Publ. RIMS, Kyoto Univ.
20 (1984), 107-117

Quasi-Invariant Measures on the Orthogonal Group over the Hilbert Space

By

Hiroaki SHIMOMURA*

§ 1. Introduction

Let H be a real separable Hilbert space and $O(H)$ be the orthogonal group over H. In this paper, we shall discuss left, right or both translationally quasi-invariant probability measures on a σ-field $\mathfrak{B}$ derived from the strong topology on $O(H)$. Invariant (rather than quasi-invariant) measures have been considered by several authors. For example in [3], [7] and [4] such measures were constructed as suitable limits of Haar measures on $O(n)$ by methods of Schmidt's orthogonalization or of Cayley transformation. And in [6] some approach based on Gaussian measures on infinite-dimensional linear spaces was attempted. However these measures are defined on larger spaces rather than $O(H)$ and invariant under a sense that "$O(H)$ acts on these spaces." This is reasonable, because it is impossible to construct measures on $O(H)$ which are invariant under all translations of elements of G, if G is a suitably large subgroup of $O(H)$. For example, let $e_1, \cdots, e_n, \cdots$ be a c. o. n. s. in H, and for each n consider a subgroup consisting of $T \in O(H)$ which leaves e_p invariant for all $p > n$. We may identify this subgroup with $O(n)$. Put $O_0(H) = \cup_{n=1}^{\infty} O(n)$. Then $O_0(H)$-invariant finite measure does not exist on $O(H)$. (See, [6]). However replacing invariance with quasi-invariance, the above situation becomes somewhat different. One but main purpose of this paper is to indicate this point. We will show that "*there does not exist any σ-finite G-quasi-invariant measure on $\mathfrak{B}$, as far as G acts transitively on the unit sphere S of H. While $O_0(H)$-quasi-invariant probability measures certainly exist.*

Communicated by S. Matsuura, December 9, 1982.
* Department of Mathematics, Fukui University, Fukui 910, Japan.

We can construct one of them by the Schmidt's orthogonalization method using a suitable family of measures on H." In the remainder parts, we will state basic properties, especially ergodic decomposition of $O_0(H)$-quasi-invariant probability measures. These arguments are carried out in parallel with them for quasi-invariant measures on linear spaces. (See, [5]).

§ 2. Non Existence of G-Quasi-Invariant Measures

Let $e_1, \cdots, e_n, \cdots$ be an arbitrarily fixed c.o.n.s. in H, and define a metric $d(\cdot, \cdot)$ on $O(H)$ such that $d(U, V) = \sum_{n=1}^{\infty} 2^{-n}\{\|Ue_n - Ve_n\| + \|U^{-1}e_n - V^{-1}e_n\|\}$, where $\|\cdot\|$ is the Hilbertian norm on H. A map $U \in O(H) \longmapsto ((Ue_1, \cdots, Ue_n, \cdots), (U^{-1}e_1, \cdots, U^{-1}e_n, \cdots)) \in H^{\infty} \times H^{\infty}$ is a into homeomorphism from $(O(H), d)$ to $H^{\infty} \times H^{\infty}$ equipped with the product-topology. Hence $(O(H), d)$ is a separable metric space. The topology derived from d coincides with the strong topology on $O(H)$, so $(O(H), d)$ is a topological group and $\mathfrak{B}$ is a σ-field generated by open sets of $(O(H), d)$. Moreover, since inverse terms $\|U^{-1}e_n - V^{-1}e_n\|$ are added to the definition of d, $(O(H), d)$ is a complete metric space and therefore a Polish space. Now let μ be a measure on $\mathfrak{B}$ and $T \in O(H)$. We shall define measures $L_T\mu(R_T\mu)$ by $L_T\mu(B) = \mu(T^{-1}\cdot B)$ $(R_T\mu(B) = \mu(B\cdot T^{-1}))$ for all $B \in \mathfrak{B}$, and call them left translation (right translation) of μ by T, respectively. If for a fixed subgroup $G \subset O(H)$, $L_T\mu(R_T\mu)$ is equivalent to μ, $L_T\mu \simeq \mu$, for all $T \in G$, μ is said to be left (right) G-quasi-invariant, respectively. Left and right G-quasi-invariant measures are defined in a similar manner. Since results for right G-quasi-invariant measures are formally derived from them for left G-quasi-invariant measures, we shall omit the "right" case for almost everywhere.

