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Period Mapping Associated to a Primitive Form

By

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§ 0. Introduction

The aim of this note is to give a summary of the study of primitive forms and period mappings associated with a universal unfolding F of an isolated hypersurface singularity, which was studied in [35]. There is already a summary [36] on the subject. Compared to that, this note contains mainly two new topics, covering some recent developments in the subject. The details will appear elsewhere.

First, in Section 5 of this note we give a description of the construction of period mappings using a sequence of $\mathscr{D}_S$-modules $\mathscr{M}_F^{(k)}$, $k \in \boldsymbol{Z}$ instead of $\mathscr{E}(0)$-modules $\mathscr{H}_F^{(k)}$, $k \in \boldsymbol{Z}$. This construction might give another insight into the relationship between Poincaré duality of the Milnor fiber and the period mapping of the family F. This part is based on the lectures by the author at R. I. M. S., Kyoto University in the springs '81 and '82.

Secondly, in Section 4 of this note, the existence of primitive forms is reduced to the existence of certain good sections of the "principal symbol module" $q_*\Omega_F$ into the $\mathscr{E}(0)$-module $\mathscr{H}_F^{(0)}$. The technique of the proof comes from the solution of Riemann-Hilbert problem on $\boldsymbol{P}^1$ by B. Malgrange [23] [24] [25] and Birkhoff [4]. The author is grateful to Professor Malgrange for helpful discussions at the Institute Fourier in Grenoble, March '83. Combining this result with a recent result by M. Saito [46], we are now able to construct primitive forms for a large class of singularities.

For the moment the period mapping associated to a primitive form has been studied explicitely only for the cases of simple singularities and simple elliptic singularities (cf. [33] [42] [43]) which might give another approach to studying universal family of such singularities by E. Brieskorn [6] [7],

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E. Looijenga [18] [19] [20] [21], P. Slodowy [47] [48] [49] and H. Pinkham [32] and others.

It might be noted that some recent works of V. Varchenko [56] [57] [58] seem to have a close relation with those in this note.

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Contents

§1. Hamiltonian System F and the Gauß-Manin Connection

In this paragraph, we recall from [41] the notations and definitions concerning the Gauß-Manin connection of a Hamiltonian system F, which are necessary in this note. For general references on $\mathscr{D}_S$- and $\mathscr{E}_S$-modules, cf. [14] [16] [29].

(1.1) *Frame* (Z, X, S, T). Let

(1.1.1)
$$\begin{array}{ccc} (Z, 0) & \xrightarrow{\hat{\pi}} & (X, 0) \\ \downarrow p & & \downarrow q \\ (S, 0) & \xrightarrow{\pi} & (T, 0) \end{array}$$

be a Cartesian product of complex manifolds with reference points 0 such that $\dim S = \dim T + 1 = m$, $\dim Z = \dim X + 1 = n + m + 1$ and π, q are submersions.

Furthermore, let δ_1 and $\hat{\delta}_1$ be holomorphic nonsingular vector fields on S and Z respectively, such that

$$\pi^{-1}\mathcal{O}_T := \{g \in \mathcal{O}_S;\ \delta_1 g = 0\}$$
$$\hat{\pi}^{-1}\mathcal{O}_X := \{g \in \mathcal{O}_Z;\ \hat{\delta}_1 g = 0\}$$
$$p_* \hat{\delta}_1 = \delta_1 .$$

(Here we denote by $\mathcal{O}_A$ the structure sheaf for a complex manifold A and by Der_A the $\mathcal{O}_A$-module of holomorphic vector fields on A and by $\mathscr{D}_A$ the $\mathcal{O}_A$

enveloping algebra of Der_A.)

Define,

$$\mathscr{G}:=\{\delta\in\pi_*\mathrm{Der}_S;\ [\delta_1,\delta]=0\} \tag{1.1.2}$$

Then $\mathscr{G}$ is an $\mathcal{O}_T$-free module of rank m such that

$$0\longrightarrow\mathcal{O}_T\delta_1\hookrightarrow\mathscr{G}\longrightarrow\mathrm{Der}_T\longrightarrow 0 \tag{1.1.3}$$

is exact. (i.e. As a Lie algebra $\mathscr{G}$ is a central extension of Der_T.)

(1.2) *Hamiltonian systems.*

Definition. A Hamiltonian system F on the frame (1.1.1) is one of the following equivalent data i)–iii).

i) A holomorphic function F on Z such that

$$F(0)=0 \quad\text{and}\quad \hat{\delta}_1F=1\,.$$

ii) A section $\iota:(X,0)\to(Z,0)$ of $\hat{\pi}$. (i.e. ι is a holomorphic map s.t. $\iota(0)=0$ and $\hat{\pi}\circ\iota=id_X$.

iii) A holomorphic map $\varphi:(X,0)\to(S,0)$ which commutes with (1.1.1).

If a Hamiltonian system F is given, we shall identify X with a hypersurface of Z given by

$$\iota(X)=(\varphi\times id_X)X=\{x\in Z:\ F(z)=0\}\subset Z\,.$$

(1.3) *Critical set C and discriminant D.* The map $\varphi: X\to S$ may be naturally regarded as an unfolding of a function of $f:=\varphi|_{q^{-1}(0)}=F|_{p^{-1}(0)}$ of $n+1$ variables by the parameter $t'\in T$. In this note we shall always assume that f has an isolated critical point at 0.

Let $C:=C_\varphi$ be the set of critical points of the map φ with the structure sheaf $\mathcal{O}_C:=\mathcal{O}_X/\left(\frac{\partial F}{\partial z_0},\dots,\frac{\partial F}{\partial z_n}\right)\mathcal{O}_X$ where $z=(z_0,\dots,z_n)$ are coordinates of X for the fibers of $q: X\to T$. Then $q_*\mathcal{O}_C$ is an $\mathcal{O}_T$-free module of rank $\mu:=$ Milnor number of f at 0. The image $D:=\varphi(C)\subset S$ is called the discriminant of F. The first fitting ideal of C defines an ideal $\mathscr{I}_D\subset\mathcal{O}_S$ for D, which has a single generator Δ such that $(\delta_1)^\mu\Delta=\mu!$.

For convenience let us choose a coordinate system $t=(t_1,t')=(t_1,t_2,\dots,t_m)$ for S such that $\delta_1t_1=1$, $\delta_1t_i=0$, $i=2,\dots,m$. Then $(z,t)=(z_0,\dots,z_n,t_1,\dots,t_m)$, $(z,t')=(z_0,\dots,z_n,t_2,\dots,t_m)$ and $t'=(t_2,\dots,t_m)$ are coordinates for Z, X and S respectively and δ_1 and $\hat{\delta}_1$ are described by $\frac{\partial}{\partial t_1}$ in these coordinates.

There is an $\mathcal{O}_T$-homomorphism,

$$(1.3.1)\qquad \mathscr{G} \longrightarrow q_*\mathcal{O}_C, \qquad \frac{\partial}{\partial t_i} \longmapsto \left.\frac{\partial F}{\partial t_i}\right|_C,$$

which does not depend on the choice of coordinates above.

Definition. The Hamiltonian system F is called a universal unfolding of f, if (1.3.1) is bijective. In this case, $\mathscr{G}$ naturally inherits a $\pi_*\mathcal{O}_S$-algebra structure of $q_*\mathcal{O}_C$. The product in the algebra will be denoted by $*$. Namely,

$$(\delta*\delta')F|_C=\delta F|_C\cdot\delta' F|_C, \quad (t_1*\delta)F|_C=t_1|_C\cdot\delta F|_C.$$

(1.4) *De-Rham cohomology* $\mathscr{H}_F^{(0)}$ *and the Gauß-Manin connection* ∇. Let $(\Omega^{\cdot}_{X/T}, d)$ be the de-Rham complex relative to q: $X\to T$. Let us define $\mathcal{O}_S$-modules, (cf. Brieskorn [5]),

$$(1.4.1)\qquad \begin{aligned} \mathscr{H}_F^{(0)} &:= \varphi_*\Omega^{n+1}_{X/T}/dF_1\wedge d(\varphi_*\Omega^{n-1}_{X/T}) \\ \mathscr{H}_F^{(-1)} &:= \varphi_*\Omega^{n}_{X/T}/dF_1\wedge \varphi_*\Omega^{n-1}_{X/T}+d(\varphi_*\Omega^{n-1}_{X/T}) \\ &= \varphi_*\Omega^{n}_{X/S}/d(\varphi_*\Omega^{n-1}_{X/S}) \\ \mathscr{H}_F^{(-2)} &:= \ker(d\colon \varphi_*\Omega^{n}_{X/S}\longrightarrow \varphi_*\Omega^{n+1}_{X/S})/d(\varphi_*\Omega^{n-1}_{X/S}) \\ &= \boldsymbol{R}^n\varphi_*(\Omega^{\cdot}_{X/S}, d) \end{aligned}$$

where F_1 is a function on X defined by $F=t_1-F_1$. These modules are $\mathcal{O}_S$-free modules of rank μ.

There are natural injective homomorphisms, which we regard as inclusions,

$$(1.4.2)\qquad \begin{aligned} &\mathscr{H}_F^{(-1)} \hookrightarrow \mathscr{H}_F^{(0)}, \qquad [\omega] \longmapsto [dF_1\wedge\omega] \\ &\mathscr{H}_F^{(-2)} \hookrightarrow \mathscr{H}_F^{(-1)}, \qquad [\omega] \longmapsto [\omega]. \end{aligned}$$

(Here $[\omega]$ means a class in $\mathscr{H}_F^{(-k)}$ represented by a differential form ω.)

Let us denote by $\{\zeta\}$ the image of $\zeta\in\Omega^{n+1}_{X/T}$ in $\Omega^{n+1}_{X/S}$. The module $\Omega^{n+1}_{X/S}$ is $\mathcal{O}_C$-free of rank 1. We shall denote it by Ω_F for short. Then one has the short exact sequences of $\mathcal{O}_S$-modules:

$$(1.4.3)\qquad 0 \hookrightarrow \mathscr{H}_F^{(-k-1)} \hookrightarrow \mathscr{H}_F^{(-k)} \xrightarrow{r^{(-k)}} \varphi_*\Omega_F \longrightarrow 0 \qquad k=0,1,$$

where $r^{(0)}([\zeta])=\{\zeta\}$ and $r^{(-1)}([\omega])=\{d\omega\}$.

The exterior differentiation d induces an integrable covariant differential operator ∇, called the Gauß-Manin connection,

$$(1.4.4)\qquad \nabla: \mathscr{H}_F^{(-k-1)} \longrightarrow \Omega^1_S\otimes\mathscr{H}_F^{(-k)} \qquad k=0,1.$$

In particular, covariant differentiation by $\delta_1=\frac{\partial}{\partial t_1}$ induces $\mathcal{O}_T$-isomorphisms, (1.4.5) $$\nabla_{\delta_1}: \pi_*\mathscr{H}_F^{(-k-1)} \simeq \pi_*\mathscr{H}_F^{(-k)} \qquad k=0,\ 1$$

where $\nabla_{\delta_1}[\zeta]=[dF_1^{-1}d\zeta]$ for $k=1$ and $\nabla_{\delta_1}[\zeta]=[d\zeta]$ for $k=0$, with commutative diagrams,

(1.4.6) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_*\mathscr{H}_F^{(-k-2)} & \longrightarrow & \pi_*\mathscr{H}_F^{(-k-1)} & \xrightarrow{r^{(-k-1)}} & q_*\Omega_F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\nabla_{\delta_1}} & & \downarrow{\scriptstyle\nabla_{\delta_1}} & & \| & & \\ 0 & \longrightarrow & \pi_*\mathscr{H}^{(-k-1)} & \longrightarrow & \pi_*\mathscr{H}_F^{(-k)} & \xrightarrow{r^{(-k)}} & q_*\Omega_F & \longrightarrow & 0 \end{array} \qquad \text{for } k=0.$$

(1.4.7) $$\begin{array}{ccc} \nabla: \mathscr{G}\times\pi_*\mathscr{H}_F^{(-k-2)} & \longrightarrow & \pi_*\mathscr{H}_F^{(-k-1)} \\ \downarrow{\scriptstyle\nabla_{\delta_1}} & & \downarrow{\scriptstyle\nabla_{\delta_1}} \\ \nabla: \mathscr{G}\times\pi_*\mathscr{H}^{(-k-1)} & \longrightarrow & \pi_*\mathscr{H}_F^{(-k)} \end{array} \qquad \text{for } k=0.$$

Define a decreasing $\mathcal{O}_S$-filtration of $\mathscr{H}_F^{(0)}$ by,

$$\mathscr{H}_F^{(-k-1)} := \{\omega\in\mathscr{H}_F^{(-k)}:\ \nabla_{\delta_1}\ \omega\in\mathscr{H}_F^{(-k)}\}$$

$$r^{(-k-1)} := r^{(-k)}\circ\nabla_{\delta_1} \qquad \text{for } k\in\boldsymbol{N}.$$

Then by induction on k, one sees that (1.4.3)-(1.4.7) hold for all $k\in\boldsymbol{N}$. In particular,

(1.4.8) $$\mathrm{gr.}(\pi_*\mathscr{H}_F^{(0)}) \simeq q_*\Omega_F \underset{\mathcal{O}_T}{\otimes} \mathcal{O}_T[\delta_1^{-1}].$$

We shall call the image of $r^{(k)}(\zeta)$ in Ω_F the principal symbol for an element $\zeta\in\mathscr{H}^{(k)}$.

