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A Definition of Boundary Values of Solutions of Partial Differential Equations with Regular Singularities

By

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§0. Introduction

In a symposium on hyperfunctions and partial differential equations held at Research Institute for Mathematical Science, Okamoto introduced the following Helgason's conjecture: Any simultaneous eigenfunction of the invariant differential operators of a Riemannian symmetric space of non-compact type has a Poisson integral representation of a hyperfunction on its maximal boundary. There had been several affirmative results in some cases. But the method used there was hard to apply to general cases. On the other hand, taking this introduction by Okamoto, we constructed the concept of boundary value problems for differential equations with regular singularities in [5] to solve this conjecture and then it was completely solved in [4].

Recently the conjecture, which is now solved, reveals many important applications in the theory of unitary representations. But the method in [5] is not always easy to be understood by every person. In this note we introduce another definition of the boundary values in an elementary way. Also we give several results concerning the definition, which are sufficient to solve "the conjecture" and moreover Corollary 5.5 in [6] which determines the image of the Poisson transformation of Schwartz's distributions on the boundary. Such applications of this note will appear in another paper.

§1. Differential Operators with Regular Singularities

As usual, $\boldsymbol{Z}$ is the ring of integers, $\boldsymbol{N}$ the set of non-negative integers, $\boldsymbol{Z}_+$

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the set of strictly positive integers, $\boldsymbol{Q}$ (resp. $\boldsymbol{R}$, resp. $\boldsymbol{C}$) the field of rational (resp. real, resp. complex) numbers and $\boldsymbol{R}_+$ the multiplicative group of strictly positive real numbers.

Let n be a non-negative integer, X a domain in $\boldsymbol{C}^{1+n}$ (or a $(1+n)$-dimensional complex manifold) and Y a 1-codimensional closed submanifold of X. Then for any point p of X there exists a local coordinate system $(t, x)=(t, x_1,\cdots, x_n)$ defined in a neighborhood U of p such that

$$Y\cap U=\{(t, x)\in U \mid t=0\}.$$

Unless otherwise stated, we use the above local coordinate system (t, x) and do not write U. Moreover in many cases in this section we may assume $X=U$ because our concerns are reduced to essentially local problems.

Let $\mathcal{O}(X)$ (resp. $\mathcal{O}(Y)$) denote the ring of holomorphic functions on X (resp. Y) and $\mathscr{D}(X)$ the ring of holomorphic differential operators on X which are of finite order. For simplicity we use the following notation

$$\begin{aligned}
&D_t=\partial/\partial t,\\
&\Theta=tD_t=t\partial/\partial t,\\
&D_x=(D_{x_1},\cdots, D_{x_n})=(\partial/\partial x_1,\cdots, \partial/\partial x_n).
\end{aligned}$$

For a multi-index $\alpha=(\alpha_1,\cdots, \alpha_n)\in \boldsymbol{N}^n$ we denote

$$\begin{aligned}
&|\alpha|=\alpha_1+\cdots+\alpha_n,\\
&\alpha!=\alpha_1!\cdots\alpha_n!,\\
&D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},\\
&x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}.
\end{aligned}$$

Any $P=P(t, x; D_t, D_x)$ in $\mathscr{D}(X)$ has the form

$$P=\sum_{\substack{(k,\alpha)\in \boldsymbol{N}^{1+n}\\ k+|\alpha|\leq m}} p_{k,\alpha}(t, x)D_t^kD_x^\alpha \quad (p_{k,\alpha}\in \mathcal{O}(X)) \tag{1.1}$$

with a suitable $m\in \boldsymbol{N}$. Its symbol $\sigma_m(P)$ of order m is defined by

$$\sigma_m(P)(t, x, \tau, \xi)=\sum_{\substack{(k,\alpha)\in \boldsymbol{N}^{1+n}\\ k+|\alpha|=m}} p_{k,\alpha}(t, x)\tau^k\xi^\alpha. \tag{1.2}$$

Here $\xi=(\xi_1,\cdots, \xi_n)$. Then $\sigma_m(P)$ belongs to the polynomial ring $\mathcal{O}(X)[\tau, \xi]$ of (τ, ξ) with coefficients in $\mathcal{O}(X)$. If $\sigma_m(P)\neq 0$, m is called the order of P (or $\mathrm{ord}(P)$) and $\sigma_m(P)$ is called the principal symbol of P (or $\sigma(P)$). Furthermore we denote by $\mathscr{D}^{(m)}(X)$ the module of differential operators in $\mathscr{D}(X)$ whose order are at most m.

The following lemma is easy but will be frequently used.

Lemma 1.1. 1) $t^{-s}\Theta t^s=\Theta+s \quad (s\in \boldsymbol{C})$,

2) $t^kD_t^k=\Theta(\Theta-1)\cdots(\Theta-k+1) \quad (k\in \boldsymbol{N}-\{0\})$,

3) $D_t^kt^k=(\Theta+1)(\Theta+2)\cdots(\Theta+k)$,

4) $tD_t=\frac{1}{k}t'D_{t'}$, *i.e.* $\Theta=\frac{\Theta'}{k}$ *if* $t=t'^k$.

Proof. 1) For a function $\phi(t, x)$

$$\begin{aligned}(t^{-s}\cdot tD_t\cdot t^s)\phi&=(t^{-s}t)(st^{s-1}+t^sD_t)\phi\\&=(s+tD_t)\phi.\end{aligned}$$

2) We will prove the equation by the induction on k. Since it is clear when $k=1$, we have by the induction hypothesis

$$\begin{aligned}t^{k+1}D_t^{k+1}&=t(t^kD_t^k)D_t\\&=t\Theta(\Theta-1)\cdots(\Theta-k+1)D_t\\&=t\Theta t^{-1}\cdot t(\Theta-1)t^{-1}\cdots t(\Theta-k+1)t^{-1}\cdot tD_t\\&=(\Theta-1)(\Theta-2)\cdots(\Theta-k)\Theta.\end{aligned}$$

3) The equation reduces to 1) when $k=1$. Using the induction on k, we have

$$\begin{aligned}D_t^{k+1}t^{k+1}&=D_t(D_t^kt^k)t\\&=D_t(\Theta+1)\cdots(\Theta+k)t\\&=t^{-1}\Theta t\cdot t^{-1}(\Theta+1)t\cdots t^{-1}(\Theta+k)t\\&=(\Theta+1)(\Theta+2)\cdots(\Theta+k+1).\end{aligned}$$

4) Since $\frac{dt}{dt'}=kt'^{k-1}$, $D_{t'}=kt'^{k-1}D_t$ and $t'D_{t'}=k't'^kD_t=ktD_t$.

Q. E. D.

Now we prepare the following lemma to define differential operators with regular singularities.

Lemma 1.2. *For a differential operator* $P\in\mathscr{D}^{(m)}(X)$ *the following conditions* a) *and* b) *are equivalent*:

a) *P has the following form*:

$$P(t, x: D_t, D_x)=\sum_{j=0}^{m}a_j(x)\Theta^j+tQ(t, x;\Theta, D_x). \tag{1.3}$$

b) *There exists* $a(x, s)\in\mathscr{O}(Y)[s]$ *such that for any* $\phi\in\mathscr{O}(X)$ *and* $s\in \boldsymbol{C}$

$$(t^{-s}Pt^s)\phi\in\mathscr{O}(X), \tag{1.4}$$

$$(t^{-s}Pt^s)\phi|_Y=a(x, s)\cdot(\phi|_Y). \tag{1.5}$$

Proof. a)⇒b): Assume P has the form as in a). Since $t^{-s}\Theta^j t^s = t^{-s}\Theta t^s \cdot t^{-s}\Theta t^s \cdots t^{-s}\Theta t^s = (\Theta+s)^j$, we have

$$\begin{aligned} t^{-s}P(t, x; D_t, D_x)t^s &= \sum_{j=0}^{m} a_j(x) t^{-s}\Theta^j t^s + t\cdot t^{-s}Q(t, x; \Theta, D_x)t^s \\ &= \sum_{j=0}^{m} a_j(x)(\Theta+s)^j + tQ(t, x; \Theta+s, D_x) \\ &= \sum_{j=0}^{m} a_j(x)s^j + tR(t, x; D_t, D_x) \end{aligned}$$

with a suitable $R(t, x; D_t, D_x) \in \mathscr{D}(X)$. Hence b) is clear by putting $a(x, s) = \sum_{j=0}^{m} a_j(x)s^j$.

b)⇒a): Assume b). Putting

$$R(t, x; D_t, D_x) = P(t, x; D_t, D_x) - a(x, \Theta),$$

we have only to prove that R has the form $R(t, x; D_t, D_x) = tQ(t, x; \Theta, D_x)$. Since for any $\phi \in \mathscr{O}(X)$

$$\begin{aligned} (t^{-s}Rt^s)\phi|_Y &= (t^{-s}Pt^s - a(x, \Theta+s))\phi|_Y \\ &= a(x, s)\phi|_Y - a(x, s)\phi|_Y \\ &= 0, \end{aligned}$$

we have $(t^{-s}Rt^s)\mathscr{O}(X) \subset \mathscr{O}(X)t$ for any $s \in \boldsymbol{C}$. Next we put

$$R(t, x; D_t, D_x) = \sum_{j=0}^{m} R_j(t, x; D_x)D_t^j$$

with $R_j \in \mathscr{D}^{(m-j)}(X)$. Then for any $\phi(x)$

$$\begin{aligned} (t^{-s}Rt^s)\phi(x) &= \sum_{j=0}^{m} t^{-s}R_j(t, x; D_x)D_t^j t^s\phi(x) \\ &= \sum_{j=0}^{m} t^{-s}R_j(t, x; D_x)s(s-1)\cdots(s-j+1)t^{s-j}\phi(x) \\ &= \sum_{j=0}^{m} s(s-1)\cdots(s-j+1)t^{-j}R_j(t, x; D_x)\phi(x). \end{aligned}$$

Putting $s = 0, 1, 2, \cdots$, we have

$$t^{-j}R_j(t, x; D_x)\phi(x) \in \mathscr{O}(X)t \quad \text{for any} \quad \phi(x).$$

Hence $R_j(t, x; D_x) = t^{j+1}Q_j(t, x; D_x)$ with suitable $Q_j \in \mathscr{D}(X)$ and then

$$\begin{aligned} R(t, x; D_t, D_x) &= \sum_{j=0}^{m} R_j(t, x; D_x)D_t^j \\ &= t\sum_{j=0}^{m} Q_j(t, x; D_x)t^jD_t^j \\ &= t\sum_{j=0}^{m} Q_j(t, x; D_x)\Theta(\Theta-1)\cdots(\Theta-j+1). \end{aligned}$$

This means a). Q.E.D.