Theorem 1. *There does not exist any left (right) G-quasi-invariant σ-finite measure on $\mathfrak{B}$, as far as G acts transitively on the unit sphere S of H.*

Proof. Suppose that it would be false, and let μ be a such one of left G-quasi-invariant measures. As $(O(H), d)$ is a Polish space, there exists a sequence of compact sets $\{K_n\}$ of $(O(H), d)$ such that $\mu(K_n)>0$ $(n=1, 2, \cdots)$ and $\mu(\cap_{n=1}^{\infty}K_n^c)=0$. From the assumption, we have $0<\mu(gK_1)=\mu(\cup_{n=1}^{\infty}K_n\cap gK_1)$ for all $g\in G$, and therefore $K_n\cap gK_1\neq\phi$ for some n. It follows that $G\subset\cup_{n=1}^{\infty}K_nK_1^{-1}$. Take an $e\in S$ and consider a continuous map f; $U\in(O(H), d)\longmapsto Ue\in S$. Then we have $S=f(G)=\cup_{n=1}^{\infty}f(K_nK_1^{-1})\subset S$. Hence S is a σ-compact set. However it is impossible in virtue of Baire's category theorem.

Q. E. D.

§ 3. A Construction of $O_0(H)$-Quasi-Invariant Measures

As it was stated in the Introduction, let us form $O_0(H)$ from an arbitrarily fixed c. o. n. s. $e_1, \cdots, e_n, \cdots$. For the purpose of the above title, it is enough to regard H as ℓ^2 and the above base as $e_n=(0, \cdots, \overset{n}{\breve{1}}, 0\cdots)\in\ell^2$. First we shall consider left quasi-invariant probability measures, and shall state some lines fo the construction. Let $\tilde{x}=(x_1, \cdots, x_n, \cdots)$ be a sequence of ℓ^2. If they are linearly independent, we have an orthonormal system $G(x_1), G(x_1, x_2), \cdots, G(x_1, \cdots, x_n), \cdots$, operating on $x_1, \cdots, x_n, \cdots$ Schmidt's orthogonalization process. Moreover, they form a c. o. n. s., if a subspace $L(\tilde{x})$ spanned by $x_1, \cdots, x_n, \cdots$ is dense in ℓ^2. And then we can define an orthogonal operator $U(\tilde{x})$ on ℓ^2 as $e_n\longmapsto G(x_1, \cdots, x_n)$ for all n. Now for each $T\in O(\ell^2)$, we shall define a map $\tilde{T}$ on the ℓ^2-sequence space $(\ell^2)^\infty$ such that $\tilde{T}(x_1, \cdots, x_n, \cdots)=(Tx_1, \cdots, Tx_n, \cdots)$. Then it is easy to see that $L_T\circ U=U\circ\tilde{T}$, namely, $TU(\tilde{x})=U(\tilde{T}\tilde{x})$. Hence one of left $O_0(H)$-quasi-invariant measures λ on $\mathfrak{B}$ is defined as $\lambda(B)=\tilde{\nu}(\tilde{x}\,|\,U(\tilde{x})\in B)$ for all $B\in\mathfrak{B}$, if we can construct a probability measure $\tilde{\nu}$ on the usual Borel field $\mathfrak{B}((\ell^2)^\infty)$ on $(\ell^2)^\infty$ satisfying following three properties,

(a) *$x_1, x_2, \cdots, x_n, \cdots$ are linearly independent for $\tilde{\nu}$-a. e. $\tilde{x}=(x_1, \cdots, x_n, \cdots)$,*

(b) *"$L(\tilde{x})$ is dense in ℓ^2" holds for $\tilde{\nu}$-a. e. $\tilde{x}$,*

(c) *$\tilde{T}\tilde{\nu}$ $(\tilde{T}\tilde{\nu}(B)=\tilde{\nu}((\tilde{T})^{-1}(B))$ for all $B\in\mathfrak{B}((\ell^2)^\infty))$ is equivalent to $\tilde{\nu}$ for all $T\in O_0(\ell^2)$.*

Now let p be a probability measure on $\mathfrak{B}(\boldsymbol{R}^1)$ which is equivalent to the Lebesgue measure and satisfies $\int_{-\infty}^{\infty}t^{-2}dp(t)=1$. 1-dimensional Gaussian measures with mean 0 and variance c will be denoted by g_c. And take positive sequences $\{v_n\}_{n=2}^{\infty}$ and $\{c_n\}_{n=2}^{\infty}$ such that

$$\sum_{n=1}^{\infty} n v_{n+1} + \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} c_j^2 < 1.$$

