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Functional Dependence between the Hamiltonian and the Modular Operator Associated with a Faithful Invariant State of a W^*-Dynamical System

By

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Abstract

Let H and Δ be the hamiltonian, resp. the modular operator associated with an invariant faithful normal state ω of a W^*-dynamical system (A, α). Then $\Delta=f(H)$ for some decreasing function f if and only if (roughly speaking) ω is 2-passive with respect to α. It follows that under certain conditions a 3-passive state is an equilibrium (i.e. KMS) state.

§ 1. Introduction and Basic Definitions

In this paper, as in an earlier one [7], we investigate certain spectral properties of invariant states of noncommutative dynamical systems closely related to the KMS condition.

Let $\mathscr{A}$ be a von Neumann algebra, and $\alpha=\{\alpha_t\}$ $(t\in\boldsymbol{R})$ an ultra weakly continuous one-parameter group of *-automorphisms of $\mathscr{A}$. The action α of $\boldsymbol{R}$ on $\mathscr{A}$ can be described by the "spectral resolution" of $\mathscr{A}$ it induces. Let us recall that each closed interval $[\lambda, \mu]$ in $\boldsymbol{R}$ determines a spectral subspace $M[\lambda, \mu]$, a generic element x of which is characterized by the property that $\int f(t)\alpha_t(x)\,dt=0$ whenever $f\in L^1(\boldsymbol{R})$ and the (inverse) Fourier transform $\hat{f}$ of f is supported in the complement of $[\lambda, \mu]$ [3, Definition 2.1 and Remark on p. 225]. Here $\hat{f}$ is defined by

$$\hat{f}(\lambda)=\int e^{i\lambda t}f(t)\,dt \quad (\lambda\in\boldsymbol{R}).$$

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It is also useful to consider the ultraweakly closed linear span $R(\lambda, \mu)$ of the set of all elements of $\mathscr{A}$ of the form $\int f(t)\alpha_t(y)\,dt$, where $y\in\mathscr{A}$, $f\in L^1(\boldsymbol{R})$ and the support of $\hat{f}$ is contained in the open interval (λ, μ) [11, Definition 2. 3. 2] (clearly $R(\lambda, \mu)\subset M[\lambda, \mu]$). The knowledge of the collection of all the $M[\lambda, +\infty)$, or all the $R(\lambda, +\infty)$ (which we call a spectral resolution of $\mathscr{A}$), is equivalent with the knowledge of α, at least in principle [3, Lemma 2 on p. 233].

In [7, Definition 1. 2] we introduced the notion of spectral passivity:

Definition 1.1. An α-invariant normal state ω of $\mathscr{A}$ is said to be *n-spectrally passive* (where n is a positive integer) if

$$\prod_{i=1}^{n}\omega(x_ix_i^*)\leq\prod_{i=1}^{n}\omega(x_i^*x_i)$$

for all n-tuples $x_1, x_2, \dots, x_n$ of elements of $\mathscr{A}$ such that $x_i\in R(\lambda_i, +\infty)$ $(i=1, 2, \dots, n)$ for some n-tuple $\lambda_1, \lambda_2, \dots, \lambda_n$ of real numbers satisfying $\sum_{i=1}^{n}\lambda_i\geq 0$. The term "1-spectral passivity" will be discarded in favour of "spectral passivity".

The interest of this notion lies in its relationship with the concept of *passivity* introduced by Pusz and Woronowicz [12 ; 7, Theorem 3. 3], but also appears immediately in view of the following formulation of the KMS condition, obtained in [7]: ω is β-KMS with respect to α (where $0\leq\beta\leq+\infty$) if and only if it is α-invariant and

$$(1)\qquad x\in R(\lambda, +\infty)\Rightarrow\omega(xx^*)\leq e^{-\beta\lambda}\omega(x^*x)$$

[7, Theorem 1. 1 and Remark 2. 1(i)].

It is clear, then, that a β-KMS state is n-spectrally passive for all n (i. e. completely spectrally passive [7, Definition 1. 1]), whatever the value of β. Quite remarkably the converse is true as well: a completely spectrally passive state is in fact a KMS state for some β. In this form, this was stated and proved in [7], but it already appears implicitly as Theorem 1. 4 in [12]. Recently, Batty provided a truly elementary proof, with (1) as a starting point [4, Section 2].

The argument developed in [7] retains some interest, however, for two reasons. On the one hand it was generalized in a significant way by Batty, who obtained important results concerning the existence of one-parameter subgroups of arbitrary abelian groups acting on a

C^*-algebra for which a given invariant state is KMS [5, Theorem 3.2]. On the other hand, the constructions of [7, Section 4] can be used to analyze in detail the mechanism by which complete spectral passivity forces a state to be KMS. This is the purpose of the present paper. In particular it will be shown that *2-spectral passivity* (or rather a naturally arising stronger version of it) already imposes so much structure on the triple $(\mathscr{A}, \alpha, \omega)$ that it implies the existence of a decreasing real function $f(\lambda)$ on the "energy spectrum" of the system generalizing, in a certain sense, the Boltzmann factor $e^{-\beta\lambda}$.

We shall have to make this statement precise. But at this point the reader should be warned that the setup is somewhat different from the one in [7]. Indeed we assume that $(\mathscr{A}, \alpha)$ is a W^*-dynamical system rather than a C^*-dynamical system. Moreover we suppose throughout the paper (except in Remark 2.13) that ω is a *faithful* normal α-invariant state of $\mathscr{A}$. Consequently we can consider $\mathscr{A}$ to act on a Hilbert space $\mathscr{H}$ with cyclic and separating vector Ω such that $\omega(x) = (\Omega, x\Omega)$ $(x \in \mathscr{A})$. We also know that there is a unique strongly continuous one-parameter group $U = \{U_t\}$ $(t \in \boldsymbol{R})$ of unitaries implementing α and leaving Ω invariant. The hamiltonian of the system is the self-adjoint operator H on $\mathscr{H}$ defined by the equation $U_t = e^{itH}$ $(t \in \boldsymbol{R})$.

