a C^∞ function $\varphi: X \to \mathbf{R}$ which is plurisubharmonic and exhaustive. We shall often say that (X, φ) is weakly 1-complete, and set $X_c = \{x \in X: \varphi(x) < c\}$.

Proposition 1.1.1. *Let X and Y be complex manifolds. Assume that there exists a proper holomorphic map $\pi: X \to Y$ and that Y is weakly 1-complete. Then X is weakly 1-complete, too.*

Proof. Let Φ be a C^∞ plurisubharmonic function on Y which is exhaustive. Then $\pi^*\Phi$ is also C^∞, plurisubharmonic, and exhaustive.

Proposition 1.1.2. *Let X be a strongly pseudoconvex manifold, i.e. a complex manifold provided with an exhaustive function of class C^2 which is strictly plurisubharmonic outside a compact subset. Then X is weakly 1-complete.*

Proof. Let ψ be an exhaustion function of X satisfying the above conditions. Then, regularizing ψ if necessary, we may assume that ψ is of class C^∞. Let c be a real number such that ψ is strictly plurisubharmonic on $\{x \in X | \psi(x) > c\}$, and let $\lambda: \mathbf{R} \to \mathbf{R}$ be a C^∞ function such that $\lambda(t) = 0$ for $t \leqq c$, and $\lambda'(t) > 0$, $\lambda''(t) > 0$ for $t > c$. We put $\varphi(x) = \lambda(\psi(x))$. Then, φ is a C^∞, plurisubharmonic, and exhaustive function on X.

We shall give a relevant generalization of the following theroem in Section 4.

Theorem (*Grauert's finiteness theorem.* cf. [10]). *Let X be a strongly* *pseudoconvex manifold and let $\mathscr{F}$ be a coherent analytic sheaf over X. Then, for any $q\geqq 1$, $H^q(X, \mathscr{F})$ is finite dimensional.*

Let us recall the basic terminologies in the theory of complex manifolds.

Let T_X be the tangent bundle to X and let $T_X\otimes\mathbf{C}=T_X^{1,0}\oplus T_X^{0,1}$ be the splitting into the $\pm\sqrt{-1}$ -eigenspaces $T_X^{1,0}$, $T_X^{0,1}$ of the complex structure of T_X. Let σ be a hermitian metric of X, i.e., a C^∞ section of $(T_X^{1,0})^*\otimes(T_X^{0,1})^*$ such that $\bar{\sigma}=\sigma$ and $\sigma(v, \bar{v})>0$ for any $v\in T_X^{1,0}$ with $v\neq 0$. We shall often regard σ as a C^∞ section of $Hom(T_X^{1,0}, (T_X^{0,1})^*)$. Let ω be the image of σ under the natural inclusion $(T_X^{1,0})^*\otimes(T_X^{0,1})^*\hookrightarrow\overset{2}{\wedge}(T_X^*\otimes\mathbf{C})$. Then we say that (X, σ) is Kählerian if ω is a d-closed form. A hermitian metric provides X with a structure of a metric space. (X, σ) is said to be complete if every ball is relatively compact. Here the distance between two points are defined as the infimum of the lengths $\int_0^1\sqrt{2\gamma^*(\sigma)}$ of differentiable curves γ: $[0, 1]\to X$ connecting them.

Proposition 1.1.3. *Let (X, φ) be a weakly 1-complete manifold with a Kähler metric σ. Then X has a complete Kähler metric.*

Proof. Let λ: $\mathbf{R}\to\mathbf{R}$ be a C^∞ convex increasing function such that

$$\int_0^\infty\sqrt{\lambda''(t)}dt=\infty\,. \tag{1}$$

Then the metric

$$\begin{aligned}\sigma_\lambda:&=\sigma+\partial\bar{\partial}\lambda(\varphi)\\&=\sigma+\lambda''(\varphi)\partial\varphi\otimes\bar{\partial}\varphi+\lambda'(\varphi)\partial\bar{\partial}\varphi\end{aligned}$$

is clearly Kählerian. Since φ is exhaustive, the completeness follows from (1).

Since every submanifold of $\mathbf{P}^n$ admits a Kähler metric, weakly 1-complete submanifolds of $\mathbf{P}^n$ admit complete Kähler metrics. In Section 6 we shall take up the problem of projective embeddability of weakly 1-complete manifolds.

2. Cohomology Groups

Let X be a paracompact complex manifold of dimension n, and let $E\to X$ be a holomorphic vector bundle of rank r. We set $C^{p,q}(X)=\{C^\infty(p, q)$-forms on $X\}$, $C^{p,q}(X, E)=\{E$-valued $C^\infty(p, q)$-forms on $X\}$, $C_0^{p,q}(X, E)=\{f\in C^{p,q}(X, E)|$ support of f is compact$\}$, and $L_{loc}^{p,q}(X, E)=\{$locally square integrable E-valued (p, q)-forms$\}$.

We put $W_{loc}^{p,q}(X, E)=\{f\in L_{loc}^{p,q}(X, E)|\bar{\partial}f\in L_{loc}^{p,q+1}(X, E)\}$. Then the correspondence

$$\begin{array}{ccc}\{\text{open sets of } X\} & \xrightarrow{\Gamma} & \{\text{abelian groups}\}\\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ U & \longmapsto & W^{p,q}_{loc}(U, E)\end{array}$$

with natural restriction maps ρ: $W^{p,q}_{loc}(U, E) \to W^{p,q}_{loc}(V, E)$ for $V \subset U$ defines a sheaf $\mathscr{W}^{p,q}(E)$ over X. Thus we have a complex

$$(\sharp) \qquad 0 \longrightarrow \Omega^p(E) \longrightarrow \mathscr{W}^{p,0}(E) \xrightarrow{\bar\partial} \mathscr{W}^{p,1}(E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{W}^{p,n}(E) \longrightarrow 0,$$

where $\Omega^p(E)$ denotes the sheaf of E-valued holomorphic p-forms. The proof of the following theorem can be found in [14], but we shall prove it later under a generalized situation.

Theorem 1.2.1. *($\sharp$) is an exact sequence of sheaves.*

Since $\mathscr{W}^{p,q}(E)$ are fine sheaves (cf. [32]), we have

Corollary 1.2.2.

$$\begin{aligned} &H^q(X, \Omega^p(E)) \\ &\cong \Gamma(X, \bar\partial\mathscr{W}^{p,q-1}(E))/\bar\partial W^{p,q-1}_{loc}(X, E) \\ &= \frac{\{f \in L^{p,q}_{loc}(X, E) \mid \bar\partial f = 0\}}{\{f \in L^{p,q}_{loc}(X, E) \mid \bar\partial g = f \text{ for some } g \in L^{p,q-1}_{loc}(X, E)\}}. \end{aligned}$$

3. Abstract Vanishing Theorem

We shall recall here fundamental lemmas due to Hörmander [14].

Let H_1, H_2 H_3 be three Hilbert spaces with inner products $(,)_1$, $(,)_2$, $(,)_3$, and T: $H_1 \to H_2$, S: $H_2 \to H_3$ be densely defined closed linear operators. We denote by N_S the kernel, by R_S the range, and by D_S the domain of S. We shall always assume that $N_S \supset R_T$. Let T^*, S^* be the adjoints of T, S. Recall that $N_S \perp R_{S^*}$, hence $R_{S^*} \perp R_T$. Furthermore,

Lemma 1.3.1. *Under the above situation, we have the orthogonal decomposition*

$$(2) \qquad H_2 = (N_S \cap N_{T^*}) \oplus \bar{R}_T \oplus \bar{R}_{S^*}.$$

Here, $\bar{R}_T$, $\bar{R}_{S^*}$ denote the closures of R_T, R_{S^*} respectively.

Proof. Clearly, $N_S \cap N_{T^*}$, R_T, R_{S^*} are mutually orthogonal. Let $f \perp R_T$. Then, for any $u \in D_T$, $(Tu, f)_2 = 0$. Hence $f \in N_{T^*}$. If moreover $f \perp R_{S^*}$, then for any $v \in D_{S^*}$, $(S^*v, f) = 0$. Hence $Sf = (S^*)^*f = 0$, so $f \in N_{T^*} \cap N_S$.

Theorem 1.3.2 (*Abstract vanishing theorem*). *Let $f \in N_S$. Assume that there exists a constant C depending on f such that for any $g \in D_S \cap D_{T^*}$,*

(3) $$|(f, g)_2|^2 \leqq C(\|T^*g\|_1^2 + \|Sg\|_3^2).$$

Then there exists u satisfying $Tu=f$ and $\|u\|_1 \leqq C$. Here $\| \ \|_i$ denote the norms in H_i.

