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A Note on a Theorem of A. Connes on Radon-Nikodym Cocycles

By

Tetsuya MASUDA*

Abstract

We give an alternative proof of a theorem of A. Connes that every unitary cocycle (relative to a modular automorphism of a weight ϕ_0) is a Radon-Nikodym unitary cocycle $(D\phi : D\phi_0)_t$ for some faithful normal semifinite weight ϕ.

§ 1. Statement of Theorem

We prove the following theorem of A. Connes ([3], Theorem 1.2.4) by a method different from his method.

Theorem. *Let ϕ_0 be a faithful normal semifinite weight on a von Neumann algebra M. Let $\{u_t\}_{t\in\boldsymbol{R}}$ be a strongly continuous one parameter family of unitaries in M satisfying the cocycle condition with respect to the modular automorphism group $\{\sigma_t^{\phi_0}\}_{t\in\boldsymbol{R}}$ associated with the weight ϕ_0 i.e.*

$$u_{s+t}=u_s\sigma_s^{\phi_0}(u_t),\quad s,\ t\in\boldsymbol{R}. \tag{1.1}$$

Then there exists a unique faithful normal semifinite weight ϕ on M satisfying

$$(D\phi : D\phi_0)_t=u_t,\quad t\in\boldsymbol{R}. \tag{1.2}$$

In the construction of L_p-spaces ([2], [6]), a generalized version of this theorem for not necessarily unitary u and not necessarily faithful weight ϕ plays an important role and is obtained from this theorem, see Appendix of [2] (see also [4]). The construction of L_p-spaces in [2] and [6] is carried out directly on the relevant von Neumann

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* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

algebra (in contrast to the construction of the same L_p-spaces using the crossed product of the relevant von Neumann algebra by the modular automorphism group [5]) except for the above theorem, for which the original proof by A. Connes utilizes the tensor product of the relevant von Neumann algebra with a type I factor, a procedure closely related to the crossed product by modular action through Takesaki duality [8]. This has been a motivation for looking for the present alternative proof.

Throughout the paper, we use the standard notion of the Tomita-Takesaki theory (for example, see [7]) and relative modular operators (for example, see [1]).

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§ 2. Proof of the Theorem

We prove the theorem in several steps. We introduce a σ-weakly continuous one parameter group of isometric linear transformations $\{\alpha_t\}_{t\in\boldsymbol{R}}$ and automorphisms $\{\beta_t\}_{t\in\boldsymbol{R}}$ as follows:

$$\alpha_t(x)=u_t\sigma_t^{\phi_0}(x), \tag{2.1}$$

$$\beta_t(x)=u_t\sigma_t^{\phi_0}(x)u_t^*,\ x\in M,\ t\in\boldsymbol{R}. \tag{2.2}$$

By the equality $\alpha_t(x)^*\alpha_t(x)=\sigma_t^{\phi_0}(x^*x)$, $x\in M$, $t\in\boldsymbol{R}$, α_t leaves N_{ϕ_0} invariant.

Lemma 2. 1. *There exists a positive self-adjoint operator* T *on* H_{ϕ_0} *satisfying*

$$u_t=T^{it}\Delta_{\phi_0}^{-it}, \tag{2.3}$$

$$T^{it}\eta_{\phi_0}(x)=\eta_{\phi_0}(\alpha_t(x)),\ x\in N_{\phi_0},\ t\in\boldsymbol{R}. \tag{2.4}$$

Proof. Let $T(t)$ be a strongly continuous one parameter family of unitaries on H_{ϕ_0} defined by

$$T(t) = u_t \Delta_{\phi_0}^{it}, \quad t \in \boldsymbol{R}. \tag{2.5}$$

By the cocycle condition (1.1), $\{T(t)\}_{t\in\boldsymbol{R}}$ is a group. Hence there exists a self-adjoint operator T such that $T(t) = T^{it}$ by Stone's theorem and (2.3) follows. The formula (2.4) follows from (2.1) and (2.5). Q. E. D.

