Publ. RIMS, Kyoto Univ.
20 (1984), 57-64

Classical Lie Group Actions on π-Manifolds

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

By

Hiromichi MATSUNAGA*

§ 1. Introduction

The purpose of this paper is to prove some non existence theorems of large group actions on certain π-manifolds. We write G for the classical group $SU(m+1)$ or $Sp(m+1)$, and accordingly denote by d the integer 2 or 4. Let M^{2dm-1} be a $(2dm-1)$-dimensional compact connected π-manifold, $m\geqq 8$. Suppose that the first Pontrjagin class vanishes and its $(dm-1)$-dimensional integral homology group has a nontrivial cyclic subgroup of even or infinite order. Then we shall prove that the manifold M^{2dm-1} can not admit a nontrivial G-action (Theorem 3 in Section 3).

Next, let M^{2n-1} be a compact simply connected $(2n-1)$-dimensional π-manifold. Suppose that $n\geqq 10$ and its $(n-1)$-dimensional homology group is nonzero. Then we shall prove that the manifold M^{2n-1} can not admit a nontrivial $SO(n+1)$-action with exceptions of the real Stiefel manifold $V_{n+1,2}$ and a product manifold $S^n\times X^{n-1}$, where X^{n-1} is an $(n-1)$-dimensional simply connected π-manifold without boundary (Theorem 4 in Section 3).

Further we shall apply these results to study group actions on sphere bundles over spheres (Corollaries to Theorems).

§ 2. Preliminaries

We write G for $SU(m+1)$ or $Sp(m+1)$, and denote by d the integer 2 or 4 in case of $G=SU(m+1)$ or $Sp(m+1)$ respectively. Let M be a smooth G-manifold. Write G_x for the isotropy group of $x\in M$ and $(G_x)_0$ for the identity component of G_x. First we have

Communicated by N. Shimada, January 10, 1983. Revised March 2, 1983.
* Department of Mathematics, Shimane University, Matsue 690, Japan.

Proposition 1 (cf. Remark 2.2 in [3]). *Let M^{2dm-1} be a $(2dm-1)$-dimensional smooth π-manifold, where $m \geqq 8$. Suppose that the first Pontrjagin class of M^{2dm-1}, $P_1(M^{2dm-1})$ vanishes. If G acts non trivially on M^{2dm-1}, then any quotient $G_x/(G_x)_0$, $x \in M^{2dm-1}$ contains no element of order 2.*

Proof. By Theorems 2.2 and 2.3 in [3], any $(G_x)_0$ is conjugate to a standard subgroup $SF(k)$ for some $k \geqq \frac{2}{3}(m+1)$, where $SF(k)$ denotes $SU(k)$ or $Sp(k)$ respectively. On the other hand we have $\dim SF(m+1)/SF(k) \leqq 2dm-1$, then $k=m$ or $m+1$. Suppose that $G_x/(G_x)_0$ contains an element g_0 of order 2, then we have a group extension $1 \longrightarrow (G_x)_0 \longrightarrow H \longrightarrow Z_2(g_0) \longrightarrow 1$ with $H \subset G_x$. We write $\tau(M)$ and ν for the tangent vector bundle of a manifold M and the normal bundle of the embedding $G/G_x \subset M$. We have a covering map $p: G/H = P^{d(m+1)-1} \longrightarrow G/(G_x)$, where $P^{d(m+1)-1}$ denotes the $(d(m+1)-1)$-dimensional real projective space. Consider the commutative diagram

$$\begin{array}{ccc} \tau(P^{d(m+1)-1}) \oplus p^!\nu & \xrightarrow{p^*} & \tau(G/G_x) \oplus \nu \\ \downarrow & & \downarrow \\ P^{d(m+1)-1} & \xrightarrow{p} & G/G_x. \end{array}$$