Proof. In virtue of Hahn-Banach's theorem and Riesz's representation theorem, we have only to prove that

(4) $$|(f, v)_2|^2 \leqq C\|T^*v\|_1^2, \qquad \text{for any} \quad v \in D_{T^*}.$$

Let us decompose $v \in D_{T^*}$ into the sum $v=v_1+v_2+v_3$, where $v_1 \in N_S \cap N_{T^*}$, $v_2 \in \bar{R}_T$ and $v_3 \in \bar{R}_{S^*}$. Since $f \in N_S$, $(f, v_3)_2=0$. By (3), $(f, v_1)_2=0$. Hence $(f, v)_2=(f, v_2)_2$. Note that $T^*v=T^*v_2$ and that $Sv_2=0$. Thus we have $|(f, v)_2|^2 \leqq C\|T^*v\|_1^2$

Lemma 1.3.3. *Assume that from every sequence $\{g_k\}_{k=1}^{\infty} \subset D_{T^*} \cap D_S \cap \{\|g\|=1\}$ with $\|T^*g_k\| \to 0$ and $\|Sg_k\| \to 0$, one can select a strongly convergent subsequence. Then, $\bar{R}_T=R_T$, $\overline{R_{T^*}}=R_{T^*}$, and $N_S \cap N_{T^*}$ is a finite dimensional vector space.*

Proof. Assume that $R_{T^*} \neq \bar{R}_{T^*}$. Then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset D_{T^*}$ such that $\|u_k\|=1$, $\|T^*u_k\| \to 0$ and $u_k \perp N_{T^*}$. Since $H_2=\bar{R}_T \oplus N_{T^*}$, and $ST=0$, $u_k \in N_S$. Hence, by assumption $\{u_k\}_{k=1}^{\infty}$ has a subsequence $\{u_{k_\nu}\}_{\nu=1}^{\infty}$ which strongly converges to some u. Clearly $\|u\|=1$, $Su=0$ and $u \perp N_{T^*}$. Moreover, for any $f \in D_T$, $(Tf, u)=\lim (Tf, u_k)=\lim (f, T^*u_k)=0$. Therefore $T^*u=0$, which contradicts the fact that $u \neq 0$ and $u \to N_{T^*}$. Thus we have proved that $R_{T^*}=\overline{R_{T^*}}$. Next, assume that $R_T \neq \overline{R_T}$. Then, there exists a sequence $\{v_k\}_{k=1}^{\infty} \subset D_T$ such that $\|v_k\|=1$, $\|Tv_k\| \to 0$, and $v_k \perp N_T$. Since $R_{T^*}=\overline{R_{T^*}}$, we can choose a sequence $\{w_k\} \subset D_T$ so that $v_k=T^*w_k$ and $\|w_k\| \leqq C$ for some constant C. Then we have $0=\lim (Tv_k, w_k)=\lim (TT^*w_k, w_k)=\lim (T^*w_k, T^*w_k)$. Hence $\|v_k\|=\|T^*w_k\| \to 0$, which contradicts that $\|v_k\|=1$. Thus $R_T=\overline{R_T}$. Lastly, by assumption the unit ball in $N_S \cap N_{T^*}$ is compact, hence $N_S \cap N_{T^*}$ should be finite dimensional.

4. Quadratic Forms

Let V be a real vector space of dimension $2n$ with a complex structure J, and let $V \otimes_{\mathbf{R}} \mathbf{C}=V_+ \oplus V_-$ be the decomposition into the eigenspaces V_+, V_- of J for the eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$, respectively. Let $\sigma \in Hom(V_+, \bar{V}_+^*)=V_+^* \otimes V_-^*$ be a hermitian metric of V_+ and let $v_1, \dots, v_n \in V_+^*$ be a basis such that $\sigma=\sum_{i=1}^{n} v_i \otimes \bar{v}_i$. We define the hermitian metrics of $\overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^*$ associated to σ

by the rule that the norms of $v_I \otimes \bar{v}_J$, $I=(i_1,\dots,i_p)$, $J=(j_1,\dots,j_q)$ are 1, where we put $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$. We shall often identify $v_I \otimes \bar{v}_J$ with $v_I \wedge \bar{v}_J$ via the natural inclusion $\overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^* \hookrightarrow \overset{r}{\wedge}(V \otimes_{\mathbf{R}} \mathbf{C})$. We put $G = (\sqrt{-1})^{n^2} v_1 \wedge \cdots \wedge v_n \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_n$. Then G does not depend on the choice of the basis and is left invariant by the complex conjugation. Recalling Laplace's formula for determinants we see that we can define a conjugate linear map $\bar{*}$ from $\overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^*$ to $\overset{n-p}{\wedge} V_+^* \otimes \overset{n-q}{\wedge} V_-^*$ by the rule that $(v_I \wedge \bar{v}_J) \wedge \bar{*}(v_{I'} \wedge \bar{v}_{J'}) = \operatorname{sgn}\begin{pmatrix} I \\ I' \end{pmatrix} \operatorname{sgn}\begin{pmatrix} J \\ J' \end{pmatrix} G$. Here we put $\operatorname{sgn}\begin{pmatrix} i_1 \cdots i_p \\ i'_1 \cdots i'_p \end{pmatrix} = 0$ if $\{i_1,\dots,i_p\} \neq \{i'_1,\dots,i'_p\}$. Note that $\bar{*}1 = G$. Let $f \in \overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^*$. We denote by $e(f)$ the left multiplication by f, and let $L = e(\sqrt{-1}\sigma)$. Let Λ be the adjoint of L. Then we have

Proposition 1.4.1. *For any* $f \in \overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^*$, $\bar{*}(\bar{*}f) = (-1)^{p+q} f$ *and* $\Lambda f = (-1)^{p+q} \bar{*} L \bar{*} f$.

Proof. Immediate from the definition.

Let $i(f)$ denote the adjoint of $e(f)$. Then we have $L = \sqrt{-1} \sum_{k=1}^{n} e(v_k) e(\bar{v}_k)$ and $\Lambda = -\sqrt{-1} \sum_{k=1}^{n} i(\bar{v}_k) i(v_k)$. Noting that $i(v_k)(v_k \wedge v_I \wedge \bar{v}_J) = v_I \wedge \bar{v}_J$ provided that $k \notin I$, we have

Proposition 1.4.2. *For any* $f \in \overset{p}{\wedge} V_+^* \otimes \overset{q}{\wedge} V_-^*$,

$$[L, \Lambda] f = (p+q-n) f, \quad \textit{where} \quad [L, \Lambda] = L\Lambda - \Lambda L.$$

Proof. An easy computation.

Let W be a complex vector space of dimension m with a hermitian metric h, and let Θ be an element of $Hom(V_+, V_-^*) \otimes Hom(W, W) = V_+^* \otimes V_-^* \otimes Hom(W, W)$. Then the multiplication $e(\Theta)$, as well as L and Λ, naturally operates on $\sum_{r=0}^{2n} \overset{r}{\wedge}(V \otimes_{\mathbf{R}} \mathbf{C}) \otimes W$. We put $\tilde{\Theta} = (\sigma^{-1} \otimes id_W)\Theta$. Here we regard $\sigma^{-1} \in Hom(V_-^*, V_+)$. Then, $\tilde{\Theta} \in Hom(V_+ \otimes W, V_+ \otimes W)$. We assume that $\tilde{\Theta}$ is self-adjoint and positive semi-definite. Let γ be the smallest eigenvalue of $\tilde{\Theta}$. Then we have

Proposition 1.4.2′. *For any* $f \in (\overset{n}{\wedge} V_+^*) \otimes V_-^* \otimes W$,

$$\langle \sqrt{-1} e(\Theta) \Lambda f, f \rangle \geqq \gamma \langle f, f \rangle .$$

Here, $\langle\ ,\ \rangle$ *denotes the inner product with respect to* σ *and* h.