Definition 1.3. We denote by $\mathscr{D}_Y^*(X)$ the subspace of $\mathscr{D}(X)$ formed by the differential operators which satisfy the equivalent conditions in Lemma 1.2 and for $P \in \mathscr{D}_Y^*(X)$ we denote by $\sigma_*(P)$ the polynomial $a(x, s)$ $(=\sum a_j(x)s^j \in \mathscr{O}(Y)[s])$ using the notation in the lemma. Here we remark that if we put $\phi \equiv 1$, it is clear that $a(x, s)$ is uniquely determined. Moreover the condition b) says that $\mathscr{D}_Y^*(X)$ is a subalgebra of $\mathscr{D}(X)$ and that the map

$$\sigma_*\colon \mathscr{D}_Y^*(X) \longrightarrow \mathscr{O}(Y)[s]$$

defines a $\boldsymbol{C}$-algebra homomorphism. Now we denote by $\mathscr{D}_Y(X)$ the subalgebra of $\mathscr{D}_Y^*(X)$ generated by $\mathscr{D}_Y^*(X) \cap \mathscr{D}^{(1)}(X)$ $(=\{P \in \mathscr{D}^{(1)}(X) \mid \sigma_1(P)|_Y = 0\})$. Then a differential operator P of order m is said to have *regular singularities* (or *R.S.* for short) *along* Y if P belongs to $\mathscr{D}_Y(X)$ and if $\sigma_*(P)(x, s)$ is of just degree m for any $x \in Y$. In the above if P belongs to $\mathscr{D}_Y^*(X)$ in place of $\mathscr{D}_Y(X)$ we say that P has R.S. along Y *in a weak sense.* In both cases $\sigma_*(P)$ is called the *indicial polynomial* of P and the roots of the equation $\sigma_*(P)(x, s) = 0$, which we denote by $s_1(x), \cdots, s_m(x)$, are called the *characteristic exponents* of P. Similarly the differential equation $Pu = 0$ is said to have R.S. (or R.S. in a weak sense) along Y if P is so. In this case the equation $\sigma_*(P) = 0$ is called the *indicial equation* of the differential equation and the roots $s_1(x), \cdots, s_m(x)$ are called its characteristic exponents.

Proposition 1.4. *Let P be a differential operator of order m. Then the following three conditions are equivalent:*

a) *P has R.S. along Y.*

b) *P has the form*

$$P = \sum_{j=0}^{m} t^j P_j(t, x; D_t, D_x) \quad \text{with} \quad P_j \in \mathscr{D}^{(j)}(X) \tag{1.6}$$

and $\sigma_m(P_m)(0, x; 1, 0) \neq 0$ for any $x \in Y$.

c) *P has the form*

$$\begin{aligned} P &= P(t, x; \Theta, tD_x) \\ &= \sum_{\substack{(k,\alpha) \in N^{1+n} \\ k+|\alpha| \leq m}} c_{k,\alpha}(t, x)\Theta^k (tD_x)^\alpha \end{aligned} \tag{1.7}$$

and $c_{m,0}(0, x) \neq 0$ for any $x \in Y$, where $(tD_x)^\alpha = (tD_{x_1})^{\alpha_1} \cdots (tD_{x_n})^{\alpha_n}$.

Proof. It is clear that b) and c) are equivalent (cf. Lemma 1.1. 2)). Since

$\mathscr{D}_Y(X)$ is the subalgebra of $\mathscr{D}(X)$ generated by Θ, $tD_{x_1},\cdots, tD_{x_n}$ and $\mathscr{O}(X)$, the relations

$$[\Theta, tD_{x_i}]=tD_{x_i}, \quad [tD_{x_i}, tD_{x_j}]=0,$$
$$[\Theta, \phi]\in\mathscr{O}(X), \quad [tD_{x_i}, \phi]\in\mathscr{O}(X) \quad \text{for any} \quad \phi\in\mathscr{O}(X)$$

imply the equivalence of the condition $P\in\mathscr{D}_Y(X)$ and the condition that P has the form (1.7). Hence a) is equivalent to c) because $\sigma_*(P)=\sum_{k=0}^{m} c_{k,0}(0, x)s^k$.

Q. E. D.

Remark 1.5. Suppose a differential operator P has R.S. in the weak sense. Then if we put $t=t'^k$ for a sufficiently large k, then P has R.S. in the coordinate system (t', x). In fact if P is of the form (1.3), then

$$P=\sum_{j=0}^{m} a_j(x)\left(\frac{\Theta'}{k}\right)^j+t'^kQ\left(t'^k, x; \frac{\Theta'}{k}, D_x\right).$$

Hence P is of the form (1.6) and $a(x, s)$ changes into $a\left(x, \frac{s}{k}\right)$ by the transformation $(t, x)=(t'^k, x)$.

Example 1.6. The differential operator

$$P=y^2(D_x^2+D_y^2)+\frac{1}{4}-\lambda^2$$

has R.S. along the hypersurface defined by $y=0$. Its indicial polynomial equals $s(s-1)+\frac{1}{4}-\lambda^2=\left(s-\frac{1}{2}+\lambda\right)\left(s-\frac{1}{2}-\lambda\right)$ and the characteristic exponents are $\frac{1}{2}-\lambda$ and $\frac{1}{2}+\lambda$. The operator

$$Q=tD_x^2+4t^2D_t^2+2tD_t+\frac{1}{4}-\lambda^2$$

has R.S. in the weak sense. If we put $t=y^2$, then $P=Q$. On the other hand the operator

$$R=\frac{1}{4}(1-x^2-y^2)^2(D_x^2+D_y^2)+\frac{1}{4}-\lambda^2$$

has R.S. along the hypersurface defined by $x^2+y^2=1$. Then the coordinate transformation

$$x\longrightarrow\frac{2y}{(1+x)^2+y^2}, \quad y\longrightarrow\frac{1-x^2-y^2}{(1+x)^2+y^2}$$

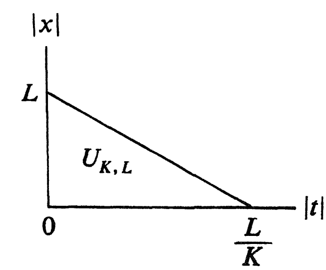
changes P into R.

§ 2. Operations on the Space of Holomorphic Functions

Retain the notation in Section 1. Then the space $\mathcal{O}(X)$ of holomorphic functions is a Fréchet space by the topology of uniform convergences on compact subsets of X and the element of $\mathcal{D}(X)$ defines a continuous linear transformation on $\mathcal{O}(X)$.

In general for positive integers N and N' and a ring R we denote by $M(N, N'; R)$ the space of matrices of size $N \times N'$ whose components are in R and for simplicity $M(N, N; R)$ is denoted by $M(N; R)$. If $r_{ij} \in R$ for $i=1,\cdots, N$ and $j=1,\cdots, N'$, then (r_{ij}) denotes the matrix whose (i, j)-component equals r_{ij}. Under this notation the map

$$\begin{array}{ccc} M(N, N'; \mathcal{D}_Y(X)) & \longrightarrow & M(N, N'; \mathcal{O}(Y)[s]) \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ (P_{ij}) & \longmapsto & (\sigma_*(P_{ij})) \end{array}$$



is also denoted by σ_*. For $K, L \in \boldsymbol{R}_+$ we put

$$U_{K,L} = \{(t, x) \in \boldsymbol{C}^{1+n} \mid K|t| + |x| < L\}$$

and

$$V_L = \{x \in \boldsymbol{C}^n \mid |x| < L\}$$

where $|x| = |x_1| + \cdots + |x_n|$. We identify V_L with the submanifold of $U_{K,L}$ defined by $t=0$. Then we can state the main theorem in this section.

Theorem 2.1. *Let X be a domain in $\boldsymbol{C}^{1+n}$ containing the closure of $U_{1,L}$ and Y a submanifold of X defined by $t=0$. We define the following conditions for a matrix $P=(P_{ij})$ in $M(N; \mathcal{D}_Y(X))$:*

a) *There exist non-negative integers m'_i and m''_j such that* $\operatorname{ord} P_{ij} \leqq m'_i + m''_j$ *for $i=1,\cdots, N$ and $j=1,\cdots, N$.*

b) *If we put $m = \sum_{i=1}^{N} m'_i + \sum_{j=1}^{N} m''_j$, the determinant* $\det(\sigma_*(P))$ *of the matrix $\sigma_*(P)$ is a polynomial of degree m for any x in Y.*

c) *Let $s_1(x),\cdots, s_m(x)$ be the solutions of the equation* $\det(\sigma_*(P))=0$. *Then*

$$\text{(A.1)} \qquad s_\nu(x) \notin \boldsymbol{N} = \{0, 1, 2,\cdots\} \quad \textit{for any} \quad x \in V_L \quad \textit{and} \quad \nu = 1,\cdots, m.$$

If the matrix P satisfies the above three conditions, there exists a positive number $K_0 \geqq 1$ such that the linear map

$$\text{(2.1)} \qquad P\colon \mathcal{O}(U_{K,L})^N \longrightarrow \mathcal{O}(U_{K,L})^N$$

is a surjective homeomorphism for any positive number K which is larger than K_0. *Here* $\mathcal{O}(U_{K,L})^N$ *denotes the linear space formed by column vectors of length N whose elements belong to* $\mathcal{O}(U_{K,L})$.

Before the proof of the theorem we give a remark which will be easily seen by our proof. It will not be referred later and the precise argument is left to reader's exercise.

Remark 2.2. 1) The number K_0 in the theorem depends only on $\det(\sigma_{m'_i+m''_j}(P_{ij}))$ and moreover the assumption that the coefficients of P_{ij} of order less than $m'_i+m''_j$ are holomorphic only on $U_{K,L}$ in place of X is sufficient to assure that the map (2.1) is a surjective homeomorphism.

2) Suppose that we omit the condition c) in the theorem. Then if we put

$$r=\max\{0\}\cup(\{1+s_\nu(x)\,|\,x\in V_L \quad \text{and} \quad \nu=1,\cdots,m\}\cap \boldsymbol{N}),$$

P induces the isomorphism

$$t^r\mathcal{O}(U_{K,L})^N \xrightarrow{\sim} t^r\mathcal{O}(U_{K,L})^N. \tag{2.2}$$

(Here the condition c) equals to the condition $r=0$). Hence we can naturally define the map

$$\bar{P}\colon \mathcal{O}(U_{K,L})^N/t^r\mathcal{O}(U_{K,L})^N \longrightarrow \mathcal{O}(U_{K,L})^N/t^r\mathcal{O}(U_{K,L})^N \tag{2.3}$$

and we have

$$\operatorname{Ker} P \xrightarrow{\sim} \operatorname{Ker} \bar{P}, \tag{2.4}$$

and

$$\begin{aligned}\operatorname{Coker} P &\xrightarrow{\sim} \operatorname{Coker} \bar{P} \\ &\xrightarrow{\sim} \bigoplus_{j=0}^{\infty} \mathcal{O}(V_L)^N/\sigma_*(P)(x,j)\mathcal{O}(V_L)^N.\end{aligned} \tag{2.5}$$

Proof of Theorem 2.1. Put

$$P(t,x;\Theta,tD_x)=A(x,\Theta)+tR(t,x;\Theta,D_x). \tag{2.6}$$

Let $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}_x$ be the rings of formal power series of (t, x) and that of x, respectively. Then P defines the map

$$P\colon \hat{\mathcal{O}}^N \longrightarrow \hat{\mathcal{O}}^N. \tag{2.7}$$

For u in $\hat{\mathcal{O}}^N$ we put

$$u=\sum_{(i,\alpha)\in \boldsymbol{N}^{1+n}} u_{i,\alpha}t^i x^\alpha=\sum_{i=0}^{\infty} u_i(x)t^i \quad (u_{i,\alpha}\in \boldsymbol{C}^N),$$

where

$$u_i(x)=\sum_{\alpha\in N^n} u_{i,\alpha}x^\alpha \in \hat{\mathcal{O}}_x^N.$$

Then for any $l \in N$ we have

$$\begin{aligned}Pu&=(A(x,\Theta)+tR(t,x;\Theta,D_x))(\sum_{i=0}^{\infty}u_i(x)t^i)\\&=\sum_{i=0}^{\infty}A(x,\Theta)u_i(x)t^i+\sum_{i=0}^{\infty}tR(t,x;\Theta,D_x)u_i(x)t^i\\&\equiv\sum_{i=0}^{l}A(x,\Theta)u_i(x)t^i+\sum_{i=0}^{l-1}tR(t,x;\Theta,D_x)u_i(x)t^i \quad \mathrm{mod}\ t^{l+1}\hat{\mathcal{O}}^N\end{aligned}$$

and hence

$$Pu\equiv A(x,l)u_l(x)t^l+P(\sum_{i=0}^{l-1}u_i(x)t^i) \quad \mathrm{mod}\ t^{l+1}\hat{\mathcal{O}}^N. \tag{2.8}$$

Here we remark that the condition (A.1) implies

$$A(x,l) \text{ is invertible in } M(N,\mathcal{O}(Y)) \quad \text{for any } l\in N. \tag{2.9}$$

First we want to prove that the map (2.7) is injective. Suppose $Pu=0$. Then (2.8) shows

$$0\equiv A(x,0)u_0(x) \quad \mathrm{mod}\ t\hat{\mathcal{O}}^N,$$

which means $u_0=0$ because of (2.9). By (2.8) and the induction on l we have

$$0\equiv A(x,l)u_l(x)t^l \quad \mathrm{mod}\ t^{l+1}\hat{\mathcal{O}}^N$$

and hence (2.9) proves $u_l=0$. Therefore (2.7) is injective, which assures that (2.1) is also injective.

