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A Remark on Almost-Quaternion Substructures on the Sphere

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By

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In [4] T. Önder has solved the existence problem of almost-quaternion k-substructures on the n-sphere S^n for all n and k except for $n \equiv 1 \pmod 4 \geqq 5$ and $k=(n-1)/4$. The purpose of this note is to solve it for this exceptional case.

Theorem 1. *Let* $n \equiv 1 \pmod 4 \geqq 5$ *and* $k=(n-1)/4$. *Then* S^n *has an almost-quaternion* k*-substructure if and only if* $n=5$.

We use natural embeddings for the classical groups (see [2]): $Sp(m) \longrightarrow SU(2m)$ and $SU(m) \longrightarrow U(m) \longrightarrow SO(2m)$. We embed, respectively, $SU(m)$ and $SO(m)$ in $SU(m+1)$ and $SO(m+1)$ as the upper left hand blocks. We embed also $SO(m) \times SO(n)$ in $SO(m+n)$ by

$$(A, B) \longrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let n and k be positive integers with $4k \leqq n$. An almost-quaternion k-substructure on an orientable n-manifold M is defined to be a reduction of the structural group of the tangent bundle $T(M)$ from $SO(n)$ to the subgroup $Sp(k) \times SO(n-4k)$. Since the principal $SO(n)$-bundle associated with $T(S^n)$ is

$$SO(n) \longrightarrow SO(n+1) \longrightarrow SO(n+1)/SO(n) = S^n,$$

it follows that S^n has an almost-quaternion k-substructure if and only if the associated fibration

$$SO(n)/Sp(k) \times SO(n-4k) \longrightarrow SO(n+1)/Sp(k) \times SO(n-4k) \xrightarrow{p} S^n$$

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has a cross section. Hence Theorem 1 is equivalent to

Theorem 2. *Let* $n\equiv 1 \pmod 4 \geqq 5$ *and* $k=(n-1)/4$. *Then* p *has a cross section if and only if* $n=5$.

Proof of Theorem 2. In our case the fibration takes the form:

$$SO(n)/Sp(k)\longrightarrow SO(n+1)/Sp(k)\xrightarrow{p} S^n.$$

We have an inclusion map $j: S^5=SU(3)/SU(2)\longrightarrow SO(6)/Sp(1)$ induced by the embeddings $Sp(1)=SU(2)\longrightarrow SU(3)\longrightarrow SO(6)$. It is easily seen that j is a cross section of p for $n=5$.

Thus we will always assume that $n\geqq 9$. By the covering homotopy property of p, the fibration has a cross section if and only if

$$p_*: \pi_n(SO(n+1)/Sp(k))\longrightarrow \pi_n(S^n)=Z$$

is an epimorphism. We will prove that

$$\mathrm{Image}(p_*)=2\pi_n(S^n)$$

so that p does not have a cross section.

Consider the commutative diagram of the fibrations:

$$\begin{array}{ccccc}
Sp(k) & = & Sp(k) & & \\
\downarrow i & & \downarrow & & \\
SO(n) & \longrightarrow & SO(n+1) & \xrightarrow{p_2} & S^n \\
\downarrow & & \downarrow p_1 & & \downarrow = \\
SO(n)/Sp(k) & \xrightarrow{t} & SO(n+1)/Sp(k) & \xrightarrow{p} & S^n.
\end{array}$$

Applying the homotopy functor $\pi_*(-)$ to this, we obtain a commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & \pi_n(Sp(k)) & & & & & & \\
 & & \downarrow & & & & & & \\
\pi_n(SO(n)) & \to & \pi_n(SO(n+1)) & \xrightarrow{p_{2*}} & \pi_n(S^n) & \to & \pi_{n-1}(SO(n)) & \to & \pi_{n-1}(SO(n+1)) \to 0 \\
\downarrow & & \downarrow p_{1*} & & \downarrow = & & & & \\
\pi_n(SO(n)/Sp(k)) & \xrightarrow{t_*} & \pi_n(SO(n+1)/Sp(k)) & \xrightarrow{p_*} & \pi_n(S^n)=Z & & & & \\
\downarrow \partial & & \downarrow \Delta & & & & & & \\
\pi_{n-1}(Sp(k)) & = & \pi_{n-1}(Sp(k)) & & & & & & \\
\downarrow i_* & & \downarrow & & & & & & \\
\pi_{n-1}(SO(n)) & \to & \pi_{n-1}(SO(n+1)). & & & & & &
\end{array}$$