Proof. Let $f = \sum_{k=1}^{n} (v_1 \wedge \cdots \wedge v_n) \wedge \bar{v}_k \otimes w_k$, where $w_k \in W$. Then, $e(\Theta)\Lambda f = \sum_{k,l} \Theta_{kl}(w_k) \otimes (v_1 \wedge \cdots \wedge v_n) \wedge \bar{v}_l$, where $\Theta = \sum_{k,l} \Theta_{kl} v_k \wedge \bar{v}_l$, $\Theta_{kl} \in Hom(W, W)$. Hence,

$$\langle \sqrt{-1}e(\Theta)\Lambda f, f\rangle = \sum_{k,l} \langle \Theta_{kl}(w_k), w_l\rangle .$$

On the other hand we have

$$\langle \tilde{\Theta}(\sum v_k^* \otimes w_k), \sum v_k^* \otimes w_k\rangle = \sum \langle \Theta_{kl}(w_k), w_l\rangle , \tag{5}$$

where $v_1^*, \dots, v_n^*$ denotes the dual basis to $v_1, \dots, v_n$. Therefore,

$$\begin{aligned} &\langle \sqrt{-1}e(\Theta)\Lambda f, f\rangle \\ &\geqq \gamma\langle \sum v_k^* \otimes w_k, \sum v_k^* \otimes w_k\rangle \\ &= \gamma\langle f, f\rangle . \end{aligned}$$

Generalizing the above proposition we have

Proposition 1.4.3. *Let γ_q be the supremum of*

$$\inf_{u \in S\otimes W, u\neq 0} \langle \tilde{\Theta}(u), u\rangle / \langle u, u\rangle ,$$

where S runs over $(q-1)$-codimensional linear subspaces of V_+. Then, $\langle \sqrt{-1}e(\Theta)\Lambda f, f\rangle \geqq \gamma_q\langle f, f\rangle$, for any $f \in (\overset{n}{\wedge} V_+^*)\otimes(\overset{q}{\wedge} V_-^*)$.

Proof. Similar as above. For the detail the reader is referred to [26].

Let $\sigma' \in Hom(V_+, V_-^*) = V_+^* \otimes V_-^*$. Assume that $\bar{\sigma}' = \sigma'$ and $\sigma'(v, \bar{v}) \geqq 0$ for any $v \in V_+$. Let γ' be the smallest eigenvalue of $\tilde{\Theta}' := ((\sigma + \sigma')^{-1} \otimes id_W) \circ (\Theta + \sigma' \otimes id_W)$.

Proposition 1.4.4. *Under the above situation, we have*

$$\gamma' \geqq \min(\gamma, 1) .$$

Proof. Given any σ' as above, we can choose $v_1, \dots, v_n \in V_+^*$ so that $\sigma = \sum v_i \otimes \bar{v}_i$ and $\sigma' = \sum \lambda_i v_i \otimes \bar{v}_i$, $\lambda_i \geqq 0$. By (5), we have

$$\begin{aligned} &\langle \tilde{\Theta}'(\sum v_k^* \otimes w_k), \sum v_k^* \otimes w_k\rangle \\ &= \sum_{k,l} \left\langle \Theta_{kl}\left(\sqrt{\frac{1}{1+\lambda_k}} w_k\right), \sqrt{\frac{1}{1+\lambda_l}} w_l \right\rangle + \sum \frac{\lambda_k}{1+\lambda_k} \langle w_k, w_k\rangle , \end{aligned}$$

where the inner product in the left hand side is with respect to $\sigma + \sigma'$. Noting that

$$\begin{aligned} &\sum_{k,l} \left\langle \Theta_{kl}\left(\sqrt{\frac{1}{1+\lambda_k}} w_k\right), \sqrt{\frac{1}{1+\lambda_l}} w_l \right\rangle \\ &\geqq \gamma \sum \frac{1}{1+\lambda_k} \|w_k\|^2 , \end{aligned}$$

we have

$$\begin{aligned}&\langle\tilde{\Theta}'(\sum v_k^*\otimes w_k),\ \sum v_k^*\otimes w_k\rangle\\ &\geqq \sum \frac{\gamma+\lambda_k}{1+\lambda_k}\|w_k\|^2\\ &\geqq \min(1,\gamma)\sum \|w_k\|^2.\end{aligned}$$

Clearly the above propositions are applicable to hermitian vector bundles. In the following sections we apply the above propositions for T_X and E in place of V and W.

§2. A Priori Estimates on Complete Kähler Manifolds

1. Approximation Principle of Andreotti-Vesentini

Let (X, σ) be a hermitian manifold, let (E, h) be a hermitian vector bundle over X, and let $\{e_{ij}\}$ be a system of transition functions of E associated to a trivializing covering $\{U_i\}$. Then h is represented by a system $\{h_i\}$ of hermitian matrix-valued C^∞ functions satisfying $h_i={}^t\bar{e}_{ji}h_je_{ji}$ on $U_i\cap U_j$. Let dv be the volume form with respect to the Riemannian metric $2\,\mathrm{Re}\,\sigma$ on the underlying differentiable manifold X. Then, $dv=\bar{*}1$ and $|f|^2dv={}^tf_i\wedge\bar{*}h_if_i$, where $f=\{f_i\}\in C^{p,q}(X, E)$ and f_i are vectors of (p, q)-forms on U_i satisfying $f_i=e_{ij}f_j$ on $U_i\cap U_j$. Therefore the (formal) adjoint ϑ_h of $\bar{\partial}$ is given by

$$\vartheta_hf=-\bar{*}{}^th_i^{-1}\bar{\partial}\bar{h}_i\bar{*}f.$$

We define a norm $\|\ \|$ in $C_0^{p,q}(X, E)$ by $\|f\|^2=\int_X|f|^2dv$. Let x_0 be a point of X and let $\rho(x)=\mathrm{dist}\,(x_0, x)$, the distance between x_0 and x. Then, by the triangle inequality ρ is a Lipschitz continuous function with Lipschitz constant 1. Let $L^{p,q}(X, E)$ be the completion of $C_0^{p,q}(X, E)$ with respect to $\|\ \|$, and let $\bar{\partial}$: $L^{p,q}(X, E)\to L^{p,q+1}(X, E)$ be the extension of $\bar{\partial}$ with domain $D_{\bar{\partial}}^{p,q}=\{f\in L^{p,q}(X, E)|\bar{\partial}f\in L^{p,q+1}(X, E)\}$. Here $\bar{\partial}f\in L^{p,q+1}(X, E)$ should read "there exists $u\in L^{p,q+1}(X, E)$ such that $(u, \varphi)=(f, \vartheta_h\varphi)$ for any $\varphi\in C_0^{p,q+1}(X, E)$". Then, recalling the usual regularization method we see that, for any $f\in D_{\bar{\partial}}^{p,q}$ one can find a sequence $\{\varphi\}_{k=1}^\infty\subset C_0^{p,q}(X, E)$ such that on any compact subset $K\subset X$, φ_k and $\bar{\partial}\varphi_k$ strongly converge to f and $\bar{\partial}f$, respectively. Thus, regularizing $\{\rho(x/r)\varphi_{k_r}\}_{r=1}^\infty$ $(k_1\ll k_2\ll\cdots)$ again, we obtain the following

Proposition 2.1.1. *If (X, σ) is a complete hermitian manifold, then $C_0^{p,q}(X, E)$ is dense in $D_{\bar{\partial}}^{p,q}$ with respect to the norm $\|u\|+\|\bar{\partial}u\|$.*

Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}: L^{p,q}(X, E) \to L^{p,q+1}(X, E)$. Then, similarly we have

Proposition 2.1.2. *If (X, σ) is a complete hermitian manifold, then $C_0^{p,q}(X, E)$ is dense in $D_{\bar{\partial}^*}^{p,q}$ with respect to the norm $\|u\| + \|\bar{\partial}^* u\|$. Moreover, $C_0^{p,q}(X, E)$ is dense in $D_{\bar{\partial}}^{p,q} \cap D_{\bar{\partial}^*}^{p,q}$ with respect to the norm $\|u\| + \|\bar{\partial} u\| + \|\bar{\partial}^* u\|$.*

For the detail of the proof, the reader is referred to [5]. We shall call Proposition 2.1.1 and Proposition 2.1.2 the approximation principle.

When we need to indicate σ and h, we denote $\|f\|_{h,\sigma}$, $L^{p,q}(X, E, h, \sigma)$, etc.

2. A Priori Estimates

Let the notations be as above. We set $\Theta_i = -\bar{\partial}(h_i^{-1} \partial h_i)$. Then $\{\Theta_i\}$ defines an element Θ_h of $C^{1,1}(X, Hom(E, E))$. Θ_h is called the curvature form of h.

Proposition 2.2.1. *Let (X, σ) be a Kähler manifold and let (E, h) be a hermitian vector bundle over X. Then we have*

$$\begin{aligned} \|\bar{\partial} f\|^2 + \|\vartheta_h f\|^2 \\ \geqq (\sqrt{-1}[e(\Theta_h), \Lambda] f, f), \end{aligned} \tag{6}$$

for any $f \in C_0^{p,q}(X, E)$.