Then for each n, a measure of product-type $\mu_n = \underbrace{g_{v_n} \times \cdots \times g_{v_n}}_{n-1 \text{ times}} \times p \times g_{c_{n+1}^2} \times \cdots \times g_{c_j^2} \times \cdots$ is defined on $\mathfrak{B}(\ell^2)$, in virtue of the choice of $\{c_n\}$. Moreover, from the rotational-invariance of $g_{v_n} \times \cdots \times g_{v_n}$, μ_n is $O(n-1)$-invariant for all n. Now let us consider a measure of product-type $\tilde{\mu} = \mu_1 \times \cdots \times \mu_n \times \cdots$ on $\mathfrak{B}((\ell^2)^\infty)$. It is fairly easy that $\tilde{\mu}$ satisfies (a). Since for all n and for all $T \in O(n-1)$, we have

$$\begin{aligned} \tilde{T}\tilde{\mu} &= T\mu_1 \times \cdots \times T\mu_{n-1} \times T\mu_n \times \cdots \times T\mu_j \times \cdots \\ &= T\mu_1 \times \cdots \times T\mu_{n-1} \times \mu_n \times \cdots \times \mu_j \times \cdots \\ &\simeq \mu_1 \times \cdots \times \mu_n \times \cdots = \tilde{\mu}, \end{aligned}$$

so $\tilde{\mu}$ satisfies (c) too. We shall consider for (b). Let $<\cdot, \cdot>$ be the scalar product on ℓ^2. Then,

$$\begin{aligned} &\int_{y \in \ell^2} \| <y, e_n>^{-1} y - e_n \|^2 d\mu_n(y) \\ &= \int <y, e_n>^{-2} \sum_{j \neq n} <y, e_j>^2 d\mu_n(y) \\ &= (n-1) v_n + \sum_{j=n+1}^{\infty} c_j^2, \end{aligned}$$

it follows that

$$\begin{aligned} &\int_{(\ell^2)^\infty} \sum_{n=1}^{\infty} \| <x_n, e_n>^{-1} x_n - e_n \|^2 d\tilde{\mu}(\tilde{x}) \\ &= \sum_{n=1}^{\infty} n v_{n+1} + \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} c_j^2 < 1. \end{aligned}$$

Hence putting

$$E = \{\tilde{x} \mid \sum_{n=1}^{\infty} \| <x_n, e_n>^{-1} x_n - e_n \|^2 < 1\},$$

we have $\tilde{\mu}(E) > 0$. Thus for all $\tilde{x} \in E$, $L(\tilde{x})$ is dense in ℓ^2 by the following lemma.

Lemma 1. *Suppose that* $\sum_{n=1}^{\infty} \|t_n - e_n\|^2 < 1$ *for a sequence* $\{t_n\} \subset \ell^2$. *Then a subspace spanned by* $t_1, \cdots, t_n, \cdots$ *is dense in* ℓ^2.

Proof. By the assumption, we can define an operator A such that $Ae_n = t_n$ for all n and $I - A$ is a Hilbert-Schmidt operator whose Hilbert-Schmidt norm is strictly less than 1. Hence we have $\|I - A\|_{op} < 1$. It implies A is an isomorphic operator. Consequently, $Ae_1, \cdots, Ae_n, \cdots$ span a dense linear subspace. Q. E. D.

At the same time we shall prove the measurability of the set

$\{\tilde{x} \mid L(\tilde{x})$ is dense$\} \equiv F$. Consider a set $(\ell^2)^\infty \times S \supset \Omega \equiv \{(\tilde{x}, a) \mid \langle x_n, a \rangle = 0$ for all $n\}$ and let p be a projection to the first coordinate. It is evident that $F^c = p(\Omega)$, and the later is a Souslin set. Therefore F is universally-measurable. As $\tilde{\mu}(F) \geqq \tilde{\mu}(E) > 0$, so we can put $\tilde{\nu}(B) = \frac{\tilde{\mu}(B \cap F)}{\tilde{\mu}(F)}$ for all $B \in \mathfrak{B}((\ell^2)^\infty)$. Clearly, $\tilde{\nu}$ satisfies (a) and (b). Moreover (c) is also satisfied, because F is an invariant set for all $\tilde{T}$, $T \in O(\ell^2)$. By the above, there exist left $O_0(H)$-quasi-invariant probability measures on $\mathfrak{B}$. Next, if we wish to construct left and right $O_0(H)$-quasi-invariant measures, we shall prepare such $\tilde{\nu}_1$ and $\tilde{\nu}_2$ and form a product-measure $\tilde{\nu}_1 \times \tilde{\nu}_2$ on $(\ell^2)^\infty \times (\ell^2)^\infty$. Then for $\tilde{\nu}_1 \times \tilde{\nu}_2$-a. e. $(\tilde{x}, \tilde{y})$, $U(\tilde{x}, \tilde{y})$; $G(x_1, \cdot\cdot, x_n) \longmapsto G(y_1, \cdot\cdot, y_n)$ $(n=1, 2 \cdot\cdot)$ is an orthogonal operator on ℓ^2 which satisfies $U(\tilde{T}\tilde{x}, \tilde{S}\tilde{y}) = SU(\tilde{x}, \tilde{y})T^{-1}$ for all T, $S \in O(\ell^2)$. It follows by similar arguments that a measure $\lambda = U(\tilde{\nu}_1 \times \tilde{\nu}_2)$ on $\mathfrak{B}$ is a left and right $O_0(\ell^2)$-quasi-invariant probability measure.

