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Vanishing Theorems on Complete Kähler Manifolds

By

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§ 0. Introduction

Let $\mathbf{X}$ be a complex manifold of dimension n and let $\mathbf{E}$ be a holomorphic vector bundle over $\mathbf{X}$. We shall here try to continue the study on the vanishment of the sheaf cohomology groups $H^q(\mathbf{X}, \mathcal{O}(\mathbf{E}))$ which has been performed by Kodaira [10], [11], Grauert-Riemenschneider [5], Andreotti-Vesentini [1], [2], Nakano [14], [15], Kazama [9], and others.

The purpose of the present paper is to study the cohomology groups on complete Kähler manifolds. Although the spirit is the same as in [1] and [14], we restrict ourselves to 'L^2-cohomology groups' and aim at finding a proper subspace of L^2-forms for which $\bar{\partial}$-equation is solvable. We shall prove the following theorem.

L^2-vanishing theorem (cf. Theorem 2.8). *Let $\mathbf{X}$ be a complete Kähler manifold of dimension n, let $(\mathbf{E}, h)$ be a hermitian bundle over $\mathbf{X}$, and let σ be a d-closed semipositive $(1, 1)$-form on $\mathbf{X}$. Assume that the curvature form for h is equal to or greater than σ. Then, for any $\bar{\partial}$-closed $\mathbf{E}$-valued (n, q)-form f which is square integrable with respect to σ (for the definition see Section 2), we can find an $\mathbf{E}$-valued $(n, q-1)$-form g which is square integrable with respect to σ satisfying $\bar{\partial}g=f$. Here $q\geqq 1$.*

This is a generalization of theorem 1.5 in [16]. We apply it here to obtain the following two vanishing theorems.

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Theorem (cf. Theorem 3.1). *Let* $\mathbf{X}$ *be a compact Kähler manifold, let* $\mathbf{Y}$ *be an analytic space, let* $f: \mathbf{X} \to \mathbf{Y}$ *be a holomorphic map, and let* $(\mathbf{E}, h)$ *be a hermitian bundle over* $\mathbf{X}$. *Assume that the curvature form for* h *is equal to or greater than the pull-back of a Kähler metric on* $\mathbf{Y}$. *Then,*

$$H^q(\mathbf{Y}, f_* \mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})) = 0, \qquad \text{for } q \geqq 1.$$

Here $\mathbf{K_X}$ *denotes the canonical bundle of* $\mathbf{X}$ *and* $f_* \mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})$ *denotes the direct image sheaf of* $\mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})$.

Theorem (cf. Theorem 4.5). *Let* $\mathbf{X}$ *be a 1-convex manifold with maximal compact analytic set* $\mathbf{A}$, *and let* $\mathbf{E} \to \mathbf{X}$ *be a holomorphic vector bundle. Assume that the restriction of* $\mathbf{E}$ *to* $\mathbf{A}$ *is Nakano-semipositive. Then*

$$H^q(\mathbf{X}, \mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})) = 0, \qquad \text{for } q \geqq 1.$$

Fortunately these theorems have applications. Namely, Theorem 3.1 provides a simple proof of Fujita's semipositivity theorem [3] for relative canonical sheaves, and Theorem 4.5 establishes the converse statement to Laufer's theorem $\mathbf{P}^1$ as an exceptional set [13].

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§ 1. Preliminaries

Let $\mathbf{X}$ be a complex manifold of dimension n with a hermitian metric ω, and let $\mathbf{E} \to \mathbf{X}$ be a holomorphic vector bundle with a hermitian metric h along the fibers. We say $(\mathbf{E}, h)$ a hermitian bundle over $\mathbf{X}$. We shall regard ω as a (1, 1)-form on $\mathbf{X}$, and h as a C^∞ section of $\mathrm{Hom}(\mathbf{E}, \bar{\mathbf{E}}^*)$. We denote by $C_0^{p,q}(\mathbf{X}, \mathbf{E})$ the space of $\mathbf{E}$-valued (p, q)-forms on $\mathbf{X}$ whose supports are compact. The length of $f \in C_0^{p,q}(\mathbf{X}, \mathbf{E})$ with respect to ω and h is denoted by $|f|$. Let dv be the volume form on $\mathbf{X}$ with respect to ω and set

$$\|f\| := \left\{ \int_{\mathbf{X}} |f|^2 dv \right\}^{1/2},$$

which is the usual L^2-norm. The L^2-norm $||f||$ determines a hermitian inner product in $C_0^{p,q}(\mathbf{X}, \mathbf{E})$ which we denote by (f, g). Let $<f, g>$ be the pointwise inner product with respect to ω and h. Then,

$$(f, g) = \int_{\mathbf{X}} <f, g> dv.$$

When we need to be more precise, we write h and ω explicitly, e. g. $<f, g>_h$ or $<f, g>_{h,\omega}$. Let $L^{p,q}(\mathbf{X}, \mathbf{E}, \omega, h)$ be the completion of $C_0^{p,q}(\mathbf{X}, \mathbf{E})$ with respect to the above norm. Then, by the theorem of Riesz-Fischer, $L^{p,q}(\mathbf{X}, \mathbf{E}, \omega, h)$ is naturally identified with the space of $\mathbf{E}$-valued integrable (p, q)-forms.

Proposition 1. 1. *Let ω_1 and ω_2 be two hermitian metrics satisfying $\omega_1 \geqq \omega_2$. Then,*

(1) $$||f||_{\omega_1} \leqq ||f||_{\omega_2}, \text{ for } f \in C_0^{n,q}(\mathbf{X}, \mathbf{E}).$$

Proof. Let $x \in \mathbf{X}$ be any point, and represent ω_1 and ω_2 at x as follows:

(2) $$\begin{cases} \omega_1 = \sum_{i=1}^{n} \sigma_i \bar{\sigma}_i \\ \omega_2 = \sum_{i=1}^{n} \lambda_i \sigma_i \bar{\sigma}_i, \qquad \lambda_i > 0. \end{cases}$$

Let f_x denote the value of f at x. We set

(3) $$f_x = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} f_{x i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \sigma_{i_1} \wedge \dots \wedge \sigma_{i_p} \wedge \bar{\sigma}_{j_1} \wedge \dots \wedge \bar{\sigma}_{j_q}.$$

Then,

(4) $$\begin{cases} |f_x|^2_{\omega_1} = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} |f_{x i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}|^2 \\ |f_x|^2_{\omega_2} = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \dfrac{|f_{x i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}|^2}{\lambda_{i_1} \dots \lambda_{i_p} \lambda_{j_1} \dots \lambda_{j_q}}. \end{cases}$$

Since

(5) $$dv_{\omega_1} = \frac{1}{\lambda_1 \dots \lambda_n} dv_{\omega_2} \quad at\ x,$$

we have

(6) $$|f_x|^2_{\omega_1} dv_{\omega_1} = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \frac{\lambda_{i_1} \dots \lambda_{i_p} \lambda_{j_1} \dots \lambda_{j_q}}{\lambda_1 \dots \lambda_n} |f_{x i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}|^2 dv_{\omega_2}.$$