An explicit calculation gives the principal symbol of the Gauß-Manin connection ∇:

(1.4.9) $$r^{(-k)}(\nabla_\delta\zeta)=\delta F|_C\cdot r^{(-k-1)}(\zeta) \qquad \text{for}\quad \delta\in\mathrm{Der}_S,\quad \zeta\in\mathscr{H}_F^{(-k-1)}.$$

(1.5) *Completion* $\widehat{\pi_*\mathscr{H}_F^{(0)}}$ *and* $\hat{\mathscr{E}}_F(0)$*-module structure.* The Gauß-Manin connection ∇ is regular singular. Hence the Δ-adic topology on $\pi_*\mathscr{H}_F^{(0)}$ and the topology on $\pi_*\mathscr{H}_F^{(0)}$ defined by the filtration $\pi_*\mathscr{H}_F^{(-k)}(=(\nabla_{\delta_1})^{-k}\pi_*\mathscr{H}_F^{(0)})$ are homeomorphic. In particular, $\bigcap_k\pi_*\mathscr{H}_F^{(-k)}=\bigcap_k\Delta^k\pi_*\mathscr{H}_F^{(0)}=\{0\}$.

Let us define the completion,

(1.5.1) $$\widehat{\pi_*\mathscr{H}_F^{(0)}} := \varprojlim_k \pi_*\mathscr{H}^{(0)}/\pi_*\mathscr{H}_F^{(-k)}.$$

It is obvious from the definition that the completion has following structures:

i) $\mathcal{O}_T[[\delta_1^{-1}]]$-module structure,

$$\mathcal{O}_T[[\delta_1^{-1}]]\times\widehat{\pi_*\mathscr{H}_F^{(0)}} \longrightarrow \widehat{\pi_*\mathscr{H}_F^{(0)}}, \quad (\sum_i a_i\delta_1^{-i})\times\{\omega_k\}_k \longmapsto \{\sum_{i+j=k} a_i\nabla_{\delta_1}^{-i}\omega_j\}_k$$

such that $\delta_1^{-k}(\widehat{\pi_*\mathscr{H}_F^{(0)}})=\widehat{\pi_*\mathscr{H}_F^{(-k)}}$.

ii) $\pi_*\mathscr{O}_S$-module structure

$$\pi_*\mathscr{O}_S\times\widehat{\pi_*\mathscr{H}_F^{(0)}}\longrightarrow\widehat{\pi_*\mathscr{H}_F^{(0)}},\quad g(t)\times\{\omega_k\}_k\longmapsto\{g\omega_k\}_k$$

iii) $\mathscr{G}\times\delta_1^{-1}$-module structure

$$\mathscr{G}\times\delta_1^{-1}\times\widehat{\delta_*\mathscr{H}_F^{(0)}}\longrightarrow\widehat{\pi_*\mathscr{H}_F^{(0)}},\quad \delta\times\delta_1^{-1}\{\omega_k\}\longmapsto\{\nabla_\delta\nabla_{\delta_1}^{-1}\omega_k\}_k.$$

To summarize these structures, it may be convenient to introduce the following notion of the algebra of formal pseudo differential operators $\hat{\mathscr{E}}_{S/T}$.

Let $U(\mathscr{G})$ be the universal enveloping algebra of $\mathscr{G}$ so that $\mathscr{O}_T[t_1]\underset{\mathscr{O}_T}{\otimes}U(\mathscr{G})$ is the universal enveloping algebra of $\mathscr{O}_T[t_1]\underset{\mathscr{O}_T}{\otimes}\mathscr{G}$ contained in $\pi_*\mathscr{D}_S$ and containing $\mathscr{O}_T[\delta_1]$. Put

$$\hat{\mathscr{E}}_{S/T}:=\mathscr{O}_T[t_1]\underset{\mathscr{O}_T}{\otimes}U(\mathscr{G})\underset{\mathscr{O}_T[\delta_1]}{\otimes}\mathscr{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle\,, \tag{1.5.2}$$

where $\mathscr{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$ is the algebra of formal Laurent series in δ_1^{-1} with coefficients in $\mathscr{O}_T$.

$\hat{\mathscr{E}}_{S/T}$ has a natural graded algebra structure,

$$\hat{\mathscr{E}}_{S/T}(k):=\{p\in\hat{\mathscr{E}}_{S/T}\colon \deg p\le k\}\,,\quad k\in\boldsymbol{Z}\,, \tag{1.5.3}$$

where $\deg(g(t)\otimes\delta^1\cdots\delta^m\otimes\delta_1^l)=m+l$ for $\delta^1,\dots,\delta^m\in\mathscr{G}$.

Then $\widehat{\pi_*\mathscr{H}_F^{(0)}}$ is an $\hat{\mathscr{E}}_{S/T}(0)$-module such that

$$\hat{\mathscr{E}}_{S/T}(k)\times\widehat{\pi_*\mathscr{H}_F^{(l)}}\longrightarrow\widehat{\pi_*\mathscr{H}_F^{(k+l)}}\,. \tag{1.5.4}$$

(1.6) *Joint $F+\sum_{i=1}^l x_i^2$ of F and $\sum_{i=1}^l x_i^2$.* Let F be a Hamiltonian system on the frame (1.1.1) as above. Then $F+\sum_{i=1}^l x_i^2$ defines a Hamiltonian system on the frame,

$$\begin{array}{ccc}(Z\times\boldsymbol{C}^l,0) & \xrightarrow{\tilde{\pi}\times id} & (X\times\boldsymbol{C}^l,0)\\ \big\downarrow{\scriptstyle p\times 0} & & \big\downarrow{\scriptstyle q\times 0}\\ (S,0) & \xrightarrow{\pi} & (T,0)\end{array} \tag{1.6.1}$$

where $(x_1,\dots,x_l)$ is a coordinate system of $\boldsymbol{C}^l$ for an $l\in\boldsymbol{N}$.

For even l the following diagram is well defined and commutative:

$$\begin{array}{ccc}\pi_*\mathscr{H}_F^{(-l/2)} & \xrightarrow{(\nabla_{\delta_1})^{l/2}} & \pi_*\mathscr{H}_F^{(0)}\\ \big\downarrow{\scriptstyle dx_1\wedge\cdots\wedge dx_l} & & \big\downarrow{\scriptstyle dx_1\wedge\cdots\wedge dx_l}\\ \pi_*\mathscr{H}^{(-l/2)}_{F+\sum_{i=1}^l x_i^2} & \xrightarrow{(\nabla_{\delta_1})^{l/2}} & \pi_*\mathscr{H}^{(0)}_{F+\sum_{i=1}^l x_i^2}\end{array} \tag{1.6.2}$$

Theorem ([41] (6.4) Theorem). *Let us denote by ρ the diagonal $\mathscr{O}_T$-homomorphism of* (1.6.2).

(1.6.3) $$\rho: \pi_*\mathscr{H}_F^{(-l/2)} \xrightarrow{\sim} \pi_*\mathscr{H}^{(0)}_{F+\sum_{i=1}^{l} x_i^2}.$$

Then ρ is an $\pi_*\mathcal{O}_S$-isomorphism, which commutes with the action of δ_1^{-1} and $\mathscr{G}\times\delta_1^{-1}$. In particular ρ induces an $\hat{\mathscr{E}}_{S/T}(0)$-isomorphism on the completions,

(1.6.4) $$\hat{\rho}: \widehat{\pi_*\mathscr{H}_F^{(-l/2)}} \simeq \widehat{\pi_*\mathscr{H}^{(0)}}_{F+\sum_{i=1}^{l} x_i^2}.$$

(1.7) *Logarithmic vector fields* $\mathrm{Der}_S(\log\Delta)$. Define the algebra of logarithmic vector fields (cf. [34]) as,

(1.7.1) $$\mathrm{Der}_S(\log\Delta) := \{\delta\in\mathrm{Der}_S : \delta\Delta\in(\Delta)\mathcal{O}_S\}.$$

In case Δ is reduced (which is the case if F is a universal unfolding), Δ is a generator of the ideal $\ker(\mathcal{O}_S\to\varphi_*\mathcal{O}_C)$. Then one checks easily that for a $\delta\in\mathrm{Der}_S$, $\delta\Delta\in(\Delta)\mathcal{O}_S$ is equivalent to $\delta F|_C=0$. Hence one gets,

Assertion. *For a $\delta\in\mathrm{Der}_S$, $\nabla_\delta\mathscr{H}_F^{(-k)}\subset\mathscr{H}_F^{(-k)}$ iff $\delta\in\mathrm{Der}_S(\log\Delta)$.*

(*Proof.* cf. (1.4.9) and (1.4.3)).

Assume that F is a universal unfolding (1.3.1). A similar reasoning gives the following $\mathcal{O}_T$-splitting,

(1.7.2) $$\pi_*\mathrm{Der}_S = \mathscr{G}\oplus\pi_*\mathrm{Der}_S(\log\Delta).$$

Let us define the $\mathcal{O}_T$-homomorphism,

(1.7.3) $$w: \mathscr{G} \longrightarrow \pi_*\mathrm{Der}_S(\log\Delta),$$

$w(\delta) := t_1\delta - t_1*\delta =$ the projection of $t_1\delta$ to the second factor of (1.7.2). One sees easily that if $\delta_1,\dots,\delta_\mu\in\mathscr{G}$ from an $\mathcal{O}_T$-free basis for $\mathscr{G}$, then $w(\delta_i) =: \sum_{j=1}^{\mu}(t_1\delta_i - a_{ij}(t'))\delta_j$, $i=1,\dots,\mu$ give an $\mathcal{O}_S$-free basis for $\mathrm{Der}_S(\log\Delta)$ and $\det(t_1\delta_{ij} - a_{ij}(t'))_{i.j} = \Delta$.

In particular, put

(1.7.4) $$E := w(\delta_1)\in\Gamma(S, \mathrm{Der}_S(\log\Delta)),$$

and call it the Eulor operator. One has the identity

(1.7.5) $$E\Delta = \mu\Delta.$$

(1.8) *Note.* The study of logarithmic vector field is closely related to a combinatorial study of arrangements of hyperplanes (for details, cf. Terao [52] [53] [54]). The case of Coxeter arrangements, which corresponds to the discriminant of simple singularities, it was studied in [33], which connects the

Killing form (=the Poincaré duality of the Milnor fiber of a simple singularity) with the residue pairing J_F introduced in the next paragraph.

§2. Higher Residue Pairings K_F

The de-Rham cohomology module $\pi_*\mathscr{H}_F^{(0)}$ introduced in Section 1, has an infinite sequence of higher residue pairings,

$$K_F^{(k)}:\ \pi_*\mathscr{H}_F^{(0)} \times \pi_*\mathscr{H}_F^{(0)} \longrightarrow \mathcal{O}_T \qquad k \in \boldsymbol{Z}$$

which was introduced in [35] (see also Namikawa [27]). In this paragraph, we recall some of their basic properties ((2.3) Theorem). For the proofs and details, cf. [35], [38].

(2.1) *The first residue pairing* J_F. Let F be a Hamiltonian system as above with the assumptions of (1.3). Then the critical set C is a complete intersection in X defined by the equations $\frac{\partial F}{\partial z_0} = \cdots = \frac{\partial F}{\partial z_n} = 0$, which is finite over T. Thus the residue symbol $\mathrm{Res}_{X/T}\begin{bmatrix} \omega \\ \frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n} \end{bmatrix}$ associates an element of $\Gamma(T, \mathcal{O}_T)$ to an $\omega \in \Gamma(X, \Omega_{X/T}^{n+1})$.

Define an $\mathcal{O}_T$-symmetric bilinear form,

$$\text{(2.1.1)} \qquad \begin{aligned} &J_F:\ q_*\Omega_F \times q_*\Omega_F \longrightarrow \mathcal{O}_T \\ &\{\psi_1(z, t')dz_0 \wedge \cdots \wedge dz_n\} \times \{\psi_2(z, t')dz_0 \wedge \cdots \wedge dz_n\} \\ &\qquad \longmapsto \mathrm{Res}_{X/T}\begin{bmatrix} \psi_1 \cdot \psi_2 dz_0 \wedge \dots \wedge dz_n \\ \frac{\partial F}{\partial z_0}, \dots\dots\dots\dots, \frac{\partial F}{\partial z_n} \end{bmatrix}. \end{aligned}$$

One sees easily that the above J_F is well-defined independently of the coordinates $z_0, \dots, z_n$. Furthermore,

i) As an $\mathcal{O}_T$-bilinear form J_F is non-degenerate.

ii) The multiplication by an element of $q_*\mathcal{O}_C$ is self-adjoint with respect to J_F.

iii) The above i) and ii) imply the existence of a non-degenerate $\mathcal{O}_T$-bilinear form,

$$\begin{aligned} &q_*\mathcal{O}_C \times (q_*\Omega_F \underset{q_*\mathcal{O}_C}{\otimes} q_*\Omega_F) \longrightarrow \mathcal{O}_T \\ &\psi \times \omega \otimes \omega' \longmapsto J_F(\psi\omega, \omega') = J_F(\omega, \psi\omega'). \end{aligned}$$

Hence we have an $\mathcal{O}_T$-isomorphism,

$$\text{(2.1.2)} \qquad J_F^*:\ q_*\Omega_F \underset{q_*\mathcal{O}_C}{\otimes} q_*\Omega_F \simeq (q_*\mathcal{O}_C)^\vee\ (:= \mathrm{Hom}_{\mathcal{O}_T}(q_*\mathcal{O}_C, \mathcal{O}_T)).$$

iv) Consider an element $\left(d\frac{\partial F}{\partial z_0}\wedge\cdots\wedge d\frac{\partial F}{\partial z_n}\right)\otimes(dz_0\wedge\cdots\wedge dz_n)\in\Gamma(T,\ q_*\Omega_F\otimes q_*\Omega_F)$, which is independent of the coordinates $z_0,\dots,z_n$. Then through J_F^* of (2.1.2), this element corresponds to $\mu\cdot$tr., where tr.: $q_*\mathcal{O}_C\to\mathcal{O}_T$ is the usual trace morphism.