§ 4. Basic Results and Ergodic Decomposition of $O_0(H)$-Quasi-Invariant Measures

From now on, we put $\mathfrak{A}_n = \{E \in \mathfrak{B} \mid T \cdot E = E$ for all $T \in O(n)\}$,

$\mathfrak{B}_n = \{E \in \mathfrak{B} \mid T \cdot E \cdot S = E$ for all T, $S \in O(n)\}$ $(n=1, 2, \cdots)$,

$\mathfrak{A}_\infty = \{E \in \mathfrak{B} \mid T \cdot E = E$ for all $T \in O_0(H)\}$

and

$\mathfrak{B}_\infty = \{E \in \mathfrak{B} \mid T \cdot E \cdot S = E$ for all T, $S \in O_0(H)\}$.

Then we have $\mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_n \supset \cdots$, $\mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_n \supset \cdots$, $\cap_{n=1}^\infty \mathfrak{A}_n = \mathfrak{A}_\infty$, and $\cap_{n=1}^\infty \mathfrak{B}_n = \mathfrak{B}_\infty$. $\mathfrak{A}_\infty (\mathfrak{B}_\infty)$ plays an essential role for left (left and right) $O_0(H)$-quasi-invariant measures.

Lemma 2. (a) *Let μ be a left $O_0(H)$-quasi-invariant probability measure on $\mathfrak{B}$, and let $E \in \mathfrak{B}$ satisfy $\mu(E \ominus T \cdot E) = 0$ for all $T \in O_0(H)$. Then there exists an $E_0 \in \mathfrak{A}_\infty$ such that $\mu(E \ominus E_0) = 0$.*

(b) *Let μ be a left and right $O_0(H)$-quasi-invariant probability measure on $\mathfrak{B}$, and let $E \in \mathfrak{B}$ satisfy $\mu(E \ominus T \cdot E \cdot S) = 0$ for all T, $S \in O_0(H)$. Then there exists an $E_0 \in \mathfrak{B}_\infty$ such that $\mu(E \ominus E_0) = 0$.*

Proof. (a) Put $f_n(U) = \int_{O(n)} \chi_E(T \cdot U) dT$, where dT is the normalized Haar measure on $O(n)$ and χ_E is the indicator function of E. Then $f_n(U)$ is an $O(n)$-invariant function and

$$\begin{aligned}&\int|f_n(U)-\chi_E(U)|d\mu(U)\\&\leqq\iint|\chi_E(T\cdot U)-\chi_E(U)|dTd\mu(U)\\&=\int_{O(n)}\mu(E\ominus T^{-1}\cdot E)dT=0.\end{aligned}$$

Hence we have $f_n(U)=\chi_E(U)$ for μ-a.e. U. Put $f(U)=\lim_n f_n(U)$, if the limit exists and $f(U)=0$, otherwise. Since $f(U)$ is $O_0(H)$-invariant, so putting $E_0=\{U|f(U)=1\}$, it holds $\mu(E\ominus E_0)=0$.

(b) It is carried out in a similar manner, only changing the integral into $\iint_{O(n)\times O(n)}\chi_E(T\cdot U\cdot S)dTdS$. Q. E. D.