Proof. We put $\bar{\vartheta} := -\bar{*}\partial\bar{*}: C^{p,q}(X, E) \to C^{p-1,q}(X, E)$. Let ∂_h be the adjoint of $\bar{\partial}$ with respect to σ and h. Then we have $(\partial_h f)_i = h_i^{-1} \partial(h_i f_i)$, $[\Lambda, \bar{\partial}] = \sqrt{-1}\,\bar{\vartheta}$, and $[\Lambda, \partial_h] = -\sqrt{-1}\,\vartheta_h$. Hence we have

$$\begin{aligned} &\bar{\partial}\vartheta_h + \vartheta_h \bar{\partial} \\ &= \bar{\partial}(\sqrt{-1}[\Lambda, \partial_h]) + (\sqrt{-1}[\Lambda, \partial_h])\bar{\partial} \\ &= \sqrt{-1}[\bar{\partial}, \Lambda]\partial_h + \sqrt{-1}\Lambda\bar{\partial}\partial_h - \sqrt{-1}\bar{\partial}\partial_h\Lambda \\ &\quad + \sqrt{-1}\partial_h[\bar{\partial}, \Lambda] - \sqrt{-1}\partial_h\bar{\partial}\Lambda + \sqrt{-1}\Lambda\partial_h\bar{\partial} \\ &= \vartheta\partial_h + \partial_h\vartheta + [-\sqrt{-1}(\bar{\partial}\partial_h + \partial_h\bar{\partial}), \Lambda], \end{aligned}$$

and

$$\begin{aligned} &-\sqrt{-1}(\bar{\partial}\partial_h + \partial_h\bar{\partial})f \\ &= \sqrt{-1}(-\bar{\partial}h_i^{-1}\partial(h_i f_i) - h_i^{-1}\partial(h_i\bar{\partial}f_i)) \\ &= \sqrt{-1}(-\bar{\partial}\partial f_i - \bar{\partial}(h_i^{-1}\partial h_i f_i) - \partial\bar{\partial}f_i - (h_i^{-1}\partial h_i)\bar{\partial}f_i) \\ &= e(\Theta_h)f. \end{aligned}$$

Thus we obtain

$$\begin{aligned}&(\bar\partial f, \bar\partial f)+(\vartheta_h f, \vartheta_h f)\\&\quad=(\bar\vartheta f, \bar\vartheta f)+(\partial_h f, \partial_h f)+(\sqrt{-1}[e(\Theta_h), \Lambda]f, f)\\&\quad\geqq(\sqrt{-1}[e(\Theta_h), \Lambda]f, f),\end{aligned}$$

for $f\in C_0^{p,q}(X, E)$.

By the approximation principle we have

Proposition 2.2.2. *If (X, φ) is complete and Kählerian, then for any hermitian bundle (E, h) over X,*

$$\begin{aligned}&\|\bar\partial f\|^2+\|\bar\partial^* f\|^2\\&\quad\geqq(\sqrt{-1}[e(\Theta_h), \Lambda]f, f), \qquad \text{for} \quad f\in D_{\bar\partial}^{p,q}\cap D_{\bar\partial^*}^{p,q}.\end{aligned}$$

Combining Proposition 2.2.2 with Abstract vanishing theorem (Theorem 1.3.2), we obtain

Theorem 2.2.3. *Let (X, σ) be a complete Kähler manifold, and let (E, h) be a hermitian vector bundle over X. Assume that for some (p, q) we have*

$$(\sqrt{-1}[e(\Theta_h), \Lambda]f, f)\geqq(c(x)f, f), \qquad \text{for any} \quad f\in C_0^{p,q}(X, E),$$

where $c(x)$ is a positive continuous function on X. Then, for any $g\in L^{p,q}(X, E)$ satisfying $\bar\partial g=0$ and $\int_X c(x)^{-1}|g|^2dv<\infty$, we can find $u\in L^{p,q-1}(X, E)$ **such** *that $\bar\partial u=g$ and $\|u\|^2\leqq\int_X c(x)^{-1}|g|^2dv$.*

Let the smallest eigenvalue of $(\sigma^{-1}\otimes id_E)\Theta_h$ at $x\in X$ be $\gamma_h(x)$. Clearly $(\sigma^{-1}\otimes id_E)\Theta_h$ is self-adjoint. Then γ_h is a continuous function on X. (E, h) is said to be Nakano-positive if $\gamma_h>0$ everywhere. There is another notion of positivity due to Griffiths [12]. They agree when $r=1$ and coincides with the classical notion of positivity due to Kodaira [16], so we say simply 'positive' for line bundles. Note that Nakano-positivity does not depend on the choice of σ, so that we can say "(E, h) is Nakano-positive". We say Θ_h is Nakano-positive at x if $\gamma_h(x)>0$.

Theorem 2.2.4. *If (E, h) is a Nakano-positive bundle over a complete Kähler manifold (X, σ), then for any $g\in L^{n,q}(X, E)$, $q\geqq 1$, satisfying $\bar\partial g=0$ and $\int_X \gamma_h^{-1}|g|^2dv<\infty$, we can find $u\in L^{n,q-1}(X, E)$ such that $\bar\partial u=g$ and $\|u\|^2\leqq\int_X \gamma_h^{-1}|g|^2dv$.*

Proof. Immediate from Proposition 1.4.3.

§3. Vanishing Theorems on Weakly 1-Complete Manifolds

Let (X, φ) be a weakly 1-complete manifold of dimension n with a Kähler metric σ, and (E, h) a hermitian bundle over X. Let the notations γ_h, Θ_h, etc. be as in Section 2.

Lemma 3.1. *For any C^∞ convex increasing function λ, $\gamma_h \leqq \gamma_{h\exp(-\lambda(\varphi))}$.*

Proof. Immediate from the definition.

Lemma 3.2. *For any positive continuous function μ: $X \to \mathbf{R}$, we can find a C^∞ convex increasing function λ: $\mathbf{R} \to \mathbf{R}$ satisfying $\int_X e^{-\lambda(\varphi)}\mu dv < \infty$.*

Proof. Trivial.

From these two Lemmas we obtain

Proposition 3.3. *Assume that (E, h) is Nakano-positive. Then, for any $g \in L^{p,q}_{loc}(X, E)$, there exists a convex increasing C^∞ function λ: $\mathbf{R} \to \mathbf{R}$ such that $\int_X \gamma_{h\exp(-\lambda(\varphi))}{}^{-1} e^{-\lambda(\varphi)}|g|^2 dv < \infty$.*

Since $e^{-\lambda(\varphi)/2}|g|$ is the length of g with respect to σ and $he^{-\lambda(\varphi)}$, Theorem 2.2.4 implies now immediately the following

Theorem 3.4. *Let X be a weakly 1-complete Kähler manifold of dimension n, and let (E, h) be a Nakano-positive bundle over X. Then, for any $g \in L^{n,q}_{loc}(X, E)$, $q \geqq 1$, satisfying $\bar\partial g = 0$, there exist $u \in L^{n,q-1}_{loc}(X, E)$ such that $\bar\partial u = g$.*

Remark 3.5. If (E, h) is Nakano-positive, then the line bundle $(\det E, \det h)$ is also positive, so that $\Theta_{\det h}$ defines a Kähler metric on X. Thus Kähler-condition is implicit in the positivity assumption of (E, h).

The ball $\mathbf{B}^n = \{z \in \mathbf{C}^n \mid \|z\| < 1\}$ is weakly 1-complete with respect to $\varphi = -\log(1 - \|z\|^2)$. Moreover the trivial bundle over $\mathbf{B}^n$ is clearly positive. Thus we have proved Theorem 1.2.1, and hence Theorem 3.4 implies the following theorem which is first due to Kazama [15] (cf. also Nakano [20] and Suzuki [28]).

Theorem 3.6. *Let X be a weakly 1-complete manifold of dimension n, and let (E, h) be a Nakano-positive bundle. Then,*

$$H^q(X, \Omega^n(E)) = 0, \qquad \text{for} \quad q \geqq 1.$$

For positive line bundles we can say more.

Theorem 3.7 (*Nakano* [21]). *Let (X, φ) be a weakly 1-complete manifold of dimension n, and let (B, a) be a positive line bundle over X. Then,*

$$H^q(X, \Omega^p(B))=0, \quad \text{when} \quad p+q>n.$$

Proof. First we prepare sublemmas.

