Publ. RIMS, Kyoto Univ.
20 (1984), 277-296

On the XY-Model on Two-Sided Infinite Chain

By

Huzihiro ARAKI*

Abstract

The XY-model on the one-dimensional lattice, infinitely extended to both directions, is studied by a method of C^*-algebras. Return to equilibrium is found for any vector state in the cyclic representation of the equilibrium state.

A known relation between the algebras of Pauli spins and the algebra of canonical anticommutation relations (CARs) is used to obtain an explicit solution. However the C*-algebras generated by the two sets of operators become dissociated in the thermodynamic limit of an infinite one-dimensional lattice extending in both directions (in contrast to one-sided chain) and this causes a mathematical complication.

In particular, we find three features different from the case of one-sided infinite chain: (1) There are no non-trivial constant observables. (2) The (twisted) asymptotic abelian property holds only partially and not in general. (3) Return to equilibrium occurs for all values of the parameter γ and is proved by a method different from the case of one-sided chain.

§ 1. Introduction

The XY-model with the Hamiltonian

(1.1) $$H=-J\sum\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)}+(1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)}\}$$

will be studied in the C^*-algebra approach, where

$$\sigma_x^{(j)}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y^{(j)}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_z^{(j)}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are Pauli spin matrices at the lattice site $j\in\boldsymbol{Z}$ (mutually commuting for different sites j), J is real and $-1<\gamma<1$. For an observable Q belonging to the C^*-algebra $\mathfrak{A}$ generated by all Pauli spins, we study the asymptotic behavior of its time translation

(1.2) $$\alpha_t(Q)=\lim_{N\to\infty}\alpha_t^{(N)}(Q),$$

(1.3) $$\alpha_t^{(N)}(Q)=e^{itH(-N,N)}Qe^{-itH(-N,N)},$$

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, JAPAN.
* Received February 25, 1983.

where $H(-N, N)$ is H of (1. 1) with the sum extending over $j=-N, -N+1, \cdots, N-1$. (The limit $N\to\infty$ of two-sided infinite chain.) We study its expectation value in a state ψ of $\mathfrak{A}$ given by an arbitrary vector $\Psi\in\mathscr{H}_\beta$ in the cyclic representation π_β of the unique equilibrium state φ_β with the inverse temperature β.

$$\psi(\alpha_t(Q))=(\Psi,\ \pi_\beta(\alpha_t(Q))\Psi). \tag{1.4}$$

(We refer, for example, to [1] for a standard material.)

Our main result is the return to equilibrium:

Theorem 1. *For any* $Q\in\mathfrak{A}$

$$\lim_{t\to\infty}\psi(\alpha_t(Q))=\varphi_\beta(Q). \tag{1.5}$$

Such a return to equilibrium for the XY-model has been discussed in [2] and [3], but the discussion has been limited to the so-called even part of $\mathfrak{A}$ (to be defined later). More recently [4], asymptotic behavior of $\psi(\alpha_t(Q))$ for large time t has been found for arbitrary $Q\in\mathfrak{A}$ in the case of the XY-model on the one-sided infinite chain. The return to equilibrium (1. 5) does not occur in general (for $\gamma\neq 0$) due to the existence of a constant observable $B_\gamma\in\mathfrak{A}$ (i.e. $\alpha_t(B_\gamma)=B_\gamma$) given by

$$B_\gamma=\sum_{j=1}^{\infty}(-\alpha)^{j-1}\sigma_z^{(1)}\cdots\sigma_z^{(2j-2)}\times\begin{cases}\sigma_x^{(2j-1)} & \text{if}\quad 0<\gamma<1\\ \sigma_y^{(2j-1)} & \text{if}\quad -1<\gamma<0\end{cases} \tag{1.6}$$

which is in $\mathfrak{A}$ due to $\alpha\equiv(1-|\gamma|)/(1+|\gamma|)\in(0,1)$ for $\gamma\neq 0$. However the return to equilibrium do occur in any one of the following cases:

(a) $\gamma=0$, any $\Psi\in\mathscr{H}_\beta$, any $Q\in\mathfrak{A}$.
(b) $\gamma\neq 0$, any $\Psi\in\mathscr{H}_\beta$ satisfying $\psi(B_\gamma)=0$, any $Q\in\mathfrak{A}$.
(c) $\gamma\neq 0$, any $\Psi\in\mathscr{H}_\beta$, any Q satisfying $\Theta(Q)=Q$.

Here Θ is the automorphism of $\mathfrak{A}$ satisfying

$$\Theta(\sigma_x^{(j)})=-\sigma_x^{(j)},\ \Theta(\sigma_y^{(j)})=-\sigma_y^{(j)},\ \Theta(\sigma_z^{(j)})=\sigma_z^{(j)} \tag{1.7}$$

for all j (i.e. 180° rotation of all spins around z-axis), where j is restricted to the natural numbers $\boldsymbol{N}$ for one-sided chain. In contrast, the result given by Theorem 1 for the two-sided chain is simple. However it disguises the complexity in its derivation.

