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Extending Derivations

By

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§ 1. Introduction

If τ is the action of a compact abelian group G on a C^*-algebra $\mathscr{A}$, and δ is a derivation on $\mathscr{A}$ commuting with τ, then there has been much interest recently in the problem of deciding when δ is a generator, under some conditions on the C^*-dynamical system ($\mathscr{A}$, G, τ) and on the restriction of the derivation to the fixed point algebra, see e.g. [1-6, 8-10, 12]. Here we consider the problem of deciding when a given derivation on the fixed point algebra extends to a derivation on $\mathscr{A}$ which commutes with the group action. In particular, let $\mathscr{D}$ be a *-subalgebra of $\mathscr{A}$, (assumed unital), which contains a unitary $u(\gamma)$ in each spectral subspace $\mathscr{A}^{\tau}(\gamma)$, $\gamma\in\hat{G}$, and such that δ_0 is a densely defined derivation on $\mathscr{D}\cap\mathscr{A}^{\tau}$. Suppose that there exists a family of traces on $\mathscr{A}$ which separate its centre. Then we show that δ_0 extends to a derivation on $\mathscr{D}$ if and only if both of the following conditions hold:

(1. 1) $u(\gamma)\delta_0(u(\gamma)^*(\cdot)u(\gamma))u(\gamma)^*-\delta_0(\cdot)$ is a bounded inner derivation on $\mathscr{A}$ for all γ in $\hat{G}$.

(1. 2) $\varphi[\delta_0(u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)]=0$ for any trace φ on $\mathscr{A}$, γ_1, $\gamma_2\in\hat{G}$.

Our technique is to produce a cohomological obstruction to extending δ_0 and to show that this obstruction vanishes in the circumstances of the preceding paragraph. We note further that in

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fact the extension problem for δ_0 is equivalent to a problem on group extensions.

§ 2. Preliminaries

If G is a compact abelian group, an action τ of G on a C^*-algebra $\mathscr{A}$ will be a homomorphism τ from G into $\mathrm{Aut}(\mathscr{A})$, the group of all $*$-automorphisms of $\mathscr{A}$, which is strongly continuous in the sense that $g\to\tau(g)(x)$ is norm continuous for each x in $\mathscr{A}$. If $\gamma\in\Gamma$, the dual group of G, the spectral subspace corresponding to γ is

$$\mathscr{A}^\tau(\gamma)=\{x\in \mathrm{A} : \tau(g)(x)=\langle\gamma, g\rangle x, g\in G\}.$$

We write $\mathscr{A}^\tau$ for the fixed point algebra. Then

$$\mathscr{A}^\tau(\gamma_1)\mathscr{A}^\tau(\gamma_2)\subseteq\mathscr{A}^\tau(\gamma_1+\gamma_2),\ \mathscr{A}^\tau(\gamma)^*=\mathscr{A}^\tau(-\gamma).$$

If $\mathscr{A}$ is a C^*-algebra, always assumed unital, $\mathscr{Z}(\mathscr{A})$ will denote its centre and $\mathscr{A}_h$ its hermitian elements. A derivation on $\mathscr{A}$ will be a linear map δ defined on a dense $*$-subalgebra $\mathscr{D}$ of $\mathscr{A}$, containing the unit of $\mathscr{A}$, into $\mathscr{A}$ satisfying

$$\delta(xy)=\delta(x)y+x\delta(y)$$
$$\delta(x^*)=\delta(x)^*,\quad x, y\in\mathscr{A}.$$

If φ is a trace on $\mathscr{A}$, and u, v unitaries in $\mathscr{D}$, then [13]:

$$\varphi[\delta(uv)v^*u^*]=\varphi[\delta(u)u^*]+\varphi[\delta(v)v^*]. \tag{2.1}$$

Moreover

$$\delta(u^*)=-u^*\delta(u)u^*. \tag{2.2}$$

§ 3. Extending Derivations

Let $(\mathscr{A}, G, \tau)$ be a C^*-dynamical system where τ is an action of a compact abelian group G on a unital C^*-algebra $\mathscr{A}$, and δ_0 a derivation on $\mathscr{A}^\tau$ with domain $\mathscr{D}_0$.

Suppose that there exist unitaries $u(\gamma)$ in $\mathscr{A}^\tau(\gamma)$ for each γ in Γ such that

$$u(0)=1 \tag{3.1}$$

$$u(\gamma_1)u(\gamma_2)u(\gamma_1+\gamma_2)^*\in\mathscr{D}_0,\quad \gamma_1, \gamma_2\in\Gamma \tag{3.2}$$

$$u(\gamma)\mathscr{D}_0u(\gamma)^*\subseteq\mathscr{D}_0,\quad \gamma\in\Gamma. \tag{3.3}$$

Then

$$u(\gamma)u(-\gamma)\in\mathscr{D}_0$$

and

$$u(\gamma)^*\mathscr{D}_0u(\gamma)=u(-\gamma)[u(\gamma)u(-\gamma)]^*\mathscr{D}_0[u(\gamma)u(-\gamma)]u(-\gamma)^* \subseteq\mathscr{D}_0$$

and so

$$u(\gamma)\mathscr{D}_0u(\gamma)^*=\mathscr{D}_0=u(\gamma)^*\mathscr{D}_0u(\gamma).$$

Moreover

$$u(\gamma)^*=[u(-\gamma)u(\gamma)]^*u(-\gamma).$$

Thus

$$\mathscr{D}=\text{lin}\{\mathscr{D}_0u(\gamma)\ :\ \gamma\in\hat{G}\} \tag{3.4}$$

is a G-invariant $*$-subalgebra of $\mathscr{A}$, and $\mathscr{D}\cap\mathscr{A}^\tau=\mathscr{D}_0$.