As a consequence of our faithfulness assumption the Tomita-Takesaki theory is available to us [13]. Let Δ be the modular operator associated with Ω. In general, the self-adjoint operators Δ and H are not related in any particular way, except for the fact that they commute strongly. On the other hand, as is well known, ω is β-KMS (with $0 \leq \beta < +\infty$) if and only if $\Delta = e^{-\beta H}$. Suppose more generally that ω is *2-spectrally passive*. The main result cf this paper (Theorem 2.8) asserts that, under certain additional hypotheses, *there exists a decreasing positive function f on the spectrum $\sigma(H)$ of H such that $\Delta = f(H)$*.*) Conversely, the existence cf such a function implies that ω is 2-spectrally passive (Theorem 2.12). If in fact ω satisfies the 3-spectral passivity condition, and if moreover $\sigma(H) = \boldsymbol{R}$, then the function f turns out to be a decreasing exponential, so that ω is KMS (Theorem 3.2).**)

*) In fact this still holds, in a somewhat generalized sense, when ω is not faithful. Cf. Remark 2.13.

**) This result was first obtained by H. A. M. Daniëls in [6b]. Cf. Remark 3.3.

Having spelled out the principal results in sufficient detail, we would now like to present a motivation for the stronger form of the 2-spectral passivity condition needed to obtain them. To that end we consider the simple case where $\mathscr{A}=\mathscr{L}(\mathfrak{H})$ with $\mathfrak{H}\cong \boldsymbol{C}^q$. In this case the dynamical group α is determined by a hermitian operator h on $\mathfrak{H}$ via the formula $\alpha_t(x)=e^{iht}xe^{-iht}$ $(x\in\mathscr{L}(\mathfrak{H}),\ t\in\boldsymbol{R})$, whereas a faithful α-invariant state ω is given by a positive invertible operator ρ, commuting with h, such that $\omega(x)=\tau(\rho x)$ for all x in $\mathscr{L}(\mathfrak{H})$ (here τ denotes the trace on $\mathscr{L}(\mathfrak{H})$).

Let $\{\phi_j\}_{j=1}^q$ be an orthonormal basis of $\mathfrak{H}$ in which both h and ρ are diagonal. Let $h_1, h_2, \dots, h_q$ and $\rho_1, \rho_2, \dots, \rho_q$ be the corresponding eigenvalues of h and ρ, respectively. It is not difficult to show that ω is 2-spectrally passive with respect to α if and only if the following holds [7, Example 4. 9]:

$$(2)\qquad \begin{cases} \text{whenever } h_j-h_k+h_{j'}-h_{k'}>0 \\ \qquad \text{for some } j,\ k,\ j',\ k'\in\{1,\ 2,\ \dots,\ q\}, \\ \text{one has } \rho_j\rho_{j'}\le\rho_k\rho_{k'}. \end{cases}$$

In order to determine the implications of the condition (2) on the modular structure induced by ω, we have to identify $\mathscr{A}$ with $\mathscr{A}\otimes 1$ acting on $\mathscr{H}=\mathfrak{H}\otimes\overline{\mathfrak{H}}$, and to put $\Omega=\sum_{j=1}^q \rho_j^{1/2}\phi_j\otimes\phi_j$. One easily computes that $H=h\otimes 1-1\otimes h$ and that $\Delta=\rho\otimes\rho^{-1}$. Both H and Δ are clearly diagonal in the basis $\{\phi_j\otimes\phi_k\}_{j,k=1}^q$, and the corresponding eigenvalues are h_j-h_k (for H) and $\rho_j\rho_k^{-1}$ (for Δ). Hence if we replace (2) with the stronger requirement that

$$(3)\qquad h_j-h_k+h_{j'}-h_{k'}\ge 0\Rightarrow \rho_j\rho_{j'}\le\rho_k\rho_{k'},$$

we can unambiguously define a decreasing function f on $\sigma(H)$ by $f(h_j-h_k)=\rho_j\rho_k^{-1}$, so that $f(H)=\Delta$. Clearly this conclusion could not have been obtained in general without some strengthening of (2).

Passing from (2) to (3) really amounts to replacing the spectral subspaces $R(\lambda, +\infty)$ in Definition 1. 1 by the larger spaces $M[\lambda, +\infty)$ corresponding to the closed half lines $[\lambda, +\infty)$. Hence we are led to introduce the following definition in the general case:

Definition 1. 2. An α-invariant normal state ω of $\mathscr{A}$ is called *strongly n-spectrally passive* if $x_i\in M[\lambda_i, +\infty)$ $(i=1, 2, \dots, n)$ implies $\prod_{i=1}^n \omega(x_ix_i^*)\le\prod_{i=1}^n \omega(x_i^*x_i)$ whenever $\sum_{i=1}^n \lambda_i\ge 0$.

Is it possible to justify the substitution of strong spectral passivity for spectral passivity on physical grounds? In the above example of a "finite spin system" the answer seems to be affirmative, in view of Lenard's analysis in [10] cf what he calls "structural stability". Extrapolating from the finite dimensional case one may conjecture that strong spectral passivity should result from a combination of spectral passivity and stability in the sense of Haag, Kastler and Trych-Pohlmeyer [9]. We give some substance to that claim in Lemma 2.10 and Remark 2.11. Another question is, whether strong spectral passivity implies passivity in the original sense of Pusz and Woronowicz [12]. This is by no means clear a priori (the converse is false).

We now return briefly to the case $\mathscr{A}=\mathscr{L}(\mathfrak{H})$ considered above to make a different point. In this case strong spectral passivity reduces to the condition that $h_j-h_k\geq 0$ implies $\rho_j\leq\rho_k$, which means exactly that ρ is a decreasing function of h. For general quantum statistical systems there is no density matrix, of course. However we are able to derive a formally analogous relation $\Delta=f(H)$, with Δ in place of ρ and h replaced by H (the spectrum of which consists of energy differences rather than energy levels). Thus our result provides an illustration of the heuristic principle that the modular operator Δ can be used as a partial substitute for the density matrix ρ in the general case, albeit in a "relative" rather than in an "absolute" sense. In some of Araki's papers (e.g. [1, 2]), which inspired us, the same philosophy is at work.

It is unclear at present what physical meaning (if any) should be attributed to the relation $\Delta=f(H)$.*) By comparison with the exponentials $e^{-\beta\lambda}$, the function $f(\lambda)$ can conceivably provide us with a quantitative measure of the deviation of the state ω from equilibrium. More specifically one might expect $\log f$, or some quantity derived from it, to have certain entropy-like properties (indeed passivity is an expression of the second law of thermodynamics [12], which itself gives rise to the notion of entropy).

Finally it is worthwhile observing that a decreasing dependence

*) The same type of relation appears in a recent paper by J. S. Cohen, H. A. M. Daniëls and M. Winnink [Commun. Math. Phys., **84**, 449-458 (1982)], where it is studied as a consequence of a modified KMS equation. I am grateful to the authors for pointing this out to me.

between passive states and hamiltonians has also been shown to exist in the framework of classical statistical mechanics [8, Proposition 2 ; 6a, Theorem 1 ; 6b, Theorem 6.2].