The derivation of the results described above uses an explicit solution of the model on a finite chain in terms of a relation between the algebra $\mathfrak{A}^s$ of spins and the algebra $\mathfrak{A}^{\mathrm{CAR}}$ of CAR's [5]. In the case of the two-sided chain, the two algebras become distinct C^*-subalgebras of a bigger C^*-algebra $\hat{\mathfrak{A}}$ and this brings about some complication, which we will describe in the next section for a general one-dimensional spin lattice.

§ 2. Spin-Fermion Correspondence

Let $\mathfrak{A}^{\mathrm{CAR}}$ be the C^*-algebra generated by c_j and c_j^* ($j \in \boldsymbol{Z}$) satisfying the following CAR's:

(2.1) $$[c_j, c_k]_+ = [c_j^*, c_k^*]_+ = \mathbf{0},$$

(2.2) $$[c_j, c_k^*]_+ = \delta_{jk}\mathbf{1}. \quad ([A, B]_+ = AB + BA.)$$

Let Θ and Θ_- be automorphisms of $\mathfrak{A}^{\mathrm{CAR}}$ satisfying

(2.3) $$\Theta(c_j) = -c_j, \ \Theta(c_j^*) = -c_j^* \quad (j \in \boldsymbol{Z}),$$

(2.4) $$\Theta_-(c_j) = \begin{cases} c_j \\ -c_j \end{cases} \quad \Theta_-(c_j^*) = \begin{cases} c_j^* & (j \geqq 1), \\ -c_j^* & (j \leqq 0). \end{cases}$$

They satisfy $\Theta^2 = \Theta_-^2 = \mathrm{id.}$, $\Theta\Theta_- = \Theta_-\Theta$. Let $\hat{\mathfrak{A}}$ be the C^*-algebra generated by $\mathfrak{A}^{\mathrm{CAR}}$ and an element T satisfying

(2.5) $$T = T^*, \ T^2 = \mathbf{1},$$

(2.6) $$TxT = \Theta_-(x), \ x \in \mathfrak{A}^{\mathrm{CAR}}.$$

(The C^* crossed product of $\mathfrak{A}^{\mathrm{CAR}}$ by the $\boldsymbol{Z}_2$-action Θ_-.)

Let $\mathfrak{A}^s$ be the C^*-subalgebra of $\hat{\mathfrak{A}}$ generated by the following Pauli spin matrices $\sigma_\alpha^{(j)}$ ($\alpha = x, y, z$) on lattice sites $j \in \boldsymbol{Z}$:

(2.7) $$\sigma_z^{(j)} = 2c_j^* c_j - \mathbf{1},$$

(2.8) $$\sigma_x^{(j)} = TS^{(j)}(c_j + c_j^*), \ \sigma_y^{(j)} = TS^{(j)} i(c_j - c_j^*),$$

(2.9) $$S^{(j)} \equiv \begin{cases} \sigma_z^{(1)} \cdots \sigma_z^{(j-1)} & \text{if } j > 1, \\ \mathbf{1} & \text{if } j = 1, \\ \sigma_z^{(0)} \cdots \sigma_z^{(j)} & \text{if } j < 1. \end{cases}$$

They satisfy the following relations which characterize $\mathfrak{A}^s$ as a C^*-algebra.

(2.10) $$(\sigma_\alpha^{(j)})^2 = \mathbf{1} \quad (\alpha = x, y, z),$$

(2.11) $$\sigma_\alpha^{(j)} \sigma_\beta^{(j)} = -\sigma_\beta^{(j)} \sigma_\alpha^{(j)} = i\sigma_\gamma^{(j)}$$
$$((\alpha, \beta, \gamma) = \text{any cyclic permutation of } (x, y, z)),$$

(2.12) $$[\sigma_\alpha^{(j)}, \sigma_\beta^{(k)}]=0 \quad \text{if} \quad j\neq k \quad (\alpha, \beta=x, y, z).$$

The automorphisms Θ and Θ_- are extended to $\hat{\mathfrak{A}}$ such that $\Theta(T)=T$, $\Theta_-(T)=T$. We define even $(+)$ and odd $(-)$ parts:

(2.13) $$\hat{\mathfrak{A}}_\pm=\{x\in\hat{\mathfrak{A}},\ \Theta(x)=\pm x\},$$

(2.14) $$\mathfrak{A}_\pm^{\mathrm{CAR}}=\mathfrak{A}^{\mathrm{CAR}}\cap\hat{\mathfrak{A}}_\pm,\quad \mathfrak{A}_\pm^s=\mathfrak{A}^s\cap\hat{\mathfrak{A}}_\pm.$$

We have

(2.15) $$\mathfrak{A}_+^s=\mathfrak{A}_+^{\mathrm{CAR}},\quad \mathfrak{A}_-^s=T\mathfrak{A}_-^{\mathrm{CAR}}.$$

Clearly T and $\mathfrak{A}^s$ generates $\hat{\mathfrak{A}}$.