Consider the following condition on the family $\{u(\gamma)\ :\ \gamma\in\Gamma\}$:

Hypothesis 3.1. *For each γ in Γ, there exists a self adjoint $b(\gamma)$ in $\mathscr{A}^\tau$ such that*

$$u(\gamma)\delta_0(u(\gamma)^*xu(\gamma))u(\gamma)^*-\delta_0(x)=i[b(\gamma),\ x],\quad x\in\mathscr{D}_0 \tag{3.5}$$

$$b(0)=0. \tag{3.6}$$

The following shows that this condition is essentially independent of the choice of unitaries in the spectral subspaces. Let $\{v(\gamma)\ :\ \gamma\in\Gamma\}$ be another family of unitaries in $\mathscr{A}^\tau(\gamma)$ with $c(\gamma)=u(\gamma)v(\gamma)^*\in\mathscr{D}_0$, $c(0)=1$, so that v satisfies (3.1-3). Then $\mathscr{D}=\text{lin}\{\mathscr{D}_0v(\gamma)\ :\ \gamma\in\Gamma\}$. (Equivalently, for each γ in Γ, let $v(\gamma)$ be a unitary in $\mathscr{D}\cap\mathscr{A}^\tau(\gamma)$, with $v(0)=1$).

Lemma 3.2. *Let $u(\cdot)$ and $v(\cdot)$ be as above. Then Hypothesis* 3.1 *holds for $\{u(\gamma)\ :\ \gamma\in\Gamma\}$ if and only if it holds for $\{v(\gamma)\ :\ \gamma\in\Gamma\}$.*

Proof. For $x\in\mathscr{D}_0$:

$$\begin{aligned}
&u(\gamma)\delta_0[u(\gamma)^*xu(\gamma)]u(\gamma)^*-\delta_0(x)\\
&\quad=c(\gamma)v(\gamma)\delta_0[v(\gamma)^*c(\gamma)^*v(\gamma)v(\gamma)^*xv(\gamma)v(\gamma)^*c(\gamma)v(\gamma)]v(\gamma)^*c(\gamma)^*\\
&\qquad-\delta_0(x)\\
&\quad=c(\gamma)v(\gamma)\,\{\delta_0[v(\gamma)^*c(\gamma)^*v(\gamma)]v(\gamma)^*xv(\gamma)v(\gamma)^*c(\gamma)v(\gamma)\\
&\qquad+v(\gamma)^*c(\gamma)^*v(\gamma)\delta_0[v(\gamma)^*xv(\gamma)]v(\gamma)^*c(\gamma)v(\gamma)\\
&\qquad+v(\gamma)^*c(\gamma)^*v(\gamma)v(\gamma)^*xv(\gamma)\delta_0[v(\gamma)^*c(\gamma)v(\gamma)]\}\,v(\gamma)^*c(\gamma)^*\\
&\qquad-\delta_0(x)
\end{aligned}$$

$$=v(\gamma)\delta_0[v(\gamma)^*xv(\gamma)]v(\gamma)^*-\delta_0(x)+[d(\gamma),\ x]$$

if

$$d(\gamma)=c(\gamma)v(\gamma)\delta_0[v(\gamma)^*c(\gamma)^*v(\gamma)]v(\gamma)^*$$
$$=-v(\gamma)\delta_0[v(\gamma)^*c(\gamma)v(\gamma)]v(\gamma)^*c(\gamma)^*\in\mathscr{A}^\tau,$$

where $d(0)=0$.

Under the conditions of Hypothesis 3.1, define $\Phi=\Phi_u:\Gamma^2\to\mathscr{A}^\tau$ by

$$\Phi(\gamma_1,\ \gamma_2)=b(\gamma_1+\gamma_2)-b(\gamma_1)-u(\gamma_1)b(\gamma_2)u(\gamma_1)^* + iu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^* \tag{3.7}$$

for $\gamma_1,\ \gamma_2\in\Gamma$.

Lemma 3.3. *$\Phi(\gamma_1,\gamma_2)$ is self adjoint and lies in the centre $\mathscr{Z}(\mathscr{A}^\tau)$ of $\mathscr{A}^\tau$, for all γ_1, γ_2 in Γ.*