Next we want to prove the surjectivity of the map (2.1). Let A_{ij} be the (i,j)-component of the matrix $A(=A(x,\Theta))$ and $\tilde{A}_{ij}$ the cofactor of A_{ji} in A. Putting $\tilde{A}=(\tilde{A}_{ij})$, we have

$$\begin{aligned}&A\tilde{A}=\tilde{A}A=\det(A)I_N=\det(\sigma_*(P))(x,\Theta)I_N,\\&\mathrm{ord}\,A_{ij}\leqq m_i'+m_j'' \quad\text{and}\quad \mathrm{ord}\,\tilde{A}_{ij}\leqq m-m_j'-m_i'',\end{aligned}$$

where I_N is the identity matrix of size N. Hence if we put $P'\ (=(P'_{ij}))=P\tilde{A}$, then $P'\in M(N;\mathscr{D}_Y(X))$ and

$$\sigma_*(P')=\det(\sigma_*(P))I_N,$$

and

$$\begin{aligned}\mathrm{ord}\,P'_{ij}&\leqq\max_\nu(\mathrm{ord}\,P_{i\nu}+\mathrm{ord}\,\tilde{A}_{\nu j})\\&\leqq\max_\nu(m_i'+m_\nu''+m-m_j'-m_\nu'')\\&=m+m_i'-m_j'.\end{aligned}$$

Using the diagonal matrices $E^{(1)}$ and $E^{(2)}$ with the i-th diagonal components $(\Theta+1)^{m-m_i'}$ and $(\Theta+1)^{m_i'}$, respectively, we put

(2.10) $$P''=E^{(1)}P'E^{(2)} \quad (=E^{(1)}P\tilde{A}E^{(2)}\in M(N;\mathscr{D}_Y(X)))\,.$$

Then

$$\begin{aligned}\sigma_*(P'')&=(s+1)^m\cdot\det(\sigma_*(P))I_N\,,\\ \operatorname{ord}P''_{ij}&\leqq(m-m_i')+(m+m_i'-m_j')+m_j'\\ &=2m\,.\end{aligned}$$

Here we remark that $E^{(1)}$ defines an injective transformation on $\hat{\mathcal{O}}^N$, which is clear by the same argument as before. Hence if P'' defines a surjective transformation on $\mathcal{O}(U_{K,L})^N$ we can conclude that the map (2.1) is surjective. For simplicity, denoting P'' and $2m$ by P and m, respectively, we may assume the following:

(2.11) $$\operatorname{ord}P_{ij}\leqq m$$

and

(2.12) $$\sigma_*(P)=a(x,s)I_N$$

where $a(x,s)$ $(\in\mathcal{O}(Y)[s])$ is a polynomial of degree m for any $x\in Y$. Let $s_1(x),\cdots,s_m(x)$ be the solutions of $a(x,s)=0$. Then they satisfy (A.1).

Let $f\in\hat{\mathcal{O}}^N$. We put $f=\sum_{(i,\alpha)\in\boldsymbol{N}^{1+n}}f_{i,\alpha}t^ix^\alpha=\sum_{i=0}^\infty f_i(x)t^i$ $(f_{i,\alpha}\in\boldsymbol{C}^N)$. Then the equation

(2.13) $$Pu=f$$

means

$$a(x,l)u_l(x)t^l+P\Big(\sum_{i=0}^{l-1}u_i(x)t^i\Big)\equiv f \mod t^{l+1}\hat{\mathcal{O}}^N$$

(cf. (2.6)–(2.8)). Hence if we inductively define $u_l(x)$ by

(2.14) $$u_l(x)=a(x,l)^{-1}\Big(\frac{D_t^l}{l!}\Big(f-P\Big(\sum_{i=0}^{l-1}u_i(x)t^i\Big)\Big)\Big)\Big|_{t=0}\,,$$

then $u=\sum u_i(x)t^i$ satisfies (2.13). Therefore (2.7) is surjective.

Next assume $f\in\mathcal{O}(U_{K,L})^N$, where K will be determined later. We want to prove that u in $\hat{\mathcal{O}}^N$ which satisfies (2.13) belongs to $\mathcal{O}(U_{K,L})^N$. It is sufficient to show $u\in\mathcal{O}(U_{K,L-\delta})^N$ for any $\delta\in\boldsymbol{R}_+$. Since $f_l(x)$ is holomorphic in V_L and P is defined on $U_{K,L}$ for any $K\geqq 1$, it follows inductively by (2.14) that $u_l(x)$ is also holomorphic in V_L. Putting $u^r=\sum_{i=0}^r u_i(x)t^i$ for $r\in\boldsymbol{N}$, $u^r\in\mathcal{O}(U_{K,L})^N$ and

$$f-Pu^r=P(u-u^r)\in t^r\hat{\mathcal{O}}^N\cap\mathcal{O}(U_{K,L})^N=t^r\mathcal{O}(U_{K,L})^N.$$

Hence replacing f by $f-Pu^r$ and u by $u-u^r$, we have only to prove that there exists a positive integer r such that $u\in\mathcal{O}(U_{K,L-2\delta})^N$ if $f\in t^r\mathcal{O}(U_{K,L})^N$. To prove it by the method of majorant we prepare

Lemma 2.3. 1) *Any function in $\mathcal{O}(U_{K,L})$ can be expressed in a power series of (t, x) which converges at any point in $U_{K,L}$.*

2) *Let $\phi(t, x)\in\hat{\mathcal{O}}$. Then $\phi(t, x)\in\mathcal{O}(U_{K,L})$ if and only if for any $L'\in\boldsymbol{R}_+$ satisfying $L'<L$ there exists $R_{L'}\in\boldsymbol{R}_+$ such that*

$$\phi(t, x)\ll R_{L'}\left(1-\frac{Kt+x_1+\cdots+x_n}{L'}\right)^{-1}. \tag{2.15}$$

Here for $\psi=\sum C_{i,\alpha}t^ix^\alpha$ and $\psi'=\sum C'_{i,\alpha}t^ix^\alpha$ in $\hat{\mathcal{O}}$, $\psi\ll\psi'$ means that ψ' is a majorant series of ψ, that is, $C'_{i,\alpha}\geqq|C_{i,\alpha}|$ for any i and α.

We continue the proof of the theorem and the proof of the lemma is given after that. We put

$$P_{ij}=a(x, \Theta)\delta_{ij}+t\sum_{k=0}^{m-1}b_{ij,k}(t, x)\Theta^k+\sum_{\substack{k+|\alpha|\leqq m\\|\alpha|\geqq 1}}b_{ij,k\alpha}(t, x)\Theta^k(tD_x)^\alpha,$$

$$a(x, \Theta)=a_0(x)\Theta^m+a_1(x)\Theta^{m-1}+\cdots+a_m(x).$$

Here we can assume $a_0(x)\equiv 1$ because $a_0(x)$ nowhere vanishes on V_L. Since $\sigma_m(P_{ij})$ is defined on a neighborhood of $\overline{U}_{1,L}$, there exists $L'\in\boldsymbol{R}$ such that $L'>L$ and that $\sigma_m(P_{ij})$ is defined on $U_{1,L'}$. Hence putting $L''=\dfrac{L+L'}{2}$, we have $L''>L$ and it follows from Lemma 2.3 that there exists an $M_0>0$ satisfying

$$b_{ij,k\alpha}(t, x)\ll M_0\left(1-\frac{t+x_1+\cdots+x_n}{L''}\right)^{-1}\qquad(k+|\alpha|=m). \tag{2.16}$$

Similarly we have

$$\begin{cases} b_{ij,k\alpha}(t, x)\ll M_\delta\left(1-\dfrac{Kt+x_1+\cdots+x_n}{L-\delta}\right)^{-1} & (k+|\alpha|<m),\\ (a_i(x)-a_i(0))\delta_{ij}+tb_{ij,k}(t, x) & \\ \qquad\ll M_\delta(t+x_1+\cdots+x_n)\left(1-\dfrac{Kt+x_1+\cdots+x_n}{L-\delta}\right)^{-1} & (0\leqq k\leqq m-1),\\ f\ll(R_{\delta,f})t^r\left(1-\dfrac{Kt+x_1+\cdots+x_n}{L-\delta}\right)^{-1} & \end{cases} \tag{2.17}$$

with suitable M_δ and $R_{\delta,f}$ in $\boldsymbol{R}_+$ because $b_{ij,k}$ and $b_{ij,k\alpha}\in\mathcal{O}(U_{K,L})$, $f\in t^r\mathcal{O}(U_{K,L})^N$ and $a_i\in\mathcal{O}(V_L)$. Here $K\,(\geqq 1)$ and $r\,(\geqq 1)$ will be determined later and $(R_{\delta,f})$

denotes the column vector of length N whose components are $R_{\delta,f}$ and for ϕ and ψ in $\hat{\mathcal{O}}^N$, $\phi \gg \psi$ means that any component of ϕ is a majorant series of the corresponding component of ψ. Moreover (A.1) means

$$|a(0, l)| \geqq Cl^m \quad (l \in \boldsymbol{N}) \tag{2.18}$$

with $C \in \boldsymbol{R}_+$. Putting

$$z = Kt + x_1 + \cdots + x_n \tag{2.19}$$

and

$$\begin{aligned} \hat{P}_{ij} = C\Theta^m \delta_{ij} &- \sum_{i=0}^{m-1} M_\delta z\Big(1 - \frac{z}{L-\delta}\Big)^{-1} \\ &- \sum_{\substack{k+|\alpha|=m \\ |\alpha| \geqq 1}} M_0\Big(1 - \frac{z}{L''}\Big)^{-1} \Theta^k (tD_x)^\alpha \\ &- \sum_{\substack{k+|\alpha|<m \\ |\alpha| \geqq 1}} M_\delta\Big(1 - \frac{z}{L-\delta}\Big)^{-1} \Theta^k (tD_x)^\alpha , \end{aligned} \tag{2.20}$$

we will show that $\phi \gg u$ if $\phi = \sum \phi_{i,\alpha} t^i x^\alpha$ in $\hat{\mathcal{O}}^N$ satisfies

$$\begin{cases} C\Theta^m \phi \gg (C\Theta^m - \hat{P})\phi + (R_{\delta,f}) t^r \Big(1 - \dfrac{z}{L-\delta}\Big)^{-1}, \\ \hat{P} = (\hat{P}_{ij}) \quad \text{and} \quad \phi \in t^r \hat{\mathcal{O}}^N . \end{cases} \tag{2.21}$$