Thus, if $p=n$, then $\lambda_{i_1}\dots\lambda_{i_p}$ and $\lambda_1\dots\lambda_n$ cancel each other so that

(7) $$|f_x|^2_{\omega_1} dv_{\omega_1} = \sum_{\substack{i_1<\dots<i_p \\ j_1<\dots<j_q}} \lambda_{j_1}\dots\lambda_{j_q} |f_{x i_1\dots i_p \bar{j}_1\dots \bar{j}_q}| \, dv_{\omega_2}.$$

Since $\lambda_i>1$, we obtain from (7),

(8) $$|f_x|^2_{\omega_1} dv_{\omega_1} \leqq |f_x|^2_{\omega_1} dv_{\omega_2}.$$

Therefore,

(9) $$\|f\|_{\omega_1} \leqq \|f\|_{\omega_2}. \qquad \text{Q. E. D.}$$

As usual we denote by $\bar{\partial}$ the exterior differentiation with respect to the conjugate of the local coordinates of $\mathbf{X}$ and by $\theta(=\theta_{\omega,h})$ the adjoint of $\bar{\partial}$ with respect to the inner product of $L^{p,q}(\mathbf{X}, \mathbf{E}, \omega, h)$. We denote by $L(=L_\omega)$ the multiplication of $\sqrt{-1}\omega$ from the left and by $\Lambda(=\Lambda_\omega)$ the adjoint to L. Let Θ_h be the curvature form for h. Recall that $\Theta_h=\bar{\partial}h^{-1}\partial h$ and that Θ_h is a $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued (1, 1)-form. Thus the left multiplication by Θ_h, which we denote by $e(\Theta_h)$, operates on $L^{p,q}(\mathbf{X}, \mathbf{E}, \omega, h)$. The following facts are basic for our purpose.

Proposition 1.2 (cf. [17]). *If ω is a Kähler metric on $\mathbf{X}$, then*

(10) $$\|\bar{\partial}f\|^2+\|\theta f\|^2 \geqq (\sqrt{-1}e(\Theta_h)\Lambda f, f),$$
for any $f\in C_0^{n,q}(\mathbf{X}, \mathbf{E})$, where $q\geqq 1$.

Proposition 1.3 (cf. Theorem 1.1 in [18]). *If ω is a complete hermitian metric on $\mathbf{X}$, then $C_0^{p,q}(\mathbf{X}, \mathbf{E})$ is dense in the space $\{f\in L^{p,q}(\mathbf{X}, \mathbf{E}, \omega, h);\ \bar{\partial}f\in L^{p,q+1}(\mathbf{X}, \mathbf{E}, \omega, h),\ \theta f\in L^{p,q-1}(\mathbf{X}, \mathbf{E}, \omega, h)\}$ with respect to the norm $\|f\|+\|\bar{\partial}f\|+\|\theta f\|$.*

§ 2. L^2-Vanishing Theorem

Let $\mathbf{X}$, ω, $\mathbf{E}$ and h be as in Section 1.

Definition 2.1. *Let Θ be a* $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$*-valued* (1, 1)*-form on* $\mathbf{X}$. *Θ is said to be semipositive (positive) if Θ satisfies*

(11) $$\langle\Theta(u), u\rangle_h(\xi, \bar{\xi}) \geqq 0 \ \ (\textit{resp.} > 0)$$

for any $u \in \mathbf{E}$ *and* $\xi \in \mathrm{T}\mathbf{X}$ *with* $u \neq 0$ *and* $\xi \neq 0$. *Here* $\mathrm{T}\mathbf{X}$ *denotes the holomorphic tangent bundle of* $\mathbf{X}$.

Given two $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued $(1, 1)$-forms Θ_1 and Θ_2, we denote $\Theta_1 \geqq \Theta_2$ if $\Theta_1 - \Theta_2$ is semipositive. A scalar $(1, 1)$-form is identified with a $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued $(1, 1)$-form when we compare it with $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued forms.

Proposition 2. 2. *Let* Θ *be a semipositive* $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$*-valued* C^∞ $(1, 1)$*-form. Then,*

$$\langle \sqrt{-1}e(\Theta)\Lambda f, f \rangle_h \geqq 0, \tag{12}$$

for any $f \in C_0^{n,q}(\mathbf{X}, \mathbf{E})$.

Proof. The reader is referred to [17].

Definition 2. 3. *Given a* C^∞ *semipositive* $(1, 1)$*-form* σ *on* X, *we set*

$$\begin{cases} \mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \sigma, h) \\ := \{f \in \mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \omega+\sigma, h) ;\ \lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega+\sigma} \text{ exists}\}, \\ \|f\|_\sigma := \lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega+\sigma}, \quad \text{for } f \in \mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \sigma, h). \end{cases} \tag{13}$$

Proposition 2. 4. $\mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \sigma, h)$ and $\|f\|_\sigma$ *do not depend on the choice of the metric* ω.

Proof. Let ω' be another hermitian metric on $\mathbf{X}$ and let K be any compact subset of $\mathbf{X}$. Then, for any $\varepsilon > 0$, we can find $\delta > 0$ so that $\varepsilon\omega' + \sigma \geqq \delta\omega + \sigma$ on K. Hence, in virtue of Proposition 1. 1, we have

$$\int_K |f|^2_{\varepsilon\omega'+\sigma} dv_{\varepsilon\omega'+\sigma} \leqq \int_K |f|^2_{\delta\omega+\sigma} dv_{\delta\omega+\sigma}. \tag{14}$$

From (14) we observe that if $\lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega+\sigma}$ exists, then $\|f\|_{\varepsilon\omega'+\sigma}$ is bounded by $\lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega+\sigma}$. Therefore, $\lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega'+\sigma} \leqq \lim\limits_{\varepsilon \searrow 0} \|f\|_{\varepsilon\omega+\sigma}$, which implies independence of $\mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \sigma, h)$ and $\|f\|_\sigma$ from the metric ω.

Q. E. D.

Clearly $\mathrm{L}^{n,q}(\mathbf{X}, \mathbf{E}, \sigma, h)$ is a Hilbert space with norm $\|f\|_\sigma$ which we write $||f||$ when there is no fear of confusion.

Definition 2. 5.

$$(15)\begin{cases} N^{n,q}(X, E, \sigma, h) := \{f \in L^{n,q}(X, E, \sigma, h) ;\ \bar{\partial} f = 0\}, \\ R^{n,q}(X, E, \sigma, h) := \{f \in L^{n,q}(X, E, \sigma, h) ;\ \textit{there exist} \\ \qquad\qquad g \in L^{n,q-1}(X, E, \sigma, h)\ \textit{satisfying}\ \bar{\partial} g = f\}, \\ H^{n,q}(X, E, \sigma, h) := N^{n,q}(X, E, \sigma, h) / R^{n,q}(X, E, \sigma, h). \end{cases}$$

Definition 2. 6. X *is called a complete Kähler manifold if there exists a complete Kähler metric on* X.