Proof. That $\Phi(\gamma_1,\ \gamma_2)$ is self adjoint follows easily from (2.2). Then for $x\in\mathscr{D}_0$, using (3.5):

$$\begin{aligned}i[\Phi(\gamma_1,\ \gamma_2),\ x]&=u(\gamma_1+\gamma_2)\delta_0(u(\gamma_1+\gamma_2)^*xu(\gamma_1+\gamma_2))u(\gamma_1+\gamma_2)^*-\delta_0(x)\\&\quad-u(\gamma_1)\delta_0(u(\gamma_1)^*xu(\gamma_1))u(\gamma_1)^*+\delta_0(x)\\&\quad-u(\gamma_1)u(\gamma_2)\delta_0(u(\gamma_2)^*u(\gamma_1)^*xu(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*\\&\quad+u(\gamma_1)\delta_0(u(\gamma_1)^*xu(\gamma_1))u(\gamma_1)^*\\&\quad-u(\gamma_1+\gamma_2)\delta_0(u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*x\\&\quad+xu(\gamma_1+\gamma_2)\delta_0(u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*\\&=u(\gamma_1+\gamma_2)\delta_0\{[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]\\&\qquad\cdot[u(\gamma_2)^*u(\gamma_1)^*xu(\gamma_1)u(\gamma_2)]\\&\qquad\cdot[u(\gamma_2)^*u(\gamma_1)^*u(\gamma_1+\gamma_2)]\}u(\gamma_1+\gamma_2)^*\\&\quad-u(\gamma_1)u(\gamma_2)\delta_0[u(\gamma_2)^*u(\gamma_1)^*xu(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^*\\&\quad-u(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^*x\\&\quad+xu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^*\\&=0.\end{aligned}$$

Lemma 3.4. *Φ is a u-twisted 2-cocycle, i.e.,*

$$\Phi(\gamma_1+\gamma_2,\ \gamma_3)+\Phi(\gamma_1,\ \gamma_2)=\Phi(\gamma_1,\ \gamma_2+\gamma_3)+u(\gamma_1)\Phi(\gamma_2,\ \gamma_3)u(\gamma_1)^* \tag{3.8}$$

for all γ_1, γ_2, γ_3 in Γ.

Proof.

$$\Phi(\gamma_1+\gamma_2,\ \gamma_3)+\Phi(\gamma_1,\ \gamma_2)-\Phi(\gamma_1,\ \gamma_2+\gamma_3)-u(\gamma_1)\Phi(\gamma_2,\ \gamma_3)u(\gamma_1)^*$$

$$
\begin{aligned}
&= b(\gamma_1+\gamma_2+\gamma_3) - b(\gamma_1+\gamma_2) - u(\gamma_1+\gamma_2) b(\gamma_3) u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1+\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad + b(\gamma_1+\gamma_2) - b(\gamma_1) - u(\gamma_1) b(\gamma_2) u(\gamma_1)^* \\
&\quad + iu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^* u(\gamma_1) u(\gamma_2)] u(\gamma_2)^* u(\gamma_1)^* \\
&\quad - b(\gamma_1+\gamma_2+\gamma_3) + b(\gamma_1) + u(\gamma_1) b(\gamma_2+\gamma_3) u(\gamma_1)^* \\
&\quad - iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1) u(\gamma_2+\gamma_3)] u(\gamma_2+\gamma_3)^* u(\gamma_1)^* \\
&\quad - u(\gamma_1) b(\gamma_2+\gamma_3) u(\gamma_1)^* + u(\gamma_1) b(\gamma_2) u(\gamma_1)^* \\
&\quad + u(\gamma_1) u(\gamma_2) b(\gamma_3) u(\gamma_2)^* u(\gamma_1)^* \\
&\quad - iu(\gamma_1) u(\gamma_2+\gamma_3)\delta_0[u(\gamma_2+\gamma_3)^* u(\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* \\
&= u(\gamma_1) u(\gamma_2) [b(\gamma_3),\ u(\gamma_2)^* u(\gamma_1)^* u(\gamma_1+\gamma_2)] u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1+\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^* u(\gamma_1) u(\gamma_2)] u(\gamma_2)^* u(\gamma_1)^* \\
&\quad - iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1) u(\gamma_2+\gamma_3)] u(\gamma_2+\gamma_3)^* u(\gamma_1)^* \\
&\quad - iu(\gamma_1) u(\gamma_2+\gamma_3)\delta_0[u(\gamma_2+\gamma_3)^* u(\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* \\
&= iu(\gamma_1) u(\gamma_2)\delta_0[u(\gamma_2)^* u(\gamma_1)^* u(\gamma_1+\gamma_2)] u(\gamma_1+\gamma_2)^* \\
&\quad - iu(\gamma_1) u(\gamma_2) u(\gamma_3)\delta_0[u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* u(\gamma_1+\gamma_2) u(\gamma_3)] \\
&\qquad \cdot u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1+\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^* u(\gamma_1) u(\gamma_2)] u(\gamma_2)^* u(\gamma_1)^* \\
&\quad - iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1) u(\gamma_2+\gamma_3)] u(\gamma_2+\gamma_3)^* u(\gamma_1)^* \\
&\quad - iu(\gamma_1) u(\gamma_2+\gamma_3)\delta_0[u(\gamma_2+\gamma_3)^* u(\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* \\
&= -iu(\gamma_1)u(\gamma_2)u(\gamma_3)\delta_0[u(\gamma_3)^*u(\gamma_2)^*u(\gamma_1)^*u(\gamma_1+\gamma_2)u(\gamma_3)]u(\gamma_3)^*u(\gamma_1+\gamma_2)^* \\
&\quad + iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1+\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad - iu(\gamma_1+\gamma_2+\gamma_3)\delta_0\{[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1) u(\gamma_2+\gamma_3)] \\
&\qquad \cdot [u(\gamma_2+\gamma_3)^* u(\gamma_2) u(\gamma_3)]\} u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* \\
&= iu(\gamma_1+\gamma_2+\gamma_3)\delta_0[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1+\gamma_2) u(\gamma_3)] u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&\quad - iu(\gamma_1+\gamma_2+\gamma_3)\delta_0\{[u(\gamma_1+\gamma_2+\gamma_3)^* u(\gamma_1) u(\gamma_2) u(\gamma_3)] \\
&\qquad \cdot [u(\gamma_3)^* u(\gamma_2)^* u(\gamma_1)^* u(\gamma_1+\gamma_2) u(\gamma_3)]\} u(\gamma_3)^* u(\gamma_1+\gamma_2)^* \\
&= 0.
\end{aligned}
$$