By (2.14) we have

$$a(x, l) \cdot \sum_{\alpha \in \boldsymbol{N}^n} u_{l,\alpha} x^\alpha = \Big(\frac{D_t^l}{l!}(f - P(\sum_{i=0}^{l-1} u_i(x) t^i))\Big)\Big|_{t=0}$$

and

$$\begin{aligned} a(0, l) u_{l,\beta} &= \Big(\frac{D_t^l D_x^\beta}{l!\beta!}\{f - a(x, l) \sum_{|\alpha|<|\beta|} u_{l,\alpha} t^l x^\alpha - P(\sum_{i=0}^{l-1} u_i(x) t^i)\}\Big)\Big|_{t=0,x=0} \\ &= \Big(\frac{D_t^l D_x^\beta}{l!\beta!}\{f + (a(0, \Theta) - P) \sum_{(i,\alpha) \in I(l,\beta)} u_{i,\alpha} t^i x^\alpha\}\Big)\Big|_{t=0,x=0} \end{aligned} \tag{2.22}$$

with $I(l, \beta) = \{(i, \alpha) \in \boldsymbol{N}^{1+n} | i < l\} \cup \{(l, \alpha) \in \boldsymbol{N}^{1+n} | \, |\alpha| < |\beta|\}$. Similarly by (2.21) we have

$$\begin{aligned} Cl^m \phi_{l\beta} \geqq \Big| \Big(\frac{D_t^l D_x^\beta}{l!\beta!}\Big\{R_{\delta,f} t^r \Big(1 - \frac{z}{L-\delta}\Big)^{-1} \\ + (C\Theta^m - \hat{P}) \sum_{(i,\alpha) \in I(l,\beta)} \phi_{i,\alpha} t^i x^\alpha\Big\}\Big)\Big|_{t=0,x=0}\Big| . \end{aligned} \tag{2.23}$$

Now we will prove $\phi_{l,\beta} t^l x^\beta \gg u_{l,\beta} t^l x^\beta$ by the induction with respect to the lexicographic order on $(l, |\beta|)$. If $l = 0$, this is clear because $\phi_{0,\beta} = u_{0,\beta} = 0$. By the hypothesis of the induction we have

$$\sum_{(i,\alpha)\in I(l,\beta)} \phi_{i,\alpha} t^i x^\alpha \gg \sum_{(i,\alpha)\in I(l,\beta)} u_{i,\alpha} t^i x^\alpha .$$

In general if $v \gg v'$ and $w \gg w'$ for v, v', w and w' in $\hat{\mathcal{O}}$, then $v+w \gg v'+w'$, $vw \gg v'w'$, $\Theta v \gg \Theta v'$ and $tD_{x_i} v \gg tD_{x_i} v'$. Hence by (2.16), (2.17), (2.19) and (2.20) we have

$$\begin{aligned}&(R_{\delta,f})t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1} + (C\Theta^m - \hat{P})\Big(\sum_{(i,\alpha)\in I(l,\beta)} u_{i,\alpha} t^i x^\alpha\Big)\\ &\gg f(t, x) + (a(0, \Theta) - P)\Big(\sum_{(i,\alpha)\in I(l,\beta)} u_{i,\alpha} t^i x^\alpha\Big)\end{aligned}$$

and therefore by (2.19), (2.22) and (2.23) we have $\phi_{l,\beta} t^l x^\beta \gg u_{l,\beta} t^l x^\beta$. Thus we have proved $\phi \gg u$.

Next we will construct a solution ϕ of (2.21) with the form $\phi = (v)$, where (v) denotes the column vector of length N whose components are the same v in $\hat{\mathcal{O}}$. We assume $v = w(z)t^r$ with a formal power series w of one variable satisfying $w \gg 0$. Then

$$\Theta v = (\Theta w)t^r + w(\Theta t^r) \gg rwt^r = rv \gg v$$

and

$$tD_{x_i} v = t(D_{x_i} w)t^r = K^{-1}(\Theta w)t^r \ll K^{-1}\Theta v \ll \Theta v .$$

Let M be a positive number which is larger than the number of the elements of the set $\{(k, \alpha) \in \boldsymbol{N}^{1+n} | k+|\alpha| \leqq m,\ |\alpha| \geqq 1\}$. Then

$$\begin{aligned}(C\Theta^m - \hat{P})(v) &= \sum_{i=0}^{m-1} M_\delta N z\Big(1-\frac{z}{L-\delta}\Big)^{-1}\Theta^i(v)\\ &\quad + \sum_{\substack{k+|\alpha|=m\\ |\alpha|\geqq 1}} M_0 N\Big(1-\frac{z}{L''}\Big)^{-1}\Theta^k(tD_x)^\alpha(v)\\ &\quad + \sum_{\substack{k+|\alpha|<m\\ |\alpha|\geqq 1}} M_\delta\Big(1-\frac{z}{L-\delta}\Big)^{-1}\Theta^k(tD_x)^\alpha(v)\\ &\ll mM_\delta N z\Big(1-\frac{z}{L-\delta}\Big)^{-1}\Theta^{m-1}(v) + MM_0N\Big(1-\frac{z}{L''}\Big)^{-1}K^{-1}\Theta^m(v)\\ &\quad + MM_\delta N\Big(1-\frac{z}{L-\delta}\Big)^{-1}\Theta^{m-1}(v)\\ &\ll g\Theta^m(v)\end{aligned}$$

with

$$g = r^{-1}(mz+M)M_\delta N\Big(1-\frac{z}{L-\delta}\Big)^{-1} + K^{-1}MM_0N\Big(1-\frac{z}{L''}\Big)^{-1} .$$

Now we choose K_0, K and r such that

$$\begin{cases} K_0 > 2C^{-1}MM_0N\Big(1-\dfrac{L}{L''}\Big)^{-1}, \ K > K_0 \\ r > 2C^{-1}(mL+M)M_\delta N\Big(1-\dfrac{L-2\delta}{L-\delta}\Big)^{-1} \end{cases}$$

and put

$$h=(C-g)^{-1}R_{\delta,f}t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1}.$$

Then h is holomorphic in $U_{K,L-2\delta}$ because $|g(z)|<C$ if $|z|<L-2\delta$. Moreover putting

$$\begin{cases} h=\sum_{i=r}^{\infty}\sum_{\alpha\in N^n} h_{i,\alpha}t^i x^\alpha, \\ v=\sum_{i=r}^{\infty}\sum_{\alpha\in N^n} i^{-m}h_{i,\alpha}t^i x^\alpha, \end{cases}$$

we have $h \gg v$, $\Theta^m v=h$ and

$$\begin{aligned} C\Theta^m(v) &= \Big(1-\frac{C}{g}\Big)^{-1}(R_{\delta,f})t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1} \\ &= (R_{\delta,f})t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1}+g(C-g)^{-1}(R_{\delta,f})t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1} \\ &\gg (R_{\delta,f})t^r\Big(1-\frac{z}{L-\delta}\Big)^{-1}+(C\Theta^m-\hat{P})(v). \end{aligned}$$

This means $\phi=(v)$ is a solution of (2.21) and therefore $u \ll (v)$. Since $h(Kt+x_1+\cdots+x_n)$ converges for $(t, x)\in U_{K,L-2\delta}$ and since $h\gg v$ and $(v)\gg u$, u also converges there and therefore u is holomorphic in $U_{K,L-2\delta}$.

Thus we have proved that (2.1) is bijective. Since the linear map (2.1) is continuous and since $\mathcal{O}(U_{K,L})^N$ is a Fréchet space, it follows from the open mapping theorem that the map is a homeomorphism. Q. E. D.

Proof of Lemma 2.3. Putting $x_{n+1}=Kt$ and replacing $n+1$ by n, we may replace $U_{K.L}$ by V_L, $\phi(t, x)$ by $\phi(x)$ and $Kt+x_1+\cdots+x_n$ by $x_1+\cdots+x_n$ in the proof of the lemma.

1) Let $\phi\in\mathcal{O}(V_L)$ and $x\in V_L$. Put $c_i=|x_i|+\dfrac{L-|x|}{2}$ for $i=1,\cdots, n$. Then $c_i>|x_i|$ and $c_1+\cdots+c_n<L$. Hence by Cauchy's integral formula we have

$$\begin{aligned} \phi(x) &= (2\pi\sqrt{-1})^{-n}\int_{|z_n|=c_n}\cdots\int_{|z_1|=c_1}\frac{\phi(z)dz_1\cdots dz_n}{(z_1-x_1)\cdots(z_n-x_n)} \\ &= (2\pi\sqrt{-1})^{-n}\int_{|z_n|=c_n}\cdots\int_{|z_1|=c_1}\Big(\sum_{\alpha\in N^n}\frac{x^\alpha}{z^{\alpha+1}}\phi(z)\Big)dz_1\cdots dz_n, \end{aligned}$$

where $\alpha+1=(\alpha_1+1,\cdots, \alpha_n+1)$. Since the sum of the integrand converges uniformly on the pathes of the integration,

$$\phi(x)=\sum_{\alpha\in \boldsymbol{N}^n}\Big((2\pi\sqrt{-1})^{-n}\int_{|z_n|=c_n}\cdots\int_{|z_1|=c_1}\frac{\phi(z)}{z^{\alpha+1}}dz_1\cdots dz_n\Big)x^\alpha .$$

2) Let $L\in \boldsymbol{R}_+$ and $\phi=\sum_{\alpha\in \boldsymbol{N}^n} a_\alpha x^\alpha\in\hat{\mathcal{O}}_x$. For any $L'\in \boldsymbol{R}_+$ with $L'<L$ we assume the existence of $R_{L'}\in \boldsymbol{R}_+$ such that $\phi\ll R_{L'}\Big(1-\frac{x_1+\cdots+x_n}{L'}\Big)^{-1}$. Let $x\in V_L$ and put $L'=\frac{|x|+L}{2}$ and $b_\alpha=R_{L'}\cdot\frac{|\alpha|!}{\alpha!L'^{|\alpha|}}$. Since

$$R_{L'}\Big(1-\frac{x_1+\cdots+x_n}{L'}\Big)^{-1}=R_{L'}\sum_{k=0}^{\infty}\Big(\frac{x_1+\cdots+x_n}{L'}\Big)^k=\sum_{\alpha\in \boldsymbol{N}^n} b_\alpha x^\alpha ,$$

we have $\sum_{\alpha\in \boldsymbol{N}^n}|a_\alpha x^\alpha|\leqq\sum_{\alpha\in \boldsymbol{N}^n} b_\alpha|x|^\alpha=R_{L'}\Big(1-\frac{|x|}{L}\Big)^{-1}<\infty$. Hence the series $\sum a_\alpha x^\alpha$ converges.