Proposition 2. 7. *Let* ω *be a complete Kähler metric on* X. *Then,*

$$||\bar{\partial} f||^2 + ||\theta f||^2 \geqq (\sqrt{-1} e(\Theta_h) \Lambda f, f),$$

for any $f \in L^{n,q}(X, E, \omega, h)$ *such that* $\bar{\partial} f \in L^{n,q+1}(X, E, \omega, h)$ *and* $\theta f \in L^{n,q-1}(X, E, \omega, h)$.

Proof is immediate from Proposition 1. 2 *and Proposition* 1. 3.

Theorem 2. 8. *Let* X *be a complete Kähler manifold, let* (E, h) *be a hermitian bundle over* X, *and let* σ *be a d-closed semipositive* (1, 1)*-form on* X. *If* $\Theta_h \geqq \sigma$, *then*

$$H^{n,q}(X, E, \sigma, h) = 0, \text{ for } q \geqq 1.$$

Proof. Let $f \in N^{n,q}(X, E, \sigma, h)$. We have to find $g \in L^{n,q-1}(X, E, \sigma, h)$ satisfying $\bar{\partial} g = f$. We first fix a complete Kähler metric ω on X and prove that for each $\varepsilon > 0$ there exists $g_\varepsilon \in L^{n,q-1}(X, E, \sigma + \varepsilon\omega, h)$ such that $\bar{\partial} g_\varepsilon = f$ and $||g_\varepsilon|| \leqq C_q ||f||$, where C_q is a constant depending only on q. In virtue of Hahn-Banach's theorem, the existence of such g_ε is assured by the following estimate:

$$(16)\qquad |(f, u)_{\varepsilon\omega+\sigma}|^2 \leqq C_q^2 ||f||^2 (||\bar{\partial} u||^2 + ||\theta u||^2),$$
for any $u \in L^{n,q}(X, E, \varepsilon\omega + \sigma, h)$ belonging to the domains of $\bar{\partial}$ and θ.

Let $\varphi \in C_0^{n,q}(X, E)$ and let δ be a positive number less than ε. By Cauchy-Schwarz' inequality we have

$$(17)\qquad |(\varphi, u)_{\varepsilon\omega+\sigma}|^2 \leqq (e(\varepsilon\omega+\sigma)\Lambda_{\delta\omega+\sigma}\varphi, \varphi)_{\varepsilon\omega+\sigma} (e(\delta\omega+\sigma)\Lambda_{\varepsilon\omega+\sigma} u, u)_{\varepsilon\omega+\sigma}.$$

Let $x\in\mathbf{X}$ be any point. We express φ, $\sigma+\varepsilon\omega$ and $\sigma+\delta\omega$ at x as follows:

$$(18)\qquad \begin{cases} \varphi=\sum_{i_1<\dots<i_q}\varphi_{i_1\dots i_q}\tau_1\wedge\dots\wedge\tau_n\wedge\bar\tau_{i_1}\wedge\dots\wedge\bar\tau_{i_q}, \\ \sigma+\varepsilon\omega=\sum_{i=1}^{n}\tau_i\bar\tau_i, \\ \sigma+\delta\omega=\sum_{i=1}^{n}\lambda_i\tau_i\bar\tau_i, \qquad 0<\lambda_i<1. \end{cases}$$

Then we have

$$(19)\qquad <e(\varepsilon\omega+\sigma)\Lambda_{\delta\omega+\sigma}\varphi,\ \varphi>_{\varepsilon\omega+\sigma}dv_{\varepsilon\omega+\sigma}=\sum_{\substack{i_1<\dots<i_q\\ 1\leqq\alpha\leqq q}}\frac{|\varphi_{i_1\dots i_q}|^2}{\lambda_{i_\alpha}}dv_{\varepsilon\omega+\sigma}$$

and

$$(20)\qquad <\varphi,\ \varphi>_{\delta\omega+\sigma}dv_{\delta\omega+\sigma}=\sum_{i_1<\dots<i_q}\frac{|\varphi_{i_1\dots i_q}|^2}{\prod_{\alpha=1}^{q}\lambda_{i_\alpha}}dv_{\varepsilon\omega+\sigma}.$$

Comparing (19) and (20) we have

$$(21)\qquad <e(\varepsilon\omega+\sigma)\Lambda_{\delta\omega+\sigma}\varphi,\ \varphi>_{\varepsilon\omega+\sigma}dv_{\varepsilon\omega+\sigma}\leqq q<\varphi,\ \varphi>_{\delta\omega+\sigma}dv_{\delta\omega+\sigma}.$$

Therefore,

$$(22)\qquad \int_{\mathbf{X}}<e(\varepsilon\omega+\sigma)\Lambda_{\delta\omega+\sigma}f,\ f>_{\varepsilon\omega+\sigma}dv_{\varepsilon\omega+\sigma}\leqq q||f||^2.$$

Hence,

$$(23)\qquad |(f,\ u)_{\varepsilon\omega+\sigma}|^2\leqq q||f||^2(e(\delta\omega+\sigma)\Lambda_{\varepsilon\omega+\sigma}u,\ u)_{\varepsilon\omega+\sigma}.$$

Letting $\delta\to0$, we have

$$(24)\qquad |(f,\ u)_{\varepsilon\omega+\sigma}|^2\leqq q||f||^2(e(\sigma)\Lambda_{\varepsilon\omega+\sigma}u,\ u)_{\varepsilon\omega+\sigma}.$$

By Proposition 2.2 and the assumption that $\Theta_h\geqq\sigma$, we have

$$(25)\qquad (\sqrt{-1}e(\sigma)\Lambda_{\varepsilon\omega+\sigma}u,\ u)\leqq(\sqrt{-1}e(\Theta_h)\Lambda_{\varepsilon\omega+\sigma}u,\ u).$$

Note that $\varepsilon\omega+\sigma$ is a complete Kähler metric on $\mathbf{X}$ so that by Proposition 2.7 we have

$$(26)\qquad (\sqrt{-1}e(\Theta_h)\Lambda_{\varepsilon\omega+\sigma}u,\ u)\leqq||\bar\partial u||^2+||\theta u||^2.$$

Combining (26) with (24) and (25) we obtain (16).

Thus, there exists $g_\varepsilon\in\mathrm{L}^{n,q-1}(\mathbf{X},\ \mathbf{E},\ \varepsilon\omega+\sigma,\ h)$ satisfying $\bar\partial g_\varepsilon=f$ and $||g_\varepsilon||\leqq q||f||$. Note that $||g_\varepsilon||_{\omega+\sigma}\leqq||g_\varepsilon||$ for $\varepsilon<1$ so that we can choose a subsequence of $\{g_\varepsilon\}_{\varepsilon>0}$ converging weakly in $L^{n,q-1}(\mathbf{X}, \mathbf{E}, \omega+\sigma, h)$. Let the weak limit be g. Then we have $\bar{\partial}g=f$. Moreover,

$$\lim_{\varepsilon\searrow 0}\|g\|_{\varepsilon\omega+\sigma}\leqq\lim_{\varepsilon\searrow 0}\|g_\varepsilon\|\leqq q\|f\|. \tag{27}$$

Therefore $g\in L^{n,q-1}(\mathbf{X}, \mathbf{E}, \sigma, h)$. Q. E. D.