§ 2. Proof of $\Delta=f(H)$

We assume throughout the paper*) that $\mathscr{A}$ is a (σ-finite) von Neumann algebra acting on a Hilbert space $\mathscr{H}$ with cyclic and separating vector Ω. Furthermore $\{U_t\}$ $(t\in\boldsymbol{R})$ is a strongly continuous one-parameter group of unitary operators on $\mathscr{H}$ satisfying $U_t\Omega=\Omega$ and $U_t\mathscr{A}U_t^{-1}=\mathscr{A}$. The state of $\mathscr{A}$ defined by Ω is denoted ω, the restriction of Ad U_t to $\mathscr{A}$ is called α_t, and $U_t=e^{itH}$ $(t\in\boldsymbol{R})$.

Let P be the unique projection-valued measure such that $H=\int\lambda dP_\lambda$. For every x in $\mathscr{A}$ we define bounded positive Radon measures μ_x and ν_x on $\boldsymbol{R}$ by their (inverse) Fourier transforms:

$$(4)\qquad \begin{cases} \omega(x^*\alpha_t(x))=\int e^{i\lambda t}d\mu_x(\lambda) \\ \text{and} \\ \omega(\alpha_t(x)x^*)=\int e^{i\lambda t}d\nu_x(\lambda) \qquad (t\in\boldsymbol{R}). \end{cases}$$

Equivalently we have

$$(5)\qquad \begin{cases} d\mu_x(\lambda)=d(x\Omega,\ P_\lambda x\Omega) \\ \text{and} \\ d\nu_x(\lambda)=d(x^*\Omega,\ P_{-\lambda}x^*\Omega). \end{cases}$$

The following proposition gives a simple but very useful reformulation of the relation $\Delta=f(H)$ in terms of the measures μ_x and ν_x.

Proposition 2.1. *Let* $f:\boldsymbol{R}\to[0,+\infty]$ *be Borel measurable and P-almost everywhere finite. Then the following statements are equivalent :*

(i) $\Delta=f(H)$.

(ii) *For all* x *in* $\mathscr{A}$, ν_x *is absolutely continuous with respect to* μ_x, *and* $\dfrac{d\nu_x}{d\mu_x}=f$ (μ_x*-almost everywhere*).

*) Except, as pointed out previously, in Remark 2.13.

Proof. (i) $\Rightarrow$ (ii). Suppose $\Delta=f(H)$ and $x\in\mathscr{A}$. For g in $L^1(\boldsymbol{R})$ we define $y=\int g(t)\alpha_t(x)\,dt$. It is easy to see that $d\mu_y(\lambda)=|\hat{g}(\lambda)|^2 d\mu_x(\lambda)$ and $d\nu_y(\lambda)=|\hat{g}(\lambda)|^2 d\nu_x(\lambda)$. Using this we have

$$\begin{aligned}\int|\hat{g}(\lambda)|^2 d\nu_x(\lambda) &= \omega(yy^*) \text{ by (4)}\\ &= \|y^*\Omega\|^2=\|\Delta^{1/2}y\Omega\|^2\\ &= \|f(H)^{1/2}y\Omega\|^2 \text{ by hypothesis}\\ &= \int f(\lambda)\,d\mu_y(\lambda) \text{ by (5)}\\ &= \int|\hat{g}(\lambda)|^2 f(\lambda)\,d\mu_x(\lambda).\end{aligned}$$

As $x\Omega\in\operatorname{dom}\Delta^{1/2}=\operatorname{dom} f(H)^{1/2}$, it follows that $f\in L^1(\mu_x)$. Hence by uniform density of $L^1(\boldsymbol{R})^\wedge$ in $C_0(\boldsymbol{R})$ we actually obtain

$$\int h(\lambda)\,d\nu_x(\lambda)=\int h(\lambda)f(\lambda)\,d\mu_x(\lambda)$$

for all nonnegative h in $C_0(\boldsymbol{R})$. This yields (ii) by the Lebesgue-Radon-Nikodým theorem.

(ii) $\Rightarrow$ (i). We assume that $\nu_x\ll\mu_x$ and $\dfrac{d\nu_x}{d\mu_x}=f$ for all x in $\mathscr{A}$. Since $f\in L^1(\mu_x)$ by the boundedness of ν_x, it follows that $x\Omega\in\operatorname{dom} f(H)^{1/2}$ and

$$\|f(H)^{1/2}x\Omega\|^2=\int f(\lambda)\,d\mu_x(\lambda)=\int d\nu_x(\lambda)=\|x^*\Omega\|^2=\|\Delta^{1/2}x\Omega\|^2.$$

As $\{x\Omega\,|\,x\in\mathscr{A}\}$ is a core for $\Delta^{1/2}$, the above equation easily implies that $\operatorname{dom}\Delta^{1/2}\subset\operatorname{dom} f(H)^{1/2}$, and that $\|\Delta^{1/2}\Phi\|=\|f(H)^{1/2}\Phi\|$ for all Φ in $\operatorname{dom}\Delta^{1/2}$. For the remainder of the proof we set $f(H)=A$, and show the following:

2.2. *Let Δ and A be strongly commuting positive selfadjoint operators on a Hilbert space $\mathscr{H}$ such that* $\operatorname{dom}\Delta^{1/2}\subset\operatorname{dom}A^{1/2}$ *and*

$$\Phi\in\operatorname{dom}\Delta^{1/2}\Rightarrow\|\Delta^{1/2}\Phi\|=\|A^{1/2}\Phi\|. \quad \textit{Then } \Delta=A.$$

First we prove that $\operatorname{dom}A^{1/2}\subset\operatorname{dom}\Delta^{1/2}$. So suppose $\Psi\in\operatorname{dom}A^{1/2}$, and let E_n be the spectral projection of $\Delta^{1/2}$ corresponding to the interval $[0, n]$ $(n\in\boldsymbol{N})$. As $\Delta^{1/2}$ commutes strongly with $A^{1/2}$, one has $E_n\Psi\in\operatorname{dom}A^{1/2}$ and $A^{1/2}E_n\Psi=E_nA^{1/2}\Psi$. But on the other hand $E_n\Psi\in\operatorname{dom}\Delta^{1/2}$, and

$$\|\Delta^{1/2}E_n\Psi-\Delta^{1/2}E_m\Psi\|=\|A^{1/2}E_n\Psi-A^{1/2}E_m\Psi\| \quad \text{by assumption,}$$
$$=\|(E_n-E_m)A^{1/2}\Psi\|,$$

which tends to zero as $n, m\to\infty$. As $\Delta^{1/2}$ is closed, we conclude that $\Psi=\lim_{n\to\infty}E_n\Psi\in\text{dom }\Delta^{1/2}$.