We denote by $\tilde{N}_{\phi_0}$ the set of all entire analytic elements of N_{ϕ_0} with respect to $\{\alpha_t\}_{t\in\boldsymbol{R}}$. We also denote by M_{ϕ_0} the set of all entire analytic elements of $N_{\phi_0}^* \cap N_{\phi_0}$ with respect to the modular action $\{\sigma_t^{\phi_0}\}_{t\in\boldsymbol{R}}$. We denote by $\tilde{N}_{\phi_0}\tilde{N}_{\phi_0}^*$ the set of all $\boldsymbol{C}$-linear combinations of the elements xy^*, x, $y \in \tilde{N}_{\phi_0}$. By (2.1),

$$\alpha_t(x)^* \alpha_t(y) = \sigma_t^{\phi_0}(x^* y) \tag{2.6}$$

and hence, $\tilde{N}_{\phi_0}^* \tilde{N}_{\phi_0} \subset M_{\phi_0}$.

Now, we define a mapping $\eta : \tilde{N}_{\phi_0}\tilde{N}_{\phi_0}^* \to H_{\phi_0}$ by

$$\eta(\sum_{k=1}^{n} x_k y_k^*) = \sum_{k=1}^{n} x_k J_{\phi_0} T^{1/2} \eta_{\phi_0}(y_k), \tag{2.7}$$

where x_k, $y_k \in \tilde{N}_{\phi_0}$, $k = 1, \cdots, n$.

Lemma 2.2. *The formula* (2.7) *gives a well-defined injective linear mapping* η *from* $\tilde{N}_{\phi_0}\tilde{N}_{\phi_0}^*$ *to* H_{ϕ_0} *with a dense range.*

Proof. If we show that η is well-defined, then the linearity of η follows. Suppose $\sum_{k=1}^{n} x_k y_k^* = 0$, x_k, $y_k \in \tilde{N}_{\phi_0}$, $k = 1, \cdots, n$. Then

$$\begin{aligned} &\| \sum_{k=1}^{n} x_k J_{\phi_0} T^{1/2} \eta_{\phi_0}(y_k) \|^2 \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} (x_k J_{\phi_0} T^{1/2} \eta_{\phi_0}(y_k), \; x_l J_{\phi_0} T^{1/2} \eta_{\phi_0}(y_l)) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} (J_{\phi_0} x_k^* x_l J_{\phi_0} \eta_{\phi_0}(\alpha_{-i/2}(y_l)), \; T^{1/2} \eta_{\phi_0}(y_k)) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} (\alpha_{-i/2}(y_l) J_{\phi_0} \eta_{\phi_0}(x_k^* x_l), \; T^{1/2} \eta_{\phi_0}(y_k)) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} (\alpha_{-i/2}(y_l) \, \eta_{\phi_0}(\sigma_{-i/2}^{\phi_0}(x_l^* x_k)), \; T^{1/2} \eta_{\phi_0}(y_k)) \\ &= \sum_{k=1}^{n} (\eta_{\phi_0}(\alpha_{-i/2}\{[\sum_{l=1}^{n} x_l y_l^*]^* x_k\}), \; T^{1/2} \eta_{\phi_0}(y_k)) \\ &= 0, \end{aligned} \tag{2.8}$$

where we used $J_{\phi_0}xJ_{\phi_0}\eta_{\phi_0}(y)=yJ_{\phi_0}\eta_{\phi_0}(x)$, $x, y\in N_{\phi_0}$ for the third equality, $\tilde{N}^*_{\phi_0}\tilde{N}_{\phi_0}\subset M_{\phi_0}$ for the fourth equality and the analytic continuation of $\alpha_t(x)\sigma_t^{\phi_0}(y)=\alpha_t(xy)$, $x\in\tilde{N}_{\phi_0}$, $y\in M_{\phi_0}$ $(t\longmapsto -i/2)$ for the fifth equality. The density of the range follows from $T^{1/2}\eta_{\phi_0}(y)=\eta_{\phi_0}(\alpha_{-i/2}(y))$, $\alpha_{-i/2}(\tilde{N}_{\phi_0})=\tilde{N}_{\phi_0}$, the density of $\eta_{\phi_0}(\tilde{N}_{\phi_0})$ in H_{ϕ_0}, $J^2_{\phi_0}=1$ and the strong density of $\tilde{N}_{\phi_0}$ in M (so that we may take $n=1$ and approximate 1 by x_1).