By the proof of Theorem 10 in [1], we can conclude that any principal isotropy subgroup is conjugate to $SF(m)$, because any principal orbit must be a π-manifold, where we notice that since $(G_x)_0$ is conjugate to $SF(m)$, the dimensional restriction in the theorem above is not necessary. Since M^{2dm-1} is a π-manifold, we have $\tau(G/G_x) \oplus \nu \oplus \theta_R = 2dm\theta_R$, where θ_R denotes the trivial real line bundle. Then we have $\tau(P^{d(m+1)-1}) \oplus p^!\nu \oplus \theta_R = 2dm\theta_R$. We have $G/(G_x)_0 = S^{d(m+1)-1}$. The principal isotropy representation $(G_x)_0 \longrightarrow D^{d(m-1)}$ is trivial, then by the differentiable slice theorem, we have $\nu \approx G \times_{G_x} D^{d(m-1)} \approx (G/(G_x)_0) \times_K D^{d(m-1)}$, where K denotes $G_x/(G_x)_0$. Since $H/(G_x)_0 = Z_2$, we have a commutative diagram

$$\begin{array}{ccc} p^!\nu = S^{d(m+1)-1} \times_{Z_2} D^{d(m-1)} & \longrightarrow & \nu = S^{d(m+1)-1} \times_K D^{d(m-1)} \\ \downarrow & & \downarrow \\ P^{d(m+1)-1} & \longrightarrow & G/G_x. \end{array}$$

Let the representation $Z_2 \longrightarrow O(d(m-1))$ in $p^!\nu$ be given by $(-I_a) \times$

(I_b), then $p^!\nu = a\xi \oplus b\theta_R$ and $\tau(P^{d(m+1)-1}) \oplus p^!\nu \oplus \theta_R = (d(m+1)+a)\xi \oplus b\theta_R = 2dm\theta_R$, where ξ denotes the canonical line bundle over $P^{d(m+1)-1}$. Therefore $(d(m+1)+a)(\xi-1)=0$ in $\widetilde{KO}(P^{d(m+1)-1}) = Z_{2^\ell}$, where $\ell \geqq d(m+1)/2-1$. On the other hand $d(m+1)+a \leqq 2dm < 2^\ell$ for $d=2, 4$ and $m \geqq 8$, which is a contradiction. Thus we have proved the proposition.

For an abelian group M, we say that M holds the property (P) if M does not contain Z_{2k} $(k=0, 1, \cdots)$ as a subgroup, where Z_0 denotes the infinite cyclic group Z. we have

Lemma. *Let $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ be an exact sequence of finitely generated abelian groups. Suppose that M_1 and M_3 hold the property (P), then M_2 holds the property (P).*

Proof. Evidently, M_2 does not contain the group Z as a subgroup. Suppose that $M_2 \supset Z_{2k}$ for some k. We restrict β on Z_{2k}, say $\beta|Z_{2k}$ and write H for the kernel of $\beta|Z_{2k}$. Then Z_{2k}/H is isomorphic to a subgroup of M_3. Since M_3 holds the property (P), H is isomorphic to $Z_{2\ell}$ for some ℓ. By the exactness, M_1 admits a finite cyclic subgroup K such that $\alpha(K)=H$. Then K is isomorphic to Z_{2n} for some n which contradicts to the assumption.

Next we have

Proposition 2. *Let M^{2dm-1} be a $(2dm-1)$-dimensional compact G-manifold such that $H_{dm-1}(M^{2dm-1}, Z)$ contains a cyclic subgroup Z_{2k}, where k is an integer. Suppose that $(G_x)_0$ is conjugate to $SF(m)$ or $SF(m+1)$ and $G_x/(G_x)_0$ is a cyclic group of odd order for each $x \in M^{2dm-1}$. Then G_x is connected for each $x \in M^{2dm-1}$.*

Proof. If the fixed point set F is not empty, we choose a closed invariant tublar neighborhood $U(F)$ of F. We put $L(q) = \{x \in M^{2dm-1};\ G_x/(G_x)_0 = Z_q,\ q>1\}$. Let Z_{q_i}, $i=1, 2, \cdots, s$ be all of nontrivial cyclic groups $G_x/(G_x)_0$, and assume that $q_1 > q_2 > \cdots > q_s$. We write M_0 for $M^{2dm-1} - \mathrm{Int}\ U(F)$, then $L_1(q_1) = M_0 \cap L(q_1)$ is an invariant closed submanifold of M_0, and we can choose a closed invariant tublar neighborhood $U(L_1(q_1))$ of $L_1(q_1)$. Inductively we have an invariant closed submanifold $L_{i+1}(q_{i+1})=M_i\cap L(q_{i+1})$ of M_i and a closed invariant tublar neighborhood $U(L_{i+1}(q_{i+1}))$ with $M_{i+1}=M_i-\mathrm{Int}(U(L_{i+1}(q_{i+1})))$, $i=0, 1, \cdots, s-1$. We shall begin by proving that $H_{dm-1}(M_s)$ holds the property (P) and finally prove that $H_{dm-1}(M^{2dm-1})$ holds the property (P). We have a fibre space $L^{d(m+1)-1}(q_i)\longrightarrow L_i(q_i)\longrightarrow \pi(L_i(q_i))$, where $L^{d(m+1)-1}(q_i)$ is the lens space $S^{d(m+1)-1}/Z_{q_i}$ and π denotes the orbit map $M^{2dm-1}\longrightarrow M^{2dm-1}/G$. Since $\dim L_i(q_i)\leqq 2dm-1-1$ ([6]), we have $\dim \pi(L_i(q_i))\leqq 2dm-2-(d(m+1)-1)=dm-d-1$. We have the homology spectral sequence