Finally, take $\Phi\in\text{dom }\Delta$ and $\Psi\in\text{dom }A^{1/2}$. Then $\Phi\in\text{dom }\Delta^{1/2}=\text{dom }A^{1/2}$ and

$$(A^{1/2}\Phi,\ A^{1/2}\Psi)=(\Delta^{1/2}\Phi,\ \Delta^{1/2}\Psi) \qquad \text{by polarization}$$
$$=(\Delta\Phi,\ \Psi).$$

It follows that $A^{1/2}\Phi\in\text{dom }A^{1/2}$, hence $\Phi\in\text{dom }A$, and that $A\Phi=A^{1/2}(A^{1/2}\Phi)=\Delta\Phi$. Consequently $\Delta\subset A$. By selfadjointness $A\subset\Delta$. This ends the proof of both (2. 2) and Proposition 2. 1. □

Next we define the function f that appears in the relation $\Delta=f(H)$, as well as an auxiliary function g (these are similar, but not equal to objects with the same name defined in [7]).

Definition 2. 3. For λ in $\boldsymbol{R}$ we put

$$f(\lambda)=\sup\{\|x^*\Omega\|^2\,\|x\Omega\|^{-2}\,|\,x\in M[\lambda,+\infty),\ x\neq0\}$$
$$g(\lambda)=\inf\{\|x^*\Omega\|^2\,\|x\Omega\|^{-2}\,|\,x\in M(-\infty,\lambda],\ x\neq0\},$$

with the convention that $\sup\phi=0$, $\inf\phi=+\infty$.

The following lemma will be used to study the properties of the functions f and g (Lemma 2. 5 below), and again in the proofs of the lemmas 2. 7 and 2. 10.

Lemma 2. 4. *If $x\in\mathscr{A}$ and $\lambda\in\boldsymbol{R}$, then there exist y and z in $\mathscr{A}$ such that*

(i) $x=y+z,\ y\in M(\{\lambda\}),\ z\in R(\boldsymbol{R}\backslash\{\lambda\})$.

If $x\in M[\lambda,+\infty)$, then z can be chosen in $R(\lambda,+\infty)$. Moreover, if (i) *holds, then one also has*

(ii) $P(\{\lambda\})x\Omega=y\Omega,\ P(\{-\lambda\})x^*\Omega=y^*\Omega,$

and

(iii) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}\omega(x^*\alpha_t(x))\,dt=\|y\Omega\|^2,$$
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}\omega(\alpha_t(x)x^*)\,dt=\|y^*\Omega\|^2.$$

Proof. Let f in $L^1(\boldsymbol{R})$ be such that $\hat{f}(\xi)=1$ if $|\xi|\leq 1$, $\hat{f}(\xi)=0$ if $|\xi|\geq 2$. For $n=1, 2, \dots$, define f_n by

$$f_n(t)=e^{-i\lambda t}n^{-1}f(n^{-1}t) \qquad (t\in\boldsymbol{R}),$$

i.e.

$$\hat{f}_n(\xi)=\hat{f}(n(\xi-\lambda)) \qquad (\xi\in\boldsymbol{R}).$$

Put $y_n=\int f_n(t)\alpha_t(x)\,dt$ and $z_n=x-y_n$. Then $y_n\in R(\lambda-3n^{-1}, \lambda+3n^{-1})$ and $z_n\in M(\boldsymbol{R}\backslash(\lambda-n^{-1}, \lambda+n^{-1}))$. If $x\in M[\lambda,+\infty)$ then in fact $z_n\in M[\lambda+n^{-1}, +\infty)$.

Since $\|y_n\|\leq\|f\|_1\|x\|$, the ultraweak compactness of the unit ball of $\mathcal{A}$ implies that the sequence $\{y_n\}_{n=1}^{\infty}$ has an ultraweak accumulation point y, which belongs to $M(\{\lambda\})$. Since $x-y$ is an accumulation point of $\{z_n\}_{n=1}^{\infty}$, it is an element of $R(\boldsymbol{R}\backslash\{\lambda\})$, and even of $R(\lambda,+\infty)$ if $x\in M[\lambda,+\infty)$. This concludes the proof of (i).

To show (ii) it is sufficient to observe that $y\Omega\in P(\{\lambda\})\mathcal{H}$, $y^*\Omega\in P(\{-\lambda\})\mathcal{H}$, $z\Omega\in P(\boldsymbol{R}\backslash\{\lambda\})\mathcal{H}$ and $z^*\Omega\in P(\boldsymbol{R}\backslash\{-\lambda\})\mathcal{H}$ [7, Lemma 1.4].

Finally, according to the mean ergodic theorem,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}\omega(x^*\alpha_t(x))\,dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}(x\Omega,\ U_t x\Omega)\,dt$$
$$=(x\Omega,\ P(\{\lambda\})x\Omega),$$

and similarly

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}\omega(\alpha_t(x)x^*)\,dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}(x^*\Omega,\ U_{-t}x^*\Omega)\,dt$$
$$=(x^*\Omega,\ P(\{-\lambda\})x^*\Omega).$$

Then (iii) follows from (ii). □

We note that the existence of a cyclic and separating vector is not required for the validity of (i) above. On the other hand the decomposition $x=y+z$ need not be unique. For instance, if $\mathcal{A}=L^\infty(\boldsymbol{R})$ and α is the group of translations, then the functions $t\longmapsto e^{i\lambda t}$ belong to $R(\boldsymbol{R}\backslash\{0\})$ if $\lambda\neq 0$, but they tend ultraweakly to the constant function 1 when $\lambda\to 0$. Hence $1\in M(\{0\})\cap R(\boldsymbol{R}\backslash\{0\})$.

Lemma 2.5.

(i) *f and g are decreasing functions.*

(ii) *$f(-\lambda)=g(\lambda)^{-1}$ for all λ in $\boldsymbol{R}$ (with $0^{-1}=+\infty$, $(+\infty)^{-1}=0$).*

(iii) $f(0)\geq 1$, $g(0)\leq 1$.

(iv) $f(\lambda)=0$ (*or, equivalently*, $g(-\lambda)=+\infty$) *if and only if* $\sigma(H)\subset[-\lambda, \lambda]$ *and* $\pm\lambda$ *is not an eigenvalue of* H.