Let us show several examples of (noncompact) complete Kähler manifolds.

Example 1. $\mathbf{C}^n$ is a complete Kähler manifold.

Example 2. Every Stein manifold is a complete Kähler manifold. More generally, a Kähler manifold provided with a C^∞ exhaustive plurisubharmonic function is a complete Kähler manifold.

Example 3. Given a complete Kähler manifold $\mathbf{X}$,

i) every closed submanifold is a complete Kähler manifold.

ii) Complements of discrete sets are complete Kählerian.

The author does not know whether complements of closed analytic subsets of complete Kähler manifolds are complete Kählerian or not.

Example 4. Let $\mathbf{D}$ be a bounded domain with a smooth pseudoconvex boundary in a Kähler manifold. Then, $\mathbf{D}$ is a complete Kähler manifold.

§ 3. A Generalization of Kodaira's Vanishing Theorem

Let $\mathbf{Y}$ be a paracompact analytic space over $\mathbf{C}$. By a hermitian metric on $\mathbf{Y}$, we mean a hermitian metric σ defined on the regular points of $\mathbf{Y}$ satisfying the following property: for any point $y\in\mathbf{Y}$, there exist a neighbourhood U, a holomorphic embedding ι: $U\to\mathbf{C}^N$ for some N, and a C^∞ positive (1, 1)-form $\tilde{\sigma}$ defined on a neighbourhood of $\iota(U)$ for which $\sigma=\iota^*\tilde{\sigma}$ on the regular points of U. We say σ is a Kähler metric if we can choose $\tilde{\sigma}$ to be d-closed. For any holomorphic map f: $\mathbf{X}\to\mathbf{Y}$ from a complex manifold $\mathbf{X}$, $f^*\sigma$ is extended uniquely to a C^∞ semipositive (1, 1)-form on $\mathbf{X}$. We shall not distinguish $f^*\sigma$ from its extension.

Theorem 3.1. *Let* $\mathbf{X}$ *be a compact Kähler manifold, let* $f: \mathbf{X} \to \mathbf{Y}$ *be a holomorphic map to an analytic space* $\mathbf{Y}$ *with a Kähler metric* σ, *and let* $(\mathbf{E}, h)$ *be a hermitian bundle over* $\mathbf{X}$. *Assume that* $\Theta_h \geqq f^*\sigma$, *then*

$$\mathrm{H}^q(\mathbf{Y}, f_* \mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})) = 0, \ \textit{for} \ q \geqq 1.$$

Before going into the proof we note the following

Lemma 3.2. *Let* $\pi: \mathbf{X} \to \mathbf{Y}$ *be a holomorphic map between complex manifolds* $\mathbf{X}$ *and* $\mathbf{Y}$ *provided with hermitian metrics* $\omega_\mathbf{X}$ *and* $\omega_\mathbf{Y}$, *respectively. Then, for any form* g *on* $\mathbf{Y}$,

$$|(\pi^* g)_x|_{\omega_\mathbf{X} + \pi^* \omega_\mathbf{Y}} \leqq |g_{\pi(x)}|_{\omega_\mathbf{Y}},$$

at any point $x \in \mathbf{X}$.

Proof is trivial.

Proof of Theorem 3.1. Let $\mathscr{V} = \{V_i\}_{i \in I}$ be a finite system of Stein open subsets covering of $\mathbf{Y}$ and let $\{c_{i_0 \cdots i_q}\}$ be a q-cocycle of $f^* \mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})$ associated to $\mathscr{V}$ $(q \geqq 1)$. We set

$$c^*_{i_0 \cdots i_q} = f^* c_{i_0 \cdots i_q}. \tag{28}$$

Then $\{c^*_{i_0 \cdots i_q}\}$ is a q-cocycle of $\mathcal{O}(\mathbf{K_X} \otimes \mathbf{E})$ associated to the covering $\{f^{-1}(V_i)\}_{i \in I}$. We regard $c^*_{i_0 \cdots i_q}$ as holomorphic n-forms on $f^{-1}(V_{i_0} \cap \cdots \cap V_{i_q})$ with values in $\mathbf{E}$. Let $\{p_i\}$ be a partition of unity associated to $\mathscr{V}$. We define $\mathbf{E}$-valued $(n, q\text{-}k)$-forms $b_{i_0 \cdots i_{k-1}}$ on $V_{i_0} \cap \cdots \cap V_{i_{k-1}}$ in such a way that

$$b_{i_0 \cdots i_{k-1}} = \sum_{i_k \in I} f^* p_{i_k} \cdot \Big(\bar\partial \Big(\sum_{i_{k+1} \in I} f^* p_{i_{k+1}} \cdot \Big(\cdots \bar\partial \sum_{i_q \in I} f^* p_{i_q} \cdot c^*_{i_0 \cdots i_q} \Big) \cdots \Big) \Big). \tag{29}$$

Then we have

$$\sum_{\alpha=0}^{k-1} (-1)^i \bar\partial b_{i_0 \cdots \check{i}_\alpha \cdots i_{k-1}} = 0, \tag{30}$$

and in particular we can define an $\mathbf{E}$-valued $\bar\partial$-closed (n, q)-form b on $\mathbf{X}$ by $b = \bar\partial b_i$. By Lemma 3.2 $|\bar\partial p_i|_\sigma$ are bounded above. Let ω be a Kähler metric on $\mathbf{X}$. Then, again by Lemma 3.2, for any $\varepsilon > 0$, $|\bar\partial (f^* p_i)|_{\varepsilon\omega + f^*\sigma}$ are bounded above by $|\bar\partial p_i|_\sigma$. Since $c^*_{i_0 \cdots i_q}$ are $(n, 0)$-

forms with values in $\mathbf{E}$, $|c^*_{i_0\cdots i_q}|^2_{\varepsilon\omega+f^*\sigma}dv_{\varepsilon\omega+f^*\sigma}$ are independent of ε. Therefore,

$$\begin{cases} b\in \mathrm{L}^{n,q}(\mathbf{X},\ \mathbf{E},\ f^*\sigma,\ h) \\ b_{i_0\cdots i_k}\in \mathrm{L}^{n,q-k+1}(V_{i_0}\cap\dots\cap V_{i_k},\ \mathbf{E},\ f^*\sigma,\ h). \end{cases} \tag{31}$$

Thus, in virtue of Theorem 2.8, there exists $a\in \mathrm{L}^{n,q-1}(\mathbf{X},\ \mathbf{E},\ f^*\sigma,\ h)$ satisfying $\bar\partial a=b$. Let $c_i^*=b_i-a$. Then we have

$$\begin{cases} c_i^*\in \mathrm{L}^{n,q-1}(f^{-1}(V_i),\ \mathbf{E},\ f^*\sigma,\ h), \\ \bar\partial c_i^*=0, \\ \bar\partial b_{ij}=c_i^*-c_j^*. \end{cases} \tag{32}$$