Lemma 3.5. *Suppose*

$$(3.9)\qquad \Phi(\gamma_1, \gamma_2) = z(\gamma_1+\gamma_2) - z(\gamma_1) - u(\gamma_1) z(\gamma_2) u(\gamma_1)^*, \quad \gamma_1, \gamma_2 \in \Gamma$$

for some self adjoint family $\{z(\gamma) : \gamma \in \Gamma\}$ *in* $\mathscr{L}(\mathscr{A}^\tau)$, *with* $z(0)=0$. *Let* $b_0(\gamma) = b(\gamma) - z(\gamma)$, $\gamma \in \Gamma$. *Then*

$$(3.10)\qquad \delta(xu(\gamma)) = \delta_0(x) u(\gamma) - ixb_0(\gamma) u(\gamma),\ \gamma \in \Gamma,\ x \in \mathscr{D}_0$$

defines a derivation δ *on* $\mathscr{D}$ *which commutes with* G, *and extends* δ_0.

Proof. The map $b_0 : \Gamma \to \mathscr{A}_h^\tau$ satisfies

(3.11) $u(\gamma)\delta_0(u(\gamma)^* x u(\gamma))u(\gamma)^* - \delta_0(x) = i[b_0(\gamma), x], \ x \in \mathscr{D}_0$

(3.12) $b_0(\gamma_1+\gamma_2) = b_0(\gamma_1) + u(\gamma_1)b_0(\gamma_2)u(\gamma_1)^*$
$$- iu(\gamma_1+\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^* u(\gamma_1)u(\gamma_2)]u(\gamma_2)^* u(\gamma_1)^*$$

(3.13) $$b_0(0) = 0.$$

Define δ by (3.10). Now by (3.12):

(3.14) $0 = b_0(\gamma) + u(\gamma)b_0(-\gamma)u(\gamma)^* - i\delta_0(u(\gamma)u(-\gamma))u(-\gamma)^* u(\gamma)^*.$

Then for $x \in \mathscr{D}_0$, $\gamma \in \Gamma$:

$$\begin{aligned}
\delta[xu(\gamma)]^* &= \delta[(u(\gamma)^* x^* u(-\gamma)^*)u(-\gamma)] \\
&= \delta_0[u(\gamma)^* x^* u(-\gamma)^*]u(-\gamma) - iu(\gamma)^* x^* u(-\gamma)^* b_0(-\gamma)u(-\gamma) \\
&= \delta_0(u(\gamma)^* x^* u(-\gamma)^*)u(-\gamma) \\
&\quad - iu(\gamma)^* x^* u(-\gamma)^*[-u(\gamma)^* b_0(\gamma)u(\gamma) \\
&\quad + iu(\gamma)^*\delta_0(u(\gamma)u(-\gamma))u(-\gamma)^*]u(-\gamma) \\
&= u(\gamma)^*\delta_0(x^* u(-\gamma)^* u(\gamma)^*)u(\gamma)u(-\gamma) \\
&\quad + iu(\gamma)^* b_0(\gamma)x^* - iu(\gamma)^* x^* u(-\gamma)^* u(\gamma)^* b_0(\gamma)u(\gamma)u(-\gamma) \\
&\quad + iu(\gamma)^* x^* u(-\gamma)^* u(\gamma)^* b_0(\gamma)u(\gamma)u(-\gamma) \\
&\quad + u(\gamma)^* x^* u(-\gamma)^* u(\gamma)^*\delta_0(u(\gamma)u(-\gamma)) \\
&= u(\gamma)^*\delta_0(x^*) + iu(\gamma)^* b_0(\gamma)x^* \\
&= [\delta(xu(\gamma))]^*.
\end{aligned}$$

Hence δ is a $*$-map. To show that δ is a derivation, consider $x_1, x_2 \in \mathscr{D}_0$, $\gamma_1, \gamma_2 \in \Gamma$:

$$\begin{aligned}
&\delta(x_1u(\gamma_1)x_2u(\gamma_2)) - \delta(x_1u(\gamma_1))x_2u(\gamma_2) - x_1u(\gamma_1)\delta(x_2u(\gamma_2)) \\
&= \delta(x_1u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^* u(\gamma_1+\gamma_2)) - \delta(x_1u(\gamma_1))x_2u(\gamma_2) \\
&\quad - x_1u(\gamma_1)\delta(x_2u(\gamma_2)) \\
&= \delta_0[x_1u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2) \\
&\quad - ix_1u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^* b_0(\gamma_1+\gamma_2)u(\gamma_1+\gamma_2) \\
&\quad - \delta_0(x_1)u(\gamma_1)x_2u(\gamma_2) + ix_1b_0(\gamma_1)u(\gamma_1)x_2u(\gamma_2) \\
&\quad - x_1u(\gamma_1)\delta_0(x_2)u(\gamma_2) + ix_1u(\gamma_1)x_2b_0(\gamma_2)u(\gamma_2) \\
&= x_1\{\delta_0[u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2) \\
&\quad - iu(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^* b_0(\gamma_1)u(\gamma_1+\gamma_2) \\
&\quad - iu(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^* u(\gamma_1)b_0(\gamma_2)u(\gamma_1)^* u(\gamma_1+\gamma_2) \\
&\quad - u(\gamma_1)x_2u(\gamma_2)\delta_0(u(\gamma_1+\gamma_2)^* u(\gamma_1)u(\gamma_2))u(\gamma_1)^* u(\gamma_2)^* u(\gamma_1+\gamma_2) \\
&\quad + ib_0(\gamma_1)u(\gamma_1)x_2u(\gamma_2) - u(\gamma_1)\delta_0(x_2)u(\gamma_2) \\
&\quad + iu(\gamma_1)x_2b_0(\gamma_2)u(\gamma_2)\} \\
&= x_1\{\delta_0[u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2)
\end{aligned}$$