(v) *If* $\lambda\in\sigma(H)$ *then* $g(\lambda+)\leq f(\lambda-)$.*)

Proof. (i), (ii) and (iii) are obvious. Suppose $f(\lambda)=0$. By (i) and (iii), $\lambda>0$. The definition of f implies $M[\lambda,+\infty)=\{0\}$. In particular $R(\lambda, +\infty)=\{0\}$, hence $\sigma(H)=sp(\alpha)\subset(-\infty, \lambda]$ [7, Remark 1.5]. But since $sp(\alpha)=-sp(\alpha)$, we actually have the inclusion $\sigma(H)\subset[-\lambda, \lambda]$. Furthermore $M(\{\lambda\})=\{0\}$. As Ω is cyclic, Lemma 2.4 implies $P(\{\pm\lambda\})=0$, i.e. $\pm\lambda$ is not an eigenvalue of H. Suppose conversely that $\sigma(H)\subset[-\lambda, \lambda]$ and that $\pm\lambda$ is not an eigenvalue of H. If $x\in M[\lambda,+\infty)$, then as Ω is separating $sp(x)\subset[-\lambda, \lambda]\cap[\lambda, +\infty)=\{\lambda\}$ [7, Remark 1.5]. But the fact that λ is not an eigenvalue of H clearly implies $M(\{\lambda\})=\{0\}$. Hence $M[\lambda,+\infty)=\{0\}$ and $f(\lambda)=0$. This ends the procf of (iv). To show (v), suppose $\lambda\in\sigma(H)$ and $\varepsilon>0$. Then we can find a nonzero element x in $M[\lambda-\varepsilon, \lambda+\varepsilon]$. It follows that $g(\lambda+\varepsilon)\leq\|x^*\Omega\|^2\,\|x\Omega\|^{-2}\leq f(\lambda-\varepsilon)$. Taking the limit as $\varepsilon\rightarrow 0$ gives the desired result. □

Next we show how to express the passivity of ω in terms of f and g.

Lemma 2.6. (i) *ω is strongly spectrally passive if and only if* $f(0)\leq 1$ (*or equivalently*, $g(0)\geq 1$).
(ii) *ω is strongly 2-spectrally passive if and only if* $f(\lambda)\leq g(\lambda)$ ($\lambda\in\boldsymbol{R}$).

Proof. (i) is obvious (if ω is strongly spectrally passive one actually has $f(0)=g(0)=1$, by Lemma 2.5 (iii)).
(ii) Suppose that ω is strongly 2-spectrally passive, $\lambda\in\boldsymbol{R}$, and $f(\lambda)>0$, $g(\lambda)<+\infty$. For every nonzero x in $M[\lambda,+\infty)$ and nonzero y in $M(-\infty, \lambda]$, Definition 1.2 implies that

$$\omega(xx^*)\omega(y^*y)\leq\omega(x^*x)\omega(yy^*),$$

or

$$\omega(xx^*)\omega(x^*x)^{-1}\leq\omega(yy^*)\omega(y^*y)^{-1}.$$

*) $g(\lambda+)=\lim_{\mu\overset{>}{\rightarrow}\lambda} g(\mu)$, $f(\lambda-)=\lim_{\mu\overset{<}{\rightarrow}\lambda} f(\mu)$.

It follows that $f(\lambda) \leq g(\lambda)$. If $f(\lambda)=0$ or $g(\lambda)=+\infty$ this inequality holds trivially. Conversely, if $0<f(\lambda) \leq g(\lambda)<+\infty$, $x \in M[\lambda,+\infty)$, $z \in M[-\lambda, +\infty)$, then

$$\omega(xx^*) \leq f(\lambda)\omega(x^*x), \ \omega(zz^*) \leq g(\lambda)^{-1}\omega(z^*z).$$

As $f(\lambda)g(\lambda)^{-1} \leq 1$ we obtain $\omega(xx^*)\omega(zz^*) \leq \omega(x^*x)\omega(z^*z)$. If $f(\lambda)=0$ (or $g(\lambda)=+\infty$) the same inequality is valid, because $x=0$ (or $z=0$).

□

Lemma 2.7. *Suppose that ω is strongly 2-spectrally passive, $x \in \mathscr{A}$, $\lambda \in \boldsymbol{R}$.*

(i) $\mu_x(\{\lambda\})=0$ *if and only if* $\nu_x(\{\lambda\})=0$.

(ii) *If* $\mu_x(\{\lambda\}) \neq 0$, *then* $\dfrac{\nu_x(\{\lambda\})}{\mu_x(\{\lambda\})}=f(\lambda)=g(\lambda)$.

Proof. Given λ and x, choose y in $\mathscr{A}$ as in Lemma 2.4. Then $\mu_x(\{\lambda\})=(x\Omega, P(\{\lambda\})x\Omega)=\|P(\{\lambda\})x\Omega\|^2=\|y\Omega\|^2$ (by (5)), and similarly $\nu_x(\{\lambda\})=\|P(\{-\lambda\})x^*\Omega\|^2=\|y^*\Omega\|^2$ (again by (5)). Hence (i) follows because Ω is separating for $\mathscr{A}$. Moreover, if $\mu_x(\{\lambda\}) \neq 0$, we have

$$g(\lambda) \leq \frac{\|y^*\Omega\|^2}{\|y\Omega\|^2}=\frac{\nu_x(\{\lambda\})}{\mu_x(\{\lambda\})} \leq f(\lambda),$$

since $y \in M(-\infty, \lambda] \cap M[\lambda, +\infty)$. Then (ii) is a consequence of the fact that $f(\lambda) \leq g(\lambda)$ (Lemma 2.6 (ii)). □

The above lemma, taken together with Proposition 2.1, already implies that $\Delta=f(H)$ in case H has a countable spectrum. The idea of the proof of our main theorem 2.8 is that Lemma 2.7 allows us to discard arbitrary finite (even countable) subsets of $\sigma(H)$. Notice that Lemma 2.7 does *not* hold if ω is merely 2-spectrally passive (see Section 1).