(2.2) Roughly speaking, our aim now is the following. The bilinear form J_F of (2.1.1) is defined on the principal symbol $q_*\Omega_F$ of $\pi_*\mathscr{H}_F^{(0)}$. We want to extend J_F to a $\mathcal{O}_T[[\delta_1^{-1}]]$ bilinear form K_F on $\widehat{\pi_*\mathscr{H}_F^{(0)}}$ so that the "leading term of K_F" is equal to J_F (see Theorem (2.3)).

For the purpose, it is more convenient to extend the $\mathcal{O}_T[[\delta_1^{-1}]]$-module $\widehat{\pi_*\mathscr{H}_F^{(0)}}$ to an $\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$-module,

(2.2.1) $$\widehat{\pi_*\mathscr{H}_F}:\underset{\text{def}}{=}\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle\underset{\mathcal{O}_T[[\delta_1^{-1}]]}{\otimes}\widehat{\pi_*\mathscr{H}_F^{(0)}}\,.$$

Since $\widehat{\pi_*\mathscr{H}_F^{(0)}}$ has an $\hat{\mathscr{E}}_{S/T}(0)$-module structure (cf. (1.5)), we have a natural $\hat{\mathscr{E}}_{S/T}$-module structure,

(2.2.2) $$\hat{\mathscr{E}}_{S/T}\times\widehat{\pi_*\mathscr{H}_F}\longrightarrow\widehat{\pi_*\mathscr{H}_F}\,,$$

as follows.

i) It is obvious by Definition (2.2.1) that $\widehat{\pi_*\mathscr{H}_F}$ is an $\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$-module.

ii) $\widehat{\pi_*\mathscr{H}_F}$ has a $\pi_*\mathcal{O}_S$-module structure as follows:

$$g(t)\ (\delta_1^m\otimes\omega):\underset{\text{def}}{=}\sum_{i=1}^{m}(-1)^i\binom{m}{r}\delta_1^{m-r}\otimes(\delta_1^r g)\omega\,.$$

iii) $\delta\in\mathscr{G}$ operates on $\widehat{\pi_*\mathscr{H}_F}$ as follows:

$$\delta(\delta_1^m\otimes\omega):\underset{\text{def}}{=}\delta_1^{m+1}\otimes\nabla_\delta\nabla_{\delta_1}^{-1}\omega$$

(2.3) Let us denote by $*$ the $\mathcal{O}_T$-involution of $\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$, such that $\delta_1^*=-\delta_1$. Using above notations, we state our main theorem on the residue pairing.

Theorem ([38] (4.10) Theorem). *There exists an $\mathcal{O}_T$-bilinear map,*

(2.3.1) $$K_F:\widehat{\pi_*\mathscr{H}_F}\times\widehat{\pi_*\mathscr{H}_F}\longrightarrow\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$$

with the following properties.

i) $K_F(\omega_1,\omega_2)=K_F(\omega_2,\omega_1)^*$ *for* $\omega_1,\omega_2\in\widehat{\pi_*\mathscr{H}_F}$.

ii) $K_F(P\omega_1,\omega_2)=K_F(\omega_1,P^*\omega_2)=PK_F(\omega_1,\omega_2)$,

for $P\in\mathcal{O}_T\langle\!\langle\delta_1^{-1}\rangle\!\rangle$ *and* $\omega_1,\omega_2\in\widehat{\pi_*\mathscr{H}_F}$.

iii) $\delta K_F(\omega_1,\omega_2)=K_F(\delta\omega_1,\omega_2)+K_F(\omega_1,\delta\omega_2)$, *for* $\delta\in\mathscr{G}$, $\omega_1, \omega_2\in\widehat{\pi_*\mathscr{H}_F}$. *(Here* $\mathscr{G}$ *acts on* $\mathcal{O}_T$ *as a derivation through the morphism* (1.1.3)).

iv) $\frac{\partial}{\partial\delta_1}K_F(\omega_1, \omega_2)=K_F(t_1\omega, \omega_2)-K_F(\omega_1, t_1\omega_2)$,

for $t_1\in\pi_*\mathcal{O}_S$ *s.t.* $\delta_1 t_1=1$ *and* $\omega_1, \omega_2\in\widehat{\pi_*\mathscr{H}_F}$.

v) *The restriction of* K_F *to the subset* $\widehat{\pi_*\mathscr{H}_F^{(0)}}\times\widehat{\pi_*\mathscr{H}_F^{(0)}}$ *takes values in* $\mathcal{O}_T[[\delta_1^{-1}]]\delta_1^{-n-1}$, *such that the following diagram is commutative.*

$$\begin{array}{ccc} K_F:\widehat{\pi_*\mathscr{H}_F^{(0)}}\times\widehat{\pi_*\mathscr{H}_F^{(0)}} & \longrightarrow & \mathcal{O}_T[[\delta_1^{-1}]]\delta_1^{-n-1} \\ \downarrow{\scriptstyle r^{(0)}\times r^{(0)}} & & \downarrow{\scriptstyle \text{modulo } \delta_1^{-n-2}\mathcal{O}_T[[\delta_1^{-1}]]} \\ J_F: q_*\Omega_F\times q_*\Omega_F & \longrightarrow & \mathcal{O}_T \end{array}$$

(2.4) The K_F introduced above behaves naturally with respect to the joint by $\sum_{i=1}^{l}x_i^2$ for even l studied in (1.6).

The isomorphism $\hat{\rho}$ of (1.6.4) extends naturally to an $\hat{\mathscr{E}}_{S/T}$-isomorphism

$$\hat{\rho}:\widehat{\pi_*\mathscr{H}_F}\simeq\widehat{\pi_*\mathscr{H}_{F+\sum_{i=1}^{l}x_i^2}}. \tag{2.4.1}$$

Theorem added to (2.3) ([35] (11.5) Corollary). *The following diagram is commutative.*

$$\begin{array}{ccc} K_F:\widehat{\pi_*\mathscr{H}_F}\underset{\mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle}{\otimes}\widehat{\pi_*\mathscr{H}_F} & \longrightarrow & \mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle \\ \wr\downarrow{\scriptstyle\hat{\rho}\times\hat{\rho}} & & \wr\downarrow \\ K_{F+\sum_{i=1}^{l}x_i^2}:\widehat{\pi_*\mathscr{H}_{F+\sum_{i=1}^{l}x_i^2}}\underset{\mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle}{\otimes}\widehat{\pi_*\mathscr{H}_{F+\sum_{i=1}^{l}x_i^2}} & \longrightarrow & \mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle. \end{array} \tag{2.4.2}$$

(2.5) *Note* 1. For odd $l\in\boldsymbol{N}$, even if we do not have the isomorphism (2.4.1), we have the following isomorphism,

$$\begin{aligned} \widehat{\pi_*\mathscr{H}_F}\underset{\mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle}{\otimes}\widehat{\pi_*\mathscr{H}_F} &\simeq\widehat{\pi_*\mathscr{H}_{F+\sum_{i=1}^{l}x_i^2}}\underset{\mathcal{O}_T\langle\langle\delta_1^{-1}\rangle\rangle}{\otimes}\widehat{\pi_*\mathscr{H}_{F+\sum_{i=1}^{l}x_i^2}}, \\ \omega_1\otimes\omega_2 &\longmapsto(\nabla_{\delta_1})^l(\omega_1\wedge dx_1\wedge\cdots\wedge dx_l)\otimes\omega_2\wedge dx_1\wedge\cdots\wedge dx_l \\ &=\omega_1\wedge dx_1\wedge\cdots\wedge dx_l\otimes(-\nabla_{\delta_1})^l\omega_2\wedge dx_1\wedge\cdots\wedge dx_l. \end{aligned} \tag{2.5.1}$$

Then using the isomorphism (2.5.1), the same diagram as (2.4.2) holds for odd l.

(2.6) *Note* 2. In the previous notes [35], [38], we have computed the higher residue pairing K_F in the form of a power series in δ_1,

$$K_F(\omega, \omega')=\sum_{k=0}^{\infty}K_F^{(-k)}(\omega, \omega')\delta_1^{-n+1+k} \quad \text{for} \quad \omega, \omega'\in\pi_*\mathscr{H}_F^{(0)}. \tag{2.6.1}$$

There we have represented the pairings $K_F^{(-k)}$ explicitly using residues for small k. For instance,

$$K_F^{(0)}([\psi_1 d\underline{z}],[\psi_2 d\underline{z}])=\mathrm{Res}_{X/T}\begin{bmatrix} \psi_1\psi_2 d\underline{z} \\ \dfrac{\partial F}{\partial z_0},\dots,\dfrac{\partial F}{\partial z_n}\end{bmatrix},$$

$$K_F^{(-1)}([\psi_1 d\underline{z}],[\psi_2 d\underline{z}])=\sum_{i=0}^{n}\mathrm{Res}_{X/T}\begin{bmatrix} \left(\dfrac{\partial\psi_1}{\partial z_i}\psi_2-\dfrac{\partial\psi_2}{\partial z_i}\psi_1\right)d\underline{z} \\ \dfrac{\partial F}{\partial z_0},\dots,\left(\dfrac{\partial F}{\partial z_i}\right)^2,\dots,\dfrac{\partial F}{\partial z_n}\end{bmatrix}, \text{ etc.}$$

These representations of $K_F^{(-k)}$ by residues, were the reason why we call them the higher residue pairings.

§3. Primitive Forms

In this paragraph, we give a definition of a primitive form $\zeta^{(0)}$ and explain some elementary consequences of the existence of a primitive form.

(3.1) **Definition.** Let F be a universal unfolding. An element $\zeta^{(0)}\in\Gamma(S,\mathscr{H}_F^{(0)})=\Gamma(T,\pi_*\mathscr{H}_F^{(0)})$ is called a primitive form if it satisfies the following conditions 0)-iv).

0) *Invertibility.* The image $\{\zeta^{(0)}\}\in\Gamma(C,\Omega_F)$ is an $\mathcal{O}_C$-generator of Ω_F.

i) *Homogeneity.* $\nabla_E\zeta^{(0)}=(r-1)\zeta^{(0)}$ for a constant r.

ii) *Orthogonarity.* $K_F(\nabla_\delta\zeta^{(0)},\ \nabla_{\delta'}\zeta^{(0)})\in\mathcal{O}_T\delta_1^{-n+1}$ for $^\forall\delta,\delta'\in\mathscr{G}$.

iii) $K_F(\nabla_\delta\nabla_{\delta'}\zeta^{(0)},\ \nabla_{\delta''}\zeta^{(0)})\in\mathcal{O}_T\delta_1^{-n+2}+\mathcal{O}_T\delta_1^{-n+1}$ for $^\forall\delta,\delta',\delta''\in\mathscr{G}$.

iv) $K_F(t_1\nabla_\delta\zeta^{(0)},\ \nabla_{\delta'}\zeta^{(0)})\in\mathcal{O}_T\delta_1^{-n+1}+\mathcal{O}_T\delta_1^{-n}$ for $^\forall\delta,\delta'\in\mathscr{G}$.

Examples. If F is a universal unfolding of a weighted homogeneous singularity of weights $r_0, r_1,\dots, r_n$. Then F has naturally a $\boldsymbol{C}^*$-action so that $\mathscr{H}_F^{(0)}$ is a graded module.

1) If F is an unfolding of a rational double point, then there exists a complex 1-dimensional subspace of $\mathscr{H}_F^{(0)}$ spanned by $[dx\wedge dy\wedge dz]$ of lowest degree $r=r_x+r_y+r_z$ (>1) elements. An element $\zeta^{(0)}\neq 0$ is a primitive form if and only if it belongs to the space.

2) If F is a universal unfolding of a simple elliptic singularity, then there exists rank 1 submodule of $\mathscr{H}_F^{(0)}$ over $\mathcal{O}_{\boldsymbol{C}}$ spanned by $\omega=[dx\wedge dy\wedge dz]$ of lowest degree $r=r_x+r_y+r_z=1$ elements, where $\mathcal{O}_{\boldsymbol{C}}$ is a subalgebra of $\mathcal{O}_S$ consisting of all degree zero elements.

One may compactify the family $X\to S$ to $\overline{X}\to S$, so that the boundary $E=\overline{X}-X$ is a family of elliptic curves over S (see E. Looijenga [18]). Since the residue res $_E(\omega)$ on the boundary defines a family of elliptic integral of the first kind, the integrals $\int_{\gamma_i}$ res $_E(\omega)$ are holomorphic functions on S, where $\gamma_i(t) \in H_1(E_t, \boldsymbol{Z})$, $i=1, 2$ are basis of horizontal family of the homology of the elliptic curves. (Note that these functions are homogeneous of degree zero on S so that they belong to $\mathcal{O}_{\boldsymbol{C}}$.) Then any primitive form can be expressed as

$$\zeta^{(0)} := \omega \Big/ \left(c \int_{\gamma_1} \mathrm{res}_E(\omega) + d \int_{\gamma_2} \mathrm{res}_E(\omega) \right),$$

for c, $d \in \boldsymbol{C}$ with $(c, d) \neq (0, 0)$.

(3.2) *An explanation of the definition.* For the primitive form $\zeta^{(0)} \in \Gamma(S, \mathscr{H}_F^{(0)})$ and an integer $k \in \boldsymbol{Z}$, let us denote,

(3.2.1) $$\zeta^{(k)} := \nabla_{\delta_1}^k \zeta^{(0)} \in \Gamma(S, \mathscr{H}_F^{(k)}).$$

(3.2.2) $$\begin{aligned} v^{(k)} : \mathscr{G} &\longrightarrow \pi_* \mathscr{H}^{(k)} \\ \delta &\longmapsto \nabla_\delta \zeta^{(k-1)} = \nabla_\delta (\nabla_{\delta_1})^{k-1} \zeta^{(0)}. \end{aligned}$$

The principal symbol of $v^{(k)}$ is computed using (1.4.9) as follows:

(3.2.3) $$r := r^{(k)} \cdot v^{(k)} : \mathscr{G} \xrightarrow{\sim} g_* \Omega_F, \ \delta \longmapsto \delta F|_C \{\zeta^{(0)}\}$$

which is independent of k and depends only on $\{\zeta^{(0)}\}$. By (3.1) 0), r is an $\mathcal{O}_T$-isomorphism, since F is a universal unfolding and (1.3.1) is an isomorphism.