Lemma 3. *Let μ be a left $O_0(H)$-quasi-invariant probability measure on $\mathfrak{B}$. Then for any $B\in\mathfrak{B}$ there exists a countable set $\{T_n\}_{n=1}^{\infty}\subset O_0(H)$ such that $\hat{B}\equiv\cup_{n=1}T_n\cdot B$ satisfies $\mu(T\cdot\hat{B}\ominus\hat{B})=0$ for all $T\in O_0(H)$. If μ is a left and right $O_0(H)$-quasi-invariant probability measure, then it holds $\mu(T\cdot\hat{B}\cdot S\ominus\hat{B})=0$ for all T, $S\in O_0(H)$, replacing the above set with $\hat{B}=\cup_{n=1}T_n\cdot B\cdot S_n$ for some $\{T_n\}_n$, $\{S_n\}_n\subset O_0(H)$.*

Proof. As $L_\mu^1(O(H))$ is separable, we can take a countable dense set $\{\chi_{T_n\cdot B}(U)\}_{n=1}^{\infty}$ of $\{\chi_{T\cdot B}(U)\}_{T\in O_0(H)}$ in the left case and $\{\chi_{T_n\cdot B\cdot S_n}(U)\}_{n=1}^{\infty}$ of $\{\chi_{T\cdot B\cdot S}(U)\}_{T,S\in O_0(H)}$ in the left and right case. It is easily checked that $\cup_{n=1}T_n\cdot B$ and $\cup_{n=1}T_n\cdot B\cdot S_n$ are desired ones respectively.

Q. E. D.

Proposition 1. *Two left $O_0(H)$-quasi-invariant probability measures μ and ν are equivalent, if and only if $\mu\simeq\nu$ on $\mathfrak{A}_\infty$. In the case of left and right $O_0(H)$-quasi-invariant measures, it is necessary and sufficient that they are equivalent on $\mathfrak{B}_\infty$.*

Proof. The necessity is obvious. So let μ and ν be left $O_0(H)$-quasi-invariant and suppose that they are not equivalent, for example, $\mu(B)>0$ and $\nu(B)=0$ for some $B\in\mathfrak{B}$. Then applying Lemma 3 for μ, there exists $\{T_n\}_n\subset O_0(H)$ such that $\hat{B}=\cup_{n=1}T_n\cdot B$ satisfies $\mu(T\cdot\hat{B}\ominus\hat{B})=0$ for all $T\in O_0(H)$. Clearly we have $\nu(T\cdot\hat{B}\ominus\hat{B})=0$ for all

$T\in O_0(H)$. Thus applying Lemma 2 for $\lambda=2^{-1}(\mu+\nu)$, there exists an $A\in\mathfrak{A}_\infty$ such that $\lambda(A\ominus\hat{B})=0$. It follows that $\mu(A)=\mu(\hat{B})>0$ and $\nu(A)=\nu(\hat{B})=0$. Therefore μ and ν are not equivalent on $\mathfrak{A}_\infty$. The left and right case is discussed in a similar way. Q. E. D.

Now we shall introduce a notion of ergodicity. A left (left and right) $O_0(H)$-quasi-invariant probability measure μ is said to be left (left and right) $O_0(H)$-ergodic, if $\mu(A)=1$ or 0 for every subset $A\in\mathfrak{B}$ satisfying $\mu(T\cdot A\ominus A)=0$ for all $T\in O_0(H)$ ($\mu(T\cdot A\cdot S\ominus A)=0$ for all $T, S\in O_0(H)$), respectively. In virtue of Lemma 2, it is equivalent that μ takes only the values 0 or 1 on $\mathfrak{A}_\infty(\mathfrak{B}_\infty)$, respectively.

Corollary. *Two left (left and right) $O_0(H)$-ergodic measures are equivalent, if and only if they agree on $\mathfrak{A}_\infty(\mathfrak{B}_\infty)$, respectively.*

Proposition 2. *Let μ and ν be left $O_0(H)$-quasi-invariant probability measures on $\mathfrak{B}$, and put $\lambda(B)=\int_{g\in O(H)}\mu(Bg)d\nu(g)$ for all $B\in\mathfrak{B}$. Then λ is left and right $O_0(H)$-quasi-invariant. Moreover, if μ and ν are left $O_0(H)$-ergodic, then λ is left and right $O_0(H)$-ergodic.*

Proof. Let $S\in O_0(H)$. Then we have $\lambda(B)=0 \Leftrightarrow \mu(Bg)=0$ for ν-a. e. $g \Leftrightarrow \mu(Bg)=0$ for $L_S\nu$-a. e. $g \Leftrightarrow \lambda(B\cdot S)=\int\mu(Bg)dL_S\nu(g)=0$. This shows that λ is right $O_0(H)$-quasi-invariant. Left $O_0(H)$-quasi-invariance of λ is clear. Next, let μ and ν be left $O_0(H)$-ergodic, and let $A\in\mathfrak{B}_\infty$. As $Ag\in\mathfrak{A}_\infty$ for all $g\in O(H)$, we have $\mu(Ag)=1$ or 0 for all $g\in O(H)$. Put $E=\{g\in O(H)\mid\mu(Ag)=1\}$. Then it follows from $E\in\mathfrak{A}_\infty$ that we have $\nu(E)=1$ or 0. Hence $\lambda(A)=1$, if $\nu(E)=1$ and $\lambda(A)=0$, if $\nu(E)=0$. Q. E. D.