Sublemma 1. *Let $\mu(t)$ be a continuous function on $\mathbf{R}$. Then there exists an entire analytic function $f: \mathbf{C} \to \mathbf{C}$ such that f is real valued on $\mathbf{R}$ and $f(t)>\mu(t)$.*

Proof. Choose a sequence $\{\mu_k\}_{k=0}^{\infty}$ of integers such that $\mu_k>k$ and $t^{\mu_k}>2^{(k-1)\mu_k}\mu(t)$, for $2^k \leqq t \leqq 2^{k+1}$. Then the power series $\sum_{k=0}^{\infty} 2^{(1-k)\mu_k} z^{\mu_k} + \sup_{-1 \leqq t \leqq 1} \mu(t)$ defines an entire function f satisfying the requirement.

Sublemma 2. *Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of positive real numbers. Assume that there exists an integer m such that $\{c_k\}_{k \geqq m}$ is monotonically decreasing and that $\lim c_k^{1/k}=0$. Then, $nc_n \leqq \sum_{k=0}^{n-1} c_k c_{n-k-1}$, for $n \gg 0$.*

Proof. Easy.

Note that for any entire function f we have

$$|f(t)| \leqq \sum_{k \geqq 0} \frac{|f^{(k)}(0)|}{k!} t^k \leqq \sum_{k \geqq 0} c_k t^k \qquad \text{for} \quad t>0.$$

Here we set

$$c_k = \sup_{m \geqq k} \left(m\sqrt{\frac{f^{(m)}(0)}{m!}} \right)^k.$$

Thus, combining these two sublemmas we obtain

Sublemma 3. *For any continuous function $\mu(t)$ on $\mathbf{R}$, we can find a convex increasing C^∞ function f on $\mathbf{R}$ such that $f(t)>\mu(t)$ for $t>0$, $(f(t))^2>f'(t)$ on (K, ∞), and $(f(t))^4>f''(t)$ on (K, ∞), where K is a positive number depending on $\mu(t)$.*

Returning to the proof of Theorem 3.7, let $f \in L_{loc}^{p,q}(X, B)$, $p+q>n$, and $\bar{\partial} f=0$. We put $\tilde{a}=a \exp(-\varphi^2)$. Then $\Theta_{\tilde{a}}=\Theta_a+2(\partial\varphi \otimes \bar{\partial}\varphi+\partial\bar{\partial}\varphi)$ gives a complete Kaehler metric $\tilde{\sigma}$ on X. Let $d\tilde{v}$ be the associated volume form, and fix a continuous function $\rho(t)$ on $\mathbf{R}$ such that $\int_X e^{-\rho(\varphi)}|f|_{\tilde{a},\tilde{\sigma}}^2 d\tilde{v}<\infty$. By Sublemma 3, we can find a constant K and a C^∞ convex increasing function λ: $\mathbf{R} \to \mathbf{R}$ such that $\lambda(t)>2\rho(t)$ for $t>0$, $(\lambda(t))^2>\lambda'(t)$ on (K, ∞), and $(\lambda(t))^4>\lambda''(t)$

on (K, ∞). We put $\sigma_\lambda=\tilde{\sigma}+\partial\bar{\partial}\lambda(\varphi)$, $a_\lambda=\tilde{a}\exp(-\lambda(\varphi))$, and $dv_\lambda=$the volume form with respect to σ_λ. Then we have

$$dv_\lambda=\prod_{i=1}^{n}(1+\lambda_i)d\tilde{v}\,.$$

Here λ_i denote the eigenvalues of $\partial\bar{\partial}\lambda(\varphi)$ with respect to $\tilde{\sigma}$. Since $\partial\bar{\partial}\lambda(\varphi)=\lambda''(\varphi)\partial\varphi\otimes\bar{\partial}\varphi+\lambda'(\varphi)\partial\bar{\partial}\varphi$, noting that the eigenvalues of $\partial\varphi\otimes\bar{\partial}\varphi$ and $\partial\bar{\partial}\varphi$ with respect to $\tilde{\sigma}$ are bounded, we obtain an estimate:

$$\prod_{i=1}^{n}(1+\lambda_i)\leqq C_0(\lambda'(\varphi)+\lambda''(\varphi))^n,$$

for some constant C_0. Hence $\prod_{i=1}^{n}(1+\lambda_i)\leqq C_1(\lambda(\varphi))^{4n}$, since $(\lambda(\varphi))^2>\lambda'(\varphi)$ and $(\lambda(\varphi))^4>\lambda''(\varphi)$ outside a compact subset of X, where C_1 is a constant. Therefore,

$$\begin{aligned}&\int_X|f|^2_{a_\lambda,\sigma_\lambda}dv_\lambda\\ &\leqq\int_X e^{-\lambda(\varphi)}|f|^2_{\tilde{a},\tilde{\sigma}}d\tilde{v}^{*)}\\ &\leqq\int_X(e^{-\rho(\varphi)}|f|^2_{\tilde{a},\tilde{\sigma}})(e^{-\lambda(\varphi)/2}\prod_{i=1}^{n}(1+\lambda_i))d\tilde{v}<\infty.\end{aligned}$$

Thus we obtain $f\in L^{p,q}(X, B, a_\lambda, \sigma_\lambda)$.

On the other hand, for any $g\in C_0^{p,q}(X, B)$ we have

$$\begin{aligned}&(\sqrt{-1}[e(\Theta_{a_\lambda}), \Lambda_{\sigma_\lambda}]g, g)_{a_\lambda,\sigma_\lambda}\\ &=([L_{\sigma_\lambda}, \Lambda_{\sigma_\lambda}]g, g)_{a_\lambda,\sigma_\lambda}\\ &=(p+q-n)(g, g)_{a_\lambda,\sigma_\lambda}\geqq\|g\|^2_{a_\lambda,\sigma_\lambda},\end{aligned}$$

when $p+q>n$.

Thus, in virtue of Theorem 2.2.3, we can find $u\in L^{p,q}(X, B, a_\lambda, \sigma_\lambda)$ such that $\bar{\partial}u=f$.

Remark. Note that the existence of the exhaustion function φ is crucial. For example, $\mathbf{C}^2\backslash\{0\}$ has a complete Kähler metric but $H^1(\mathbf{C}^2\backslash\{0\}, \Omega^2_{\mathbf{C}^2\backslash\{0\}})$ does not vanish.

§4. Finite-Dimensionality Theorems

Since every proper modification of a weakly 1-complete manifold is again weakly 1-complete, the following theorems would be of some interest.

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*) Since $\sigma_\lambda\geqq\tilde{\sigma}$, $|v|_{\tilde{\sigma}}\geqq|v|_{\sigma_\lambda}$ for any $v\in T_X^*\otimes\mathbf{C}$.

Theorem 4.1 (*Nakano-Rhai* [22]). *Let* (X, φ) *be a weakly* 1-*complete manifold of dimension* n, *and let* (E, h) *be a hermitian bundle over* X *whose curvature form is Nakano-positive outside a compact subset of* X. *Then* $H^q(X, \Omega^n(E))$ *is finite dimensional for* $q \geqq 1$.

Theorem 4.2 (*Ohsawa* [24], [26]). *Let* (X, φ) *be a weakly* 1-*complete manifold of dimension* n, *and let* (B, a) *be a hermitian line bundle over* X *whose curvature form is positive outside a compact subset of* X. *Then* $H^q(X, \Omega^p(B))$ *is finite dimensional when* $p+q>n$.

In fact, they are relevant generalizations of Grauert's finiteness theorem. We shall only prove Theorem 4.1, the proof of Theorem 4.2 being similar in the spirit.