We use the following known results (see [1] and [3]):

(1) $\pi_{n-1}(U(2k))=Z_{(2k)!}$;

(2) $\pi_{n-1}(Sp(k))=\pi_{n-1}(Sp(\infty))=\begin{cases} 0 & \text{if } n\equiv 1 \pmod 8 \\ Z_2 & \text{if } n\equiv 5 \pmod 8 \end{cases}$;

(3) $\pi_n(Sp(k))=\pi_n(Sp(\infty))=\begin{cases} 0 & \text{if } n\equiv 1 \pmod 8 \\ Z_2 & \text{if } n\equiv 5 \pmod 8 \end{cases}$;

(4) $\pi_{n-1}(SO(n+1))=\pi_{n-1}(SO(\infty))=\begin{cases} Z_2 & \text{if } n\equiv 1 \pmod 8 \\ 0 & \text{if } n\equiv 5 \pmod 8 \end{cases}$;

(5) $\pi_{n-1}(SO(n))=\begin{cases} Z_2\oplus Z_2 & \text{if } n\equiv 1 \pmod 8 \\ Z_2 & \text{if } n\equiv 5 \pmod 8 \end{cases}$;

(6) $\pi_n(SO(n))=\begin{cases} Z_2\oplus Z_2 & \text{if } n\equiv 1 \pmod 8 \\ Z_2 & \text{if } n\equiv 5 \pmod 8 \end{cases}$;

(7) $\pi_n(SO(n+1))=\begin{cases} Z\oplus Z_2 & \text{if } n\equiv 1 \pmod 8 \\ Z & \text{if } n\equiv 5 \pmod 8. \end{cases}$

By (4) and (5), we have

(8) Image $(p_{2*})=2\pi_n(S^n)$.

By (2) and (3), p_{1*} is an isomorphism if $n\equiv 1 \pmod 8$. It follows that Image$(p_*)=$Image$(p_{2*})=2\pi_n(S^n)$ and so p does not have a cross section if $n\equiv 1 \pmod 8$.

Let $n\equiv 5 \pmod 8$. Note that i_* is the composition of the canonical homomorphisms:

$$\pi_{n-1}(Sp(k))\longrightarrow\pi_{n-1}(U(2k))\longrightarrow\pi_{n-1}(SO(n-1))\longrightarrow\pi_{n-1}(SO(n)).$$

It follows from (1), (2), (5) and an equation $(2k)!\equiv 0 \pmod 4$ that $i_*=0$. Hence, by (2), (3), (4), (6) and (7), we have a commutative diagram with exact columns:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
Z_2 & & Z & \xrightarrow{p_{2*}} & Z \\
\downarrow & & \downarrow{\scriptstyle p_{1*}} & & \downarrow{\scriptstyle =} \\
\pi_n(SO(n)/Sp(k)) & \xrightarrow{t_*} & \pi_n(SO(n+1)/Sp(k)) & \xrightarrow{p_*} & Z \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \Delta} & & \\
Z_2 & = & Z_2 & & \\
\downarrow & & \downarrow & & \\
0 & & 0. & &
\end{array}$$

Choose $x \in \pi_n(SO(n)/Sp(k))$ such that $\partial(x)$ is the generator. Then $t_*(x)$ is of finite order and $\Delta(t_*(x)) = \partial(x)$. Hence the order of $t_*(x)$ is two, so Δ splits and p_{1*} maps Z isomorphically onto a free summand. Therefore Image(p_*) = Image(p_{2*}) which equals to $2\pi_n(S^n)$ by (8), so p does not have a cross section. This completes the proof.

References

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Added in proof. I was informed from T. Önder that he also obtained Theorem 1 by using his methods [5].