Theorem 2.8. *As before let $\mathscr{A}$ be a von Neumann algebra with cyclic and separating vector Ω, and $\{\alpha_t\}$ $(t \in \boldsymbol{R})$ a continuous group of *-automorphisms of $\mathscr{A}$ leaving the corresponding state ω of $\mathscr{A}$ invariant. If ω is strongly 2-spectrally passive with respect to α, then there exists a P-almost everywhere finite, decreasing function $f: \boldsymbol{R} \to [0, +\infty]$ such that $\Delta=f(H)$. In fact f is as defined in 2.3.*

Proof. Let λ_ℓ be the left endpoint of $\sigma(H)$ (possibly $-\infty$). If $\lambda>\lambda_\ell$, then $g(\lambda)$ is finite by Lemma 2.5 (iv), hence $f(\lambda)<+\infty$ by Lemma 2.6 (ii). For x in $\mathscr{A}$ we first show that $\nu_x \ll \mu_x$ on each bounded open interval (ξ, ξ'), where $\xi>\lambda_\ell$. Suppose $h\in L^1(\boldsymbol{R})$ and supp $\hat{h}\subset(\xi, \xi')$. Then actually supp $\hat{h}\subset(\xi+\varepsilon, \xi'-\varepsilon)$ for some strictly positive ε. Put $y=\int h(t)\alpha_t(x)dt$, so that $y\in R(\xi+\varepsilon, \xi'-\varepsilon)\subset M[\xi+\varepsilon, \xi'-\varepsilon]$. According to Definition 2.3 we have

$$g(\xi'-)\omega(y^*y)\leq g(\xi'-\varepsilon)\omega(y^*y)\leq\omega(yy^*)$$
$$\leq f(\xi+\varepsilon)\omega(y^*y)\leq f(\xi+)\omega(y^*y).$$

We rewrite these inequalities as

$$(6)\qquad g(\xi'-)\int|\hat{h}(\lambda)|^2d\mu_x(\lambda)\leq\int|\hat{h}(\lambda)|^2d\nu_x(\lambda)\leq f(\xi+)\int|\hat{h}(\lambda)|^2d\mu_x(\lambda).$$

By an argument of uniform density, the above inequality holds with $|\hat{h}|^2$ replaced by an arbitrary nonnegative continuous function supported in (ξ, ξ'). By the Lebesgue-Radon-Nikodým theorem, ν_x is absolutely continuous with respect to μ_x on (ξ, ξ'). If H is unbounded this shows at once that $\nu_x\ll\mu_x$. If H is bounded, the same conclusion is obtained using Lemma 2.7 (i) with $\lambda=\lambda_\ell$.

We now have to prove that the Radon-Nikodým derivative $d\nu_x/d\mu_x$ is essentially independent of x, and that actually $d\nu_x/d\mu_x=f=g$ μ_x-almost everywhere for all x. Fix $\xi_0>\lambda_\ell$ and $\varepsilon>0$. Since $0\leq f\leq f(\xi_0)$ on $(\xi_0, +\infty)$, there are only finitely many points ξ in $(\xi_0, +\infty)$ where $f(\xi-)-f(\xi+)$ exceeds a given positive value. Hence we can find a partition $\xi_0<\xi_1<\cdots<\xi_{n-1}<\xi_n=+\infty$ of $(\xi_0,+\infty)$ such that

$$(7)\qquad f(\xi_{j-1}+)-f(\xi_j-)<\varepsilon \text{ for } j=1, 2, \ldots, n.^{*)}$$

On the other hand, (6) implies that for μ_x-almost all λ in (ξ_{j-1}, ξ_j)

$$f(\xi_j-)\leq g(\xi_j-)\leq\frac{d\nu_x}{d\mu_x}(\lambda)\leq f(\xi_{j-1}+)$$

(the first inequality follows from Lemma 2.6 (ii)). Since clearly

$$f(\xi_j-)\leq f(\lambda)\leq f(\xi_{j-1}+)$$

for all λ in (ξ_{j-1}, ξ_j), (7) yields

*) $f(\xi_n-)=\lim_{\xi\to+\infty}f(\xi)$.

$$\left| \frac{d\nu_x}{d\mu_x}(\lambda) - f(\lambda) \right| < \varepsilon$$

for μ_x-almost all λ in $\bigcup_{j=1}^{n} (\xi_{j-1}, \xi_j)$. By Lemma 2.7 (ii) this inequality actually holds for μ_x-almost all λ in $(\xi_0, +\infty)$. Since ε was arbitrary we conclude that $d\nu_x/d\mu_x = f$ μ_x-almost everywhere on $(\xi_0, +\infty)$, hence μ_x-almost everywhere on $\boldsymbol{R}$ (one has to invoke Lemma 2.7 once more if H is bounded and λ_ℓ is an eigenvalue). Clearly a similar reasoning proves that $d\nu_x/d\mu_x = g$. □

In view of the more immediate physical meaning of passivity as compared with strong spectral passivity, the following variant of the previous result may be of interest:

Corollary 2.9. *Adopting the same general assumptions as in Theorem* 2.8, *suppose that the following properties hold for the triple* $(\mathscr{A}, \alpha, \omega)$:

(i) ω *is* 2-*spectrally passive with respect to* α.

(ii) H *has continuous spectrum except at* 0 (*i.e.* 0 *is the only eigenvalue of* H, *or* $M(\{\lambda\}) = \{0\}$ *whenever* $\lambda \neq 0$).

(iii) *For all* x *in* $\mathscr{A}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega([x^*, \alpha_t(x)])\, dt = 0.$$

Then $\Delta = f(H)$ *for some decreasing*, P-*essentially finite function* f: $\boldsymbol{R} \to [0, +\infty]$. □

This corollary follows from Theorem 2.8 and the following lemma.

Lemma 2.10. *If* ω *is* n-*spectrally passive with respect to* α, *and the above conditions* (ii) *and* (iii) *hold, then* ω *is strongly* n-*spectrally passive.*

Proof. First assume that ω is spectrally passive and $x \in M[0, +\infty)$. Writing $x = y + z$ with $y \in M(\{0\})$ and $z \in R(0, +\infty)$ (Lemma 2.4 (i)), we have $\|x^*\Omega\|^2 = \|y^*\Omega\|^2 + \|z^*\Omega\|^2$ because $y^*\Omega$ and $z^*\Omega$ are orthogonal. But $\|y^*\Omega\|^2 = \|y\Omega\|^2$ by (iii) and Lemma 2.4 (iii), and $\|z^*\Omega\|^2 \leq \|z\Omega\|^2$ by assumption. Hence

$$\|x^*\Omega\|^2 \leq \|y\Omega\|^2 + \|z\Omega\|^2 = \|x\Omega\|^2.$$

This shows that ω is strongly spectrally passive.