$$E^2_{a,b}=H_a(\pi(L_i(q_i)),\ H_b(L^{d(m+1)-1}(q_i))\Longrightarrow E^{\infty}_{a,b},\quad \text{and}$$
$$H_c(L_i(q_i))=D_{c,0}\supset D_{c-1,1}\supset\cdots\supset D_{0,c},$$
$$E^{\infty}_{a,b}=D_{a,b}/D_{a-1,b+1}.$$

It is known that

$$H_b(L^{d(m+1)-1}(q_i))=\begin{cases} Z_{q_i} & \text{for } b=1,\ 3,\ \cdots,\ d(m+1)-3,\\ Z & \text{for } b=0,\ d(m+1)-1,\\ 0 & \text{otherwise.}\end{cases}$$

Since $E^2_{a,b}=H_a(C_*(\pi(L_i(q_i))\otimes H_b(L^{d(m+1)-1}(q_i)))$, we can see that $H_t(L_i(q_i))$ holds the property (P) for $dm-d\leqq t\leqq dm+d-2$. Consider the exact sequence

$$(*)\longrightarrow H_{dm-1}(U(L_i(q_i)),\ \partial U(L_i(q_i)))\xrightarrow{\partial} H_{dm-2}(\partial U(L_i(q_i)))\longrightarrow H_{dm-2}(U(L_i(q_i)))\longrightarrow.$$

By Poincaré Lefschetz duality and universal coefficient theorem, we have

$$H_{dm-1}(U(L_i(q_i)),\partial U(L_i(q_i)))\approx H^{dm}(U(L_i(q_i)))\approx H^{dm}(L_i(q_i)),$$

and a short exact sequence

$$0\longrightarrow \mathrm{Ext}(H_{dm-1}(L_i(q_i)),\ Z)\longrightarrow H^{dm}(L_i(q_i))\longrightarrow \mathrm{Hom}(H_{dm}(L_i(q_i)),\ Z)\longrightarrow 0.$$

Since $H_{dm-1}(L_i(q_i))$ and $H_{dm}(L_i(q_i))$ hold the property (P), Hom $(H_{dm}(L_i(q_i)),\ Z)=0$, and $\mathrm{Ext}(H_{dm-1}(L_i(q_i)),\ Z)$ holds the property (P), where we use the relation $\mathrm{Ext}(Z_n, Z)\approx Z_n$, $n=1, 2, \cdots$. Therefore by the lemma and the exact sequence $(*)$ above, $H_{dm-2}(\partial U(L_i(q_i))$ holds the property (P). Next consider the fibre space $S^{d(m+1)-1}\longrightarrow M_s\longrightarrow \pi(M_s)$, then $\dim \pi(M_s)=2dm-1-(d(m+1)-1)=d(m-1)$.

Therefore the associated homology spectral sequence shows that $H_{dm-1}(\pi(M_s)) = H_{dm-1}(M_s) = 0$. Consider the Mayer-Vietoris exact sequence for the couple $(M_s, U(L_s(q_s)))$,

$$\longrightarrow H_{dm-1}(M_s) + H_{dm-1}(U(L_s(q_s))) \longrightarrow H_{dm-1}(M_{s-1}) \longrightarrow H_{dm-2}(\partial U(L_s(q_s))) \longrightarrow,$$

then by the lemma we see that the property (P) holds for $H_{dm-1}(M_{s-1})$. Inductively, we see that the property (P) holds for $H_{dm-1}(M_0)$. We have a fibre space $S^N \longrightarrow \partial U(F) \longrightarrow F$ for some $N \geqq d(m+1)-1$, then $\dim F \leqq d(m-1)-1$ and $H_{dm-2}(F) = 0$, therefore $H_{dm-2}(\partial U(F)) = 0$. Thus the property (P) holds for $H_{dm-1}(M^{2dm-1})$, which contradicts to the assumption $Z_{2k} \subset H_{dm-1}(M^{2dm-1})$.