Hence by (1.4.3), one has an $\mathcal{O}_T[[\delta_1^{-1}]]$-isomorphism,

(3.2.4) $$\begin{aligned} &\mathscr{G} \underset{\mathcal{O}_T}{\otimes} \mathcal{O}_T[[\delta_1^{-1}]]\delta_1^{-1} \simeq \widehat{\pi_* \mathscr{H}_F^{(0)}} \\ &P = \sum_{k=0}^{\infty} \delta_k \delta_1^{-k-1} \longmapsto P\zeta^{(0)} = \sum_{k=0}^{\infty} \nabla_{\delta_k} \zeta^{(-k-1)} = \sum_{k=0}^{\infty} v^{(k)}(\delta_k). \end{aligned}$$

Through this isomorphism (which depends on $\zeta^{(0)}$), the structures on $\widehat{\pi_* \mathscr{H}_F^{(0)}}$ (namely, the higher residue pairing K_F, $\mathscr{E}_{S/T}(0)$-module structure and $\pi_* \mathcal{O}_S$-module structure) should introduce some structures on $\mathscr{G} \underset{\mathcal{O}_T}{\otimes} \mathcal{O}_T[[\delta_1^{-1}]]\delta_1^{-1}$, as we shall see below.

First, the bilinear form J_F of (2.1.1) induces a non-degenerate $\mathcal{O}_T$-bilinear form on $\mathscr{G}$ via the $\mathcal{O}_T$-isomorphism r of (3.2.3),

(3.2.5) $$J_{\zeta^{(0)}} : \mathscr{G} \times \mathscr{G} \longrightarrow \mathcal{O}_T, \quad (\delta, \delta') \longmapsto J_F(v(\delta), v(\delta')).$$

To avoid complications, let us denote simply J instead of $J_{\zeta^{(0)}}$.

Then the pairing K_F is described by a use of J, due to the orthogonality (3.1) ii) as follows:

(3.2.6) $$\begin{aligned} K_F(v^{(i)}(\delta), v^{(j)}(\delta')) &= K_F(\nabla_\delta \nabla_{\delta_1}^{i-1} \zeta^{(0)}, \nabla_{\delta'} \nabla_{\delta_1}^{j-1} \zeta^{(0)}) \\ &= (-1)^j J(\delta, \delta') \delta_1^{i+j-n+1}. \end{aligned}$$

Under the assumptions (3.1) 0), ii), the condition iii) is equivalent to the existence of a bi-additive map,

(3.2.7) $$\nabla : \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$$
s.t.
$$\nabla_\delta \nabla_{\delta'} \zeta^{(k)} = \nabla_{\delta * \delta'} \zeta^{(k+1)} + \nabla_{\nabla_\delta \delta'} \zeta^{(k)} \qquad \text{for} \quad \delta, \delta' \in \mathscr{G},\ k \in \boldsymbol{Z}.$$

Under the assumptions (3.1) 0), ii), the condition iv) is equivalent to the existence of an $\mathcal{O}_T$-endomorphism,

(3.2.8) $$N : \mathscr{G} \longrightarrow \mathscr{G}$$
s.t.
$$t_1 \nabla_\delta \zeta^{(k)} = \nabla_{t_1 * \delta} \zeta^{(k)} + \nabla_{(N-k-1)\delta} \zeta^{(k-1)} \qquad \text{for} \quad \delta \in \mathscr{G},\ k \in \boldsymbol{Z}.$$

It is easily to check the following relations, which determine ∇ and N uniquely from $\{\zeta^{(0)}\} \in \Gamma(C, \Omega_F)$.

(3.2.9) i) $K_F^{(1)}(\nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)}) = 0 \qquad \text{for} \quad {}^\forall \delta, \delta' \in \mathscr{G}$
and
ii) $J(\delta, \delta') = K^{(0)}(\nabla_\delta \zeta^{(-1)}, \nabla_\delta \zeta^{(-1)})$

(3.2.10) $$J(\nabla_\delta \delta', \delta'') = K_F^{(1)}(\nabla_\delta \nabla_{\delta'} \zeta^{(-2)}, \nabla_{\delta''} \zeta^{(-1)}),$$

(3.2.11) $$J(N\delta, \delta') = K_F^{(1)}(t_1 \nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)}).$$

(3.3) Now we describe some elementary consequences of the definition of the primitive form $\zeta^{(0)}$.

1) *Flat coordinates.* By (1.1.3) the map ∇ of (3.2.7) factors through $\mathrm{Der}_T \times \mathscr{G}$ as a connection on $\mathscr{G}$,

(3.3.1) $$\nabla : \mathrm{Der}_T \times \mathscr{G} \longrightarrow \mathscr{G}.$$

The integrability of the connection ∇ implies the following relations,

(3.3.2) $$\nabla_\delta(\delta' * \delta'') + \delta * \nabla_{\delta'} \delta'' - \nabla_{\delta'}(\delta * \delta'') - \delta' * \nabla_\delta \delta'' = [\delta, \delta'] * \delta''$$
for $\delta, \delta', \delta'' \in \mathscr{G}$,

(3.3.3) $$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta, \delta']} \qquad \text{for} \quad {}^\forall \delta, \delta' \in \mathscr{G}.$$

The metric property of ∇ on K_F ((2.3) Theorem iii)) implies

(3.3.4) $$\delta J(\delta', \delta'') = J(\nabla_\delta \delta', \delta'') + J(\delta', \nabla_\delta \delta'') \qquad \text{for} \quad \delta, \delta', \delta'' \in \mathscr{G}.$$

Obviously, (3.3.3) is the integrability condition on ∇ and (3.3.4) means that ∇ is metric w.r.t. J. Furthermore, putting $\delta'' = \delta_1$ in (3.3.2), we see that ∇

is torsion free.

(3.3.5) $$\nabla_\delta \delta' - \nabla_{\delta'} \delta = [\delta, \delta'] \qquad \text{for} \quad \delta, \delta' \in \mathscr{G}.$$

Hence the space of horizontal sections of ∇ is a μ-dimensional $\boldsymbol{C}$-vector space with a non-degenerate metric J, which is integrable by the bracket product $[\ ,\ ]$. Note that

(3.3.6) $$\nabla \delta_1 = 0.$$

A function t on S which defines a linear functional on the horizontal space of ∇ is called a flat coordinate (w.r.t. ζ). A flat coordinate t is characterized by the following system of linear differential equations:

(3.3.7) $$(\delta\delta' - \nabla_\delta \delta')t = 0 \qquad \text{for} \quad {}^\forall\delta, \delta' \in \mathscr{G}.$$

Including the constant function, (3.3.7) has $\mu+1$ linearly independent solutions, which give affine coordinates on S. Flat coordinates are calculated for simple singularities in [10] [12] [33] [44] [59]. The calculation for simple elliptic singularities will appear in [42] [43].

2) *Exponents.* The relation $[\nabla_\delta, t_1] = \delta t_1$ implies the following relations, for $\delta, \delta' \in \mathscr{G}$.

(3.3.8) $$\nabla_\delta(t_1 * \delta') + \delta * (N+1)\delta' = N(\delta * \delta') + t_1 * \nabla_\delta \delta' + (\delta t_1)\delta'$$

(3.3.9) $$\nabla N = 0 \quad (\text{i.e. } \nabla_\delta(N\delta') = N(\nabla_\delta \delta')).$$

The behaviour of K_F with respect to the multiplication by t_1 ((2.3) Theorem iv)) implies,

(3.3.10) $$J(N\delta, \delta') + J(\delta, N\delta') = (n+1)J(\delta, \delta') \qquad \text{for} \quad \delta, \delta' \in \mathscr{G},$$
$$\text{i.e.} \quad N + N^* = (n+1)id.,$$

where N^* is the adjoint of N w.r.t. the metric J.

Homogeneity (3.1) i) implies that δ_1 is an eigenvector of N,

(3.3.11) $$N\delta_1 = r\delta_1.$$

In particular, putting $\delta' = \delta_1$ in (3.3.8), one gets

(3.3.12) $$N\delta = \nabla_\delta t_1 * \delta_1 + (r+1)\delta - (\delta t_1)\delta_1 \qquad \text{for} \quad \delta \in \mathscr{G}.$$

Combining (3.3.12) with (3.3.5), we get

(3.3.13) $$[E, \delta] = -(r+1)\delta - N\delta - \nabla_{t_1 * \delta_1}\delta \qquad \text{for} \quad \delta \in \mathscr{G}.$$

Now (3.3.9) implies that N is a $\boldsymbol{C}$-linear endomorphism of the horizontal vector space of ∇ having the duality property (3.3.10). Hence the bracket product with E also induces a $\boldsymbol{C}$-endomorphism of the space such that $E+N=(r+1)id=0$ ($\because$ (3.3.13)).

Definition. The eigenvalues of N will be called the exponents.

Note 1. It seems very probable that N is semi-simple such that the set of exponents coincides with the exponents defined by mixed Hodge structure. See J. Steenbrink [50], M. Saito [45] [46].

Note 2. Let $\alpha_1,\dots,\alpha_\mu$ be the exponents. To study the distribution of the exponents, we introduce the characteristic polynomial,

$$\chi(T):=\sum_{i=1}^{\mu} T^{\alpha}. \tag{3.3.14}$$

If the singularity is weighted homogeneous, the roots of the equation $\chi(T)=0$ are either zero or roots of unity. Using the computer, we studied the roots of $\chi(T)=0$, for some other examples in [37].

3) *The flat function* τ. Let $\zeta^{(0)}$ be a primitive form. Let $\omega\in\sum_{i=1}^{\mu}\mathcal{O}_T dt_i=\mathcal{G}^\vee$ be the image of $\{\zeta^{(0)}\}\otimes\{\zeta^{(0)}\}$ by the morphism J_F^* of (2.1.2) and the dual of (1.3.1). Explicitly,

$$\begin{aligned}\omega:&=\sum_{i=1}^{\mu} J_F\left(\frac{\partial F}{\partial t_i}\Big|_C\{\zeta^{(0)}\},\,\{\zeta^{(0)}\}\right)dt_i \\ &=(-1)^p\sum_{i=1}^{\mu} K_F(\nabla_{\frac{\partial}{\partial t_i}}\zeta^{(n-1-p)},\,\zeta^{(p)})dt_i \qquad \text{for}\quad p\in\boldsymbol{Z}.\end{aligned} \tag{3.3.15}$$

It is easy to see that property (3.2.9) i) implies that ω is a closed form. Choose a function τ on S s.t.

$$\omega=d\tau \tag{3.3.16}$$

and call τ the flat function associated with $\zeta^{(0)}$.

From the definition (3.3.15) one checks easily that τ satisfied the following system of equations,

$$\delta\delta'\tau-\nabla_\delta\delta'\tau=0 \qquad \text{for}\quad \delta,\,\delta'\in\mathcal{G}, \tag{3.3.17}$$

$$d(E\tau-(1-s)\tau)=0 \quad \text{where}\quad s=n+1-2r, \tag{3.3.18}$$

which imply that τ is a flat coordinate, homogeneous of degree $1-s$ (smallest possible degree for a flat coordinate).

Example 1. In case F is a universal unfolding of a simple singularity,

$\tau \in \boldsymbol{C}[\mathscr{G}^*]^W$ is identified with a constant multiple of the Killing form.

2. In case F is universal unfolding of a simple elliptic singularity, τ is of the form

$$\tau=(a\tau_0+b)/(c\tau_0+d) \qquad \text{for} \quad a, b, c, d \in \boldsymbol{C},\ ad-bc\neq 0,$$

$$\tau_0=\int_{\gamma_1}\operatorname{res}_E(\omega)\Big/\int_{\gamma_2}\operatorname{res}_E(\omega) \quad \text{for} \quad \gamma_1, \gamma_2 \in H_1(E, \boldsymbol{Z}) \text{ a free basis.}$$

4) *A uniqueness property.* Let $\{\zeta\} \in \Gamma(C, \Omega_F)$ be an invertible element (i.e. an element satisfying (3.1) 0) of the principal symbol module and let $\zeta \in \Gamma(S, \mathscr{H}^{(0)})$ be any lifting with $r^{(0)}(\zeta)=\{\zeta\}$. The condition (3.2.9) i) is equivalent to saying that the image of $\{\zeta\}\otimes\{\zeta\}$ under J_F^* of (2.1.2) in $\mathscr{G}^\vee$ is a closed form. The connection ∇ and the endomorphism N defined by (3.2.9), (3.2.10), (3.2.11) do not depend on the lifting ζ but only on the symbol $\{\zeta\}$.

Lemma. *If* ∇ *and* N *satisfies* (3.3.3) *and* (3.3.9) ii), *there exists a unique primitive form* $\zeta^{(0)}$ *such that* $\{\zeta\}=r^{(0)}(\zeta^{(0)})$.

Proof. Consider the system of equations for $u \in \mathscr{H}_F^{(0)}$.

$$\{E-(r-1)\}u=0$$

$$\{\delta\delta'-(\delta*\delta')\delta_1-\nabla_\delta\delta'\}u=0 \qquad \text{for} \quad \delta, \delta' \in \mathscr{G}$$

$$\{(t_1\delta-t_1*\delta)\delta_1-(N-2)\}u=0 \qquad \text{for} \quad \delta \in \mathscr{G}.$$

Then the conditions (3.3.3) and (3.3.9) imply the involutivity of the system so that it becomes a simple holonomic system in the sense of [14]. Hence there exists a unique solution $\zeta \in \Gamma(S, \mathscr{H}_F^{(0)})$, whose principal symbol is equal to $\{\zeta\}$. Note that we have the bijection (3.2.4). Now define,

$$K_F^*\Big(\sum_k \delta_k\delta_1^{-k-1}\zeta^{(0)}, \sum_l \xi_l\delta_1^{-l-1}\zeta^{(0)}\Big):=\sum_{k,l}(-1)^l J(\delta_k, \xi_l)\delta_1^{-n-k-l-1}.$$

From (3.1) 0) one sees that K_F^* satisfies the properties (2.3) i)–v), which implies $K_F^*=K_F$. Hence $\zeta=\zeta^{(0)}$ satisfies the conditions of the definition (3.1).