Next, we show that η is injective. Suppose $\eta(\sum_{k=1}^n x_ky_k^*)=0$, $x_k, y_k\in\tilde{N}_{\phi_0}$. By the same calculation as (2.8), we obtain $0=(\eta(\sum_{k=1}^n x_ky_k^*), \eta(xy^*))=(T^{1/2}\eta_{\phi_0}([\sum_{k=1}^n x_ky_k^*]^*x), T^{1/2}\eta_{\phi_0}(y))$ for any $x, y\in\tilde{N}_{\phi_0}$. The density of $T^{1/2}\eta_{\phi_0}(\tilde{N}_{\phi_0})$ implies $T^{1/2}\eta_{\phi_0}([\sum_{k=1}^n x_ky_k^*]^*x)=0$. Because T is nonsingular and ϕ_0 is faithful, $[\sum_{k=1}^n x_ky_k^*]^*x=0$. By taking limit $x\to 1$, we obtain $\sum_{k=1}^n x_ky_k^*=0$. Thus the injectivity of η follows. Q. E. D.

Now, we set $\mathfrak{A}_0=\eta(\tilde{N}_{\phi_0}\tilde{N}^*_{\phi_0})$ with the following algebraic operations $\eta(x)\eta(y)=\eta(xy)$, $\eta(x)^\#=\eta(x^*)$, $\Delta(z)\eta(x)=\eta(\beta_{-iz}(x))$, $z\in\boldsymbol{C}$ (note that by (2.1) and (2.2), $\alpha_t(x)\alpha_t(y)^*=\beta_t(xy^*)$ and hence the elements of $\tilde{N}_{\phi_0}\tilde{N}^*_{\phi_0}$ are entire alalytic with respect to the automorphism group $\{\beta_t\}_{t\in\boldsymbol{R}}$ and $\tilde{N}_{\phi_0}\tilde{N}^*_{\phi_0}$ is β-invariant due to the α-invariance of $\tilde{N}_{\phi_0}$).

Lemma 2.3.

(1) *The mapping* $z\in\boldsymbol{C}\longmapsto\Delta(z)\xi$ *is analytic for* $\xi\in\mathfrak{A}_0$.

(2) $(\Delta(z)\xi_1, \xi_2)=(\xi_1, \Delta(\bar{z})\xi_2)$, $\xi_1, \xi_2\in\mathfrak{A}_0$, $z\in\boldsymbol{C}$.

(3) $(\Delta(1)\xi_1, \xi_2)=(\xi_2^\#, \xi_1^\#)$, $\xi_1, \xi_2\in\mathfrak{A}_0$.

Proof. (1) Let $x, y\in\tilde{N}_{\phi_0}$. Then

$$(2.9)\qquad \begin{aligned}\Delta(z)\eta(xy^*)&=\eta(\beta_{-iz}(xy^*))=\eta(\alpha_{-iz}(x)\alpha_{i\bar{z}}(y)^*)\\&=\alpha_{-iz}(x)J_{\phi_0}T^{(1/2)-\bar{z}}\eta_{\phi_0}(y).\end{aligned}$$

Hence we obtain the analyticity of the mapping $z\longmapsto\Delta(z)\xi$, $\xi\in\mathfrak{A}_0$.