Since V_i are Stein open sets, $f^{-1}(V_i)$ are complete Kähler manifolds. Hence we can apply Theorem 2.8 to $f^{-1}(V_i)$ and find $a_i\in \mathrm{L}^{n,q-2}(f^{-1}(V_i),\ \mathbf{E},\ f^*\sigma,\ h)$ such that $c_i^*=\bar\partial a_i$. Let $c_{ij}^*=b_{ij}-a_i-a_j$. Then we have

$$\begin{cases} c_{ij}^*\in \mathrm{L}^{n,q-2}(f^{-1}(V_i\cap V_j),\ \mathbf{E},\ f^*\sigma,\ h), \\ \bar\partial c_{ij}^*=0, \\ \bar\partial b_{ijk}=c_{ij}^*+c_{jk}^*+c_{ki}^*. \end{cases} \tag{33}$$

We can continue this process until we obtain $\mathbf{E}$-valued holomorphic n-forms $c^*_{i_0\cdots i_{q-1}}$ on $f^{-1}(V_{i_0}\cap\dots\cap V_{i_{q-1}})$ satisfying

$$c^*_{i_0\cdots i_q}=\sum_{\alpha=0}^{q}(-1)^\alpha c^*_{i_0\cdots\check{i}_\alpha\cdots i_q}. \tag{34}$$

We put

$$c^*_{i_0\cdots i_{q-1}}=f^*c_{i_0\cdots i_{q-1}}, \tag{35}$$

where $c_{i_0\cdots i_{q-1}}$ are sections of $f_*\mathcal{O}(\mathbf{K_X}\otimes\mathbf{E})$ over $V_{i_0}\cap\dots\cap V_{i_{q-1}}$. (34) implies that

$$c_{i_0\cdots i_q}=\sum_{\alpha=0}^{q}(-1)^\alpha c_{i_0\cdots\check{i}_\alpha\cdots i_q}. \tag{36}$$

Q. E. D.

Corollary 3.3 (cf. Fujita [3]). *Let $\pi\colon \mathbf{X}\to\mathbf{Y}$ be a surjective holomorphic map with connected fibers from a compact Kähler manifold $\mathbf{X}$ to a nonsingular curve $\mathbf{Y}$. Then, every quotient invertible sheaf of $\pi_*\omega_{\mathbf{X/Y}}$ is of nonnegative degree. Here we put $\omega_{\mathbf{X/Y}}=\mathcal{O}(\mathbf{K_X}\otimes\pi^*\mathbf{K_Y^*})$.*

Proof. Let

$$0\longrightarrow\mathcal{S}\longrightarrow\pi_*\omega_{\mathbf{X/Y}}\longrightarrow\mathcal{L}\longrightarrow 0 \tag{37}$$

be an exact sequence of coherent analytic sheaves over $\mathbf{Y}$. Let $\mathscr{B}$ be an invertible sheaf of positive degree over $\mathbf{Y}$, then we have the following exact sequence:

$$\begin{aligned}(38)\qquad & H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\pi_*\omega_{X/Y}\otimes\mathscr{B})\\ &\longrightarrow H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\mathscr{L}\otimes\mathscr{B})\\ &\longrightarrow H^2(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\mathscr{S}\otimes\mathscr{B}).\end{aligned}$$

Since $\dim \mathbf{Y}=1$, we have $H^2(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\mathscr{S}\otimes\mathscr{B})=0$. On the other hand, by Theorem 3. 3,

$$(39)\quad H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\pi_*\omega_{X/Y}\otimes\mathscr{B})\,(=H^1(\mathbf{Y},\ \pi_*\mathcal{O}(\mathbf{K}_X\otimes\pi^*\mathscr{B})))=0.$$

Here we used the assumption that the fibers of π are connected. Hence $H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes\mathscr{L}\otimes\mathscr{B})$ also vanishes. Therefore $\mathscr{L}$ cannot be an invertible sheaf of negative degree. Otherwise $H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y)\otimes \mathscr{L}\otimes\mathscr{L}^*)=0$, which contradicts that $H^1(\mathbf{Y},\ \mathcal{O}(\mathbf{K}_Y))\cong H^0(\mathbf{Y},\ \mathcal{O}_Y)\cong\mathbf{C}$.

Q. E. D.

§ 4. A Vanishing Theorem on 1-Convex Manifolds

Let $\mathbf{X}$ be a 1-convex manifold, i. e. $\mathbf{X}$ is connected and there exists a C^∞ exhaustive function which is strictly plurisubharmonic outside a compact subset of $\mathbf{X}$. The following fact is first due to Grauert [4]: there is a compact analytic subset $\mathbf{A}\subset\mathbf{X}$ and a proper holomorphic map π from $\mathbf{X}$ onto a Stein space $\hat{\mathbf{X}}$ such that π_{X-A} is biholomorphic. If $\mathbf{A}$ is everywhere of positive dimension, $\mathbf{A}$ is called the maximal compact analytic set. By the fundamental work of Hironaka [6], [7], there is a complex manifold $\tilde{\mathbf{X}}$ obtained from $\hat{\mathbf{X}}$ by a succession of blowing up along nonsingular centers, such that the induced bimeromorphic map π': $\tilde{\mathbf{X}}\to\mathbf{X}$ is holomorphic. $\tilde{\mathbf{X}}$ can be chosen so that

(I) $\pi\circ\pi'$ is biholomorphic on $\tilde{\mathbf{X}}-\pi'^{-1}(\mathbf{A})$.

(II) $\pi'^{-1}(\mathbf{A})$ is a divisor with normal crossings whose irreducible components $\{\tilde{\mathbf{A}}_j\}_{j=1}^{\nu}$ are nonsingular.

(III) There exist ν tuple of positive integers $(p_1,\dots, p_\nu)$ so that the line bundle $\sum_{j=1}^{\nu} p_j[\tilde{\mathbf{A}}_j]^*$ is very ample.

Set $\tilde{\mathbf{A}}=\sum_{j=1}^{\nu} p_j\tilde{\mathbf{A}}_j$ and denote the support of $\tilde{\mathbf{A}}$ by $|\tilde{\mathbf{A}}|$. Since $[\tilde{\mathbf{A}}]^*$ is very ample there is a metric $\tilde{a}$ along the fibers of $[\tilde{\mathbf{A}}]^*$ such that the curvature form $\Theta_{\tilde{a}}$ gives a Kähler metric on $\mathbf{X}$. On $\mathbf{X}-|\tilde{\mathbf{A}}|$, $\tilde{a}$ is given by a positive C^∞ function ϕ satisfying

(40) $$\partial\bar{\partial}(-\log\phi)=\Theta_{\tilde{a}}$$

and that

(41) $\log\phi+\log|s|^2$ is C^∞ on $\mathbf{X}$, where s is a canonical section of $[\tilde{\mathbf{A}}]$.