Now it is easy to set up an induction proof. Suppose the lemma holds for $n=m-1$, and $x_i \in M[\lambda_i, +\infty)$ $(i=1, 2, \dots, m)$, $\sum_{i=1}^{m} \lambda_i \geq 0$. By (ii) and Lemma 2.4 (i), $M[\lambda, +\infty) = R(\lambda, +\infty)$ except possibly if $\lambda=0$. Hence if $\lambda_i \neq 0$ for all i, the inequality $\prod_{i=1}^{m} \omega(x_i x_i^*) \leq \prod_{i=1}^{m} \omega(x_i^* x_i)$ follows from the assumption of m-spectral passivity. If $\lambda_1=0$, say, then $\omega(x_1 x_1^*) \leq \omega(x_1^* x_1)$ by the first part of the proof and $\prod_{i=2}^{m} \omega(x_i x_i^*) \leq \prod_{i=2}^{m} \omega(x_i^* x_i)$ by the induction hypothesis. Multiplying these inequalities we conclude that ω is strongly m-spectrally passive. □

Remark 2.11. The condition (iii) in the statement of Corollary 2.9 is easily seen to be equivalent with

$$\mu_x(\{0\}) = \nu_x(\{0\}) \quad \text{for all } x \in \mathscr{A},$$

and is thus a necessary condition for the equality $\Delta = f(H)$ to hold. One can say that it amounts to "the KMS condition at 0 energy". This formulation is often used to describe the main direct consequence of the stability notion due to Haag, Kastler and Trych-Pohlmeyer (see [9, pp. 177-178]).

Concluding this section, we prove that strong 2-spectral passivity is not only a sufficient but also a necessary condition for the equation $\Delta = f(H)$ to hold (with decreasing f).

Theorem 2.12. *If $\Delta = f(H)$, where $f: \boldsymbol{R} \to [0, +\infty]$ is everywhere decreasing, then ω is strongly 2-spectrally passive.*

Proof. First we observe that $\Delta = f(H)$ implies that $f(-\lambda) = f(\lambda)^{-1}$ for P-almost all λ. This follows from the fact that

$$f(H)^{-1} = \Delta^{-1} = J\Delta J = Jf(H)J = f(-H),$$

where J has the usual meaning (i.e. $J\Delta^{1/2} x\Omega = x^*\Omega$ for all x in $\mathscr{A}$), and the last equality follows from $JHJ = -H$. In particular $f(0) = 1$.

As before put $\lambda_\ell = \inf \sigma(H)$. If $\sigma(H) = \{0\}$ then $\Delta = 1$ and ω is a trace, so we can suppose $\lambda_\ell < 0$. If $\lambda \in (\lambda_\ell, -\lambda_\ell)$ then $0 < f(\pm\lambda) < +\infty$, where the first inequality follows from the injectivity of Δ. Let us write $f_e(-\lambda)$ for the P-essential supremum of $\{f(\xi) \mid \xi \in [-\lambda, +\infty)\}$.

If $\xi\geq-\lambda$, then $f(-\xi)^{-1}\leq f(\lambda)^{-1}$ because f is decreasing, hence by the above observation $f(\xi)\leq f(\lambda)^{-1}$ for P-almost all ξ in $[-\lambda, +\infty)$. It follows that $f_e(-\lambda)\leq f(\lambda)^{-1}$. Thus for x in $M[\lambda, +\infty)$ and y in $M[-\lambda, +\infty)$ we obtain

$$\begin{aligned}\omega(xx^*)\omega(yy^*) &= \|\Delta^{1/2}x\Omega\|^2\,\|\Delta^{1/2}y\Omega\|^2\\ &= \int f(\xi)\,d\mu_x(\xi)\cdot\int f(\xi)\,d\mu_y(\xi)\\ &\leq f(\lambda)\omega(x^*x)\cdot f_e(-\lambda)\omega(y^*y) \text{ because supp } \mu_x\\ &\qquad \subset[\lambda, +\infty), \text{ supp } \mu_y\subset[-\lambda, +\infty) \text{ [7, Lemma 1.6]}\\ &\leq\omega(x^*x)\omega(y^*y).\end{aligned}$$

This shows that ω is strongly 2-spectrally passive, at least if H is unbounded. If H is bounded we also have to consider x in $M[-\lambda_\ell, +\infty)$ and y in $M[\lambda_\ell,+\infty)=\mathscr{A}$. Then either $\pm\lambda_\ell$ are eigenvalues of H, in which case $0<f(\pm\lambda_\ell)<+\infty$ and the previous reasoning still goes through (with $-\lambda_\ell$ in place of λ); or $\pm\lambda_\ell$ are not eigenvalues of H, and then $M[-\lambda_\ell,+\infty)=\{0\}$. □

Remark 2.13. Theorem 2.8 can be generalized to deal with a *not necessarily faithful* strongly 2-spectrally passive state ω. Let $\mathscr{A}$ be represented in the corresponding GNS representation, and let Q denote the support of ω, i.e. the projection onto the closure of $\mathscr{A}'\Omega$. For λ in $\boldsymbol{R}$ define

$$f(\lambda)=\inf\{a\geq 0\,|\,\omega(xx^*)\leq a\omega(x^*x) \quad \text{for all } x \text{ in } M[\lambda, +\infty)\}$$
$$g(\lambda)=\sup\{b\geq 0\,|\,\omega(xx^*)\geq b\omega(x^*x) \quad \text{for all } x \text{ in } M(-\infty, \lambda]\}$$

(these prescriptions coincide with Definition 2.3 if ω is faithful).

We make the following observations:

(i) If $f(\lambda)=0$ then $Q\leq P(-\infty, \lambda)$. Indeed, if $x\in M[\lambda,+\infty)$ and $y\in\mathscr{A}'$ then $(y\Omega, x\Omega)=(yx^*\Omega,\Omega)=0$ because $x^*\Omega=0$. Hence $Qx\Omega=0$. Using Lemma 2.4 (i) it follows that $QP[\lambda,+\infty)=0$.

(ii) If $g(\lambda)>0$ then $Q\geq P(-\infty, \lambda]$. To see this, consider x in $M(-\infty, \lambda]$. Clearly $(1-Q)x\in M(-\infty, \lambda]$ and $x^*(1-Q)\Omega=0$. Then $(1-Q)x\Omega=0$ because $g(\lambda)>0$. Again using Lemma 2.4 (i) we conclude that $(1-Q)P(-\infty,\lambda]=0$.