(2) First, we show the unitarity of $\Delta(it)$, $t\in\boldsymbol{R}$. Let $x_k, y_k\in\tilde{N}_{\phi_0}$, $k=1, 2$. Then

$$(2.10)\qquad \begin{aligned}&(\Delta(it)\eta(x_1y_1^*), \Delta(it)\eta(x_2y_2^*))\\&=(\eta(\alpha_t(x_1)\alpha_t(y_1)^*), \eta(\alpha_t(x_2)\alpha_t(y_2)^*))\\&=(\alpha_t(x_1)J_{\phi_0}T^{(1/2)+it}\eta_{\phi_0}(y_1), \alpha_t(x_2)J_{\phi_0}T^{(1/2)+it}\eta_{\phi_0}(y_2))\end{aligned}$$

$$
\begin{aligned}
&= (x_1\Delta_{\phi_0}^{-it}J_{\phi_0}T^{it}T^{1/2}\eta_{\phi_0}(y_1),\ x_2\Delta_{\phi_0}^{-it}J_{\phi_0}T^{it}T^{1/2}\eta_{\phi_0}(y_2))\\
&= (x_1J_{\phi_0}u_{-t}^*T^{1/2}\eta_{\phi_0}(y_1),\ x_2J_{\phi_0}u_{-t}^*T^{1/2}\eta_{\phi_0}(y_2))\\
&= (J_{\phi_0}u_{-t}^*J_{\phi_0}x_1J_{\phi_0}T^{1/2}\eta_{\phi_0}(y_1),\ J_{\phi_0}u_{-t}^*J_{\phi_0}x_2J_{\phi_0}T^{1/2}\eta_{\phi_0}(y_2))\\
&= (x_1J_{\phi_0}T^{1/2}\eta_{\phi_0}(y_1),\ x_2J_{\phi_0}T^{1/2}\eta_{\phi_0}(y_2))\\
&= (\eta(x_1y_1^*),\ \eta(x_2y_2^*)).
\end{aligned}
$$

By the density of the range and domain of $\Delta(z)$, $\Delta(it)$ is unitary. Hence, we obtain $(\Delta(it)\xi_1, \xi_2) = (\xi_1, \Delta(-it)\xi_2)$, $\xi_1, \xi_2 \in \mathfrak{A}_0$ due to the group property of $\{\Delta(z)\}_{z\in C}$. By the analyticity of two functions $z \longmapsto (\Delta(z)\xi_1, \xi_2)$ and $z \longmapsto (\xi_1, \Delta(\bar{z})\xi_2)$, (see (1)), we obtain (2),

(3) Let $x_k, y_k \in \tilde{N}_{\phi_0}$, $k=1, 2$. Then

$$
\begin{aligned}
(2.11)\qquad &(\Delta(1)\eta(x_1y_1^*),\ \eta(x_2y_2^*))\\
&= (\Delta(1/2)\Delta(1/2)\eta(x_1y_1^*),\ \eta(x_2y_2^*))\\
&= (\Delta(1/2)\eta(x_1y_1^*),\ \Delta(1/2)\eta(x_2y_2^*))\\
&= (\alpha_{-i/2}(x_1)J_{\phi_0}\eta_{\phi_0}(y_1),\ \alpha_{-i/2}(x_2)J_{\phi_0}\eta_{\phi_0}(y_2))\\
&= (J_{\phi_0}y_1J_{\phi_0}\eta_{\phi_0}(\alpha_{-i/2}(x_1)),\ J_{\phi_0}y_2J_{\phi_0}\eta_{\phi_0}(\alpha_{-i/2}(x_2)))\\
&= (y_2J_{\phi_0}T^{1/2}\eta_{\phi_0}(x_2),\ y_1J_{\phi_0}T^{1/2}\eta_{\phi_0}(x_1))\\
&= (\eta(y_2x_2^*),\ \eta(y_1x_1^*)),
\end{aligned}
$$

where we used (2) with $z=1/2$ for the second equality, and (2.9) for the third equality. Hence we obtain (3). Q. E. D.