On the contrary assume ϕ converges for any $x\in V_L$. Let $L'\in \boldsymbol{R}_+$ with $L'<L$ and put $L''=\frac{L'+L}{2}$. Then $L'<L''<L$ and $\sum|a_\alpha x^\alpha|$ converges uniformly on $\bar{V}_{L''}$ and defines a continuous function. Let $M_{L'}$ be its maximal value on $\bar{V}_{L''}$. Then $|a_\alpha c^\alpha|\leqq\sum|a_\alpha x^\alpha|\leqq M_{L'}$ for any $c=(c_1,\cdots, c_n)\in[0, L'']^n$ satisfying $c_1+\cdots+c_n=L''$. If the relation

$$\max\{c^\alpha|c_1+\cdots+c_n=L''\}\geqq L''^{|\alpha|}\frac{\alpha!}{|\alpha|!(|\alpha|+1)^{n-1}} \tag{2.25}$$

is valid, then we have

$$\begin{aligned}|a_\alpha|&\leqq M_{L'}\Big(L''^{|\alpha|}\frac{\alpha!}{|\alpha|!(|\alpha|+1)^{n-1}}\Big)^{-1}\\&=M_{L'}(|\alpha|+1)^{n-1}\Big(\frac{L'}{L''}\Big)^{|\alpha|}\frac{|\alpha|!}{\alpha!L'^{|\alpha|}}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{\alpha\in \boldsymbol{N}^n} a_\alpha x^\alpha&\ll R_{L'}\sum_{\alpha\in \boldsymbol{N}^n}\frac{|\alpha|!}{\alpha!L'^{|\alpha|}}x^\alpha\\&=R_{L'}\Big(1-\frac{x_1+\cdots+x_n}{L'}\Big)^{-1}\end{aligned}$$

with $R_{L'}=\max_{i\geqq 0} M_{L'}(i+1)^{n-1}\Big(\frac{L'}{L''}\Big)^i$ $(<\infty)$.

We will show (2.25). It is clear when $\alpha=0$. Therefore we may assume $\alpha_1\neq 0$. We define $c\in \boldsymbol{R}^n_+$ such that $\frac{c_i}{\alpha_i}=\frac{c_1}{\alpha_1}$ (resp. $c_i=0$) if $\alpha_i\neq 0$ (resp. $\alpha_i=0$). Let $\beta\in \boldsymbol{N}^n$ with $|\beta|=|\alpha|$. If $\alpha_i\geqq\beta_i$, then

$$\frac{c_i^{\alpha_i}}{\alpha_i!}=\frac{c_i^{\alpha_i-\beta_i}}{\alpha_i(\alpha_i-1)\cdots(\beta_i+1)}\frac{c_i^{\beta_i}}{\beta_i!}\geqq\Big(\frac{c_i}{\alpha_i}\Big)^{\alpha_i-\beta_i}\cdot\frac{c_i^{\beta_i}}{\beta_i!}=\Big(\frac{c_1}{\alpha_1}\Big)^{\alpha_i-\beta_i}\cdot\frac{c_i^{\beta_i}}{\beta_i!}.$$

If $\alpha_i<\beta_i$, then

$$\frac{c_i^{\alpha_i}}{\alpha_i!}=\frac{\beta_i(\beta_i-1)\cdots(\alpha_i+1)}{c_i^{\beta_i-\alpha_i}}\frac{c_i^{\beta_i}}{\beta_i!}\geqq\left(\frac{\alpha_i}{c_i}\right)^{\beta_i-\alpha_i}\cdot\frac{c_i^{\beta_i}}{\beta_i!}=\left(\frac{\alpha_1}{c_1}\right)^{\beta_i-\alpha_i}\cdot\frac{c_i^{\beta_i}}{\beta_i!}.$$

Hence

$$\frac{c^\alpha}{\alpha!}\geqq\prod_i\left\{\left(\frac{c_i}{\alpha_i}\right)^{\alpha_i-\beta_i}\cdot\frac{c_i^{\beta_i}}{\beta_i!}\right\}=\frac{c^\beta}{\beta!}$$

and

$$L''^{|\alpha|}=(c_1+\cdots+c_n)^{|\alpha|}=\sum_{\substack{\beta\in N^n\\|\beta|=|\alpha|}}\frac{|\beta|!}{\beta!}c^\beta\leqq(|\alpha|+1)^{n-1}\cdot\frac{|\alpha|!}{\alpha!}\cdot c^\alpha$$

because $\#\{\beta\in N^n\,|\,|\beta|=|\alpha|\}\leqq(|\alpha|+1)^{n-1}$. Thus we have (2.25). Q.E.D.

For the operators in $\mathscr{D}_Y^*(X)$ we have the following theorem:

Theorem 2.4. *Let X be a domain in $\boldsymbol{C}^{1+n}$ containing the origin and Y a submanifold defined by $t=0$. Let $P=(P_{ij}(t, x; \Theta, D_x))$ be a matrix in $M(N; \mathscr{D}_Y^*(X))$ and r be a positive integer such that $P_{ij}\left(t^r, x; \frac{\Theta}{r}, D_x\right)$ has R.S. along Y for $i, j=1,\cdots, N$ (cf. Remark 1.5). For K and L in $\boldsymbol{R}_+$ we put*

$$U_{K,L}^r=\{(t, x)\in\boldsymbol{C}^{1+n}|\,K|t|^{\frac{1}{r}}+|x|<L\}.$$

Let L be a positive real number such that X contains the closure of $U_{1,L}^r$. We assume P satisfies a) *and* b) *in Theorem* 2.1. *Let $s_1(x),\cdots, s_m(x)$ be the roots of* $\det(\sigma_*(P))=0$. *Here we assume*

(A.1)′ $\qquad rs_\nu(x)\notin\boldsymbol{N}\quad$ *for* $\quad\nu=1,\cdots, m$

in place of (A.1). *Then there exists $K_0\in\boldsymbol{R}_+$ such that the map*

$$P\colon\mathscr{O}(U_{K,L}^r)^N\longrightarrow\mathscr{O}(U_{K,L}^r)^N$$

is a surjective homeomorphism for any K satisfying $K>K_0$.

Proof. For any $u=\sum_{(i,\alpha)\in N^{1+n}}u_{i,\alpha}t^ix^\alpha$ in $\hat{\mathscr{O}}^N(u_{i,\alpha}\in\boldsymbol{C}^N)$, we put $u_*=\sum_{(i,\alpha)\in N^{1+n}}u_{i,\alpha}t^{ri}x^\alpha$ and $u^*=\sum_{(i,\alpha)\in N^{1+n}}u_{ri,\alpha}t^ix^\alpha$. Moreover we put $P_*=\left(P_{ij}\left(t^r, x; \frac{\Theta}{r}, D_x\right)\right)$. Since P_* satisfies the assumptions in Theorem 2.1, there exists $K_0\geqq 1$ such that if $K>K_0$ then the map $P_*\colon\mathscr{O}(U_{K,L})^N\to\mathscr{O}(U_{K,L})^N$ is bijective.

We choose K_0 and K as above and define the linear maps

$$\Phi\colon\begin{array}{ccc}\mathscr{O}(U_{K,L}^r)^N&\longrightarrow&\mathscr{O}(U_{K,L})^N\\ \cup&&\cup\\ u&\longmapsto&u_*\end{array}\quad\text{and}\quad\Psi\colon\begin{array}{ccc}\mathscr{O}(U_{K,L})^N&\longrightarrow&\mathscr{O}(U_{K,L}^r)^N\\ \cup&&\cup\\ u&\longmapsto&u^*.\end{array}$$

Let I_N be the identity map on $\mathcal{O}(U_{K,L})^N$. Then $\mathcal{O}(U_{K,L})^N = \Phi\Psi(\mathcal{O}(U_{K,L})^N) \oplus (I_N - \Phi\Psi)(\mathcal{O}(U_{K,L})^N)$, $P_* \Phi\Psi(\mathcal{O}(U_{K,L})^N) \subset \Phi\Psi\mathcal{O}(U_{K,L})^N$ and $P_*(I_N - \Phi\Psi)(\mathcal{O}(U_{K,L})^N) \subset (I_N - \Phi\Psi)(\mathcal{O}(U_{K,L})^N)$. Therefore since the map P_* is a bijective transformation on $\mathcal{O}(U_{K,L})^N$, it also defines a bijective transformation on $\Phi\Psi(\mathcal{O}(U_{K,L})^N) = \Phi(\mathcal{O}(U^r_{K,L})^N)$. Moreover since $\Psi\Phi$ is the identity transformation on $\mathcal{O}(U^r_{K,L})^N$, the map $P = \Psi P_* \Phi$ is a linear bijective transformation on $\mathcal{O}(U^r_{K,L})^N$. Hence P is a homeomorphism because of the open mapping theorem. Q. E. D.

The following theorem is fundamental in the theory of differential equations. It is an easy corollary of Theorem 2.1 but will not be referred later.

Theorem 2.5 (Cauchy-Kowalevsky). *Let $Q(t, x; D_t, D_x)$ be a differential operator of order m defined in a neighborhood of $\bar{U}_{1,L}$. Assume $\sigma_m(P)(0, x, 1, 0) \neq 0$ for any $x \in \bar{V}_L$. Then there exists a positive number K_0 such that the linear map*

$$(2.26)\qquad \begin{array}{ccc} \mathcal{O}(U_{K,L}) & \longrightarrow & \mathcal{O}(U_{K,L}) \oplus \left(\bigoplus_{i=0}^{m-1} \mathcal{O}(V_L)\right) \\ \text{Ⅲ} & & \text{Ⅲ} \\ u & \longmapsto & (Qf;\ D_t^0 f|_{t=0}, \cdots, D_t^{m-1} f|_{t=0}) \end{array}$$

is a surjective homeomorphism for any $K > K_0$.

Proof. Put $Q = \sum q_{i,\alpha}(t, x) D_t^i D_x^\alpha$ and $P = Qt^m$. Since

$$\begin{aligned} P &= \sum q_{i,\alpha}(t, x) D_t^i D_x^\alpha t^m \\ &= \sum q_{i,\alpha}(t, x)(\Theta + 1)\cdots(\Theta + i) t^{m-i} D_x^\alpha, \end{aligned}$$

P has R.S. along Y and $\sigma_*(P) = q_{m,0}(0, x)(s+1)\cdots(s+m)$. Since the characteristic exponents of P are $-1, -2, \cdots, -m$, the operator P satisfies the assumptions in Theorem 2.1 and hence there exists a positive number K_0 such that the map

$$(2.27)\qquad P = Qt^m \colon \mathcal{O}(U_{K,L}) \xrightarrow{\sim} \mathcal{O}(U_{K,L})$$

is a bijection for any $K > K_0$.

We indentify $\mathcal{O}(V_L)$ with a subspace of $\mathcal{O}(U_{K,L})$ by the coordinate system (t, x). For any $(f; v_0, \cdots, v_{m-1}) \in \mathcal{O}(U_{K,L}) \oplus (\bigoplus_{i=0}^{m-1} \mathcal{O}(V_L))$, there exists a function $w \in \mathcal{O}(U_{K,L})$ which satisfies

$$Qt^m w = f - Q\left(v_0 + \frac{t}{1!} v_1 + \cdots + \frac{t^{m-1}}{(m-1)!} v_{m-1}\right)$$

because (2.27) is surjective. Putting $u = t^m w + v_0 + \frac{t}{1!} v_1 + \cdots + \frac{t^{m-1}}{(m-1)!} v_{m-1}$, we have $(Qu;\ D_t^0 u|_{t=0},\cdots,\ D_t^{m-1}u|_{t=0})=(f;\ v_0,\cdots,\ v_{m-1})$. This means (2.26) is surjective.

Let $u\in\mathcal{O}(U_{K,L})$ such that $D_t^i u|_{t=0}=0$ for $i=0,\cdots,m-1$. Then $u=t^m w$ with a function $w\in\mathcal{O}(U_{K,L})$. Suppose $Qu=0$. Then $Qt^m w=0$ and (2.27) proves $w=0$. Hence (2.26) is bijective.

Thus we have proved that (2.26) is a continuous bijective linear map and hence (2.26) is a homeomorphism. Q.E.D.