Now we shall consider an ergodic decomposition of $O_0(H)$-quasi-invariant measures. Let μ be a probability measure on $\mathfrak{B}$. As $(O(H), d)$ is a Polish space, so for any sub-σ-field $\mathfrak{A}$ of $\mathfrak{B}$, there exists a family of conditional probability measures on $\mathfrak{B}$ relative to $\mathfrak{A}$ $\{\mu(g, \mathfrak{A}, \cdot)\}_{g\in O(H)}$ which satisfy (1) for each fixed $B\in\mathfrak{B}$, $\mu(g, \mathfrak{A}, B)$ is an $\mathfrak{A}$-measurable function and (2) $\mu(A\cap B)=\int_A\mu(g, \mathfrak{A}, B)d\mu(g)$ for all $A\in\mathfrak{A}$ and for all $B\in\mathfrak{B}$.

Lemma 4. *Under the above notation, we take an arbitrary* $B\in\mathfrak{B}$ *and fix it. Then for all* n, $\mu(g, \mathfrak{A}, T\cdot B\cdot S)$ *is a jointly* $\mathfrak{A}\times\mathfrak{B}(O(n))\times\mathfrak{B}(O(n))$*-measurable function of variables* $(g, T, S)\in O(H)\times O(n)\times O(n)$, *where* $\mathfrak{B}(O(n))$ *is a usual Borel field on* $O(n)$.

Proof. Let f be a continuous bounded function on $O(H)$. Put $h(g, T, S)=\int_{O(H)} f(T^{-1}\cdot t\cdot S^{-1})\mu(g, \mathfrak{A}, dt)$. Then (1) for a fixed $(T, S)\in O(n)\times O(n)$, $h(g, T, S)$ *is* $\mathfrak{A}$-measurable of g and (2) for a fixed $g\in O(H)$ $h(g, T, S)$ is continuous on $O(n)\times O(n)$. Hence $h(g, T, S)$ is jointly-measurable. Next, if f is an indicator function of a closed set B, then we see that $h(g, T, S)$ is again measurable, taking a family of bounded continuous functions $\{f_n\}$, $f_n\downarrow f$. Now a family of Borel subsets satisfying the assertion of this Lemma is a monotone class, and contains an algebra generated by closed sets by the above arguments. Thus it coincides with $\mathfrak{B}$. Q. E. D.

Let μ be a left $O_0(H)$-quasi-invariant probability measure on $\mathfrak{B}$. First we shall ask for conditional probability measures relative to $\mathfrak{A}_n$, using the normalized Haar measure dT on $O(n)$ for each n. We put $\mu_n(B)=\int_{T\in O(n)}\mu(T\cdot B)dT$ for all $B\in\mathfrak{B}$. Then we have $\mu_n\simeq\mu$, $\mu_n(A)=\mu(A)$ for all $A\in\mathfrak{A}_n$ and μ_n is $O(n)$-invariant. It follows that for all $A\in\mathfrak{A}_n$ and for all $B\in\mathfrak{B}$,

$$\begin{aligned}\mu(A\cap B)&=\int_{A\cap B}\frac{d\mu}{d\mu_n}(g)\,d\mu_n(g)\\&=\int_A\int_{T\in O(n)}\chi_B(T\cdot g)\frac{d\mu}{d\mu_n}(T\cdot g)\,dTd\mu_n(g)\\&=\int_A\int_{T\in O(n)}\chi_B(T\cdot g)\frac{d\mu}{d\mu_n}(T\cdot g)\,dTd\mu(g).\end{aligned}$$