Now we put $\mu=\inf\{\lambda\,|\,f(\lambda)=0\}\in\boldsymbol{R}\cup\{+\infty\}$. By (i) $Q\leq P(-\infty, \mu]$, and $Q\leq P(-\infty, \mu)$ if $f(\mu)=0$. The assumption of strong 2-spectral passivity implies $f\leq g$ (as in Lemma 2.6). Hence by (ii)

$Q\geq P(-\infty, \mu)$, and $Q\geq P(-\infty, \mu]$ if $f(\mu)>0$. We conclude that either $Q=P(-\infty, \mu)$ (if $f(\mu)=0$) or $Q=P(-\infty, \mu]$ (if $f(\mu)>0$).

If we define Δ as the direct sum of the usual modular operator associated to the cyclic and separating vector Ω of the von Neumann algebra $Q\mathscr{A}Q$ on $Q\mathscr{H}$ and the zero-operator on $(1-Q)\mathscr{H}$ (as is done e.g. in [12, p. 284]), then it is still true that $\Delta=f(H)$. Notice also that if ω is not faithful ($\mu<\infty$) then H is bounded below (by $-\mu$).

§ 3. Equilibrium and 3-Passivity

If a faithful invariant state ω of a W^*-dynamical system is known to be strongly 2-spectrally passive, so that $\Delta=f(H)$ by Theorem 2.8, then in order to decide whether ω is β-KMS it is of course sufficient to check if $f(\lambda)=e^{-\beta\lambda}$ for P-almost all λ $(0\leq\beta<+\infty)$.

Since the KMS condition is known to follow from *complete* spectral passivity [7, Theorem 1.3], it is natural, as a first step, to study the implications of 3-spectral passivity on the properties of f. Since our final result is obtained under the assumption that $\sigma(H)=\boldsymbol{R}$, we can restrict our attention at once to the case where H is unbounded. This is convenient technically because it implies $0<f, g<+\infty$.

Lemma 3.1. *Let f and g be as in Definition 2.3, and suppose H is unbounded. Then the following are equivalent* :

(i) *ω is strongly 3-spectrally passive.*

(ii) *For all λ, λ' in $\boldsymbol{R}$,*
$$f(\lambda)f(\lambda')\leq g(\lambda+\lambda').$$

(iii) *For all λ, λ' in $\boldsymbol{R}$,*
$$f(\lambda+\lambda')\leq g(\lambda)g(\lambda').$$

Proof. Straightforward, and left to the reader (compare with Lemma 2.6). □

Next we wish to turn the inequalities (ii) and (iii) above into equalities. To this end we make the assumption that $\sigma(H)=\boldsymbol{R}$, which implies that $g(\lambda+)\leq f(\lambda-)$ for all λ in $\boldsymbol{R}$ (Lemma 2.5 (v)). Let D be the (finite or countable) set of those points in $\boldsymbol{R}$ where at least one of the functions f and g is discontinuous. If none of the points λ, λ', $\lambda+\lambda'$ belongs to D, one has, under the assumption of strong 3-spectral passivity (Lemma 3.1 and 2.5 (v)):

$$f(\lambda)f(\lambda') \leq g(\lambda+\lambda') \leq f(\lambda+\lambda')$$

and

$$f(\lambda+\lambda') \leq g(\lambda)g(\lambda') \leq f(\lambda)f(\lambda')$$

i.e.

$$f(\lambda)f(\lambda') = f(\lambda+\lambda') \tag{10}$$

provided $\lambda \notin D$, $\lambda' \notin D$, $\lambda+\lambda' \notin D$.

Let E be the set of all real numbers λ for which there exists a number $\phi(\lambda)$ such that $f(\lambda+\mu)f(\mu)^{-1}=\phi(\lambda)$ for all μ outside some finite or countable subset of $\boldsymbol{R}$. Clearly E is an additive semigroup and

$$\phi(\lambda)\phi(\lambda') = \phi(\lambda+\lambda')$$

whenever $\lambda, \lambda' \in E$. Moreover the complement of E is contained in D by (10), and $\phi(\lambda)=f(\lambda)$ if $\lambda \notin D$. It follows easily that $E=\boldsymbol{R}$ and that ϕ is Borel measurable (because f is). Consequently $\phi(\lambda)=e^{-\beta\lambda}$ for all λ in $\boldsymbol{R}$, where $\beta \geq 0$.

It remains to be shown that $f(\lambda)=e^{-\beta\lambda}$ for *all* λ in $\boldsymbol{R}$. Suppose $f(\lambda)>e^{-\beta\lambda}$ for some λ. Then there exists a positive number δ such that $e^{-\beta\mu}<f(\lambda)$ whenever $\lambda-\delta \leq \mu \leq \lambda$, and hence $e^{-\beta\mu}<f(\mu)$ since f decreases. But this contradicts the fact that $f(\mu)=\phi(\mu)=e^{-\beta\mu}$ for all but at most countably many values of μ. The possibility that $f(\lambda)< e^{-\beta\lambda}$ is ruled out in a similar way. As $\sigma(H)=sp(\alpha)$ [7] we have shown:

Theorem 3.2. *If $sp(\alpha)=\boldsymbol{R}$, then any faithful, normal, strongly 3-spectrally passive state of $(\mathscr{A}, \alpha)$ is β-KMS with respect to α for some β, $0 \leq \beta < +\infty$.* □

Remark 3.3. After this work was finished it was pointed out to the author by H. Daniëls that the result of Theorem 3.2 still holds if the assumption of strong 3-spectral passivity is replaced by 3-spectral passivity. This is seen by combining [6b, Theorem 1.23] with [7, Theorem 3.3]. In fact one can also easily adapt the above proof to accomodate this more general situation. It is sufficient to redefine f and g as

$$f(\lambda)=\sup\{\|x^*\Omega\|^2\,\|x\Omega\|^{-2} \mid x \in R(\lambda,+\infty),\ x\neq 0\}$$

and

$$g(\lambda)=\inf\{\|x^*\Omega\|^2\,\|x\Omega\|^{-2} \mid x \in R(-\infty,\lambda),\ x\neq 0\}.$$

It is still true that $\lambda \in \sigma(H)$ implies $g(\lambda+) \leq f(\lambda-)$, and exactly as before one shows that $f(\lambda) = e^{-\beta\lambda}$ for some β. It follows that ω is KMS by [7, Theorem 1.1].

These results should be compared with the counterexample 4.9 in [7] of an n-spectrally passive state that is not $(n+1)$-spectrally passive. Let us finally point out that in classical mechanics (under suitable regularity assumptions) 2-passivity is already sufficient to ensure equilibrium [8, Theorems 1 and 2 ; 6b, Theorem 3.19].

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