Corollary 2.6. *Let P be a differential operator of order m defined in a neighborhood of $\bar{U}_{1,L}$. Assume P has R.S. along $\bar{V}_L$ in the weak sense. Let $s_1(x),\cdots,s_m(x)$ be its characteristic exponents. Assume moreover $s_1(x)\equiv 0$ and $rs_\nu(x)\notin \boldsymbol{Z}_+$ for any $x\in V_L$ and $\nu=2,\cdots,m$ under the notation in Theorem 2.4. Then we have the bijections*

$$\begin{array}{ccc}\mathcal{O}(U^r_{K,L}) & \xrightarrow{\sim} & t\mathcal{O}(U^r_{K,L})\oplus\mathcal{O}(V_L)\\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid\\ u & \longmapsto & (Pu,\ u|_{t=0})\end{array}\tag{2.28}$$

and

$$\begin{array}{ccc}\{u\in\mathcal{O}(U^r_{K,L})\,|\,Pu=0\} & \xrightarrow{\sim} & \mathcal{O}(V_L)\\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid\\ u & \longmapsto & u|_{t=0}\,.\end{array}\tag{2.29}$$

Proof. Under the expression (1.3), $a_0=0$ because $s_1(x)\equiv 0$. Hence $P=tQ$ with a differential operator Q. Since $Qt=t^{-1}Pt^1=P(t,\ x;\ \Theta+1,\ D_x)$, Qt has R.S. in the weak sense and any of its characteristic exponents never take values in $\boldsymbol{N}$ for any $x\in\bar{V}_L$. Therefore Theorem 2.4 implies Qt: $\mathcal{O}(U^r_{K,L})\xrightarrow{\sim}\mathcal{O}(U^r_{K,L})$.

Note that if $u\in\mathcal{O}(U^r_{K,L})$, then $Pu=tQu\in t\mathcal{O}(U^r_{K,L})$. For arbitrary functions $f\in\mathcal{O}(U^r_{K,L})$ and $v\in\mathcal{O}(V_L)$, we have $(Qt)w=f-Qv$ with $w\in\mathcal{O}(U^r_{K,L})$. Hence $P(tw+v)=tf$ and $(tw+v)|_{t=0}=v$. On the other hand if a function $u\in\mathcal{O}(U^r_{K,L})$ satisfies $u|_{t=0}=0$ and $Pu=0$. Then $u=tw$ with $w\in\mathcal{O}(U^r_{K,L})$ and $tQtw=0$. Hence $w=0$ and therefore $u=0$. Thus we have proved (2.28) is bijective and hence (2.29) is also bijective. Q.E.D.

Now we consider the following situation: There exist manifolds Ω, X' and Y' of dimension r, $1+n'$ and n', respectively, such that $Y=\Omega\times Y'$ and $X=\Omega\times X'$. Then $n=r+n'$. Let $\lambda=(\lambda_1,\cdots,\lambda_r)$, $z=(z_1,\cdots,z_{n'})$ and $(t,\ z)=(t,\ z_1,\cdots,z_{n'})$ be the coordinate systems of Ω, Y' and X', respectively, such that Y' is defined by $t=0$. A holomorphic differential operator P on X is said to be a differential operator on X' with a holomorphic parameter λ if P is of the form $P=P(\lambda, t, z; D_t, D_z)$, that is,

$$[P, \lambda_i]=0 \qquad \text{for} \quad i=1,\cdots, r.$$

Now suppose that $P(\lambda, t, z; D_t, D_z)$ has regular singularities along Y and assume the conditions:

(A.2) Any characteristic exponent $s_\nu(\lambda, z)$ of P does not depend on z.

(A.3) $s_\nu(\lambda, z)-s_{\nu'}(\lambda, z)\notin \boldsymbol{Z}$ for any $(\lambda, z)\in Y$ and $\nu\neq\nu'$.

Then consider the equation

$$Pu=0. \tag{2.30}$$

Definition 2.7. The solution u of the equation (2.30) is called an *ideally analytic solution* if u is of the form

$$u=\sum_{\nu=1}^{m} a_\nu(\lambda, t, z)t^{s_\nu(\lambda)} \tag{2.31}$$

with holomorphic functions $a_\nu(\lambda, t, z)$ defined on a neighborhood of Y.

Corollary 2.8. *We can choose a neighborhood U of Y satisfying the following: For any $(b_1(\lambda, z),\cdots, b_m(\lambda, z))\in\mathcal{O}(Y)^m$ there exists a unique $(a_1(\lambda, t, z),\cdots, a_m(\lambda, t, z))\in\mathcal{O}(U)^m$ such that the function u given by* (2.31) *is a solution of* (2.30) *with the condition $a_\nu(\lambda, 0, z)=b_\nu(\lambda, z)$ for $\nu=1,\cdots, m$.*

Proof. Put $Q_\nu=t^{-s_\nu(\lambda)}Pt^{s_\nu(\lambda)}$ for $\nu=1,\cdots, m$. Since P is of the form $P(\lambda, t, z; \Theta, tD_z)$, $Q_\nu=P(\lambda, t, z; \Theta+s_\nu(\lambda), tD_z)$. Hence Q_ν has R.S. along Y and its characteristic exponents are

$$s_\mu(\lambda)-s_\nu(\lambda) \quad (\mu=1,\cdots, m).$$

Therefore Q_ν satisfies the assumption in Corollary 2.6 and there exists a unique function $a_\nu(\lambda, t, z)$ in $\mathcal{O}(U)$ such that $P_\nu a_\nu=0$ and that $a_\nu(\lambda, 0, z)=b_\nu(\lambda, z)$. Here U is a suitable neighborhood of Y. The function u defined by (2.31) is clearly a desired solution of (2.30).

Let u be a function of the form (2.31). Since

$$Pu=\sum_{\nu=1}^{m} Pt^{s_\nu(\lambda)}a_\nu=\sum_{\nu=1}^{m} t^{s_\nu(\lambda)}Q_\nu a_\nu$$

and $t^{s_1(\lambda)},\cdots, t^{s_m(\lambda)}$ are linearly independent over $\mathcal{O}(U)$, the condition $Pu=0$ implies $Q_\nu a_\nu=0$. Thus we have the uniqueness of the solution by Corollary 2.6.

Q.E.D.

§3. Definition of Boundary Values

Let M be a $(1+n)$-dimensional real analytic manifold and N an n-dimensional submanifold of M. Assume M is devided by N into two connected components M_+ and M_-. We choose a local coordinate system $(t, x)=(t, x_1,\cdots, x_n)$ of M so that N, M_+ and M_- are defined by $t=0$, $t>0$ and $t<0$, respectively. Let Ω be a domain in $\boldsymbol{C}^r$. Let P be a differential operator on M with a holomorphic parameter $\lambda\in\Omega$. We assume that there exist a complex neighborhood X of M and a complexification Y of N in X such that P can be extended to a holomorphic differential operator on X with a holomorphic parameter $\lambda\in\Omega$. For simplicity we identify P and the extension. We moreover assume that if we regard P as a differential operator on $\Omega\times X$, P has R.S. along $\Omega\times Y$. In this case a differential operator P on M is said to have R.S. along N. We assume that any characteristic exponents of P does not depend on x but depends holomorphically on $\lambda\in\Omega$. We denote the order of P by m and the characteristic exponents by $s_1(\lambda),\cdots, s_m(\lambda)$.

For a real analytic manifold U, let $\mathscr{A}(U)$ (resp. $\mathscr{C}^\infty(U)$, $\mathscr{D}'(U)$ and $\mathscr{B}(U)$) denote the linear space of all analytic functions (resp. infinitely differentiable functions, Schwartz's distributions and Sato's hyperfunctions) on U. If $\mathscr{F}$ is $\mathscr{A}$, $\mathscr{C}^\infty$, $\mathscr{D}'$ or $\mathscr{B}$ and W is a subset of U, then we set

$$\mathscr{F}_U[W]=\{f\in\mathscr{F}(U)\mid \operatorname{supp} f\subset W\},$$
$$\mathscr{F}(U)_c=\{f\in\mathscr{F}(U)\mid \operatorname{supp} f \text{ is compact}\}.$$

Let ${}_\Omega\mathscr{B}(M)$ denote the linear space of all hyperfunctions on M with a holomorphic parameter $\lambda\in\Omega$, that is,

$${}_\Omega\mathscr{B}(M)=\{f\in\mathscr{B}(\Omega\times M)\mid \bar\partial_{\lambda_i}f=0 \text{ for } i=1,\cdots, r\}.$$

Here $\bar\partial_{\lambda_i}=\frac{1}{2}\left(\frac{\partial}{\partial\xi_i}+\sqrt{-1}\frac{\partial}{\partial\eta_i}\right)$ if $\lambda_i=\xi_i+\sqrt{-1}\eta_i$ with ξ_i, $\eta_i\in\boldsymbol{R}$.

In this section we always use the above notation and assume that P is the above differential operator. Then the following theorem is essential to define the boundary values of the solutions of the equation $Pu=0$.

Theorem 3.1. *Put*

$${}_\Omega\mathscr{B}^P(M_+)=\{u\in{}_\Omega\mathscr{B}(M_+)\mid Pu=0\},$$
$${}_\Omega\mathscr{B}^P[\overline{M}_+]=\{u\in{}_\Omega\mathscr{B}^P(M)\mid \operatorname{supp} u\subset\Omega\times\overline{M}_+,\ Pu=0\},$$
$${}_\Omega\mathscr{D}'^P(M_+)={}_\Omega\mathscr{B}^P(M_+)\cap{}_\Omega\mathscr{D}'(M_+),$$
$${}_\Omega\mathscr{D}'^P[\overline{M}_+]={}_\Omega\mathscr{B}^P[\overline{M}_+]\cap{}_\Omega\mathscr{D}'(M).$$

If any characteristic exponent of P does not take the value in negative integers, then the restriction maps

$$\iota_P : {}_{\Omega}\mathscr{B}^P[\overline{M}_+] \xrightarrow{\sim} {}_{\Omega}\mathscr{B}^P(M_+) \tag{3.1}$$

$$\cup \qquad\qquad \cup$$

$${}_{\Omega}\mathscr{D}'^P[\overline{M}_+] \xrightarrow{\sim} {}_{\Omega}\mathscr{D}'^P(M_+) \cap (\mathscr{D}'(\Omega \times M)|_{\Omega\times M_+}) \tag{3.2}$$

are bijective.