(3.4) *The intersection form I.* For any integer $p \in \boldsymbol{Z}$, put

$$\text{(3.4.1)} \qquad I_p :\underset{\text{def}}{=} \sum_{i=1}^{\mu} \nabla_{\delta_i}\zeta^{(k-1)} \otimes \nabla_{w(\delta^{i*})}\zeta^{(n-k-1)} \in \mathscr{H}_F^{(k)} \underset{\mathcal{O}_S}{\otimes} \mathscr{H}_F^{(n-k)}$$

where $\delta_1,\dots,\delta_\mu$ and $\delta^{1*},\dots,\delta^{\mu*}$ are an $\mathcal{O}_T$-basis of $\mathscr{G}$ and the dual basis w.r.t. J, respectively, and where $w\colon \mathscr{G}\to\pi_*\operatorname{Der}_S(\log \Delta)$ is defined as in (1.7.3). Note that the definition of I_p does not depend on the choice of the basis $\delta_1,\dots,\delta_\mu$ so that I_p is a global element on S.

Using (2.1) ii), (3.2.7), (3.2.8), (3.3.3), (3.3.4), (3.3.9) and (3.3.10), one computes easily, for any $p \in \boldsymbol{Z}$, that

$$I_p = {}^t I_{n-p}, \tag{3.4.2}$$

$$I_p = -{}^t I_{n-p-1}, \tag{3.4.3}$$

$$\nabla I_p = 0. \tag{3.4.4}$$

(Here ${}^t I$ means the transpose of I and ∇ is the connection on the tensor). The relations (3.4.2) and (3.4.3) show that $I_p = (-1)^p I_0$ is symmetric or skew-symmetric according as n is even or odd. The relation (3.4.4) implies that I_p induces a constant coefficient bilinear form on the local system of the solutions of $\mathscr{H}_F|_{S-D}$, which may be regarded as a bilinear form on the homology group $H_n(X_t, \boldsymbol{Z})$, $X_t = \varphi^{-1}(t)$, which is invariant under the monodromy representation of $\pi(S-D, t)$. Such a bilinear form is a constant multiple of the intersection form on $H_n(X_t, \boldsymbol{Z})$ (c.f. [39]) so that we obtain the following

Theorem. *Let n be even. For $\gamma, \gamma' \in H_n(X_t, \boldsymbol{Z})$, the intersection number of these cycles is given by,*

$$\begin{aligned} \langle \gamma, \gamma' \rangle &= (2\pi)^{-n}(-1)^{\frac{n}{2}-k} I_k(\gamma, \gamma') \\ &= (2\pi)^{-n}(-1)^{\frac{n}{2}-k} \sum_{i=1}^{\mu} \delta_i \int_\gamma \zeta^{(k-1)} \cdot w(\delta^{i*}) \int_{\gamma'} \zeta^{(\frac{n}{2}-k-1)} \\ &= (2\pi\sqrt{-1})^{-n} \sum_{i=1}^{\mu} (N-nI)(t_1 I - A)^{-1}(N-(n-1)I)\cdots \\ &\qquad \cdots (N-I)\delta_i \int_\gamma \zeta^{(-1)} \cdot \delta^{i*} \int_{\gamma'} \zeta^{(-1)}. \end{aligned} \tag{3.4.5}$$

(*Here A is the endomorphism of $\mathscr{G}$ by $t_1 *$*).

It is obvious from this expression that the intersection form degenerates iff there is an exponent in $[1, n] \cap \boldsymbol{Z}$.

(3.5) *Jacobian determinant.* Using the property (2.1) ii) of the higher residue K_F, one can compute $\det\left(\int_{\gamma_i(t)} v_j\right)_{i,j} = \text{unit } \Delta^{\frac{n-1}{2}}$ for an $\mathcal{O}_S$-basis $v_1, \dots, v_\mu$ of $\mathscr{H}_F^{(0)}$ and a $\boldsymbol{Z}$-basis $\gamma_1(t), \dots, \gamma_\mu(t) \in H_n(X_t, \boldsymbol{Z})$ of the horizontal family of the homology. (See also A. Varchenko [56]).

Using a primitive form $\zeta^{(0)}$, let us give some more precise descriptions. First, one computes,

$$\operatorname{tr}((N-k)(\delta * \cdot)) = \left(\frac{n-1}{2} - k\right) w(\delta) \log \Delta \tag{3.5.1}$$

(*Proof.* Use (3.3.10) and [41] (4.5) Assertion 4.)

$$\nabla_{w(\delta)} \nabla_{\delta'} \zeta^{(k-1)} = \nabla_{(N-k-1)(\delta * \delta')} \zeta^{(k-1)} \quad \text{for} \quad \nabla \delta' = 0. \tag{3.5.2}$$

(*Proof.* (3.2.7), (3.2.8))

Combining (3.5.1) and (3.5.2) above, one obtains

$$\det\left(\frac{\partial}{\partial t_j}\int_{\gamma_i(t)}\zeta^{(k-1)}\right)_{i,j}=const.\,\Delta^{\frac{n-1}{2}-k}, \tag{3.5.3}$$

where $t_1,\dots,t_\mu$ are flat coordinates.

Note. The constant in (3.5.3) becomes zero iff there exists an exponent in $(0, k]\cap \boldsymbol{Z}$.

Therefore for a $k\in \boldsymbol{Z}$ with $k<$smallest integral exponent, $\nabla_{\frac{\partial}{\partial t_j}}\zeta^{(k-1)}$, $j=1,\dots,\mu$ form $\mathcal{O}_S$-basis of $\mathscr{H}_F^{(k)}$. In particular, we have isomorphisms

$$\begin{array}{ccccc} \mathrm{Der}_S(\log\Delta) & \hookrightarrow & \mathrm{Der}_S & \ni & \delta \\ \downarrow\wr & & \downarrow\wr & & \downarrow \\ \mathscr{H}_F^{(k-1)} & \hookrightarrow & \mathscr{H}_F^{(k)} & \ni & \nabla_\delta\zeta^{(k-1)} \end{array} \tag{3.5.4}$$

On the contrary, if there exists an exponent in $(0, k]\cap\boldsymbol{Z}$, the functions $\int_{\gamma_j(t)}\zeta^{(k-1)}$, $j=1,\dots,\mu$ are linearly dependent over $\boldsymbol{C}$. In particular if one defines a period mapping, as conventional, by the functions $\int_{\gamma_j(t)}\zeta^{(n/2-1)}$, $(j=1,\dots,\mu)$, it defines a degenerate map if there exists an integral exponent. ($\because$ the duality of exponents). To recover μ-linearly independent functions, i.e. to construct a "good period mapping", we shall study certain holonomic $\mathscr{D}_S$-modules $\mathscr{M}^{(k)}$, $k\in\boldsymbol{Z}$ in Section 5, which contain $\int_{\gamma_i}\zeta^{(k-1)}$, $i=1,\dots,\mu$ as a part of solutions.

§4. A Reduction of the Existence of a Primitive Form to That of a Good Section

In this paragraph, we reduce the existence of a primitive form to that of a good section v (cf. (4.1) Definition and (4.3) Lemmas). The proof is based on a solution of the Riemann-Hilbert problem on $\boldsymbol{P}^1(\boldsymbol{C})$ due to B. Malgrange [23] [25] and Birkhoff [4].

The reader who wants to get to the period mapping as quickly as possible, might skip this paragraph to the next at the first reading.

(4.1) First, let us reformulate the definition of a primitive form ((3.1) Def.), in terms of good sections. For this, we define:

Definition. An $\mathcal{O}_T$-linear map, (4.1.1) $$v\colon q_*\Omega_F \longrightarrow \pi_*\mathscr{H}_F^{(0)}$$

is a good section, if the following conditions hold:

i) v is an $\mathcal{O}_T$-splitting of the exact sequence (1.4.3) for $k=0$. (i.e. $r^{(0)}\circ v=id$)

ii) $K_F(v(e), v(e'))\in \mathcal{O}_T\delta_1^{-n-1}$ for $e, e'\in q_*\Omega_F$,

iii) $\nabla_\delta v(e)\in \nabla_{\delta_1}(\mathrm{Image}\,(v))+\mathrm{Image}\,(v)$ for $e\in q_*\Omega_F$, $\delta\in\mathscr{G}$,

iv) $t_1 v(e)\in \mathrm{Image}\,(v)+(\nabla_{\delta_1})^{-1}\,\mathrm{Image}\,(v)$ for $e\in q_*\Omega_F$.

Note. Arguments similar to those in (3.2) and (3.3) show the existence of an $\mathcal{O}_T$-connection $\not\nabla\colon \mathrm{Der}_T\times q_*\Omega_F\to q_*\Omega_F$ and an $\mathcal{O}_T$-endomorphism $N\colon q_*\Omega_F\to q_*\Omega_F$ such that,

(4.1.2) $$\nabla_\delta v(e)=\nabla_{\delta_1}v(\delta F|_C e)+v(\not\nabla_\delta e) \qquad \text{for} \quad \delta\in\mathscr{G},$$

(4.1.3) $$t_1 v(e)=v(t_1|_C e)+(\nabla_{\delta_1})^{-1}v(Ne) \qquad \text{for} \quad e\in q_*\Omega_F$$

with the properties,

(4.1.4) $$\not\nabla^2=0,\ \not\nabla N=0,\ \not\nabla J_F=0, \quad \text{and} \quad N+N^*=(n+1)id.$$

Thus the horizontal space,

$$\Omega_f := \{e\in q_*\Omega_F;\ \not\nabla e=0\}$$

is a μ-dimensional $\boldsymbol{C}$-vector space with a $\boldsymbol{C}$-endomorphism $N|_{\Omega_f}$ and a non-degenerate inner product $J_F|_{\Omega_f\times\Omega_f}$.

Note. If the Hamiltonian F is classical (i.e. $T=\{0\}$), then the condition iii) of (4.1) Definition is void so that $q_*\Omega_F$ is already the horizontal space.

(4.2) **Lemma 1.** *Let F be a universal unfolding, then there exists a natural surjective mapping,*

(4.2.1) *{the set of primitive forms}→{the set of good sections}.*

Proof. Let $\zeta^{(0)}$ be a primitive form. With the notations (3.2.2) (3.2.3), let us define an $\mathcal{O}_T$-section by

(4.2.2) $$v :\underset{\mathrm{def}}{=} v^{(0)}\circ r^{-1}: q_*\Omega_F \longrightarrow \mathscr{G} \longrightarrow \pi_*\mathscr{H}_F^{(0)}$$
$$\delta F|_C\{\zeta^{(0)}\} \longmapsto \delta \longmapsto \nabla_\delta\zeta^{(-1)}.$$

By (3.2.6), (3.2.7) and (3.2.8) v is a good section.

Conversely, suppose a good section v is given. Since $q_*\Omega_F\simeq\Omega_f\otimes\mathcal{O}_T$, there exists an eigen-vector $e_0\in\Omega_f$ of N which is invertible in $q_*\Omega_{F,0}$ as a $q_*\mathcal{O}_{C,0}$-module. Thus (by shrinking Z, X, S, T to a smaller neighbourhood of base points, if necessary). $q_*\Omega_F = q_*\mathcal{O}_C e_0$. Now Put

$$\zeta^{(0)} := v(e_0).$$

Since e_0 is horizontal (4.1) iii) implies the relation

$$\nabla_\delta \zeta^{(-1)} = v(\delta * e_0) \quad \text{for} \quad \delta \in \mathscr{G}.$$

Then one sees easily that (4.1) Definition implies (3.1) Definition. q.e.d.

Note. One is led to conjecture that the correspondence

(4.2.3) {the set of primitive forms}/$\boldsymbol{C}^*$ → {the set of good sections}

defined by $\zeta^{(0)} \mapsto v$ by (4.2.2), is bijective.

For the proof one has only to show the uniqueness of e_0 (an eigenvector of N in Ω_f which is invertible in $q_*\Omega_F$) associated with a good section v. This can be shown if we show that the multiplicity of the smallest eigenvalue r of N is equal to 1. This question seems closely related to the study of Lefschetz homomorphism on the mixed Hodge structure of the singularity of f.

(4.3) Now we are going to reduce the existence of a good section for F to the existence of a good section for the generating center F_0 of F, which is defined as follows.

Let F be a Hamiltonian system over (Z, X, S, T). For a given variety $(T_\alpha, 0)$ with a holomorphic map $\alpha: (T_\alpha, 0) \to (T, 0)$, put $S_\alpha = T_\alpha \underset{T}{\otimes} S$, $X_\alpha = T_\alpha \underset{T}{\otimes} X$, $Z_\alpha = T_\alpha \underset{T}{\otimes} Z$. Then $F|Z_\alpha$ defines a Hamiltonian system on the frame $(Z_\alpha, X_\alpha, S_\alpha, T_\alpha)$, so that there exist natural isomorphisms

(4.3.1) $$\alpha^*: \alpha^{-1}(\pi_*\mathscr{H}_F^{(k)}) \otimes \mathcal{O}_{T_\alpha} \simeq (\pi_\alpha)_*\mathscr{H}_{F_\alpha}^{(k)} \quad \text{for} \quad k \in \boldsymbol{Z}$$

(4.3.2) $$\alpha^{-1}(q_*\Omega_F) \otimes \mathcal{O}_{T_\alpha} \simeq (q_\alpha)_*\Omega_{F_\alpha},$$

which commutes with $\hat{\mathscr{E}}_{S/T}$-module structure. One also obtains the commutative diagram,

(4.3.3) $$\begin{array}{ccc} K_F: \pi_*\mathscr{H}_F^{(0)} \times \pi_*\mathscr{H}_F^{(0)} & \longrightarrow & \mathcal{O}_T[[\delta_1^{-1}]] \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ K_{F_\alpha}: (\pi_\alpha)_*\mathscr{H}_{F_\alpha}^{(0)} \times (\pi_\alpha)_*\mathscr{H}_F^{(0)} & \longrightarrow & \mathcal{O}_{T_\alpha}[[\delta_1^{-1}]] \end{array}$$

In particular, if $O: \{0\} \to (T, 0)$ is the embedding of the base point of T, the induced Hamiltonian system $F_0 := F|Z_0 = t_1 - f(z)$ is what is called the generating center by R. Thom [55].