§ 3. Non Existence Theorems

In this section we use the same notations G, d, $SF(n)$ and Z_{2k} as ones in Section 2. First we have

Theorem 3. *Let M^{2dm-1} be a $(2dm-1)$-dimensional compact connected π-manifold, $m \geqq 8$. Suppose that $P_1(M^{2dm-1}) = 0$ and $H_{dm-1}(M^{2dm-1})$ contains a subgroup Z_{2k}, then M^{2dm-1} can not admit a nontrivial G-action.*

Proof. For any isotropy group G_x, we have $(G_x)_0 \subset G_x \subset N((G_x)_0)$, the normalizer of $(G_x)_0$, and $G_x/(G_x)_0 \subset S^{d-1}$. When $G = Sp(m+1)$, up to conjugacy, a non cyclic finite subgroup of $S^3 = SU(2)$ is one of

$(b.d.)$ $D^*_{4n} = \{x, y\ ;\ x^2 = (xy)^2 = y^n,\ n \geqq 2,\ x^4 = 1\}$,
$(b.t.)$ $T^* = \{x, y\ ;\ x^2 = (xy)^3 = y^3,\ x^4 = 1\}$,
$(b.o.)$ $O^* = \{x, y\ ;\ x^2 = (xy)^3 = y^4,\ x^4 = 1\}$,
$(b.i.)$ $I^* = \{x, y\ ;\ x^2 = (xy)^3 = y^5,\ x^4 = 1\}$.

Then each group above contains a normal subgroup $Z_2(x^2)$. Therefore by Proposition 1, these groups can not be contained in $G_x/(G_x)_0$. Hence by the proofs of propositions 1 and 2, we can assume that the orbit types are $SF(m+1)/SF(m)$ and possibly fixed points. By 2 of Chapter IV in [3], the orbit space $B = M^{2dm-1}/G$ is a $d(m-1)$-dimensional manifold possibly with boundary $\partial B = F$ and we have a sphere bundle $S^{d(m+1)-1} \longrightarrow E(\xi) \longrightarrow B$ with $N(SF(m))/SF(m)$ as the structure group, further $M^{2dm-1}=E(\xi)\cup U(F)$. From the homology spectral sequence associated with the sphere bundle above, we see that $H_{dm-1}(E(\xi))=0$. Consider the Mayer-Vietoris exact sequence

$$\longrightarrow H_{dm-1}(E(\xi))+H_{dm-1}(U(F)) \longrightarrow H_{dm-1}(M^{2dm-1}) \longrightarrow H_{dm-2}(\partial U(F)) \longrightarrow .$$

Since $H_{dm-1}(U(F))=H_{dm-1}(F)=0$ and $H_{dm-2}(\partial U(F))=0$ (cf. the proof of Proposition 2), we have $H_{dm-1}(M^{2dm-1})=0$, which contradicts to the assumption.

Corollary to Theorem 3. *Let $S^{dm-1} \longrightarrow E \longrightarrow S^{dm}$ be a sphere bundle, where $m \geqq 8$ and the total space E is a π-manifold, then E can not admit a nontrivial G-action.*

Proof. If the total space E is homotopically equivalent to $S^{dm}\times S^{dm-1}$, then the assumption in the theorem 3 is satisfied and the corollary is obtained. Now we consider a nontrivial bundle E. Write τ for the class of the characteristic map for the tangent bundle of S^{dm} and σ for a generator of $\pi_{4t-1}(SO(4t))$ which gives rise to a generator of the stable group $\pi_{4t-1}(SO)$. For $\chi\in\pi_{dm-1}(SO(m))$, we denote by $E(\chi)$ the sphere bundle with χ as the class of characteristic maps. By 5.5 and 5.6 in [5], when m is odd and $dm\neq 2$, $E(\chi)$ is a π-manifold if and only if $\chi=k\tau$ for some integer k, further, when m is even and $dm\neq 4, 8$, $E(\chi)$ is *a* π-manifold if and only if $\chi=k\tau+2k\ell\sigma$ for some integers k, ℓ. Let $p: SO(dm) \longrightarrow S^{dm-1}$ be the canonical projection. By 23.4 in [7], $p_*(\tau)=2\iota_{dm-1}$, therefore by 3.4 in [4] $E(\chi)$ has a cell complex structure $S^{dm-1}\cup_{2k\iota_{dm-1}} e^{dm}\cup e^{2dm-1}$, and $H_{dm-1}(E(\chi))=Z_{2k}$. Thus by Theorem 3 we obtain the corollary.