Since

$$\int_{T\in O(n)}\chi_B(T\cdot g)\frac{d\mu}{d\mu_n}(T\cdot g)\,dT\equiv\mu(g, \mathfrak{A}_n, B)$$

is an $\mathfrak{A}_n$-measurable function of g for each fixed $B\in\mathfrak{B}$, so we have $\mu(g, \mathfrak{A}_n, O(H))=1$ for μ-a. e. g and $\{\mu(g, \mathfrak{A}_n, \cdot)\}_{g\in O(H)}$ is the family of conditional probability measures relative to $\mathfrak{A}_n$. Let $A\in\mathfrak{A}_\infty$ and $B\in\mathfrak{B}$. Then

$$\mu_n(A\cap B)=\int_{T\in O(n)}\mu(A\cap T\cdot B)\,dT$$

$$=\int_{T\in O(n)}\int_A \mu(g,\ \mathfrak{A}_\infty,\ T\cdot B)\ d\mu(g)\,dT$$
$$=\int_A d\mu_n(g)\int_{T\in O(n)}\mu(g,\ \mathfrak{A}_\infty,\ T\cdot B)\,dT.$$

Therefore by Fubini's theorem and Lemma 4,

$$\int_{T\in O(n)}\mu(g,\ \mathfrak{A}_\infty,\ T\cdot B)\,dT\equiv\mu_n(g,\ \mathfrak{A}_\infty,\ B)$$

are conditional probability measures of μ_n relative to $\mathfrak{A}_\infty$. Since it holds $\mu_n\simeq\mu$, applying general discussions for conditional probability measures, it is assured that there exists an $\Omega_n\in\mathfrak{A}_\infty$ with $\mu(\Omega_n)=1$ such that

$$\mu_n(g,\ \mathfrak{A}_\infty,\ \cdot)\equiv\mu_n^g\simeq\mu(g,\ \mathfrak{A}_\infty,\ \cdot)\equiv\mu^g$$

and the Radon-Nikodim derivative $\dfrac{d\mu^g}{d\mu_n^g}$ can be taken as $\dfrac{d\mu}{d\mu_n}$ for all $g\in\Omega_n$. As μ_n^g is $O(n)$-invariant, we conclude that for all $g\in\bigcap_{n=1}^{\infty}\Omega_n\equiv\Omega_0$, μ^g is left $O_0(H)$-quasi-invariant. Moreover, from $(\mu^g)_n=\mu_n^g$ we have for all $g\in\Omega_n$,

$$\mu^g(t,\ \mathfrak{A}_n,\ B)=\int_{T\in O(n)}\chi_B(T\cdot t)\frac{d\mu^g}{d(\mu^g)_n}(T\cdot t)\,dT$$
$$=\int_{T\in O(n)}\chi_B(T\cdot t)\frac{d\mu}{d\mu_n}(T\cdot t)\,dT$$
$$=\mu(t,\ \mathfrak{A}_n,\ B)$$

for all $t\in O(H)$ and for all $B\in\mathfrak{B}$. Consequently, for all $g\in\Omega_0$, $\mu^g(t,\ \mathfrak{A}_n,\ \cdot)=\mu(t,\ \mathfrak{A}_n,\ \cdot)$ holds for all $t\in O(H)$ and for all n. In virtue of inverse martingale theorem, for all $B\in\mathfrak{B}$,

$$0=\lim_n\int|\mu(t,\ \mathfrak{A}_\infty,\ B)-\mu(t,\ \mathfrak{A}_n,\ B)|\,d\mu(t)$$
$$=\lim_n\int|\mu(t,\ \mathfrak{A}_\infty,\ B)-\mu(t,\ \mathfrak{A}_n,\ B)|\,d\mu^g(t)\,d\mu(g).$$

Taking a subsequence $\{n_j\}$ if necessary, there exists an $\Omega_B^1\in\mathfrak{A}_\infty$ with $\mu(\Omega_B^1)=1$ such that for all $g\in\Omega_B^1$,

$$\lim_j\int|\mu(t,\ \mathfrak{A}_\infty,\ B)-\mu(t,\ \mathfrak{A}_{n_j},\ B)|\,d\mu^g(t)=0.$$

Hence again using the inverse martingale theorem, we have for all $g\in\Omega_B^1\cap\Omega_0$,

$$\int|\mu(t,\ \mathfrak{A}_\infty,\ B)-\mu^g(t,\ \mathfrak{A}_\infty,\ B)|\,d\mu^g(t)=0.$$

It follows that

$$\begin{aligned}&\int |\mu(g, \mathfrak{A}_\infty, B) - \mu^g(t, \mathfrak{A}_\infty, B)|^2 d\mu^g(t) d\mu(g)\\ &= 2\int \mu(g, \mathfrak{A}_\infty, B)^2 d\mu(g) - 2\int \mu(g, \mathfrak{A}_\infty, B)\int \mu^g(t, \mathfrak{A}_\infty, B) d\mu^g(t) d\mu(g) = 0.\end{aligned}$$