The morphism α^* of (4.3.1) and (4.3.2) induces a map, (4.3.4) {good sections of F}$\to${good sections of F_α}.

Our main lemma is the following.

Lemma 2. *The correspondence* (4.3.4) *is bijective.*

Proof. Since for any α the map O: $\{0\}\to(T, 0)$ is factored through α, one has only to prove Lemma 2 for the case $\alpha=0$.

The proof used the following analytic lemma due to B. Malgrange ([23] (1, 4), [24] 4, [25] 2) (see also G. D. Birkhoff [4]).

(4.4) **Lemma.** *Let M be a holonomic $\mathscr{E}_{S/T}$-module with a good filtration $\{M^{(k)}\}$ such that $\delta_1 M^{(k)}=M^{(k+1)}$ and $\pi_* M^{(0)}/\pi_* M^{(-1)}$ is $\mathcal{O}_T$-free of rank μ. Let $e_1,\dots, e_\mu$ be an $\mathcal{O}_T\{\{\delta_1^{-1}\}\}$-basis of $M^{(0)}$, which is the lifting of an $\mathcal{O}_T$-basis of $\pi_* M^{(0)}/\pi_* M^{(-1)}$. Put*

(4.4.1)
$$t_1\underline{e}=A(t', \delta_1^{-1})\underline{e}$$
$$\frac{\partial}{\partial t_i}\underline{e}=B_i(t', \delta_1^{-1})\underline{e} \qquad i=1,\dots, m.$$

Expanding,

(4.4.2)
$$B_i(t', \delta_1^{-1})=B_i^1(t')\delta_1+B_i^0(t')+B_i^{-1}(t')\delta_1^{-1}+\cdots$$
$$A(t', \delta_1^{-1})=A^0(t')+A^{-1}(t')\delta_1^{-1}+A^{-2}(t')\delta_1^{-2}+\cdots$$

Assume that

i) $B_i^1(0)$, $i=2,\dots, \mu$ *and* $A^0(0)$ *are nilpotent,*

ii) $A(0, \delta_1^{-1})=A^0(0)+A^{-1}(0)\delta_1^{-1}$ (i.e. $A^{-i}(0)=0$ *for* $i\geq 2$).

Then there exists a unique holomorphic matrix $S(t', \delta_1^{-1})$ such that $S(0, \delta_1^{-1})=id.$, and for the new basis $\underline{f}=S\underline{e}$, we have

(4.4.3)
$$t_1\underline{f}=(\bar{A}^0(t')+\bar{A}^{-1}(t')\delta_1^{-1})\underline{f}$$
$$\frac{\partial}{\partial t_i}\underline{f}=(\bar{B}_i^1(t')\delta_1+\bar{B}_i^0(t'))\underline{f}, \quad i=1,\dots, m.$$

(The proof of this Lemma is devided into two parts. First, by a change of basis, we reduce it to the case where the series (4.4.2) are convergent in δ_1^{-1} at $\delta_1^{-1}=0$, so that the equation (4.4.1) is regarded as an equation on all of $\boldsymbol{P}^1$ with the coordinate δ_1 having a singularity at $\delta_1=\infty$ (with a deformation parameter $t'\in T$). Then we solve the Riemann-Hilbert problem on the family of $\boldsymbol{P}^1$'s).

Proof of Lemma 2. Let a good section v_0 for F_0 be given. Let $E_1,\dots, E_\mu\in$

$\pi_*\mathscr{H}_F^{(0)}$ be the liftings of $v_0(e_1),\dots, v_0(e_\mu)\in\pi_{0*}\mathscr{H}_{F_0}^{(0)}$ for a $\boldsymbol{C}$-basis $e_1,\dots, e_\mu$ of Ω_f. Applying Lemma 3, one finds an $\mathcal{O}_T\{\{\delta_1^{-1}\}\}$-basis $f_1,\dots,f_\mu$ of $\pi_*\mathscr{H}_F^{(0)}$ satisfying (4.4.3). Let us denote by $v\colon q_*\Omega_F\to\pi_*\mathscr{H}_F^{(0)}$ the $\mathcal{O}_T$-section of $r^{(0)}$, defined by the $\mathcal{O}_T$-module $\sum_{i=1}^{\mu}\mathcal{O}_T f_i$ of $\pi_*\mathscr{H}_F^{(0)}$. The section v satisfies iii) iv) of (4.1) Definition by (4.4.3).

If we show that v also satisfies ii) of (4.1) so that v is a good section for F, we will have shown the bijectivity of (4.3.4). This can be seen as follows. Since the section v is a lifting of a good section v_0, we have

$$K_F(v(e),\ v(e'))|_{t'=0}=K_{F_0}(v_0(e),\ v_0(e'))\in \boldsymbol{C}\delta_1^{-n-1}\qquad \text{for}\quad e,\ e'\in q_*\Omega_F\,.$$

Using properties iii) and v) of (2.3) of K_F, one checks easily that

$$\frac{\partial}{\partial t_{i_1}}\cdots\frac{\partial}{\partial t_{i_l}}K_F(v(e),\ v(e'))|_{t'=0}\in \boldsymbol{C}\delta_1^{-n-1}\qquad \text{for}\quad e,\ e'\in q_*\Omega_F$$

and for any sequence of derivations $\frac{\partial}{\partial t_{i_1}},\dots,\ \frac{\partial}{\partial t_{i_l}}$. This obviously implies that $K_F(v(e),\ v(e'))\in\mathcal{O}_T\delta_1^{-n-1}$. q.e.d.

(4.5) **Corollary of Lemma 1.2.** *Let F be a universal unfolding of a function $f\colon(\boldsymbol{C}^{n+1},0)\to(\boldsymbol{C},0)$. There exists a natural surjective map,*

(4.5.1) $\{$*primitive forms of* $F\}/\boldsymbol{C}^*\to\{$*good sections of* $F_0=t_1-f\}$.

Note 1. If the map (4.2.1) of Note (4.2) is bijective, then the map (4.5.1) above is bijective.

Note 2. For the Hamiltonian system F_0, the condition iii) of (4.1) for a good section holds trivially. To be explicit, the target space of (4.5.1) is

$$\{v\colon \Omega_f=\Omega^{n+1}_{\boldsymbol{C}^{n+1}}/df\wedge\Omega^n_{\boldsymbol{C}^{n+1}}\longrightarrow \Gamma(\boldsymbol{C},\ \mathscr{H}_F^{(0)}),\ \boldsymbol{C}\text{-linear}$$

$$\text{s.t.}\quad K_F(v(e),\ v(e'))\in\boldsymbol{C}\delta_1^{-n+1}\qquad\text{for}\quad e,\ e'\in\Omega_f\quad\text{and}$$

$$t\cdot v(e)=v(Ae)+\left(\frac{\partial}{\partial t}\right)^{-1}v(Ne)\qquad\text{for some}\quad A,\ N\in\operatorname{End}(\Omega_f)\}\,.$$

(4.6) **Example 1.** Let f and g be functions with isolated critical points and $f+g$ be their joint. Then the set of good sections for $f+g$ is the direct product of these for f and g (cf. [41] §5).

Example 2. Universal unfolding for a weighted homogeneous singularity has primitive forms.

Example 3. Universal unfolding for a cusp singularity,

$$f(x, y, z)=x^p+y^q+z^r+axyz \qquad \text{s.t.} \quad 1/p+1/q+1/r\leq 1$$

has primitive forms.

(4.7) *Note.* Recently, M. Saito [46] has found a one to one correspondence between the set of certain sections of F_0 satisfying (4.1.3) and the set of certain splittings of the Hodge filtration on the vanishing cohomology of f.

§5. The Period Mapping

In this paragraph, we define a period mapping associated to a primitive form $\zeta^{(0)}$ for a universal unfolding F of a hypersurface isolated singular point. It will be defined using solutions of a certain self-dual holonomic system $\mathscr{M}^{(n/2)}$. On the period domain, the flat function τ introduced in (3.4.2) plays the role of a kernel function (cf. (5.8)). A comparison with the conventional period mapping (for instance, the one used in [18]) is given in (5.9).

The only cases, in which the period mapping and its inversion are relatively well-understood, are the simple singularities and the simple elliptic singularities. In the latter case, there are studies by E. Looijenga [18] [19] [20] [21] by P. Slodowy [48], [49] and by the author [42] [43].

(5.1) Let F be a universal unfolding over a frame (Z, X, S, T) and let $\zeta^{(0)} \in \Gamma(S, \mathscr{H}_F^{(0)})$ be a primitive form for F. Let us recall the system of linear equations satisfied by $\zeta^{(k-1)}=(\nabla_{\delta_1})^{k-1}\zeta^{(0)}$ for an integer $k\in \boldsymbol{Z}$. (cf. (3.2.7), (3.2.8)).

$$(\delta\delta'-(\delta*\delta')\delta_1-\nabla_\delta\delta')\zeta^{(k-1)}=0 \qquad \text{for} \quad \delta, \delta'\in\mathscr{G},$$

$$(w(\delta)\delta_1-(N-k-1)\delta)\zeta^{(k-1)}=0 \qquad \text{for} \quad \delta\in\mathscr{G},$$

$$(E-(r-k))\zeta^{(k-1)}=0\,.$$

Definition. For any $s\in \boldsymbol{C}$, define the $\mathscr{D}_S$-module

(5.1.1) $\qquad \mathscr{M}^{(s)} :\underset{\text{def}}{=} \mathscr{D}_S/\mathscr{I}_s$, where

(5.1.2) $\qquad \mathscr{I}_s := \sum_{\delta,\delta'\in\mathscr{G}} \mathscr{D}_S P(\delta, \delta') + \sum_{\delta\in\mathscr{G}} \mathscr{D}_S Q_s(\delta)$

(5.1.3) $\qquad P(\delta, \delta') := \delta\delta'-(\delta*\delta')\delta_1-\nabla_\delta\delta' \qquad \text{for} \quad \delta, \delta'\in\mathscr{G},$

(5.1.4) $\qquad Q_s(\delta) := w(\delta)\delta_1-(N-s-1)\delta \qquad \text{for} \quad \delta\in\mathscr{G}.$

For symmetry of the defining equations for $\mathscr{M}^{(s)}$, let us introduce some additional equations.

(5.1.5) $Q_s(\delta, \delta') := \delta w(\delta') - w(\nabla_\delta \delta') - \delta * (N-s)\delta' \qquad \text{for} \quad \delta, \delta \in \mathscr{G}.$

This is contained in $\mathscr{I}_s$, since we have relations,

(5.1.6) $$Q_s(\delta, \delta') = Q_s(\delta * \delta') + t_1 P(\delta, \delta') - P(\delta, t_1 * \delta'),$$

(5.1.7) $$Q_s(\delta_1, \delta) = Q_s(\delta).$$

Note. By definition, $\mathscr{M}^{(s)}$ is a cyclic module over $\mathscr{D}_S$. We shall fix the generator corresponding to 1 in (5.1.1). We shall denote by $[P]$ the element $P \cdot 1 = P \bmod \mathscr{I}_s$ in $\mathscr{M}^{(s)}$.

(5.2) *Simple holonomicity of $\mathscr{M}^{(s)}$.* To study the singularities of $\mathscr{M}^{(s)}$, we use the following relations, which follow from an elementary calculation using (3.3.2), (3.3.3), (3.3.8), (3.3.9).

(5.2.1) $$\begin{aligned} &\delta P(\delta', \delta'') - \delta' P(\delta, \delta'') + \delta_1(P(\delta, \delta' * \delta'') - P(\delta', \delta * \delta'')) \\ &= -P(\delta, \nabla_{\delta'} \delta'') + P(\delta', \nabla_\delta \delta'') \qquad \text{for} \quad \delta, \delta', \delta'' \in \mathscr{G}. \end{aligned}$$

(5.2.2) $$\begin{aligned} &\delta Q_s(\delta') - \delta_1 Q_s(\delta * \delta') - Q_s(\nabla_\delta \delta') \\ &= EP(\delta, \delta') + \delta P(t_1 * \delta_1, \delta') - \delta_1 P(t_1 * \delta_1, \delta * \delta') \\ &\quad + P([t_1 * \delta_1, \delta], \delta') + P(\delta, \nabla_{t_1 * \delta_1} \delta') \\ &\quad - P(\delta, (N-s-1)\delta') - P(t_1 * \delta_1, \nabla_\delta \delta') \qquad \text{for} \quad \delta, \delta' \in \mathscr{G}. \end{aligned}$$

As a consequence of these relations, one can prove the following

Assertion. *Let P be an element of $\mathscr{I}_s$ of degree m. Then $m \geq 2$ and there exist R_{ij}, $i, j = 2, \ldots, \mu$ and S_i, $i = 1, \ldots, \mu$ of $\mathscr{D}_S$, of degrees less than or equal to $m-2$ such that*

(5.2.3) $$P = \sum_{i,j} R_{ij} P\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right) + \sum_i S_i Q_S\left(\frac{\partial}{\partial t_i}\right).$$

Proof. Assume P has an expression of the form (5.2.3) such that $\max(\deg(R_{ij}), \deg(S_i)) \geq m-1$. Then using the relations (5.2.1), (5.2.2), one can reduce the problem to the case when the highest degree terms of R_{ij} and S_i involving only powers of δ_1, which must necessarily be zero. q.e.d.