To prove the theorem we prepare:

Lemma 3.2. *Under the same assumption as in Theorem* 3.1, *the following maps are bijective.*

$$P: \mathscr{B}_{\Omega\times M}[\Omega \times N] \xrightarrow{\sim} \mathscr{B}_{\Omega\times M}[\Omega \times N] \tag{3.3}$$

$$\cup \qquad\qquad \cup$$

$$\mathscr{D}'_{\Omega\times M}[\Omega \times N] \xrightarrow{\sim} \mathscr{D}'_{\Omega\times M}[\Omega \times N]. \tag{3.4}$$

Proof. Let P^* be the adjoint operator of P. Put

$$P = a(\lambda, x, \Theta) + tQ(\lambda, t, x; \Theta) + \sum_{\substack{k+|\alpha|\leq m \\ |\alpha|>0}} r_{k,\alpha}(\lambda, t, x)\Theta^k(tD_x)^\alpha . \tag{3.5}$$

Then $a(\lambda, x, s)$ is the indicial polynomial of P. Since $\Theta^* = (tD_t)^* = -\Theta - 1$, P^* has also R.S. along N whose indicial polynomial equals $a(\lambda, x, -s-1)$. Therefore the characteristic exponents of P^* are $-1-s_1(\lambda), \cdots, -1-s_m(\lambda)$. The assumption implies that any of them does not take the value in non-negative integers. Hence for any compact subset K of $\Omega \times N$, we have the topological isomorphism

$$P^*: \mathscr{A}_{\Omega\times M}(K) \xrightarrow{\sim} \mathscr{A}_{\Omega\times M}(K) \tag{3.6}$$

by Theorem 2.2. Here the space $\mathscr{A}_{\Omega\times M}(K)$ has the natural topology induced by the inductive limit of the Fréchet spaces $\mathscr{A}(U)$, where U runs over fundamental neighborhoods of K in a complexification of $\Omega \times M$. Then the topological dual of the map (3.6) means

$$P: \mathscr{B}_{\Omega\times M}[K] \xrightarrow{\sim} \mathscr{B}_{\Omega\times M}[K]. \tag{3.7}$$

Now consider the map (3.3). Let f be a function in $\mathscr{B}_{\Omega\times M}[\Omega \times N]$. Since the sheaf of hyperfunctions is flabby, f can be expressed as a locally finite sum $f = \sum f_i$, where $f_i \in \mathscr{B}_{\Omega\times M}[\Omega \times N]$ and the support of f_i is compact. Then the map (3.7) implies that there exist $g_i \in \mathscr{B}_{\Omega\times M}[\operatorname{supp} f_i]$ satisfying $Pg_i = f_i$. Since $P\sum g_i = f$, we see that the map (3.4) is surjective. Suppose $Pf = 0$ to prove the injectivity of the map. For any point p of $\Omega \times N$ there exists a neighborhood V

of p such that the set $I=\{i|(\mathrm{supp} f_i)\cap V\neq\phi\}$ is finite. *Put* $f'=\sum_{i\in I} f_i$. Then it follows from (3.7) that there exists $h'\in\mathscr{B}_{\Omega\times M}[\mathrm{supp}\, Pf']$ which satisfies $Ph'=Pf'$. The injectivity of (3.7) means $f'=h'$ and hence f equals 0 in the neighborhood V of p because $f|_V=f'|_V=h'|_V$ and $\mathrm{supp}\, Pf|_V=0$. Thus we have proved that the map (3.3) is bijective.

Next consider the case of distributions. We can prove that the map (3.4) is bijective by the same way as in the case of hyperfunctions but we use a more explicit method. Let V be any relatively compact open subset of $\Omega\times M$. Then any element $f\in\mathscr{D}'_{\Omega\times M}[\Omega\times N]$ is locally of the form

$$f|_V=\sum_{i=0}^{l} f_i(\lambda, x)\delta^{(i)}(t) \tag{3.8}$$

with distributions $f_i(\lambda, x)$. Here $\delta^{(i)}(t)$ is the i-th derivative of the Dirac's δ-function of the variable t. Using the expression (3.5) we have

$$Pf_i(\lambda, x)\delta^{(i)}(t)=a(\lambda, x, -i-1)f_i(\lambda, x)\delta^{(i)}(t)+\sum_{\nu=0}^{i-1} g_\nu(\lambda, x)\delta^{(\nu)}(t) \tag{3.9}$$

with suitable distributions $g_\nu(\lambda, x)$. We note that the assumption says that $a(\lambda, x, -i-1)$ does not vanish at any point in $\Omega\times N$. Hence considering (3.8) and (3.9), we can easily prove by the induction on l that the restriction of the map (3.4) on V is both injective and surjective. This proves that the map (3.4) is bijective. Q.E.D.

Proof of Theorem 3.1. Let $f\in\mathscr{B}(\Omega\times M_+)$ which satisfies $Pf=0$. In view of the flabbiness of the sheaf of hyperfunctions there exists a function $g\in\mathscr{B}(\Omega\times M)$ such that $g|_{\Omega\times M_+}=f$ and $g|_{\Omega\times M_-}=0$. Since $Pg\in\mathscr{B}_{\Omega\times M}[\Omega\times N]$, Lemma 3.2 proves the existence of $h\in\mathscr{B}_{\Omega\times M}[\Omega\times N]$ with $Pg=Ph$. Replacing g by $g-h$, we may moreover assume the condition $Pg=0$. Thus we have proved that the restriction map

$$\mathscr{B}^P_{\Omega\times M}[\Omega\times\overline{M}_+]\longrightarrow\mathscr{B}^P(\Omega\times M_+) \tag{3.10}$$

is surjective, where $\mathscr{B}^P_{\Omega\times M}[\Omega\times\overline{M}_+]=\{u\in\mathscr{B}_{\Omega\times M}[\Omega\times\overline{M}_+]|Pu=0\}$ and $\mathscr{B}^P(\Omega\times M_+)=\{u\in\mathscr{B}(\Omega\times M_+)|Pu=0\}$. The injectivity of the map (3.10) also follows from the same property of (3.3).

Under the above notation we assume $f\in{}_\Omega\mathscr{B}^P(M_+)$. Then $P(\bar{\partial}_{\lambda_i}g)=\bar{\partial}_{\lambda_i}(Pg)=0$ and $\mathrm{supp}\,\bar{\partial}_{\lambda_i}g\subset\Omega\times N$ for $i=1,\cdots, r$. Hence the injectivity of (3.3) implies $\bar{\partial}_{\lambda_i}g=0$. Thus we can conclude that the map (3.1) is bijective.

In the same way as above we can prove that the map (3.2) is bijective if we note that for any $f \in \mathscr{D}'(\Omega \times M)|_{\Omega \times M_+}$ there exists an extension $g \in \mathscr{D}'(\Omega \times M)$ such that $g|_{\Omega \times M_+} = f$ and that $\operatorname{supp} f \subset \Omega \times \overline{M}_+$ (cf. [2]). Q.E.D.

Now we will define boundary values of the solutions of the equation

$$(3.11) \qquad P(\lambda, t, x; \Theta, tD_x)u = 0,$$

where the operator P satisfies the assumption mentioned in the first part of this section. We pay attention to one characteristic exponent of P, say $s_i(\lambda)$, and assume

$$(3.12)_i \qquad s_i(\lambda) - s_\nu(\lambda) \notin \boldsymbol{Z}_+ \quad \text{for any} \quad \lambda \in \Omega \quad \text{and} \quad \nu = 1, \cdots, m.$$

Let $u(\lambda, t, x)$ be a function in ${}_\Omega\mathscr{B}^P(M_+)$, that is, u is a hyperfunction solution of the equation (3.11) which is defined on M_+ and has the holomorphic parameter $\lambda \in \Omega$.

There exists a differential operator $Q_i(\lambda, t, x; D_t, D_x)$ which satisfies

$$(3.13) \qquad t^{-s_i(\lambda)} P t^{s_i(\lambda)} = tQ_i.$$

In fact, since $s_i(\lambda)$ is a characteristic exponent of P, P is of the form $P = (\Theta - s_i(\lambda)) \cdot b(\lambda, x, \Theta) + tB(\lambda, t, x; \Theta, D_x)$ and therefore $Q_i = D_t b(\lambda, x, \Theta + s_i(\lambda)) + B(\lambda, t, x; \Theta + s_i(\lambda), D_x)$. Here we remark that the operator $P_i = tQ_i$ has R.S. along N and its characteristic exponents are $s_\nu(\lambda) - s_i(\lambda)$ $(\nu = 1, \cdots, m)$. Moreover we have $P_i t^{-s_i(\lambda)} u = t^{-s_i(\lambda)} Pu = 0$. Then in view of Theorem 3.1 we have a unique $\tilde{u}_i \in {}_\Omega\mathscr{B}^{P_i}[\overline{M}_+]$ such that $\tilde{u}_i|_{\Omega \times M_+} = t^{-s_i(\lambda)} u$ because $(3.12)_i$ assures the assumption in the Theorem 3.1. Since $tQ_i \tilde{u}_i = 0$, we have

$$(3.14) \qquad Q_i \tilde{u}_i = \phi_i(\lambda, x)\delta(t)$$

with $\phi_i(\lambda, x) \in {}_\Omega\mathscr{B}(N)$. That is,

$$(3.15) \qquad \phi_i(\lambda, x) = \int Q_i \iota_{P_i}^{-1}(t^{-s_i(\lambda)} u(\lambda, t, x)) dt,$$

where ι_{P_i} is the map defined in Theorem 3.1.

Definition 3.3. The function $\phi_i(\lambda, x) \in {}_\Omega\mathscr{B}(N)$ defined as above is called the boundary value of the solution $u(\lambda, t, x) \in {}_\Omega\mathscr{B}^P(M_+)$ with respect to the characteristic exponent $s_i(\lambda)$.

Remark 3.4. If an equation $Q(t, x; D_t, D_x)u(t, x) = 0$ of order m is non-characteristic with respect to N, then

$$(3.16) \qquad Q(\iota_{t^m Q}^{-1}(u(t, x))) = \sum_{i=0}^{m-1} \phi_i(x)\delta^{(i)}(t)$$

with $\phi_i(x) \in \mathscr{B}(N)$. Then $(\phi_0(x), \cdots, \phi_{m-1}(x))$ is called the boundary value of $u(t, x)$ by Komatsu-Kawai [6] and Schapira [9].

Now we consider the boundary values of the distribution solutions or the ideally analytic solutions:

Theorem 3.5. *Retain the notation in Definition* 3.3.

1) *If* $u \in {}_\Omega\mathscr{D}'^P(M_+) \cap \mathscr{D}'(\Omega \times M)|_{\Omega \times M_+}$, *then* $\phi_i(\lambda, x) \in {}_\Omega\mathscr{D}'(N)$.

2) *Let* $a(\lambda, x, s)$ *be the indicial polynomial of* P. *If*

$$s_i(\lambda) - s_j(\lambda) \notin \boldsymbol{Z} \qquad \text{for any} \quad i, j = 1, \cdots, m \quad \text{and} \quad \lambda \in \Omega \tag{3.17}$$

and u *is an ideally analytic solution of the form*

$$u = \sum_{\nu=1}^{m} f_\nu(\lambda, t, x) t^{s_\nu(\lambda)}, \tag{3.18}$$

namely $f_\nu(\lambda, t, x)$ *are analytic in a neighborhood of* $\Omega \times N$, *then*

$$\phi_i(\lambda, x) = \left(\frac{a(\lambda, x, s + s_i(\lambda))}{s} \Big|_{s=0} \right) f_i(\lambda, 0, x). \tag{3.19}$$

Proof. 1) If $u \in \mathscr{D}'(\Omega \times M)|_{\Omega \times M_+}$, then $t^{-s_i(\lambda)} u \in \mathscr{D}'(\Omega \times M)|_{\Omega \times M_+}$ (cf. [2]). Hence the first part of the theorem is a direct consequence of Theorem 3.1 and the definition of the boundary value.

2) Since the map of taking the boundary values is $\boldsymbol{C}$-linear, we have only to determine the boundary value of the solution $f_\nu(\lambda, t, x) t^{s_\nu(\lambda)}$ with respect to the characteristic exponent $s_\nu(\lambda)$ $(\nu = 1, \cdots, m)$. Put $\tilde{u}_i = f_\nu(\lambda, t, x) t_+^{s_\nu(\lambda) - s_i(\lambda)}$. Here t_+^s is a distribution-valued meromorphic function of s whose poles are negative integers (cf. [1], [8]).

In general for analytic functions $f(\lambda, t, x)$ and $s(\lambda)$, where $s(\lambda)$ is never equal to any negative integer,

$$\Theta(f(\lambda, t, x) t_+^{s(\lambda)}) = (\Theta(f(\lambda, t, x)) + s(\lambda) f) t_+^{s(\lambda)},$$
$$D_{x_k}(f(\lambda, t, x) t_+^{s(\lambda)}) = (D_{x_k} f) t_+^{s(\lambda)}.$$

Hence if Q is a differential operator of the form $Q(\lambda, t, x; \Theta, D_x)$ and $Q(f(\lambda, t, x) t^{s(\lambda)}) = g(\lambda, t, x) t^{s(\lambda)}$ on $\Omega \times M_+$ with a function g, then $Q(f(\lambda, t, x) t_+^{s(\lambda)}) = g(\lambda, t, x) t_+^{s(\lambda)}$.