Proof of Theorem 4.1. Fix $c \in \mathbf{R}$ such that $X_c \supset K$, and let K_1 be a compact subset of X_c containing K in its interior. We put $h_c = h(c-\varphi)$ and fix a hermitian metric σ_c on X_c such that $\sigma_c = \Theta_{\det h} + \partial\bar{\partial}(-\log(c-\varphi))$ on $X_c \backslash K_1$. Replacing φ by $c+(\varphi-c)\varepsilon$, $0<\varepsilon \ll 1$, if necessary, we may assume that $X_{c-1} \supset K_1$. We have as in Section 2 the following estimate:

$$\|\bar{\partial}f\|_c^2 + \|\bar{\partial}^* f\|_c^2 \geqq \gamma_0 \|f\|_c^2, \qquad \text{for} \quad f \in C_0^{n,q}(X_c \backslash K_1, E),\ q \geqq 1. \tag{7}$$

Here, the norm $\| \ \|_c$, $\bar{\partial}^*$ are with respect to (h_c, σ_c), and γ_0 denotes the infimum on $X_c \backslash K_1$ of the eigenvalues of $(\sigma_c^{-1} \otimes id_E)\Theta_{h_c}$. By Proposition 1.4.4 it is clear that $\gamma_0 > 0$. Applying (7) to ρf, where $f \in C_0^{n,q}(X_c, E)$, ρ is a C^∞ function such that supp $\rho \Subset X_c$ and $\rho = 0$ on a neighbourhood of K_1, we have

$$C_1\left\{\|\bar{\partial}f\|_c^2 + \|\bar{\partial}^* f\|_c^2 + \int_{K_2} |f|_c^2 dv_c\right\} \geqq \|f\|_c^2, \qquad \text{for} \quad f \in C_0^{n,q}(X_c, E). \tag{8}$$

Here, C_1 is a constant and K_2 is a compact subset of X_c containing K_1. The hermitian metric σ_c is complete, as we can see it from the inequality $\sigma_c \geqq (c-\varphi)^{-2}\partial\varphi \otimes \bar{\partial}\varphi$. Hence by the approximation principle we have

$$C_1\left\{\|\bar{\partial}f\|_c^2 + \|\bar{\partial}^* f\|_c^2 + \int_{K_2} |f|_c^2 dv_c\right\} \geqq \|f\|_c^2, \qquad \text{for} \quad f \in D_{\bar{\partial}}^{n,q} \cap D_{\bar{\partial}^*}^{n,q}. \tag{9}$$

By strong ellipticity of $\bar{\partial}\vartheta_h + \vartheta_h\bar{\partial}$, we can apply Garding's inequality for the elements of $C_0^{p,q}(X, E)$ (cf. [17]). Hence, by a regularization argument we obtain that for any sequence $\{f_k\} \subset L^{n,q}(X_c, E, h_c, \sigma_c)$ satisfying $\|\bar{\partial}f_k\|_c \to 0$, $\|\bar{\partial}^* f_k\|_c \to 0$, $\|f_k\|_c = 1$, and for any $d<c$, 1-st order derivatives of f_k are bounded on X_d in L^2-sense. Therefore, by Rellich's lemma, we can find subsequence $\Gamma \subset \{f_k\}$ converging strongly on K_2. Moreover, by (9), Γ converges strongly on

X_c. Therefore, by Theorem 1.3.3, $R_{\bar\partial^*}^{n,q}$ is closed for $q \geqq 0$ and $H_c^{n,q} := \{\bar\partial f = 0, \bar\partial^* f = 0\}$ is finite dimensional for $q \geqq 1$.

Next, let $\{\lambda_k\}_{k\geqq 1}$ be a sequence of C^∞ convex increasing functions such that $\lambda_k(t) = -\log(c-t)$ for $t < c - \frac{1}{k}$, $\lambda_k'(t) < \frac{1}{c-t}$, $\lambda_k''(t) < \frac{1}{(c-t)^2}$, for $t < c$, and that $\int_0^\infty \lambda_k''(t)dt = \infty$. We fix hermitian metrics σ_k on X such that $\sigma_k = \sigma_c$ on K_1 and $\sigma_k = \Theta_{\det h} + \partial\bar\partial\lambda_k(\varphi)$ on $X \backslash K_1$. Then, σ_k are complete metrics on X and we have the following estimates for $f \in C_0^{n,q}(X \backslash K_1, E)$, $q \geqq 1$:

$$\|\bar\partial f\|_k^2 + \|\bar\partial^* f\|_k^2 \geqq (\sqrt{-1}e(\Theta_{h_k})\Lambda_k f, f)_k, \tag{10}$$

where we put $h_k = he^{-\lambda_k(\varphi)}$, and $\|\ \|_k$, $\bar\partial^*$, etc. are with respect to (h_k, σ_k). Thus, similarly as in the case of (h_c, σ_c), we have

$$\|\bar\partial f\|_k^2 + \|\bar\partial^* f\|_k^2 = \int_{K_2} |f|_k^2 dv_k \geqq (\gamma f, f)_k, \qquad \text{for} \quad f \in D_{\bar\partial}^{n,q} \cap D_{\bar\partial^*}^{n,q},\ q \geqq 1. \tag{11}$$

Here, γ is a positive continuous function on X. By Proposition 1.4.4, γ can be chosen to be independent of the choice of $\{\lambda_k\}$.

Sublemma. *For any $f \in L^{n,q}(X, E, h_k, \sigma_k)$, $q \geqq 0$, we have $\|f\|_k \geqq \|f|_{X_c}\|_c$.*

Proof. Immediate from the inequalities $\sigma_k \leqq \sigma_c$ and $h_k \geqq h_c$.

Assertion. *There exist an integer k_0 and a constant C_2 such that for any $k \geqq k_0$ we have*

$$C_2\{\|\bar\partial f\|_k^2 + \|\bar\partial^* f\|_k^2\} \geqq (\gamma f, f)_k, \tag{12}$$

$$\text{for} \quad f \in D_{\bar\partial}^{n,q} \cap D_{\bar\partial^*}^{n,q}\ (q \geqq 1) \quad \text{which satisfy} \quad f|_{X_c} \perp H_c^{n,q}.$$

Proof. Assume the contrary. Then we have a sequence $\{f_k\}_{k=1}^\infty$ such that $f_k \in L^{n,q}(X, E, h_k, \sigma_k)$, $\|\bar\partial f_k\|_k \to 0$, $\|\bar\partial^* f_k\|_k \to 0$, $(\gamma f_k, f_k)_k = 1$, and that $f_k|_{X_c} \perp H_c^{n,q}$. From (11), as before we can choose a subsequence $\mathscr{S} \subset \{f_k\}$ converging strongly on K_2 to a non-zero form. Choose a subsequence $\{f_{k_i}\} \subset \mathscr{S}$ such that $\{f_{k_i}|_{X_c}\}$ converges weakly in $L^{n,q}(X_c, E, h_c, \sigma_c)$ (cf. Sublemma). Let the weak limit be f. Then $f \neq 0$ and $f \perp H_c^{n,q}$. But, since $\|\bar\partial f_k\|_k \geqq \|\bar\partial f_k|_{X_c}\|_c$, we have $\bar\partial f = 0$, and moreover $(f, \bar\partial u)_c = \lim(f_k, \bar\partial u)_k = \lim(\bar\partial^* f_k, u)_k = 0$ for any $u \in C_0^{n,q-1}(X_c, E)$, so that $\bar\partial^* f = 0$. It is a contradiction.

Thus, by Theorem 1.3.2, for any $g \in N_{\bar\partial}^{n,q} \cap L^{n,q}(X, E, h_k, \sigma_k)$ $(q \geqq 1)$, satisfying $g|_{X_c} \perp H_c^{n,q}$ and $\int_X \gamma^{-1}\langle g, g\rangle_k dv_k < \infty$ for some $k \geqq k_0$, we can find $u \in L^{n,q-1}(X, E, h_k, \sigma_k)$ such that $\bar\partial u = g$. Since the growth of λ_k outside $(-\infty, c)$ can be chosen to be arbitrarily rapid, we conclude that for any $g \in L_{loc}^{n,q}(X, E)$,

$q \geqq 1$, satisfying $g|_{X_c} \perp H_c^{n,q}$ and $\bar{\partial} g = 0$, we can find $u \in L_{loc}^{n,q-1}(X, E)$ satisfying $\bar{\partial} u = g$.

Thus we have proved that the composite of restriction and harmonic projection $H^q(X, \Omega^n(E)) \to H_c^{n,q}$ is injective for $q \geqq 1$. Since $H_c^{n,q}$ is finite-dimensional as we have proved earlier, $H^q(X, \Omega^n(E))$ is also finite-dimensional.

§5. Other Results

Originally, Theorem 4.1 was proved via the following two theorems which are interesting themselves.

Theorem 5.1. *Let (X, φ) be a weakly 1-complete manifold of dimension n, and let (E, h) be a hermitian vector bundle over X whose curvature form is Nakano-positive outside a compact subset $K \subset X$. Then, for any X_c which contains K, the natural restriction maps*

$$\rho_c \colon H^q(X, \Omega^n(E)) \longrightarrow H^q(X_c, \Omega^n(E))$$

have dense images for $q \geqq 0$. Here the topology of $H^q(X_c, \Omega^n(E))$ is induced from $L_{loc}^{n,q}(X_c, E)$.