Corollary. *The module $\mathscr{M}^{(s)}$ for $s \in \boldsymbol{C}$ is a simple holonomic system, whose singular support in the cotangent space T_S^* of S is the union of the zero section of T_S^* and the conormal bundle N_D^* of D. (Here N_D^* means the closure of the conormal bundle $N_{(D-\mathrm{Sing}\,D)}^*$ of the smooth part of D).*

Proof. Note that N_D^* is equal to the cone of the embedding of C into T_S^*

by $x\in C\mapsto\left(\varphi(x), \frac{\partial F}{\partial t_1}(x),\dots, \frac{\partial F}{\partial t_\mu}(x)\right)\in T^*_X$ (for instance cf. Teissier [51]).

On the other hand, the Assertion says that the singular support of $\mathcal{M}^{(s)}$ is defined by the ideal generated by principal symbols $\sigma(P(\delta, \delta'))=\delta\delta'-(\delta*\delta')\delta_1$ and $\sigma(Q_s(\delta, \delta'))=\delta w(\delta')$ for $\delta, \delta'\in\mathcal{G}$.

Note. On $S-D$, we have $\mathcal{D}$-isomorphism

$$\mathcal{M}^{(s)}|_{S-D}\simeq\mathcal{O}_{S-D}[1]+\sum_{i=1}^{\mu}\mathcal{O}_{S-D}\left[\frac{\partial}{\partial t_i}\right].$$

(5.3) *The duality morphism.* Let us define a left $\mathcal{D}_S$-homomorphism

(5.3.1) $$\mathcal{M}^{(s+1)}\longrightarrow\mathcal{M}^{(s)},\ [P]\longmapsto[P\delta_1].$$

To show that this is well-defined, we need to verify that

(5.3.2) $$P(\delta, \delta')\delta_1=\delta_1P(\delta, \delta'),$$

(5.3.3) $$Q_{s+1}(\delta)\delta_1=\delta_1Q_s(\delta).$$

Assertion. *Let l be the corank of the endomorphism $N-s-1$ on the μ-dimensional horizontal vector space of ∇ on $\mathcal{G}$ and let $\eta_1,\dots,\eta_l$ and $\xi_1,\dots,\xi_l$ be basis of* ker $(N-s-1)$ *and* coker $(N-s-1)$, *respectively.*

Then one has an exact sequence of $\mathcal{D}$-modules,

(5.3.4) $$0\longrightarrow\mathcal{O}_S[E_{s+1}]\oplus\bigoplus_{i=1}^{l}\mathcal{O}_S[w(\eta_i)]\longrightarrow\mathcal{M}^{(s+1)}\longrightarrow\mathcal{M}^{(s)}\longrightarrow$$
$$\mathcal{O}_S\oplus\bigoplus_{i=1}^{l}\mathcal{O}_S\xi_i\longrightarrow 0\qquad\text{for}\quad r\neq s+1,\quad\text{and}$$

(5.3.5) $$0\longrightarrow\bigoplus_{i=1}^{l}\mathcal{O}_S[w(\eta_i)]\longrightarrow\mathcal{M}^{(s+1)}\longrightarrow\mathcal{M}^{(s)}\longrightarrow\mathcal{O}_S\oplus\bigoplus_{i=2}^{l}\mathcal{O}_S\xi_i\longrightarrow 0$$

for $r=s+1$, *where we put* $\delta_1=\xi_1$.

Here we used the following notation and relations for the proof:

(5.3.6) $$E_s:=E-(r-s),$$

(5.3.7) $$Q_s(\delta, \delta_1)=\delta E_s,$$

(5.3.8) $$Q_s(\delta_1)=E_{s+1}\delta_1=\delta_1E_s.$$

Notation. We denote $\mathrm{Hom}_{\mathcal{D}_S}(\mathcal{M}, \mathcal{O}_S)$ by $\mathrm{Sol}(\mathcal{M})$ for short.

Corollary. i) *The kernel and the cokernel of the morphism,*

(5.3.9) $$\mathrm{Sol}(\mathcal{M}^{(s)})\xrightarrow{\delta_1}\mathrm{Sol}(\mathcal{M}^{(s+1)}),$$

form a local system of rank $l+1$ (or l in case $r=s+1$) on S. More precisely, *the kernel is spanned by flat coordinates* (cf. (3.3.7)) *of degree* $s+1$ *and by a constant function* 1_S.

ii) $$\mathscr{E}xt^i_{\mathscr{D}_S}(\mathscr{M}^{(s)}, \mathscr{O}_S) \simeq \mathscr{E}xt^i_{\mathscr{D}_S}(\mathscr{M}^{(s+1)}, \mathscr{O}_S) \qquad \text{for} \quad i \geq 1.$$

Note 1. As in Note in (5.2), $\mathscr{M}^{(s)}$ is cyclic with the generator 1; a solution $\varphi \in \mathrm{Sol}(\mathscr{M}^{(s)})$ is identified with a function $\varphi(1)$ on S. Then the morphism (5.3.9) is the derivation by δ_1.

Let us denote by $d\,\mathrm{Sol}(\mathscr{M}^{(s)})$ the image of $\mathrm{Sol}(\mathscr{M}^{(s)})$ in Ω^1_S under the exterior differentiation d. Since the solution space $\mathrm{Sol}(\mathscr{M}^{(s)})$ always contains the constant functions, we have an isomorphism of local systems

(5.3.10) $$d\,\mathrm{Sol}(\mathscr{M}^{(s)}) \simeq \mathrm{Sol}(\mathscr{M}^{(s)})/\boldsymbol{C}_S.$$

Note 2. Let $\gamma(t) \in H_n(X_t, \boldsymbol{Z})$ be a horizontal family of homology (defined on a simply connected domain of a covering space of $S-D$).

For any integer $k \in \boldsymbol{Z}$, the integral $\int_{\gamma(t)} \zeta^{(k-1)}$ gives a solution of the system $\mathscr{M}^{(k)}$. In particular, for $k \leq 0$, the constant function 1_S on S and $\int_{\gamma_i(t)} \zeta^{(k-1)}$, $i=1,\dots,\mu$ form a $\boldsymbol{C}$-basis for the solutions of $\mathscr{M}^{(k)}$. ($\because$ The Notes of (3.5) and (3.5.4))

In other words, there exists a sequence of natural isomorphisms of local systems,

(5.3.11) $$\cdots \overset{\delta_1}{\simeq} d\,\mathrm{Sol}(\mathscr{M}^{(-2)})|_{S-D} \overset{\delta_1}{\simeq} d\,\mathrm{Sol}(\mathscr{M}^{(-1)})|_{S-D} \overset{\delta_1}{\simeq} d\,\mathrm{Sol}(\mathscr{M}^{(0)})|_{S-D} \simeq \bigcup_{t \in S-D} H_n(X_t, \boldsymbol{C}).$$

Note 3. Instead of $\mathscr{M}^{(s)}$, one may define and study

(5.3.12) $$\tilde{\mathscr{M}}^{(s)} \underset{\mathrm{def}}{:=} \mathscr{M}^{(s)}/\mathscr{D}_S[E_s] = \mathscr{D}_S/\mathscr{I}_s + \mathscr{D}_S E_s.$$

Using the relation (5.3.7), one gets a short exact sequence,

(5.3.13) $$0 \longrightarrow \mathscr{O}_S E_s \longrightarrow \mathscr{M}^{(s)} \longrightarrow \tilde{\mathscr{M}}^{(s)} \longrightarrow 0.$$

Hence the solution space is described by

i) For $r \neq s$ $\quad \mathrm{Sol}(\mathscr{M}^{(s)}) \simeq \mathrm{Sol}(\tilde{\mathscr{M}}^{(s)}) \oplus \boldsymbol{C}$,

ii) For $r = s$ $\quad \mathrm{Sol}(\mathscr{M}^{(s)}) \simeq \mathrm{Sol}(\tilde{\mathscr{M}}^{(s)}) \oplus \boldsymbol{C}\lambda$

where λ is a function on $S-D$ such that $E\lambda = \mathrm{const.} \neq 0$. Since we want to include such functions λ in our study, we use the module $\mathscr{M}^{(s)}$ instead of $\tilde{\mathscr{M}}^{(s)}$.

Note also that the homomorphism (5.3.1) is factored

(5.3.14) $$\tilde{\mathscr{M}}^{(s+1)} \longrightarrow \mathscr{M}^{(s)}$$

($\because$ (5.3.8)) so that one gets also

$$\tilde{\mathcal{M}}^{(s+1)} \longrightarrow \tilde{\mathcal{M}}^{(s)} \tag{5.3.15}$$

Note 4. *Question*, $\mathcal{E}xt^i_{\mathcal{D}_S}(\mathcal{M}^{(s)}, \mathcal{O}_S)=0$ for $i>0$?

(5.4) **Definition of** I. Let us define an $\mathcal{O}_S$-bilinear form

$$I: \Omega^1_S \times \Omega^1_S \longrightarrow \mathcal{O}_S \tag{5.4.1}$$

$$I(\omega, \omega') := \sum_{i=1}^{\mu} \langle \delta_i \omega \rangle \langle w(\delta^{i*})\omega' \rangle$$

where $\delta_1, \dots, \delta_\mu$ and $\delta^{1*}, \dots, \delta^{\mu*}$ are an $\mathcal{O}_T$-basis and the dual basis of $\mathcal{G}$ w.r.t. J and $\langle\ \rangle$ is the pairing between forms and vector fields on S. Using the definition (1.7.3) of w, one sees easily that I is symmetric.

Proposition 1. *I induces a $\boldsymbol{C}$-bilinear form,*

$$I_s: d\,\mathrm{Sol}(\mathcal{M}^{(s)})|_{S-D} \times d\,\mathrm{Sol}(\mathcal{M}^{(n-s)})|_{S-D} \longrightarrow \boldsymbol{C}_S \tag{5.4.2}$$

The induced form I_s is non-degenerate.

2. *For $u \in \mathrm{Sol}(\mathcal{M}^{(s)})$ and $v \in \mathrm{Sol}(\mathcal{M}^{(n-s-1)})$,*

$$I_s(u, \delta_1 v) = -I_{s+1}(\delta_1 u, v) \tag{5.4.3}$$

Proof. Define the element

$$I_s :=_{\mathrm{def}} \sum_{i=1}^{\mu} \delta_i \otimes w(\delta^{i*}) \in \mathcal{M}^{(s)} \underset{\mathcal{O}_S}{\otimes} \mathcal{M}^{(n-s)}, \tag{5.4.4}$$

where the right hand side of (5.4.4) is a left $\mathcal{D}_S$-module.

We have only to show that

$$\mathcal{D}_S I_s = \mathcal{O}_S I_s, \tag{5.4.5}$$

which can be shown in the same way as (3.4.4).

Since the determinant of I of (5.4.4) is equal to Δ (cf. (1.7)), it is non-degenerate on $S-D$. Since $\Omega^1_{S,t} = \mathcal{O}_{S,t} \otimes d\,\mathrm{Sol}(\mathcal{M}^{(s)})_t$ for any $t \in S-D$, the bilinear form (5.4.2) is non-degenerate. (5.4.3) can be shown similarly to (3.4.3). q.e.d.

(5.5) Proposition 1 above implies that the local systems $d\,\mathrm{Sol}(\mathcal{M}^{(s)})|_{S-D}$ and $d\,\mathrm{Sol}(\mathcal{M}^{n-s})|_{S-D}$ are $\boldsymbol{C}$-dual to each other. In view of the sequence (5.3.11), we have a sequence of isomorphisms of local systems,

$$\bigcup_{t \in S-D} H^n(X_t, \boldsymbol{C}) \simeq d\,\mathrm{Sol}\,(\mathcal{M}^{(n)})|_{S-D} \overset{\delta_1}{\simeq} d\,\mathrm{Sol}\,(\mathcal{M}^{(n+1)})|_{S-D} \simeq \cdots . \tag{5.5.1}$$

Thus for $0 \leq i \leq n$ we have remaining local systems with homomorphisms δ_1,

$$(5.5.2) \qquad d\,\mathrm{Sol}\,(\mathcal{M}^{(0)}) \xrightarrow{\delta_1} d\,\mathrm{Sol}\,(\mathcal{M}^{(1)}) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_1} d\,\mathrm{Sol}\,(\mathcal{M}^{(n-1)}) \xrightarrow{\delta_1} d\,\mathrm{Sol}\,(\mathcal{M}^{(n)}).$$

This sequence is the real object of interset in this note, where the left and the right ends are naturally identified with the local system of homology and cohomology groups of the fibers of the Milnor's fibration φ: $X-\varphi^{-1}(D)\to S-D$, as we just have seen.

Proposition. *For even n, the composition of maps in* (5.5.2),

$$(5.5.3) \qquad \left(\frac{\delta_1}{2\pi\sqrt{-1}}\right)^n : d\,\mathrm{Sol}\,(\mathcal{M}^{(0)})|_{S-D} \longrightarrow d\,\mathrm{Sol}\,(\mathcal{M}^{(n)})|_{S-D}$$

coincides with the linear mapping defined by the intersection form.

Proof. For two solutions $u, v \in \mathrm{Sol}(\mathcal{M}^{(0)})$, the number $(2\pi\sqrt{-1})^{-n} I_0 \cdot (u, \delta_1^n v)$ is equal to the intersection number (cf. The theorem of (3.4)).