Thus there exists an $\Omega_B^2 \in \mathfrak{A}_\infty$ with $\mu(\Omega_B^2) = 1$ such that

$$\int |\mu(g, \mathfrak{A}_\infty, B) - \mu^g(t, \mathfrak{A}_\infty, B)| d\mu^g(t) = 0$$

for all $g \in \Omega_B^2$. Finally we shall put $\Omega = \bigcap_{B \in \mathscr{F}} \Omega_B^2$, where $\mathscr{F}$ is a countable algebra generated by a countable open base of $(O(H), d)$. Then for all $g \in \Omega$, the above formula holds for every $B \in \mathfrak{B}$, so for all $A \in \mathfrak{A}_\infty$ and $B \in \mathfrak{B}$,

$$\mu^g(A \cap B) = \int_A \mu^g(t, \mathfrak{A}_\infty, B) d\mu^g(t) = \mu^g(B)\mu^g(A).$$

Especially, we have $\mu^g(A) = 1$ or 0 for all $A \in \mathfrak{A}_\infty$ and it implies μ^g is left $O_0(H)$-ergodic for all $g \in \Omega_0 \cap \Omega$. We shall conclude these arguments with the following theorem.

Theorem 2. *Let μ be a left $O_0(H)$-quasi-invariant probability measure on $\mathfrak{B}$. Then the conditional probability measures $\mu(g, \mathfrak{A}_\infty, \cdot)$ relative to $\mathfrak{A}_\infty$ are left $O_0(H)$-ergodic for μ-a.e.g.*

From Theorem 2, we can derive a following theorem called canonical decomposition in a quite similar way with it in pp. 372-373 in [5].

Theorem 3. *Let μ be a left $O_0(H)$-quasi-invariant probability measure. Then there exist a family of probability measures $\{\mu^\tau\}_{\tau \in \boldsymbol{R}^1}$ on $\mathfrak{B}$ and a map p from $O(H)$ to $\boldsymbol{R}^1$ which satisfy*

(a) *μ^τ is left $O_0(H)$-ergodic for all $\tau \in \boldsymbol{R}^1$,*

(b) *for each fixed $B \in \mathfrak{B}$, $\mu^\tau(B)$ is $\mathfrak{B}(\boldsymbol{R}^1)$-measurable,*

(c) *$p^{-1}(\mathfrak{B}(\boldsymbol{R}^1)) \subset \mathfrak{A}_\infty$,*

(d) *$\mu(B \cap p^{-1}(E)) = \int_E \mu^\tau(B) dp\mu(\tau)$ for all $B \in \mathfrak{B}$ and $E \in \mathfrak{B}(\boldsymbol{R}^1)$,*

(e) *there exists $E_0 \in \mathfrak{B}(\boldsymbol{R}^1)$, $\mu(p^{-1}(E_0)) = 1$ such that μ^{τ_1} and μ^{τ_2} are mutually singular for all $\tau_1, \tau_2 \in E_0 (\tau_1 \neq \tau_2)$.*

Ergodic decomposition of left and right $O_0(H)$-quasi-invariant measures is carried out in parallel with the left case, only changing the integrals $\int_{O(n)} \cdots dT$ into double integrals $\iint_{O(n)\times O(n)} \cdots dTdS$. And the statements of Theorem 3 remains valid, changing "left" and $\mathfrak{A}_\infty$ into "left and right" and $\mathfrak{B}_\infty$, respectively.

References

[1] Meyer, P. A., *Probability and potentials*, Waltham Mass. Blaisdell Publ. Co., 1966.

[2] Parthasarathy, K. R., *Probability measures on metric spaces*, Academic Press, 1967.

[3] Shale, D., Invariant integration over the infinite dimension orthogonal group and related spaces, *Trans. Amer. Math. Soc.*, **124** (1966), 148-157.

[4] Shimomura, H., On the construction of invariant measure over the orthogonal group on the Hilbert space by the method of Cayley transformation, *Publ. RIMS, Kyoto Univ.*, **10** (1975), 413-424.

[5] ———, Ergodic decomposition of quasi-invariant measures, *ibid.*, **14** (1978), 359-381.

[6] Yamasaki, Y., Invariant measure of the infinite dimensional rotation group, *ibid.*, **8** (1972) 131-140.

[7] ———, Projective limit of Haar measures on $O(n)$, *ibid.*, **8** (1972), 141-149.

[8] ———, *Measures on infinite-dimensional spaces*, Kinokuniya, 1978, in Japanese.