Sketch of Proof. Let $X_c \Supset X_{c'} \supset K$ and let $u \in L^{n,q}(X_{c'}, E, h_{c'}, \sigma_{c'})$ with $u \perp \{f|_{X_{c'}} | f \in L_{loc}^{n,q}(X, E), \bar{\partial} f = 0\}$. Then, if we extend u outside $X_{c'}$ by 0 and define $u_k \in L^{n,q}(X, E, h_k, \sigma_k)$ by "$(u, v'|_{X_{c'}})_{c'} = (u_k, v')_k$ for any $v' \in L^{n,q}(X, E, h_k, \sigma_k)$", then we have $u_k = \bar{\partial}^* v_k$ for some V_k with $\|v_k\|_k \lesssim \|u_k\|_k \leqq \|u\|_{c'}$. Let the weak limit of a subsequence of $\{v_k|_{X_{c'}}\}$ be v. Then we have $u = \bar{\partial}^* v$, so that $u \perp N^{n,q}$.

Since $H^q(X_c, \Omega^n(E))$ are finite-dimensional for $q \geqq 1$, we have thus proved that ρ_c are surjective for $q \geqq 1$. When $q = 0$, Theorem 5.1 amounts to a generalization of the classical theorem of Runge. By the same argument we can prove that the map $H^q(X, \Omega^n(E)) \to H_c^{n,q}$ have dense image for $q \geqq 0$. Hence, for $q \geqq 1$, $H^q(X, \Omega^n(E))$ is isomorphic to $H_c^{n,q}$. Thus we obtain

Theorem 5.2. *Let the situation be as above. Then, the natural restriction maps ρ_c are isomorphisms for $q \geqq 1$.*

Similarly we have

Theorem 5.3 (cf. [24]). *Let the situation be as above, and let the rank of E be 1. Then the maps $\rho_c \colon H^q(X, \Omega^p(E)) \to H^q(X_c, \Omega^p(E))$ have dense images for $p+q \geqq n$, and are isomorphisms for $p+q > n$.*

§ 6. Applications

Let M be a complex manifold and let $D\subset M$ be a relatively compact domain with C^2-smooth boundary ∂D. Let ψ be a function of class C^2 on M such that $D=\{x\in M|\psi(x)<0\}$ and that $d\psi\neq 0$ everywhere on ∂D. Then ∂D is said to be pseudoconvex (strongly pseudoconvex) if $\partial\bar{\partial}\psi(\xi, \bar{\xi})\geqq 0$ (>0) for any $\xi\in T^{1,0}_{M,x}\backslash\{0\}$, $x\in\partial D$, satisfying $\partial\psi(\xi)=0$.

Theorem 6.1 (*Grauert* [10]). *If D is a domain with strongly pseudoconvex boundary, then D is holomorphically convex.*

As a corollary we have

Theorem 6.2. *If X is a strongly pseudoconvex manifold, then X is holomorphically convex.*

Proof. Let Φ be an exhaustion function which is strictly plurisubharmonic outside a compact subset of X. Then we can choose by Sard's theorem an increasing sequence of real numbers $\{c_i\}_{i=1}^{\infty}$ such that c_i are non-critical values of Φ and that $\{\Phi=c_i\}$ are strongly pseudoconvex. By Theorem 6.1, $D_i:=\{\Phi<c_i\}$ are holomorphically convex. Hence, for any $i<j$, (D_i, D_j) is a Runge-pair, whence follows immediately the holomorph-convexity of $X=\bigcup_{i=1} D_i$.

We shall give a direct proof of Theorem 6.2 as an application of Theorem 4.1. Since every domain with strongly pseudoconvex boundary is a strongly pseudoconvex manifold, (cf. [13]), Theorem 6.1 is then a special case of Theorem 6.2, and we need no approximation theorem of Runge type.

Quadratic transformation: Let X be a complex manifold of dimension n, and let $x\in X$ be any point. Then, there exists a unique complex manifold Q_xX which have the following properties (cf. [32]).

i) There is a proper holomorphic map $\pi_x\colon Q_xX\to X$ which is one to one outside $\pi_x^{-1}(x)$.
ii) $\pi_x^{-1}(x)\cong\mathbf{P}^{n-1}$.

Q_xX is called the quadratic transform of X centered at x. Let $[\pi_x^{-1}(x)]$ be the line bundle associated to the divisor $\pi_x^{-1}(x)$. Then, $[\pi_x^{-1}(x)]|_{\pi_x^{-1}(x)}\cong H_{n-1}^{-1}$, where H_{n-1} denotes the hyperplane bundle over $\mathbf{P}^{n-1}$. For distinct points $x, y\in X$, we have $Q_{\pi_y^{-1}(x)}(Q_yX)\cong Q_{\pi_x^{-1}(y)}(Q_xX)$. Thus we can put $Q_{\{x,y\}}X:=$

$Q_{\pi_x^{-1}(y)}(Q_xX)$. Similarly we define Q_IX and $\pi_I\colon Q_IX\to X$ for any discrete set of points $I\subset X$.

Lemma 6.3. $$(\overset{n}{\wedge} T^{1,0}_{Q_xX})|_{\pi_x^{-1}(x)}\cong H^{n-1}_{n-1}.$$

Proof. See [32].

Proof of Theorem 6.2. Let (X, φ) be a non-compact strongly pseudoconvex manifold of dimension n with a C^∞ plurisubharmonic exhaustion function. Let $I=\{x_i\}_{i=1}^\infty\subset X$ be any discrete sequence. Then, by Lemma 6.3, the line bundle $(\overset{n}{\wedge} T^{1,0}_{Q_IX})\otimes\pi_I^{-1}(I)]^*$ admits a hermitian metric a whose curvature form Θ satisfies $\Theta(v, \bar v)>0$ for any $v\in T^{1,0}_X$ which are tangent to $\pi_I^{-1}(I)$. Hence we can find a C^∞ convex increasing function $\lambda\colon \mathbf{R}\to\mathbf{R}$ such that the curvature form of $a\exp(-\lambda(\pi_I^*\varphi))$ is positive outside a compact subset of Q_IX. Thus, $H^1(Q_IX, \mathcal{O}([\pi_I^{-1}(I)]^*))=H^1(Q_IX, \Omega^n(\overset{n}{\wedge} T^{1,0}_X\otimes[\pi_I^{-1}(I)]^*))$ is finite dimensional. For the structure sheaf $\mathcal{O}_{Q_IX}$ we have the following exact sequence:

$$\begin{array}{l}\Gamma(X, \mathcal{O}_X)\\ \quad\| \\ \Gamma(Q_IX, \mathcal{O}_{Q_IX})\longrightarrow\Gamma(I, \mathbf{C})\longrightarrow H^1(Q_IX, \mathcal{O}([\pi^{-1}(I)]^*)).\end{array}$$

Since $H^1(Q_IX, [\pi^{-1}(I)]^*)$ is finite dimensional, by the above exact sequence we can find $f\in\Gamma(X, \mathcal{O}_X)$ such that $|f(x_i)|\to\infty$ when $i\to\infty$.

Similarly as above we obtain

Theorem 6.4. *Let X be a weakly 1-complete manifold and let $S\subset X$ be a divisor. Assume that $\overset{n}{\wedge} T^{1,0}_X\otimes[S]^{-1}$ has a hermitian metric whose curvature form is positive. Then, every holomorphic function on S is holomorphically extendable to X.*

Proof. Immediate from the following exact sequence:

$$\begin{array}{l}\Gamma(X, \mathcal{O}_X)\longrightarrow\Gamma(S, \mathcal{O}_S)\longrightarrow H^1(X, \mathcal{O}([S]^{-1}))\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad\| \\ \qquad\qquad\qquad\qquad\qquad H^1(X, \Omega^n(\overset{n}{\wedge} T^{1,0}_X\otimes[S]^{-1}))\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad\| \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad 0,\end{array}$$

(cf. Theorem 3.6).

It follows from Theorem 6.4 that some divisors can be contracted analytically to lower dimensional analytic subsets. In particular, we have

Theorem 6.5. *Let $D\to M$ be a holomorphic $\mathbf{P}^n$-bundle over a connected complex manifold M and let $D\hookrightarrow X$ be an embedding of D as a divisor. If the* *degree of the bundle* $[D]|_{\pi^{-1}(p)}$ *is* -1 *for some (hence for any)* $p \in M$, *then there exists a complex manifold* Y *and a proper holomorphic map* $\tilde{\pi}: X \to Y$ *such that* $\tilde{\pi}$ *is one to one outside* D.

Proof. See Nakano [19].

Remark. Theorem 6.5 has been considerably generalized by Fujiki [9] and Bingener [7] (see also Ancona-Tomassini [33]).