Via π' we shall identify ϕ with a function on $\mathbf{X}-\mathbf{A}$. Let φ be a C^∞ plurisubharmonic exhaustive function on $\mathbf{X}$ which is strictly plurisubharmonic outside $\mathbf{A}$.

Proposition 4.1. *$\mathbf{X}-\mathbf{A}$ is a complete Kähler manifold.*

Proof. Let $V:=\{x\in\mathbf{X}-\mathbf{A};\ \log\phi(x)>0\}$. Then, $V\cup\mathbf{A}$ is a neighbourhood of $\mathbf{A}$ in $\mathbf{X}$. Let ρ be a C^∞ function on $\mathbf{X}$ such that $0\leqq\rho\leqq1$ on $\mathbf{X}$, $\rho=0$ on $\mathbf{X}-V$ and $\rho=1$ on a neighbourhood of $\mathbf{A}$. Then, for sufficiently large K, $\partial\bar{\partial}(K\varphi^2-\log(1+\rho\log\phi))$ is a complete Kähler metric on $\mathbf{X}-\mathbf{A}$. Q. E. D.

Definition 4.2. *Let $\mathbf{Y}$ be an analytic space which is isomorphic to an analytic subset of a domain Ω in $\mathbf{C}^n$* and *let h be a C^∞ matrix-valued function on $\mathbf{Y}$ with values in $r\times r$ positive definite hermitian matrices. We say that h has semipositive curvature if there is a C^∞ extension $\tilde{h}$ of h to a neighbourhood of $\mathbf{Y}$ in Ω such that $\Theta_{\tilde{h}}:=\bar{\partial}(\tilde{h}^{-1}\partial\tilde{h})$ is semipositive (cf. Definition* 2.1).

Proposition 4.3. *Let π: $\mathbf{Y}'\to\mathbf{Y}$ be a holomorphic map between analytic spaces and let h be a matrix-valued function on $\mathbf{Y}$ with semipositive curvature. Then, π^*h has semipositive curvature, too.*

Proof is trivial.

Definition 4.4. Let $\mathbf{Y}$ be an analytic space and let $(\mathbf{E}, h)$ be a hermitian bundle over $\mathbf{Y}$. $(\mathbf{E}, h)$ is said to be Nakano-semipositive if for any local representation h_i of h as a C^∞ matrix-valued function, h_i has semipositive curvature.

Theorem 4.5. Let $\mathbf{X}$ be a 1-convex manifold with maximal compact analytic subset $\mathbf{A}$ and let $(\mathbf{E}, h)$ be a hermitian bundle over $\mathbf{X}$. Assume that $(\mathbf{E}|_{\mathbf{A}}, h|_{\mathbf{A}})$ is Nakano-semipositive. Then,

$$\mathrm{H}^q(\mathbf{X}, \mathcal{O}(\mathbf{K_X}\otimes\mathbf{E}))=0, \qquad \text{for } q\geqq 1.$$

Proof. First we shall prove that the hermitian bundle $(\mathbf{E}|_{\mathbf{X-A}}, h(1+\rho\log\psi)e^{-L\varphi})$ is Nakano-semipositive for sufficiently large L. Note that by Proposition 4.3 $(\pi'^*\mathbf{E}|_{|\tilde{\mathbf{A}}|}, \pi'^*h|_{|\tilde{\mathbf{A}}|})$ is Nakano-semipositive. Since $|\tilde{\mathbf{A}}|$ is a divisor with normal crossings, it is clear that

$$\langle\Theta_{\pi'^*h}(u), u\rangle_{\pi'^*h}(\xi, \bar{\xi})\geqq 0, \tag{42}$$

for any $\xi\in(\sum_{j=1}^{\nu}\mathrm{T}\tilde{\mathbf{A}}_j)_x$ and $u\in\mathbf{E}_x$ at any point $x\in|\tilde{\mathbf{A}}|$. Here, $\mathrm{T}\tilde{\mathbf{A}}_j$ are regarded as subspaces of $\mathrm{T}\tilde{\mathbf{X}}$ and

$$(\sum_{j=1}^{\nu}\mathrm{T}\tilde{\mathbf{A}}_j)_x:=\{v\in\mathrm{T}_x\tilde{\mathbf{X}};\ \text{there exist } v_j\in\mathrm{T}_x\tilde{\mathbf{A}}_j,\ 1\leqq j\leqq\nu \text{ such that } v=\sum v_j\}. \tag{43}$$

We put $\sum_{j=1}^{\nu}\mathrm{T}\tilde{\mathbf{A}}_j:=\bigcup_{x\in|\tilde{\mathbf{A}}|}(\sum_{j=1}^{\nu}\mathrm{T}\tilde{\mathbf{A}}_j)_x$.

Let $x\in|\tilde{\mathbf{A}}|$ be any point, let $(z_1,\dots,z_n)$ be a local coordinate on a neighbourhood U of x such that $z_1\cdot\cdots\cdot z_k=0$ is a local equation of $|\tilde{\mathbf{A}}|$, and let η denote an element of $\mathrm{T}\tilde{\mathbf{X}}$. Then, $\sum\mathrm{T}\tilde{\mathbf{A}}_j$ is locally defined by the following two equations:

$$\begin{cases}\eta(z_1\cdot\cdots\cdot z_k)=0\\ z_1\cdot\cdots\cdot z_k=0.\end{cases} \tag{44}$$

Hence we infer from (42) that

$$\begin{aligned}&\langle\Theta_{\pi'^*h}(u), u\rangle_{\pi'^*h}(\eta, \bar{\eta})\\ &\geqq -K|\eta|^2|u|^2\Big(\frac{|\eta(z_1\cdot\cdots\cdot z_k|}{|\eta|}+|z_1\cdot\cdots\cdot z_k|\Big)\end{aligned} \tag{45}$$

on U, where K depends on $\Theta_{\tilde{a}}$, h and the choice of $(z_1,\dots,z_n)$. We compare the right hand terms of (45) with $\Theta_{(1+\rho\log\psi)}$ (cf. Proposition 4.1). Since $\log\psi=\infty$ on $|\tilde{\mathbf{A}}|$, there is a neighbourhood W of $|\tilde{\mathbf{A}}|$ such that

$$\begin{aligned}&-\partial\bar{\partial}\log(1+\log\psi)\\ &\geqq\frac{-\partial\bar{\partial}\log\psi}{2\log\psi}+\frac{\partial\psi\bar{\partial}\psi}{2\psi^2(\log\psi)^2}\end{aligned} \tag{46}$$

on $W-|\tilde{\mathbf{A}}|$. We can find a C^∞ function λ on U and negative integers n_i such that $\psi=|z_1^{n_1}\cdot\cdots\cdot z_k^{n_k}|^2\lambda$. Shrinking W if necessary we obtain

$$(47)\qquad \frac{\partial\psi\bar\partial\psi}{\psi^2(\log\psi)^2}+\frac{-\partial\bar\partial\log\psi}{\log\psi}$$
$$=\frac{1}{2(\log\psi)^2}\left(\sum_{i=1}^k n_i\frac{dz_i}{z_i}+\frac{\partial\lambda}{\lambda}\right)\left(\sum_{i=1}^k n_i\frac{d\bar z_i}{\bar z_i}+\frac{\bar\partial\lambda}{\lambda}\right)+\frac{-\partial\bar\partial\log\psi}{\log\psi}$$
$$\geqq\frac{1}{2(\log\psi)^2}\left(\sum_{i=1}^k n_i\frac{dz_i}{z_i}\right)\left(\sum_{i=1}^k n_i\frac{d\bar z_i}{\bar z_i}\right)$$
$$+\frac{-\partial\bar\partial\log\psi}{2\log\psi},\ \text{on}\ W\cap U-|\tilde{\mathbf{A}}|.$$