Corollary. *An element of $H_n(X_t, \boldsymbol{Z})$ is an invariant cycle iff it is represented by $u \in \mathrm{Sol}(\mathcal{M}^{(0)})$ which is a polynomial in t_1 of degree less than or equal to n.*

(5.6) *Picard-Lefschetz formula.* We give the Picard-Lefschetz formula for $\mathrm{Sol}(\mathcal{M}^{(s)})$, which can be proved analytically. For a generic point t' of T-{bifurcation set of $(\pi|_D)$}, put $\pi^{-1}(t') \cap D = \{p_1, \dots, p_\mu\}$. Let us fix a base point $p_0 \in \pi^{-1}(t') - \{p_1, \dots, p_\mu\}$ and simple paths $g_1, \dots, g_\mu$ in $\pi^{-1}(t')$, which combine p_0 with $p_1, \dots, p_\mu$, which are disjoint in $\pi^{-1}(t') - \{p_0\}$. For a homotopy class in $\pi_1(S-D, p_0)$ of a path, which goes along g_i near to D_i, turns once around counterclockwise and comes back along g_i, the monodromy representation in the local system $d\,\mathrm{Sol}\,(\mathcal{M}^{(k)})|_{S-D}$ is given by,

$$(5.6.1) \qquad u \longmapsto u - (-1)^{\frac{n(n+1)}{2}+k} I_k\left(u, \left(\frac{\delta_1}{2\pi\sqrt{-1}}\right)^{n-k} \gamma_i\right)\left(\frac{\delta_1}{2\pi\sqrt{-1}}\right)^k \gamma_i$$

where the $\gamma_i \in \mathrm{Sol}\,(\mathcal{M}^{(0)})$ are defined as follows.

Let S_i and X_i be suitable neighbourhoods of p_i in S, and $q_i := \varphi^{-1}(p_i) \cap C$ in X and $Z_i = S_i \times_T X_i$. Then $F_i := F|_{Z_i}$ defines a Hamiltonian system whose set of singular points $C_i = C \cap X_i$ consists only of ordinary double points. (i.e. φ_i is a non-degenerate Morse function at C_i.) In that case there exists a unique primitive form $\zeta_i^{(0)}$ for F_i and there is a standard way of constructing a solution e_i of $\mathcal{H}_{F_i}^{(0)}$ as follows. (cf. [35] § 12)

(5.6.2) $$e_i\colon \mathcal{H}_{F_i}^{(0)} \longrightarrow \mathcal{O}_{S_i}\sqrt{t_1-h_i(t')}\,,$$
$$g(t)\zeta_i^{(0)} \longmapsto \frac{\sqrt{\pi}^{n+1}}{\sqrt{-1}^{\frac{n+1}{2}}}\, g(t)\sqrt{t_1-h_i}^{\frac{n-1}{2}}\sqrt{J_{F_i}(\zeta_i,\zeta_i)}$$

where $t_1-h_i(t')$ is the defining equation for D_i in S_i. Then the natural composition map,

(5.6.3) $$\mathcal{H}_F^{(0)}|_{S_i} \longrightarrow \mathcal{H}_{F_i}^{(0)} \xrightarrow{e_i} \mathcal{O}_{S_i}[\sqrt{t_1-h_i}]$$

defines a solution of $\mathcal{H}_F^{(0)}|_{S_i}$ and hence a solution of $\mathcal{M}^{(0)}$. The analytic continuation of the solution to p_0 along the path g_i is the solution γ_i in the formula (5.6.1).

Note 1. The solution e_i of (5.6.2) is identified with the homology class of the vanishing cycle of the Morse function F_i (cf. [35] § 12).

Hence the set $\gamma_1,\dots,\gamma_\mu$ of solutions of $\mathcal{M}^{(0)}$ (at the point p_0) form an integral basis for the integral homology group $H_n(X_{p_0}, \boldsymbol{Z})$ (cf. [5] Appendix). It might be interesting to find an analytic proof of this fact.

Namely, we ask:
Find an analytic proof that

(5.6.4) $$(2\pi\sqrt{-1})^{-n}I_0(\gamma_i, \delta_1^n\gamma_j)\in \boldsymbol{Z} \qquad \text{for} \quad i, j=1,\dots,\mu.$$

Note 2. Using the Picard-Lefschetz formula (5.6.1), one may determine the monodromy representation,

$$\rho_k\colon \pi_1(S-D, p_0) \longrightarrow \mathrm{Aut}\,(\mathrm{Sol}\,(\mathcal{M}^{(k)})|_{S-D}), \quad k=0,\dots,n$$

with relations,

(5.6.5) $$\delta_1\rho_k=\rho_{k+1}\delta_1 \qquad (\because (5.4.3))$$

(5.6.6) $$\rho_k^*=\rho_{n-k} \qquad (\because (5.4.2))$$

As we shall see in an example (cf. (5.8). 2)), ρ_k, $k=0,\dots,n$, may be different from each other.

(5.7) *A period mapping.* Let us fix a base point $p_0\in S-D$ and let $\widetilde{S-D}$ be a monodromy covering space of $S-D$ w.r.t. the monodromy presentation $\rho_{n/2}$. (i.e. the smallest covering space of $S-D$ such that the lifting of $\rho_{n/2}$ to the space becomes trivial.) For any point $t\in\widetilde{S-D}$, let us define a linear functional on the $\mu+1$ dimensional $\boldsymbol{C}$-vector space $\mathrm{Sol}\,(\mathcal{M}^{(\frac{n}{2})})_{p_0}$,

(5.7.1) $$\widetilde{S-D}\times \mathrm{Sol}\,(\mathcal{M}^{(\frac{n}{2})})_{p_0} \longrightarrow \boldsymbol{C}, \quad (t, u) \longmapsto u(t),$$

where $u(t)$ means the value at t of the analytic continuation of u. Hence we obtain a map from $\widetilde{S-D}$ to a μ-dimensional affine space,

$$(5.7.2) \qquad \widetilde{S-D} \longrightarrow E :\underset{\text{def}}{=} \{\chi \in \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Sol}\,(\mathscr{M}^{(\frac{n}{2})}),\ \boldsymbol{C})_{p_0} : \chi(1_S)=1\}.$$

By definition the map (5.7.2) is holomorphic. We shall call it the period mapping associated to $\zeta^{(0)}$.

1) The domain of definition of the period mapping can be extended to a manifold $\tilde{S} \supset \widetilde{S-D}$, which is characterized as follows. The difference $\tilde{D} := \tilde{S} - (\widetilde{S-D})$ is a divisor in $\tilde{S}$. There exists a flat holomorphic map $\alpha: \tilde{S} \to S$ which extends the covering $\widetilde{S-D} \to S-D$, having the following property: For any $t \in D$ such that the fiber $X_t = \varphi^{-1}(t)$ has at most simple singularities, there exists a neighbourhood $U \subset S$ of t such that for any connected component $\tilde{U}$ of $\alpha^{-1}(U)$, the restriction $\alpha|\tilde{U}$ is a proper finite map (cf. [6]).

The extended period mapping

$$(5.7.3) \qquad P_{\zeta^{(0)}} : \tilde{S} \longrightarrow E$$

is an local isomorphism. It is obvious by definition that the map (5.7.3) is equivariant with the monodromy group action of the representation $\rho_{\frac{n}{2}}$.

Proof. By the Riemann extension theorem it is enough to extend the period map and to show the biregularity only at smooth points of $\tilde{D}$, which can be easily done by using F_i in (5.6).

2) The cotangent space of E is naturally identified with the vector space $d\,\mathrm{Sol}\cdot(\mathscr{M}^{(\frac{n}{2})})_{t_0}$. Hence the non-degenerate bilinear form $I_{n/2}$ on the space induces a bilinear map,

$$(5.7.4) \qquad I_{n/2} : \mathcal{O}_{\tilde{S}} \times \mathcal{O}_{\tilde{S}} \longrightarrow \mathcal{O}_{\tilde{S}},\ (u, v) \longmapsto I_{n/2}(du, dv)\,.$$

In particular, for linear coordinate $u_1, \ldots, u_\mu$ of E, $(I_{n/2}(du_i, du_j))_{ij}$ is a constant matrix with non zero determinant, and for a flat coordinate system $t_1, \ldots, t_\mu$ of S, $\det(I_{n/2}(dt_i, dt_j))_{ij} = \Delta$ (cf. (1.7)).

Thus the Jacobian of the map $\alpha: \tilde{S} \to S$ is calculated as

$$(5.7.5) \qquad \mathrm{Jac.}\ (\alpha) = \frac{\partial(t_1, \ldots, t_\mu)}{\partial(u_1, \ldots, u_\mu)} = c\Delta^{\frac{1}{2}},\ c \neq 0\,.$$

(5.8) *Reproducing kernel* $A(t, s)$. If $r \neq n/2$, using $\tilde{\mathscr{M}}^{(\frac{n}{2})}$ in the *Note* 3 of (5.3), one may naturally identify the affine space E (5.7.2) with the dual vector space of $\mathrm{Sol}\,(\tilde{\mathscr{M}}^{(\frac{n}{2})})_{t_0} \subset \mathrm{Sol}\,(\mathscr{M}^{(\frac{n}{2})})_{t_0}$. The bilinear form $I_{n/2}$ induces a non-degenerate bilinear form on $\mathrm{Sol}(\tilde{\mathscr{M}}^{(\frac{n}{2})})_{t_0}$. Let us denote by A the dual bilinear form of $I_{n/2}$ defined on $E \simeq \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Sol}(\tilde{\mathscr{M}}^{(\frac{n}{2})})_{t_0}, \boldsymbol{C})$

(5.8.1) $$A: E \times E \longrightarrow \boldsymbol{C}.$$

Let us denote by the same A the function $A(t, s)$ defined on $\tilde{S} \times \tilde{S}$ which is induced by the period mapping $P_{\zeta^{(0)}}$. Then A has the following properties.

i) $A(t, s) = A(s, t)$.

ii) For any fixed $s \in \tilde{S}$, $A(t, s)$ is homogeneous of degree $r - \frac{n}{2}$ as a function of t (w.r.t. the Euler operator) and $A(t, s) \in \mathrm{Sol}(\mathscr{M}^{(\frac{n}{2})})$.

iii) *reproducing property*

(5.8.2) $(2\pi)^{-n} I_{n/2}(d_t A(t, s), d_t v(t)) = v(s)$ for $v \in \mathrm{Sol}(\mathscr{M}^{(\frac{n}{2})})$.

iv) *trace formula*

(5.8.3) $$A(t, t) = \frac{(2\pi)^{-n}}{r - n/2} \tau(t).$$

Here d_t denotes the exterior differentiation in the variable t and $\tau(t)$ is the flat function τ defined in (3.3) 3) composed with the covering map $\alpha: \tilde{S} \to S$.

Proof. i) and ii) are trivial by definition. iii) and iv) are proven in the same manner as Theorem 1 of [40].

(5.9) *Note.* A comparison of $P_{\zeta^{(0)}}$ with the conventional period map is given as follows.

The left $\mathscr{D}_S$-homomorphism

(5.9.1) $$\delta_1^{\frac{n}{2}}: \mathscr{M}^{(n/2)} \longrightarrow \tilde{\mathscr{M}}^{(0)}, \ [P] \longmapsto [P\delta_1^{n/2}]$$

induces a linear map from the module of solutions to its dual.

(5.9.2) i) $\mathrm{Sol}(\tilde{\mathscr{M}}^{(0)}) \longrightarrow \mathrm{Sol}(\mathscr{M}^{(n/2)}), \quad u \longmapsto \delta_1^{n/2} u$,

ii) $E \longrightarrow \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Sol}(\tilde{\mathscr{M}}^{(0)})_{t_0}, \boldsymbol{C}) \simeq H^n(X_{t_0}, \boldsymbol{C})$.

The composition of the above projection with the period map (5.7.3),

(5.9.3) $$\tilde{S} \longrightarrow H^n(X_{t_0}, \boldsymbol{C}), \quad t \longrightarrow \left(\int_{\gamma_i(t)} \zeta^{(\frac{n}{2}-1)}\right)_{i=1,\dots,\mu}$$

is the period mapping for the family F, in the conventional sense. (See for instance [17], [36]). The mapping (5.9.2) ii) is bijective iff there is no integral exponent, in which case one may identify the period mapping in the conventional sense with the period mapping of this note.

(5.10) **Example.** Let F be a universal unfolding of a simple elliptic singularity. Fixing a primitive form $\zeta^{(0)}$ as in Example 2 in (3.1), one sees that

1) The period map (5.8.2) induces an isomorphism from $\tilde{S}$ to a half space

$$\tilde{S} \simeq E := \{u \in E : \mathrm{Im}(\tau(u)) > 0\} .$$

2) The monodromy group $\tilde{W}$ for the representation ρ_1 is a central extension of the monodromy group W for ρ_0

$$1 \longrightarrow \boldsymbol{Z} \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1 ,$$

where $\boldsymbol{Z}$ is generated by a power of the classical monodromy (i.e. the image of the generator of $\boldsymbol{Z} = \pi_1(\pi^{-1}(0) - \{0\})$).

Let T be subgroup of translations of W and let $\tilde{T}$ be the inverse image of T in $\tilde{W}$, so that one has a central extension,

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \tilde{T} \longrightarrow T \longrightarrow 1 .$$

Then this extension defines a family of polarized Abelian varieties over the upper half plane

$$\boldsymbol{E}/\tilde{T} \xrightarrow{\tau} \boldsymbol{H} = \{\tau \in \boldsymbol{C} : \mathrm{Im}\, \tau > 0\}$$

on which a finite Weyl group W/T is acting, which was studied by E. Looijenga [18] [20].

There are other aproaches using characters of Kac-Moody Lie algebra of Euclidean type by P. Slodowy [48] [49] and Kac-Peterson [13].

There is another approach to studying the space $\boldsymbol{E}/\tilde{W}$, using an extension of an affine root system (which is called the extended affine root system) (cf. [42] [43]), so that, in particular, the flat structure of the space is determined.

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