$$
\begin{aligned}
&+i[b_0(\gamma_1),\ u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2)\\
&+iu(\gamma_1)x_2[b_0(\gamma_2),\ u(\gamma_2)u(\gamma_1+\gamma_2)^*u(\gamma_1)]u(\gamma_1)^*u(\gamma_1+\gamma_2)\\
&-u(\gamma_1)x_2u(\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1+\gamma_2)\\
&-u(\gamma_1)\delta_0(x_2)u(\gamma_2)\}\\
=&x_1\{\delta_0[u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2)\\
&+u(\gamma_1)\delta_0[x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*u(\gamma_1)]u(\gamma_1)^*u(\gamma_1+\gamma_2)\\
&-\delta_0[u(\gamma_1)x_2u(\gamma_2)u(\gamma_1+\gamma_2)^*]u(\gamma_1+\gamma_2)\\
&+u(\gamma_1)x_2u(\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^*u(\gamma_1+\gamma_2)\\
&-u(\gamma_1)x_2\delta_0[u(\gamma_2)u(\gamma_1+\gamma_2)^*u(\gamma_1)]u(\gamma_1)^*u(\gamma_1+\gamma_2)\\
&-u(\gamma_1)x_2u(\gamma_2)\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1+\gamma_2)\\
&-u(\gamma_1)\delta_0(x_2)u(\gamma_2)\}\\
=&0.
\end{aligned}
$$

The following lemma shows that at least all symmetric Φ which we have considered can be decomposed.

Lemma 3.6. *Let D be a divisible abelian group and $\Phi: \Gamma\times\Gamma\to D$ a symmetric 2-cocycle, i.e.,*

(3.15) $\Phi(\gamma_1+\gamma_2,\ \gamma_3)+\Phi(\gamma_1,\ \gamma_2)=\Phi(\gamma_1,\ \gamma_2+\gamma_3)+\Phi(\gamma_2,\ \gamma_3),\ \gamma_i\in\Gamma.$

(3.16) $\Phi(\gamma_1,\ 0)=0=\Phi(0,\ \gamma_1),\quad \gamma_1\in\Gamma.$

(3.17) $\Phi(\gamma_1, \gamma_2)=\Phi(\gamma_2,\ \gamma_1),\quad \gamma_1,\ \gamma_2\in\Gamma.$

Then there exists $z:\Gamma\to D$ such that

(3.18) $\Phi(\gamma_1,\ \gamma_2)=z(\gamma_1+\gamma_2)-z(\gamma_1)-z(\gamma_2),\quad \gamma_1,\ \gamma_2\in\Gamma.$

(3.19) $z(0)=0.$

Proof. Define a new group $\tilde{\Gamma}=\{(\gamma, b):\gamma\in\Gamma, b\in D\}$ with addition

$$(\gamma_1,\ b_1)+(\gamma_2,\ b_2)=(\gamma_1+\gamma_2,\ b_1+b_2+\Phi(\gamma_1,\ \gamma_2)).$$

and inverse: $-(\gamma_1,\ b_1)=(-\gamma_1,\ -b_1-\Phi(\gamma_1,\ -\gamma_1))$. Then $\tilde{\Gamma}$ is abelian because Φ is symmetric.

Now [7] if K is a subgroup of a (discrete) abelian group H, then any homomorphism of K into a divisible abelian group D, extends to H. Taking $H=\tilde{\Gamma}$, $K=D$, this means that there exists a homomorphism $\eta:\tilde{\Gamma}\to D$ extending the identity map from D into D. Taking $z(\gamma)=-\eta(\gamma,\ 0)$, we have the result.

Remark 3.7. In order to measure the obstruction to Φ (defined by (3.7)) being a coboundary, it will be useful to look at

$$\varphi\{\delta_0[u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2)]u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)\}$$

if φ is a trace on $\mathscr{A}$. It is useful to note that this expression does not depend on the choice of family $\{u(\gamma):\gamma\in\Gamma\}$. Thus let $\{u(\gamma):\gamma\in\Gamma\}$ be a family of unitaries satisfying (3.1-3), such that Hypothesis 3.1 holds. Let $v(\gamma)$ be another family of unitaries in $\mathscr{D}\cap\mathscr{A}^\tau(\gamma)$. Then if $d(\gamma)=v(\gamma)^*u(\gamma)\in\mathscr{D}_0$:

$$\begin{aligned}&u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2)\\&\quad=d(\gamma_1)^*[v(\gamma_1)^*d(\gamma_2)^*v(\gamma_1)][v(\gamma_1)^*v(\gamma_2)^*v(\gamma_1)v(\gamma_2)]\\&\qquad\cdot[v(\gamma_2)^*d(\gamma_1)v(\gamma_2)]d(\gamma_2).\end{aligned}$$