Hence

$$(48)\qquad <\Theta_{\pi'^*h(1+\rho\log\psi)}(u),\ u>_{\pi'^*h}(\eta,\ \bar\eta)$$
$$\geqq -K|\eta|^2|u|^2\left(\frac{|\eta(z_1\cdot\cdots\cdot z_k)|}{|\eta|}+|z_1\cdot\cdots\cdot z_k|\right)$$
$$+\frac{1}{4(\log\psi)^2}\left(\sum n_i\frac{\eta(z_i)}{z_i}\right)\left(\sum n_i\frac{\overline{\eta(z_i)}}{\bar z_i}\right)|u|^2$$
$$+\frac{1}{4\log\psi}|\eta|^2|u|^2,\ \text{on}\ W\cap U-|\tilde{\mathbf{A}}|.$$

From (48) it is easy to see that

$$(49)\qquad <\Theta_{\pi'^*h(1+\rho\log\psi)}(u),\ u>_{\pi'^*h}(\eta,\ \bar\eta)\geqq 0,$$

on $W\cap U-|\tilde{\mathbf{A}}|$, where we possibly shrink U and W. Thus, by compactness argument $(\mathbf{E}|_{\mathbf{X}-\mathbf{A}},\ h(1+\rho\log\psi)e^{-L\varphi})$ is Nakano-semipositive for sufficiently large L. We set $\Phi=(1+\rho\log\psi)e^{-L\varphi}$. Then, by Theorem 2.8, we have

$$(50)\qquad \mathrm{H}^{n,q}(\mathbf{X}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)=0,\qquad \textit{for}\ q\geqq 1.$$

We are going to deduce from (50) that $\mathrm{H}^q(\mathbf{X},\ \mathcal{O}(\mathbf{K_X}\otimes\mathbf{E}))=0$ for $q\geqq 1$. Let f be any C^∞ $\mathbf{E}$-valued $\bar\partial$-closed $(n,\ q)$-form on $\mathbf{X}$. Since any power of $\log\psi$ is locally square integrable on $\mathbf{X}$, we may assume that $f\in\mathrm{L}^{n,q}(\mathbf{X}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$, if necessary replacing φ by a more rapidly increasing function. Hence we can find $g\in\mathrm{L}^{n,q-1}(\mathbf{X}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$ such that $\bar\partial g=f$. If $q=1$ we are done, since g is then locally square integrable on $\mathbf{X}$ and in view of the equality $\bar\partial g=f$ on $\mathbf{X}-\mathbf{A}$, g is extended to a C^∞ n-form with values in $\mathbf{E}$. Let $q\geqq 2$. Then we choose a locally finite covering $\{U_i\}_{i\in I}$ of $\mathbf{X}$ by Stein open sets and define $\{f_{i_1\cdots i_k}\}$, $\{g_{i_1\cdots i_k}\}$ and $\{u_{i_1\cdots i_k}\}$ inductively as follows. Let u_i be a C^∞ ($\mathbf{E}$-valued) $(n,\ q\text{-}1)$-form on U_i such that $\bar\partial u_i=f$. We set $f_i=g-u_i$. Since $\bar\partial f_i=0$ and $f_i\in\mathrm{L}^{n,q-1}(U_i-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$, (where we possibly shrink U_i and replace φ again), we can find $g_i\in\mathrm{L}^{n,q-2}(U_i-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$ such that $\bar\partial g_i=f_i$. Assume that $\{f_{i_1\cdots i_k}\}$,

$\{g_{i_1\ldots i_k}\}$ and $\{u_{i_1\ldots i_k}\}$ are already determined in such a way that

$$(51)\qquad \begin{cases} \sum_{\alpha=1}^{k}(-1)^\alpha g_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}+\sum_{\alpha=1}^{k}(-1)^\alpha u_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}=0,\\ \bar\partial f_{i_1\ldots i_k}=0,\\ u_{i_1\ldots i_k} \text{ are } C^\infty \text{ on } U_{i_1}\cap\ldots\cap U_{i_k}\\ f_{i_1\ldots i_k}\in \mathrm{L}^{n,q-k}(U_{i_1}\cap\ldots\cap U_{i_k}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2),\\ g_{i_1\ldots i_k}\in \mathrm{L}^{n,q-k-1}(U_{i_1}\cap\ldots\cap U_{i_k}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2). \end{cases}$$

If $k\leqq q-2$, we set $\{f_{i_1\ldots i_{k+1}}\}$, $\{g_{i_1\ldots i_{k+1}}\}$ and $\{u_{i_1\ldots i_{k+1}}\}$ as follows. First we take $u_{i_1\ldots i_{k+1}}$ to be C^∞ and that

$$(52)\qquad \bar\partial u_{i_1\ldots i_{k+1}}=\sum_{\alpha=1}^{k+1}(-1)^{\alpha+1}u_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}.$$

Then we set

$$(53)\qquad f_{i_1\ldots i_{k+1}}=\sum_{\alpha=1}^{k+1}(-1)^{\alpha+1}g_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}+u_{i_1\ldots i_{k+1}}.$$

We have $\bar\partial f_{i_1\ldots i_{k+1}}=0$ and may assume that $f_{i_1\ldots i_{k+1}}\in \mathrm{L}^{n,q-k-1}(U_{i_1}\cap\ldots\cap U_{i_{k+1}}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$. Hence we can find $g_{i_1\ldots i_{k+1}}\in \mathrm{L}^{n,q-k-2}(U_{i_1}\cap\ldots\cap U_{i_{k+1}}-\mathbf{A},\ \mathbf{E},\ \Theta_\Phi,\ h\Phi^2)$ such that $\bar\partial g_{i_1\ldots i_{k+1}}=f_{i_1\ldots i_{k+1}}$. By the inductive assumption we have

$$(54)\qquad \bar\partial\Big(\sum_{\alpha=1}^{k+1}(-1)^\alpha g_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}\Big)+\sum_{\alpha=1}^{k+1}(-1)^\alpha u_{i_1\ldots \check{i}_\alpha\ldots i_{k+1}}=0.$$