Remark 1. When the bundle is trivial and $d=2$, the corollary is obtained from (a) of the theorem 2.1 in [8].

Remark 2. The referee has kindly pointed out that we can prove the corollary to Theorem 3 without the assumption that the total space $E(\chi)$ is a π-manifold.

Now we consider $SO(n+1)$-actions. Then we have

Theorem 4. *Let* M^{2n-1} *be a compact simply connected* $(2n-1)$*-dimensional* π*-manifold. Suppose that the integral homology group* $H_{n-1}(M^{2n-1})\neq 0$ *and* $n\geqq 10$. *Then* M^{2n-1} *can not admit a nontrivial* $SO(n+1)$*-action with exceptions of the real Stiefel manifold* $V_{n+1,2}$ *and a product manifold* $S^n\times X^{n-1}$, *where* X^{n-1} *is an* $(n-1)$*-dimensional simply connected manifold without boundary.*

Proof. Assume that M^{2n-1} admits a nontrivial $SO(n+1)$-action. By the theorem II and the remark in [2], we have $M^{2n-1}=\partial(D^{n+1}\times X^{n-1})$, where D^{n+1} is an $(n+1)$-disk and X^{n-1} is an $(n-1)$-dimensional simply connected manifold possibly with boundary.

(1) Suppose that $\partial X\neq\phi$. Consider the homology exact sequence

$$\longrightarrow H_n(D^{n+1}\times X^{n-1},\ M^{2n-1})\longrightarrow H_{n-1}(M^{2n-1})\longrightarrow H_{n-1}(D^{n+1}\times X^{n-1})\longrightarrow.$$

We have $H_n(D^{n+1}\times X^{n-1},\ M^{2n-1})\approx H_n(S^{n+1}\wedge(X^{n-1}/\partial X^{n-1}))=0$, and $H_{n-1}(D^{n+1}\times X^{n-1})\approx H_{n-1}(X^{n-1})=0$, then $H_{n-1}(M^{2n-1})=0$, which is a contradiction.

(2) Suppose that $\partial X=\phi$, then we have $M^{2n-1}=S^n\times X^{n-1}$, which is an exceptional case.

Corollary to Theorem 4. *Let* $S^{n-1}\longrightarrow E\longrightarrow S^n$ *be a sphere bundle,* where $n\geqq 10$ *and the total space* E *is a* π*-manifold. Then* E *can not admit a nontrivial* $SO(n+1)$*-action with exceptions of* $V_{n+1,2}$ *and a trivial bundle.*

Proof. For even n, the corollary is obtained from the proof of the corollary to Theorem 3 and Theorem 4. Now we consider the case n is odd. By 5.4 in [5], E is a π-manifold only if E is the Stiefel manifold $V_{n+1,2}$, or a trivial bundle. Then we have the corollary.

References

[1] Hsiang, Wu-chung and Hsiang, Wu-yi, Classification of differentiable actions on S^n, R^n and D^n with S^k as the principal orbit type, *Ann. of Math.*, **82** (1965), 421-433.

[2] Hsiang, Wu-yi, On the classification of differentiable $SO(n)$-actions on simply connected π-manifolds, *Amer. J. Math.*, **88** (1966), 137-153.

[3] Hsiang, Wu-chung and Hsiang, Wu-yi, Differentiable actions of compact connected classical groups I, *Amer. J. Math.*, **139** (1967), 705-786.

[4] James, I. M. and Whitehead, J. H. C., The homotopy theory of sphere bundles over spheres (I), *Proc. London Math. Soc.*, **4** (1954), 196-218.
[5] Kosinski, A., On the inertia group of π-manifolds, *Amer.* J. *Math.*, **89** (1967), 227-248.
[6] Montgomery, D., Samelson H. and Yang, C. T., Exceptional orbits of highest dimensions, *Ann. of Math.*, **63** (1965), 131-141.
[7] Steenrod, N., *The topology of fibre bundles*, Princeton, 1951.
[8] Uchida, F., $SU(n)$ actions on a product of spheres, *Bull. Yamagata Univ. Nat. Sci.*, **10** (1981), 167-176.