Hence by (2.1),

$$\begin{aligned}&\varphi[\delta_0(u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)]\\&\quad=\varphi[\delta_0(d(\gamma_1)^*)d(\gamma_1)]+\varphi[\delta_0(v(\gamma_1)^*d(\gamma_2)^*v(\gamma_1))v(\gamma_1)^*d(\gamma_2)v(\gamma_1))]\\&\qquad+\varphi[\delta_0(v(\gamma_1)^*v(\gamma_2)^*v(\gamma_1)v(\gamma_2))v(\gamma_2)^*v(\gamma_1)^*v(\gamma_2)v(\gamma_1)]\\&\qquad+\varphi[\delta_0(v(\gamma_2)^*d(\gamma_1)v(\gamma_2))v(\gamma_2)^*d(\gamma_1)^*v(\gamma_2)]\\&\qquad+\varphi[\delta_0(d(\gamma_2))d(\gamma_2)^*]\\&\quad=\varphi[\delta_0(d(\gamma_1)^*)d(\gamma_1)]+\varphi[v(\gamma_1)^*\delta_0(d(\gamma_2)^*)v(\gamma_1)v(\gamma_1)^*d(\gamma_2)v(\gamma_1)]\\&\qquad+\varphi\{v(\gamma_1)^*i[e(\gamma_1),\ d(\gamma_2)^*]v(\gamma_1)v(\gamma_1)^*d(\gamma_2)v(\gamma_1)\}\\&\qquad+\varphi[\delta_0(v(\gamma_1)^*v(\gamma_2)^*v(\gamma_1)v(\gamma_2))v(\gamma_2)^*v(\gamma_1)^*v(\gamma_2)v(\gamma_1)]\\&\qquad+\varphi[v(\gamma_2)^*\delta_0(d(\gamma_1))v(\gamma_2)v(\gamma_2)^*d(\gamma_1)^*v(\gamma_2)]\\&\qquad+\varphi[v(\gamma_2)^*i[e(\gamma_2),\ d(\gamma_1)]v(\gamma_2)v(\gamma_2)^*d(\gamma_1)^*v(\gamma_2)]\\&\qquad+\varphi[\delta_0(d(\gamma_2))d(\gamma_2)^*]\\&\quad=\varphi[\delta_0(v(\gamma_1)^*v(\gamma_2)^*v(\gamma_1)v(\gamma_2))v(\gamma_2)^*v(\gamma_1)^*v(\gamma_2)v(\gamma_1)]\end{aligned}$$

where we have used

$$\delta_0(v(\gamma)^*xv(\gamma))-v(\gamma)^*\delta_0(x)v(\gamma)=v(\gamma)^*i[e(\gamma),\ x]v(\gamma)$$

for $x\in\mathscr{D}_0$ and some family $e(\gamma)$ in $\mathscr{A}_h^\tau$.

Theorem 3.8. *Let τ be an action of a compact abelian group on a C^*-algebra $\mathscr{A}$, δ_0 be a derivation on $\mathscr{A}^\tau$ with domain $\mathscr{D}_0$, and suppose that there exists a family $\{u(\gamma):\gamma\in\Gamma\}$ of unitaries in $\mathscr{A}^\tau(\gamma)$ satisfying* (3.1-3). *Suppose that*

(3.20) *there exists a family of traces on $\mathscr{A}$, which separates $\mathscr{Z}(\mathscr{A}^\tau)$.*

Then δ_0 *extends to a derivation on*

$$\mathscr{D}=\mathrm{lin}\{\mathscr{D}_0 u(\gamma)\ :\ \gamma\in\Gamma\}$$

which commutes with G, *if and only if both the following hold*:

(3.21) *There exists a family* $\{b(\gamma)\ :\ \gamma\in\Gamma\}$ *of self adjoint elements of* $\mathscr{A}^{\tau}$ *such that*

$$u(\gamma)\delta_0(u(\gamma)^*xu(\gamma))u(\gamma)^*-\delta_0(x)=i[b(\gamma),\ x]$$

for all x *in* $\mathscr{D}_0$, $\gamma\in\Gamma$.

(3.22) $\varphi[\delta_0(u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)]=0$

for any trace φ *on* $\mathscr{A}$, $\gamma_1,\gamma_2\in\Gamma$.

If δ_0 *is closable, then so is* δ.

Proof. We claim that $\mathscr{Z}(\mathscr{A}^{\tau})\subseteq\mathscr{Z}(\mathscr{A})$ (cf. [2]). Let u be any unitary in $\mathscr{A}^{\tau}(\gamma)$ for some γ in Γ. Then $uzu^*\in\mathscr{Z}(\mathscr{A}^{\tau})$ for any $z\in\mathscr{Z}(\mathscr{A}^{\tau})$. If φ is a trace on $\mathscr{A}$, then $\varphi(z)=\varphi(uzu^*)$ and so by (3.20), $z=uzu^*$, and hence $z\in\mathscr{Z}(\mathscr{A})$.