Lemma 2.4. *Equipped with the above structure, $\mathfrak{A}_0$ is a left Hilbert algebra with its left von Neumann algebra M.*

Proof. It is easy to see that $\eta(x)=0$ implies $\eta(xy)=0$, $\eta(yx)=0$ and $\eta(x^*)=0$ due to the injectivity of η in Lemma 2.2, $(\xi\eta, \zeta) = (\eta, \xi^\sharp\zeta)$ for $\xi, \eta, \zeta \in \mathfrak{A}_0$, the boundedness of the left multiplication $\mathfrak{A}_0 \ni \eta \longmapsto \xi\eta \in \mathfrak{A}_0$, $\xi \in \mathfrak{A}_0$, the density of $\mathfrak{A}_0\mathfrak{A}_0$ in $\mathfrak{A}_0$ (which follows from the σ-weak density cf $\tilde{N}_{\phi_0}$ in M). Now, we prove the preclosedness of $\xi \longmapsto \xi^\sharp$. We define $\xi \longmapsto \xi^\flat$ by $\eta(x)' = \eta(\beta_{-i}(x^*))$. Let $x_k, y_k \in \tilde{N}_{\phi_0}$, $k=1, 2$. Then,

$$
\begin{aligned}
(2.12)\qquad &(\eta(x_1y_1^*)^\flat,\ \eta(x_2y_2^*)) = (\Delta(1)\eta(y_1x_1^*),\ \eta(x_2y_2^*))\\
&= (\eta(y_2x_2^*),\ \eta(x_1y_1^*)) = (\eta(x_2y_2^*)^\sharp,\ \eta(x_1y_1^*)),
\end{aligned}
$$

where we used Lemma 2.3 (3) for the second equality. This shows the preclosedness of $\xi \longmapsto \xi^\sharp$.

By the σ-weak density of $\tilde{N}_{\phi_0}$ in M, $\mathfrak{A}_0$ is a σ-weakly dense

*-subalgebra of M through the left multiplication, hence the corresponding left von Neumann algebra is M. Q. E. D.

Now, we obtain a faithful normal semifinite weight ϕ on M given by the achieved left Hilbert algebra $\tilde{\mathfrak{A}}_0$ given by the left Hilbert algebra $\mathfrak{A}_0$ obtained above. By the construction, $\{\beta_t\}_{t\in\boldsymbol{R}}$ is the modular automorphism group of the weight ϕ.

Proof of the Theorem. By the construction of the mapping η given by (2.7),

$$\eta_\phi(xy^*) = I(\phi,\ \phi_0)xJ_{\phi_0}T^{1/2}\eta_{\phi_0}(y), \tag{2.13}$$

where $x,\ y\in\tilde{N}_{\phi_0}$ and $I(\phi,\ \phi_0)$ is the unitary operator identifying the standard representation spaces $H_{\phi_0} \longrightarrow H_\phi$. Hence we obtain,

$$\eta_\phi(xy^*) = xJ_{\phi,\phi_0}T^{1/2}\eta_{\phi_0}(y), \quad x,\ y\in\tilde{N}_{\phi_0}. \tag{2.14}$$

It follows that $\tilde{N}_{\phi_0}^*\subset N_\phi$ and

$$\Delta_{\phi,\phi_0}^{1/2}\eta_{\phi_0}(y) = T^{1/2}\eta_{\phi_0}(y),\ y\in\tilde{N}_{\phi_0}. \tag{2.15}$$

By construction, $\eta_{\phi_0}(\tilde{N}_{\phi_0})$ is a core for $T^{1/2}$, we obtain $\Delta_{\phi,\phi_0}^{1/2}\supset T^{1/2}$. Since both sides are self-adjoint, we obtain $\Delta_{\phi,\phi_0}^{1/2}=T^{1/2}$ and hence we obtain $u_t=T^{it}\Delta_{\phi_0}^{-it}=\Delta_{\phi,\phi_0}^{it}\Delta_{\phi_0}^{-it}=(D\phi : D\phi_0)_t$. Q. E. D.

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