Therefore, for any k with $1\leqq k\leqq q-1$, we have inductively determined $\{f_{i_1\ldots i_k}\}$, $\{g_{i_1\ldots i_k}\}$ and $\{u_{i_1\ldots i_k}\}$ satisfying (51). Note that in particular $g_{i_1\ldots i_{q-1}}$ are square integrable forms on $U_{i_1}\cap\ldots\cap U_{i_{q-1}}$ such that $\bar\partial(\sum(-1)^\alpha g_{i_1\ldots \check{i}_\alpha\ldots i_q})$ are C^∞ on $U_{i_1}\cap\ldots\cap U_{i_q}$. Hence there exist C^∞ forms $v_{i_1\ldots i_{q-1}}$ on $U_{i_1}\ldots U_{i_{q-1}}$ such that

$$(55)\qquad \sum_\alpha(-1)^\alpha g_{i_1\ldots \check{i}_\alpha\ldots i_q}=\sum_\alpha(-1)^\alpha v_{i_1\ldots \check{i}_\alpha\ldots i_q}.$$

Taking $\bar\partial$ of the both sides in (55) we have

$$(56)\qquad \sum_\alpha(-1)^\alpha(u_{i_1\ldots \check{i}_\alpha\ldots i_q}+\bar\partial v_{i_1\ldots \check{i}_\alpha\ldots i_q})=0.$$

Therefore, we can find $v_{i_1\ldots i_{q-2}}$ such that

$$(57)\qquad u_{i_1\ldots i_{q-1}}+\bar\partial v_{i_1\ldots i_{q-1}}=\sum_\alpha(-1)^\alpha v_{i_1\ldots \check{i}_\alpha\ldots i_{q-1}},$$

whence we obtain

$$(58)\qquad \bar\partial u_{i_1\ldots i_{q-1}}=\sum_\alpha(-1)^\alpha\bar\partial v_{i_1\ldots \check{i}_\alpha\ldots i_{q-1}}.$$

Continuing this process we arrive at the equality

(59) $$u_i - u_j = \bar{\partial} u_{ij} = \bar{\partial} v_i - \bar{\partial} v_j.$$

Thus we obtain a C^∞ form $g = u_i - \bar{\partial} v_i$ on $\mathbf{X}$ such that $\bar{\partial} g = f$.

Q. E. D.

Corollary 4.6 (Laufer [12], Kato [8]). *Let* $\mathbf{X}$ *be a* 1*-convex manifold of dimension* 2 *with maximal compact analytic set* $\mathbf{A}$, *and let* $\mathbf{L} \to \mathbf{X}$ *be a line bundle. Assume that* $\mathbf{K}_{\mathbf{X}}^* \otimes \mathbf{L}|_{\mathbf{A}_i}$ *is of nonnegative degree for every irreducible component* $\mathbf{A}_i$ *of* $\mathbf{A}$. *Then* $\mathrm{H}^1(\mathbf{X}, \mathcal{O}(\mathbf{L})) = 0$.

§ 5. A Sufficient Condition for Rationality of Isolated Singularities

Let $(\mathbf{X}, x)$ be a germ of an analytic space $\mathbf{X}$ for which x is an isolated singular point. $(\mathbf{X}, x)$ is said to be rational if for any resolution of singularity $\pi\colon \tilde{\mathbf{X}} \to \mathbf{X}$, $R^q \pi_* \mathcal{O}_{\tilde{\mathbf{X}}}$ vanishes for $q \geqq 1$. Here $R^q \pi_* \mathcal{O}_{\tilde{\mathbf{X}}}$ denotes the higher direct image sheaves of $\mathcal{O}_{\tilde{\mathbf{X}}}$. Note that the property that $R^q \pi_* \mathcal{O}_{\tilde{\mathbf{X}}} = 0$ for $q \geqq 1$ is independent of the choice of the resolution. (cf. Hironaka [6]). We can state a condition for the rationality of $(\mathbf{X}, x)$ in terms of the maximal compact analytic set of $\tilde{\mathbf{X}}$.

Theorem 5.1. *Let the notation be as above and let* $\mathbf{A}$ *be the maximal compact analytic subset of* $\tilde{\mathbf{X}}$. *Assume that* $\mathbf{K}_{\tilde{\mathbf{X}}|\mathbf{A}}$ *has a metric* h *along the fibers for which* $(\mathbf{K}_{\tilde{\mathbf{X}}|\mathbf{A}}, h)$ *is Nakano-semipositive. Then* $(\mathbf{X}, x)$ *is rational.*

Proof is immediate from Theorem 4.5.

As an application we obtain the following

Proposition 5.2. Let $\mathbf{X}$ be an analytic space of dimension 3 with an isolated singularity at x. Let $\pi\colon \tilde{\mathbf{X}} \to \mathbf{X}$ be a resolution of singularity. Suppose that $\mathbf{A} = \pi^{-1}(x)$ is isomorphic to $\mathbf{P}^1$ and that the normal bundle of $\mathbf{A}$ splits into line bundles whose chern classes are either $(-1, -1)$, $(-2, 0)$, or $(-3, 1)$. Then, $(\mathbf{X}, x)$ is a rational singularity.

The following proposition was suggested by A. Fujiki.

Proposition 5.3. Let $\mathbf{X}$ be an analytic space of dimension 3 with a rational isolated singularity at x. Let π: $\tilde{\mathbf{X}}\to\mathbf{X}$ be a resolution of the singularity. Suppose that $\mathbf{A}=\pi^{-1}(x)$ is isomorphic to $\mathbf{P}^1$ and that the degree of $\mathbf{K}_{\tilde{\mathbf{X}}|\mathbf{A}}$ is zero. Then there exist a neighbourhood U of x and a nowhere-zero holomorphic 3-form defined on $U-\{x\}$.

Proof is standard.

Combining Proposition 5.2 with Proposition 5.3 we obtain *the converse* of the following

Theorem 5.4 (Theorem 4.1 in Laufer [13]). *Let* $\mathbf{X}$ *be an analytic space of dimension* $n\geqq 3$ *with an isolated singularity at* x. *Suppose that there exists a nowhere zero holomorphic n-form* ω *on* $\mathbf{X}-x$. *Let* π: $\tilde{\mathbf{X}}\to\mathbf{X}$ *be a resolution. Suppose that* $\mathbf{A}=\pi^{-1}(x)$ *is* 1-*dimensional and irreducible. Then* $\mathbf{A}$ *is isomorphic to* $\mathbf{P}^1$ *and* $n=3$. *Also, the normal bundle of* $\mathbf{A}$ *splits into line bundles whose chern classes are* $(-1, -1)$, $(-2, 0)$, *or* $(-3, 1)$.

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Added in proof. Combining Proposition 5. 3 with a result of M. Reid (Minimal models of canonical 3-folds, Proc. Sympos. Algebraic and Analytic Varieties (Tokyo, June 1981), Sympos. in Math, vol. 1, Kinokuniya, Tokyo and North-Holland, Amsterdam), Proposition 5. 2 is strengthened so that we can conclude that $(\mathbf{X}, x)$ is a hypersurface singularity with defining equation $z_0^2=f(z_1, z_2, z_3)$. The author is grateful to Dr. M. Tomari for informing Reid's result to him.