We now prove necessity of conditions (3.21) and (3.22). If δ extends δ_0, then for $x\in\mathscr{D}_0$, $\gamma\in\Gamma$:

$$\begin{aligned}&u(\gamma)\delta_0(u(\gamma)^*xu(\gamma))u(\gamma)^*-\delta_0(x)\\&\quad=u(\gamma)[\delta(u(\gamma)^*)xu(\gamma)+u(\gamma)^*\delta_0(x)u(\gamma)+u(\gamma)^*x\delta(u(\gamma))]u(\gamma)^*\\&\qquad-\delta_0(x)\\&\quad=[-\delta(u(\gamma))u(\gamma)^*,\ x]\end{aligned}$$

noting that by (2.2)

$$\delta(u(\gamma)^*)=-u(\gamma)^*\delta(u(\gamma))u(\gamma)^*$$

and that $\delta(u(\gamma))u(\gamma)^*\in\mathscr{A}^{\tau}$ if δ commutes with τ. If φ is any trace on $\mathscr{A}$, then by (2.1), $u\to\varphi[\delta(u)u^*]$ is a homomorphism on the unitary part of $\mathscr{D}$. Hence (3.22) holds. Conversely, suppose (3.21) and (3.22) hold. Define Φ by 3.7. Then $u(\gamma_1)\Phi(\gamma_2,\gamma_3)u(\gamma_1)^*=\Phi(\gamma_2,\gamma_3)$, $\gamma_i\in\Gamma$, because $\Phi(\gamma_2,\gamma_3)\in\mathscr{Z}(\mathscr{A}^{\tau})\subset\mathscr{Z}(\mathscr{A})$ by Lemma 3.3 and the above remark. Let φ be any trace on $\mathscr{A}$. Then

$$\begin{aligned}\varphi(\Phi(\gamma_1,\gamma_2))&=\varphi(b(\gamma_1+\gamma_2))-\varphi(b(\gamma_1))-\varphi[u(\gamma_1)b(\gamma_2)u(\gamma_1)^*]\\&\quad-i\varphi[u(\gamma_1+\gamma_2)\delta_0(u(\gamma_1+\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*]\\&=\varphi(b(\gamma_1+\gamma_2))-\varphi(b(\gamma_1))-\varphi(b(\gamma_2))\end{aligned}$$

$$
\begin{aligned}
&-i\varphi\{\delta_0[u(\gamma_1+\gamma_2)^*u(\gamma_2)u(\gamma_1)u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2)] \\
&\quad \cdot u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1+\gamma_2)\} \\
&=\varphi(b(\gamma_1+\gamma_2))-\varphi(b(\gamma_1))-\varphi(b(\gamma_2)) \\
&\quad -i\varphi[\delta_0(u(\gamma_1+\gamma_2)^*u(\gamma_2)u(\gamma_1))u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1+\gamma_2)] \\
&\quad -i\varphi[\delta_0(u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1)u(\gamma_2))u(\gamma_2)^*u(\gamma_1)^*u(\gamma_2)u(\gamma_1)] \\
&=\varphi(b(\gamma_1+\gamma_2))-\varphi(b(\gamma_1))-\varphi(b(\gamma_2)) \\
&\quad -i\varphi[\delta_0(u(\gamma_1+\gamma_2)^*u(\gamma_2)u(\gamma_1))u(\gamma_1)^*u(\gamma_2)^*u(\gamma_1+\gamma_2)] \\
&=\varphi(\Phi(\gamma_2,\gamma_1))
\end{aligned}
$$

where we have used (2. 1) and (3. 22). But $\Phi(\gamma_1, \gamma_2)\in\mathscr{Z}(\mathscr{A}^\tau)_h$ by Lemma 3. 3. Hence $\Phi(\gamma_1, \gamma_2)=\Phi(\gamma_2, \gamma_1)$ by (3. 21). Then by Lemma 3. 4 and Lemma 3. 6 with $\mathscr{D}=\mathscr{Z}(\mathscr{A}^\tau)_h$, there exists a map $z:\Gamma\to\mathscr{Z}(\mathscr{A}^\tau)$ such that $z(0)=0$ and $\Phi(\gamma_1, \gamma_2)=z(\gamma_1+\gamma_2)-z(\gamma_1)-z(\gamma_2)$. Thus by Lemma 3. 5, δ extends to a derivation on $\mathscr{D}$ commuting with τ. The final remark is clear by uniqueness of Fourier decompositions.

Remark 3. 9. The hypotheses of Theorem 3. 8 eliminate the "twist" from Φ. Here we explain why this is necessary. The existence of the family of unitaries $\{u(\gamma):\gamma\in\Gamma\}$ leads to an action $\eta:\Gamma\longmapsto \mathrm{Aut}\,(\mathscr{Z}(\mathscr{A}^\tau))$ where

$$\eta(\gamma)(b)=u(\gamma)bu(\gamma)^*,\ b\in\mathscr{Z}(\mathscr{A}^\tau),\ \gamma\in\Gamma.$$

The u-twisted 2-cocycle Φ defines a group

$$\Gamma_\Phi=\{(\gamma, b):\gamma\in\Gamma,\ b\in\mathscr{Z}(\mathscr{A}^\tau)_h\}$$

by

$$(\gamma, b)+(\gamma', b')=(\gamma+\gamma', \Phi(\gamma, \gamma')+\eta(\gamma)(b')+b)$$

and hence Φ determines an element ζ_Φ of the second cohomology group $H^2_\eta(\Gamma, \mathscr{Z}(\mathscr{A}^\tau)_h)$ where we use the notation of [11]. Then (3. 9) is equivalent to ζ_Φ being zero. Thus a necessary and sufficient condition for (3. 9) to hold is that the exact sequence

$$0\to\mathscr{Z}(\mathscr{A}^\tau)_h\longrightarrow\Gamma_\Phi\longrightarrow\Gamma\longrightarrow 0 \tag{3.23}$$

split (see [11] again). Unfortunately simple criteria for Φ, as defined by (3. 7), to define a sequence (3. 23) which splits do not appear to exist except when η is trivial where we have the necessary and sufficient condition that Φ be symmetric.