Since $f_\nu(\lambda, t, x) t^{s_\nu(\lambda) - s_i(\lambda)}$ is a solution of the equation

$$(t^{-s_i(\lambda)} P t^{s_i(\lambda)}) u = 0, \tag{3.20}$$

so the function $\tilde{u}_i$. Hence $\tilde{u}_i = \iota_{tQ_i}^{-1}(t^{-s_i(\lambda)} f_\nu(t, \lambda, x) t^{s_\nu(\lambda)})$.

On the other hand, since $a(\lambda, x, s_i(\lambda)) = 0$,

$$a(\lambda, x, s+s_i(\lambda))=sb(\lambda, x, s) \tag{3.21}$$

with a polynomial $b(\lambda, x, s)$ of s. Then

$$t^{-s_i(\lambda)}Pt^{s_i(\lambda)}=\Theta b(\lambda, x, \Theta)+tR(\lambda, t, x; \Theta, D_x)$$

with a differential operator $R(\lambda, t, x; \Theta, D_x)$, which implies

$$Q_i=D_t b(\lambda, x, \Theta)+R(\lambda, t, x; \Theta, D_x). \tag{3.22}$$

Consider the case where $\nu \neq i$. It follows from this expression that

$$\begin{aligned} Q_i\tilde{u}_i &= Q_i(f_\nu(\lambda, t, x)t_+^{s_\nu(\lambda)-s_i(\lambda)}) \\ &= \psi_\nu(\lambda, t, x)t_+^{s_\nu(\lambda)-s_i(\lambda)-1} \end{aligned}$$

with an analytic function ψ_ν. Since $s_\nu(\lambda)-s_i(\lambda)-1$ does not take the value in negative integers and since $tQ_i\tilde{u}_i=0$, we have $\psi_\nu=0$ and therefore the boundary value is zero.

Next consider the case where $\nu=i$. Here we remark that t_+^0 coincides with the Heaviside's function $Y(t)$. Putting

$$f_i(\lambda, t, x)=f_i(\lambda, 0, x)+g_i(\lambda, t, x)t,$$

we have

$$\begin{aligned} Q_i\tilde{u}_i &= Q_i(f_i(\lambda, t, x)Y(t)) \\ &= Q_i(f_i(\lambda, 0, x)Y(t)+g_i(\lambda, t, x)t_+) \\ &= D_t b(\lambda, x, \Theta)(f_i(\lambda, 0, x)Y(t)) \\ &\quad + R(f_i(\lambda, 0, x)Y(t))+(D_t b+R)(g_i(\lambda, t, x)t_+) \\ &= b(\lambda, x, 0)f_i(\lambda, 0, x)\delta(t)+h_i(\lambda, t, x)Y(t) \end{aligned}$$

with an analytic function h_i. Since $0=tQ_i\tilde{u}_i=h_i(\lambda, t, x)t_+$, we can conclude that $h_i=0$ and that the boundary value equals $b(\lambda, x, 0)f_i(\lambda, 0, x)$. Q. E. D.

The following theorem relates to the induced equations on the boundary.

Theorem 3.6. *Retain the notation in Definition* 3.3. *Let* $R_1(\lambda, t, x; \Theta, D_x)$, $\cdots$, $R_l(\lambda, t, x; \Theta, D_x)$ *be differential operators which satisfy the following conditions*:

a) $R_j(\lambda, t, x; \Theta, D_x)u(\lambda, t, x)=0$ *for* $j=1,\cdots, l$.

b) *There exist differential operators* S_j^k *of the form* $S_j^k(\lambda, t, x; \Theta, tD_x)$ $(j, k=1,\cdots, l)$ *such that*

$$[P, R_j]=\sum_{k=1}^{l} S_j^k R_k, \tag{3.23}$$

$$\operatorname{ord} S_j^k<\operatorname{ord} P+\operatorname{ord} R_j-\operatorname{ord} R_k, \tag{3.24}$$

(3.25) $$\sigma_*(S_j^k)=0 \quad \textit{for} \quad j, k=1,\cdots, l.$$

Then the boundary value $\phi_i(\lambda, x)$ *satisfies*

(3.26) $$R_j(\lambda, 0, x; s_i(\lambda), D_x)\phi_i(\lambda, x)=0 \quad \textit{for} \quad j=1,\cdots, l.$$

Proof. Let R denote the column vector of length l whose k-th component equals R_k and let S denotes the square matrix of size l whose (j, k)-component equals S_j^k. Then the assumption says

(3.27) $$RP=(P+S)R,$$

where P is identified with the scalar matrix whose diagonal components equal P. We will retain the same notation which was used to define the boundary value. Put $P'=t^{-s_i(\lambda)}Pt^{s_i(\lambda)}$, $S'=t^{-s_i(\lambda)}St^{s_i(\lambda)}$ and $R'=t^{-s_i(\lambda)}Rt^{s_i(\lambda)}$. Then $(P'+S')\cdot R'\tilde{u}_i=R'P'\tilde{u}_i=R'tQ_i\tilde{u}_i=0$ and $R'\tilde{u}_i|_{\Omega\times M_+}=t^{-s_i(\lambda)}Ru=0$. On the other hand, using Theorem 2.1, we can prove that the map

(3.28) $$P'+S': \mathscr{B}_{\Omega\times M}[\Omega\times N]^l \longrightarrow \mathscr{B}_{\Omega\times M}[\Omega\times N]^l$$

is bijective in the same way as in the proof of Lemma 3.2 because $\sigma_*(P'+S')$ is a scalar matrix whose j-th component equals $\sigma_*(P)(\lambda, x, s+s_i(\lambda))$. Thus we can conclude that $R'\tilde{u}_i=0$.

Now since $\sigma_*(S')=0$, we have $S'=tT'$ with a suitable matrix T' of differential operators and therefore $(t^{-1}R't)Q_i\tilde{u}_i=(Q_i+T')R'\tilde{u}_i=0$, which means

$$\begin{aligned}R_j(\lambda, 0, x; s_i(\lambda), D_x)\phi_i(\lambda, x)\delta(t)&=R_j(\lambda, t, x; \Theta+s_i(\lambda)+1)\phi_i(\lambda, x)\delta(t)\\&=0\end{aligned}$$

for $j=1,\cdots, l$. Thus we have the theorem. Q.E.D.

The following theorem is used to define the boundary value globally on a manifold.

Theorem 3.7. *Let u be a function in ${}_\Omega\mathscr{B}^P(M_+)$ and use the notation in Definition* 3.3.

1) *The ${}_\Omega\mathscr{B}(N)$-valued section $\phi_i(\lambda, x)(dt)^{s_i(\lambda)}$ of $(T_N^*M)^{\otimes s_i(\lambda)}$ is independent of the choice of local coordinate systems. Namely, let (t', x') be another local coordinate system of M which satisfies $t'=c(t, x)t$ and $x'_j=x'_j(t, x)$ $(j=1,\cdots, n)$ with $c(t, x)>0$ and let $\phi'_i(\lambda, x')$ be the boundary value of u with respect to characteristic exponent $s_i(\lambda)$ which is defined by using the coordinate system (t', x'). Then*

(3.29) $$\phi_i(\lambda, x)=\phi'_i(\lambda, x'(0, x))(c(0, x))^{s_i(\lambda)}.$$

2) *Let* $F_1(\lambda, t, x)$ *and* $F_2(\lambda, t, x)$ *be real analytic functions on* M *which never vanish anywhere and let* $R(\lambda, t, x; \Theta, D_x)$ *be a differential operator with* $R=\sum A_jR_j$, *where* R_j *are the differential operators in Theorem* 3.6 *and* A_j *are suitable differential operators of the form* $A_j(\lambda, t, x; \Theta, D_x)$. *Let* $\phi_i''(\lambda, x)$ *be the boundary value of* F_2u *which is defined by using* $(F_1P+tR)F_2^{-1}$ *in place of* P. *Then* $\phi_i''(\lambda, x)=F_1(\lambda, 0, x)\phi_i(\lambda, x)$.

Proof. 1) Let Q_i' be the differential operator defined by the coordinate system (t', x') which corresponds to Q_i. Then

$$\begin{aligned} t'Q_i' &= t'^{-s_i(\lambda)}Pt'^{s_i(\lambda)} \\ &= c(t, x)^{-s_i(\lambda)}t^{-s_i(\lambda)}Pt^{s_i(\lambda)}c(t, x)^{s_i(\lambda)} \\ &= c(t, x)^{-s_i(\lambda)}tQ_ic(t, x)^{s_i(\lambda)}. \end{aligned}$$

Hence

$$Q_i'=c(t, x)^{-s_i(\lambda)-1}Q_ic(t, x)^{s_i(\lambda)}. \tag{3.30}$$

Since

$$t'^{-s_i(\lambda)}u=c(t, x)^{-s_i(\lambda)}t^{-s_i(\lambda)}u$$

and since

$$t'Q_i'(c(t, x)^{-s_i(\lambda)}\tilde{u}_i)=c(t, x)^{-s_i(\lambda)}tQ_i\tilde{u}_i=0,$$

we have

$$\iota_{t'Q_i'}^{-1}(t'^{-s_i(\lambda)}u)=c(t, x)^{-s_i(\lambda)}\tilde{u}_i$$

and therefore

$$\begin{aligned} Q_i'\iota_{t'Q_i'}^{-1}(t'^{-s_i(\lambda)}u) &= c(t, x)^{-s_i(\lambda)-1}Q_i\tilde{u}_i \\ &= c(t, x)^{-s_i(\lambda)-1}\phi_i(\lambda, x)\delta(t). \end{aligned}$$

Thus by Definition 3.3 we have

$$\begin{aligned} \phi_i'(\lambda, x') &= \int c(t, x)^{-s_i(\lambda)-1}\phi_i(\lambda, x)\delta(t)dt' \\ &= \int \phi_i(\lambda, x)c(t, x)^{-s_i(\lambda)-1}\delta(t)\Big(\frac{\partial c}{\partial t}(t, x)t+c(t, x)\Big)dt \\ &= \phi_i(\lambda, x)c(0, x)^{-s_i(\lambda)}, \end{aligned}$$

which is equivalent to (3.29).

2) It follows from the proof of Theorem 3.6 that $(t^{-s_i(\lambda)}(F_1P+tR)F_2^{-1}t^{s_i(\lambda)})\cdot F_2\tilde{u}_i=F_1tQ_i\tilde{u}_i+\sum tA_j(\lambda, t, x; \Theta+s_i(\lambda), D_x)R_j'\tilde{u}_i=0$.

Since $F_2\tilde{u}_i|_{\Omega\times M_+}=t^{-s_i(\lambda)}F_2u$, We have

$$\begin{aligned}\phi_i''(\lambda, x)\delta(t) &= (t^{-s_i(\lambda)-1}(F_1P+tR)t^{s_i(\lambda)}F_2^{-1})(F_2\tilde{u}_i)\\ &= F_1Q_i\tilde{u}_i+\sum A_j(\lambda, t, x; \Theta+s_i(\lambda), D_x)R'_j\tilde{u}_i\\ &= F_1(\lambda, t, x)\phi_i(\lambda, x)\delta(t)\\ &= F_1(\lambda, 0, x)\phi_i(\lambda, x)\delta(t),\end{aligned}$$

which implies the theorem. Q. E. D.

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