In the same spirit as above, we have shown in [25],

Theorem 6.6. *Let* X *be a weakly* 1*-complete manifold of dimension two. Assume that* $\overset{2}{\wedge} T_X^{1,0}$ *has a hermitian metric whose curvature form is positive. Then* X *is holomorphically convex.*

Combining Theorem 6.6 with Theorem 3.6, we can prove

Theorem 6.7 (cf. *Theorem* 2.1 *and Theorem* 2.3 *in* [25]). *Let* Y *be either a hypersurface in* $\mathbf{P}^n$ *of degree less than* 4, *or a complete intersection of type* (2,2) *in* $\mathbf{P}^n$, *and let* X *be an unramified domain over* Y. *Then,* X *is holomorphically convex if and only if* X *is weakly* 1*-complete.*

As for the projective embeddability of weakly 1-complete manifolds, we have

Theorem 6.8. *Let* (X, φ) *be a weakly* 1*-complete manifold and let* (B, a) *be a positive line bundle over* X. *Then, for any* $c \in \mathbf{R}$, *there exist an integer* m *and a holomorphic embedding of* X_c *into* $\mathbf{P}^N$, *where* N *depends on* c.

Under the situation of Theorem 6.8, whether X is embeddable into some $\mathbf{P}^N$ or no is an open problem. By now the following is the unique result in this direction.

Theorem 6.9 (*Takegoshi* [30]). *Let* $\dim X = 2$. *Assume that* X *contains only finitely many exceptional curves, then* X *is holomorphically embeddable into* $\mathbf{P}^5$.

§ 7. Variations of Vanishing Theorems

Let (X, φ) be a weakly 1-complete manifold of dimension n, let (B, a) be a hermitian line bundle over X, and let Θ be the curvature form of a. There are several variations of Theorem 3.6 and Theorem 3.7.

Theorem 7.1 (*Abdelkader* [1], [2], *Takegoshi-Ohsawa* [31], *Skoda* [27]).

Assume that X has a Kähler metric. If Θ is positive semi-definite and rank $\Theta \geqq n-k+1$ *everywhere, then*

$$H^q(X, \Omega^p(B))=0 \quad \textit{when} \quad p+q \geqq n+k.$$

Theorem 7.2 (*Ohsawa* [26]). *If Θ has at least $n-k+1$ positive eigenvalues, then, for any holomorphic vector bundle $E \to X$ and for any $c \in \mathbf{R}$, there exists an integer m_0 such that*

$$H^q(X_c, \mathcal{O}(E \otimes B^m))=0, \qquad \textit{for} \quad q \geqq k \quad \textit{and} \quad m \geqq m_0.$$

Theorem 7.3 (*Ohsawa* [26]). *Assume that X is Kählerian and that* rank $\partial\bar{\partial}(e^\varphi) \leqq r$. *If Θ is negative semi-definite and has rank at least $n-k+1$, then*

$$H^q(X, \Omega^p(B))=0, \qquad \textit{for} \quad p+q \leqq n-k-r.$$

Takegoshi applied Aronszajn's unique continuation theorem [6] for $\bar{\partial}\vartheta_h + \vartheta_h\bar{\partial}$ to obtain

Theorem7.4 (cf. [29]). *Assume that X is connected and Kählerian, and that $\Theta \geqq 0$. If $\Theta > 0$ outside a compact subset of X, then*

$$H^q(X, \Omega^n(B))=0, \qquad \textit{for} \quad q \geqq 1.$$

Finite-dimensionality theorems analoguous to Theorem 4.2 are also valid. See Ohsawa [26] and Abdelkader [3].

References

[1] Abdelkader, O., Annulation de la cohomolgie d'une variete kählerienne faiblement 1-complete a valeur dans un fibre vectoriel holomorphe semi-positif, *C. R. Acad. Sci. Paris,* **290** (1980), 75–78.

[2] ———, Un theoreme d'approximation pour les formes a valeurs dan un fibre semi-positif. *C. R. Acad. Sci. Paris,* **293** (1981), 513–515.

[3] ———, Generalisation d'un theorem de finitude, *C. R. Acad. Sci. Paris,* **293** (1981), 629–632.

[4] Andreotti, A. and Vesentini, E., Sopra un teorema di Kodaira, *Ann. Sci. Norm. Sup. Pisa,* (3), **15** (1961), 283–309.

[5] ———, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Publ. Math. IHES,* **25** (1965), 81–130.

[6] Aronszajn, N., A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, *J. Math. Pure. Appl.,* **36** (1957), 235–249.

[7] Bingener, J., On the existence of analytic contractions, *Inventiones math.* **64** (1981), 25–67.

[8] Fujiki, A. and Nakano, S., Supplement to "On the inverse of monoidal transformation" *Publ. RIMS, Kyoto Univ.,* **7** (1971–72). 637–644.

[9] Fujiki, A., On the blowing down of analytic spaces, *Publ. RIMS, Kyoto Univ.,* **10**

(1975), 473–507.

[10] Grauert, H., On Levi's Problem and the imbedding of real-analytic manifolds, *Ann. of Math.* **68**, (1958), 460–472.

[11] Grauert, R., and Riemenschneider, O., Kählersche Mannigfaltigkeiten mit hyper-q-konvexem Rand, *Problems in Analysis,* Princeton Univ. Press, Princeton, N. J., (1970), 61–70.

[12] Griffiths, P. H., Hermitian differential geometry, Chern classes, and positive vector bundles, *Global analysis,* Univ. of Tokyo Press and Princeton Univ. Press., (1969), 185–251.

[13] Gunning, R. C., and Rossi, H., *Analytic functions of several complex variables,* Englewood Cliffs, N. J., Prentice Hall Inc., 1965.

[14] Hörmander, L., L^2 estimates and existence theorems for the $\bar{\partial}$ operator, *Acta Math.,* **113** (1965), 89–152.

[15] Kazama, A., Approximation theorem and application to Nakano's vanishing theorems for weakly 1-complete manifolds, *Mem. Fac. Sci. Kyushu Univ.,* **27** (1973), 221–240.

[16] Kodaira, K., On a differntial geometric method in the theory of analytic stacks, *Pro. Nat. Acad. Sci. U. S. A.,* **39** (1953), 1269–1273.

[17] Morrey, C. B., *Multiple integrals in the calculus of variations,* Springer, 1966.

[18] Nakano, S., On complex analytic vector bundles, *J. M. S. Jap.,* **7** (1955), 1–12.

[19] ———, On the inverse of monoidal transformation, *Publ. PIMS, Kyoto Univ.,* **6** (1970/71), 483–502.

[20] ———, Vanishing theorems for weakly 1-complete manifolds, *Number theory, Algebraic geometry, and Commutative algebra,* Kinokuniya, Tokyo, 1973, 169–179.

[21] ———, Vanishing theorems for weakly 1-complete manifolds II, *Publ. RIMS, Kyoto Univ.,* **10** (1974), 101–110.

[22] Nakano, S., and Rhai, T. S., Vector bundle version of Ohsawa's finiteness theorems, *Math. Japonica,* **24** (1980), 657–664.

[23] Ohsawa, T., Finiteness theorems on weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.,* **15** (1979), 853–870.

[24] ———, On $H^{p,q}(X, B)$ of weakly 1-complete manifolds, *Publ, RIMS, Kyoto Univ.,* **17** (1981), 113–126.

[25] ———, Weakly 1-complete manifold and Levi problem, *Publ. RIMS, Kyoto Univ.,* **17** (1981), 153–164.

[26] ———, Isomorphism theorems for cohomology groups of weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.,* **18** (1982) 191–232.

[27] Skoda, H., Remarques a propos des theoremes d'annulation pour les fibres semipositifs, *Lecture Notes in Mathematics,* **822** (1980), 252–257.

[28] Suzuki, O., Simple proofs of Nakano's vanishing theorems for weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.,* **17** (1975), 201–211.

[29] Takegoshi, K., A generalization of vanishing theorems for weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.,* **17** (1981), 311–330.

[30] ———, On weakly 1-complete surfaces without non-constant holomorphic functions, *Publ. RIMS, Kyoto Univ,* **18** (1982), to appear.

[31] Takegoshi, K., and Ohsawa, T., A vanishing theorem for $H^p(X, \Omega^q(B))$ on weakly 1-complete manifolds, *Publ. RIMS, Kyoto Univ.,* **17** (1981), 723–733.

[32] Wells, R. O., *Differential analysis on complex manifolds,* Prentice Hall, Englewood Cliffs, N. J., 1973.

[33] Ancona, V. and Tomassini, G., *Modification analytiques. Lecture Notes in Mathematics,* **943** (1982).