Note that $H^2(\Gamma, \mathscr{Z}(\mathscr{A}^\tau)_h)=(0)$ if $\Gamma=\boldsymbol{Z}$.

§ 4. An Example

Let $G=\boldsymbol{T}^2$, $\Gamma=\boldsymbol{Z}^2$. For $\gamma_1=(m_1, n_1)$, $\gamma_2=(m_2, n_2)\in\Gamma$, define $[\gamma_1, \gamma_2]=m_1n_2-m_2n_1$, and $\omega(\gamma_1, \gamma_2)\in C[0, 1]$ by $\omega(\gamma_1, \gamma_2)(t)=\exp\{it[\gamma_1, \gamma_2]/2\}$, $t\in[0, 1]$. Then $\overline{\text{lin}}\{\omega(\gamma_1, \gamma_2): \gamma_i\in\Gamma\}=C[0, 1]$, and

$$(4.1)\qquad \omega(\gamma_1, \gamma_2)\omega(\gamma_1+\gamma_2, \gamma_3)=\omega(\gamma_1, \gamma_2+\gamma_3)\omega(\gamma_2, \gamma_3)$$

for $\gamma_1, \gamma_2, \gamma_3$ in Γ. Let K denote the Hilbert space $l^2(\Gamma, L^2[0, 1])$, and define unitaries $\{W(\gamma) : \gamma\in\Gamma\}$ on K by

$$(W(\gamma_2)f)(\gamma_1)=\omega(\gamma_1, \gamma_2)f(\gamma_1+\gamma_2)$$

for $f\in K$, $\gamma_1, \gamma_2\in\Gamma$, and where we let $C[0, 1]$ act on $L^2[0, 1]$ by pointwise multiplication. Then by (4.1)

$$(4.2)\qquad W(\gamma_1)W(\gamma_2)=\omega(\gamma_1, \gamma_2)W(\gamma_1+\gamma_2)$$

for $\gamma_1, \gamma_2\in\Gamma$, and where we let $C[0, 1]$ act on K through its action on $L^2[0, 1]$ alone.

Let $\mathscr{A}$ denote the C^*-algebra generated by $\{W(\gamma) : \gamma\in\Gamma\}$. Then by (4.2), $\mathscr{A}\supset C[0, 1]$, and

$$\mathscr{A}=\overline{\text{lin}}\{C[0, 1]W(\gamma) : \gamma\in\Gamma\}.$$

Define a strongly continuous unitary representation U of G on K by

$$(U_gf)(\gamma)=\langle\gamma, g\rangle^{-1}f(\gamma)$$

$f\in K$, $g\in G$, $\gamma\in\Gamma$. Then

$$U_gW(\gamma)U_g^*=\langle\gamma, g\rangle W(\gamma)$$

for $g\in G$, $\gamma\in\Gamma$. Hence

$$\tau_g=Ad(U_g)|_A$$

defines a strongly continuous action of G on $\mathscr{A}$ such that $W(\gamma)\in\mathscr{A}^\tau(\gamma)$ for each γ in Γ. It is clear from (4.2) that $C[0, 1]\subseteq\mathscr{A}^\tau$, and in fact it is easy to see using $P=\int_G\tau(g)dg$ that $\mathscr{A}^\tau=C[0, 1]$.

Let $\mathscr{D}_0=C^1[0, 1]$, and δ_0 denote differentiation on $\mathscr{D}_0$. Then W satisfies conditions (3.1-3.3). In fact, note that $W(\gamma)$ commutes with $C[0, 1]$ so that in this case

$$C[0, 1]=\mathscr{A}^\tau=\mathscr{Z}(\mathscr{A}^\tau)=\mathscr{Z}(\mathscr{A}).$$

If φ is any state on $\mathscr{A}^\tau$, $\varphi\circ P$ is a trace on $\mathscr{A}$, so that (3.22) holds.

Moreover

$$W(\gamma)\delta_0(W(\gamma)^*(\cdot)W(\gamma))W(\gamma)^*-\delta_0(\cdot)\equiv 0$$

so that one can take $b(\gamma)\equiv 0$. However, if $\gamma_1, \gamma_2\in\Gamma$, and

$$W=W(\gamma_1)^*W(\gamma_2)^*W(\gamma_1)W(\gamma_2)=\overline{\omega(\gamma_2, \gamma_1)}\omega(\gamma_1, \gamma_2)$$

then W is the function $t\in[0, 1]\to\exp(it[\gamma_1, \gamma_2])$, so that $\delta_0(W^*)W=i[\gamma_1, \gamma_2]$ and (3.22) cannot possibly hold.

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