§ 3. Time Evolution

Let $\mathfrak{A}^s(I)$ be the C^*-subalgebra of $\mathfrak{A}^s$ generated by $\sigma_{x,y,z}^{(j)}$ with j belonging to a non-empty subset I of lattice points (i.e. $I\subset\boldsymbol{Z}$). Let $\Phi(I)\in\mathfrak{A}^s(I)$ (a many-body interaction potential between spins of sites in a non-empty *finite* subset I of $\boldsymbol{Z}$) and

(3.1) $$H_N=H([-N, N]),\quad H(I)=\sum_{\Lambda\subset I}\Phi(\Lambda)$$

(the total Hamiltonian for the interval $[-N, N]$).

We make the following assumptions in general.

(1) Evenness: $\Theta(\Phi(I))=\Phi(I)\quad(I\subset\boldsymbol{Z})$.

(2) Bounded surface energy: For disjoint finite subsets I and J, we denote

(3.2a) $$W(I, J)\equiv\sum_K\{\Phi(K):\ K\subset I\cup J,\ K\not\subset I,\ K\not\subset J\}.$$

Then, either for a finite interval I_1 and any subset I_2 of the complement of I_1, or for $I_1=(-\infty, j]$ and $I_2=[j+1, \infty)$ with any $j\in\boldsymbol{Z}$, the following limit exists

(3.2b) $$\lim_{N\to\infty} W(I_1\cap[-N, N], I_2\cap[-N, N])=W(I_1, I_2),$$

and

(3.3) $$\sup_N \|W([-N, N], (-\infty, -N)\cup(N, \infty))\|<\infty.$$

Under assumption (2), the following limit exists and defines a continuous one-parameter group of automorphisms of $\hat{\mathfrak{A}}$:

(3.4) $$\alpha_t(x)=\lim_{N\to\infty} e^{iH_Nt}xe^{-iH_Nt}.\qquad (x\in\hat{\mathfrak{A}})$$

The existence of limit for $x\in\mathfrak{A}^s$ is by [6] and for T by the computation below, see (3.14) and (3.15). Due to the evenness assumption (1), $\Phi(I)$ belongs to $\mathfrak{A}^s_+ = \mathfrak{A}^{\mathrm{CAR}}_+$ and hence

$$\alpha_t(\mathfrak{A}^s) = \mathfrak{A}^s, \quad \alpha_t(\mathfrak{A}^{\mathrm{CAR}}) = \mathfrak{A}^{\mathrm{CAR}}, \tag{3.5}$$

$$\alpha_t \Theta = \Theta \alpha_t. \tag{3.6}$$

In the case of the two-sided *XY*-model, we have

$$\begin{aligned} \Phi(\{j, j+1\}) &= -J\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)}\} \\ &= 2J\{c_j^* c_{j+1} + c_{j+1}^* c_j + \gamma(c_j^* c_{j+1}^* + c_{j+1} c_j)\}. \end{aligned} \tag{3.7}$$

($\Phi(I) = \mathbf{0}$ for all other I.) A computation of [4] yields

$$\alpha_t(B(h)) = B(e^{2JiK_\gamma t} h), \tag{3.8}$$

where we have used the following notations:

$$c(f) = \sum_j f_j c_j, \quad c^*(f) = \sum_j f_j c_j^*, \tag{3.9}$$

$$f = (f_j)_{j \in \mathbf{Z}} \in l_2(\mathbf{Z}), \tag{3.10}$$

$$B(h) = c^*(f) + c(g) \quad \text{for } h = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{3.11}$$

$$K_\gamma = \begin{bmatrix} U + U^* & \gamma(U - U^*) \\ -\gamma(U - U^*) & -(U + U^*) \end{bmatrix}, \tag{3.12}$$

$$(Uf)_j = f_{j+1}, \quad (U^* f)_j = f_{j-1}. \tag{3.13}$$

The time evolution of T is given by

$$\alpha_t(T) = TV_t, \tag{3.14}$$

$$\begin{aligned} V_t &= \lim_{N \to \infty} Te^{iH_N t} Te^{-iH_N t} \\ &= \lim_{N \to \infty} e^{i\Theta_-(H_N)t} e^{-iH_N t} \\ &= \sum_{j=0}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \alpha_{t_n}(A) \cdots \alpha_{t_1}(A) \end{aligned} \tag{3.15}$$

by the theory of inner perturbation of automorphism groups (for example, see [1]), where

$$\begin{aligned} A = \lim_{N \to \infty} (\Theta_-(H_N) - H_N) &= \Theta_- W((-\infty, 0], [1, \infty)) \\ &\quad - W((-\infty, 0], [1, \infty)) \end{aligned} \tag{3.16}$$

due to the split

$$H_N = H([-N, 0]) + H([1, N]) + W([-N, 0], [1, N]), \tag{3.17}$$

and the relation $\Theta_-(x) = x$ for $x = H([1, N]) \in \mathfrak{A}^s([1, \infty))$, $\Theta_-(y) = \Theta(y) = y$ for $y = H([-N, 0]) \in \mathfrak{A}^s((-\infty, 0])$. Note that V_t is a unitary operator (both V_t and V_t^* are strong limits of unitaries) belonging to $\mathfrak{A}^s_+ = \mathfrak{A}^{\text{CAR}}_+$ and $\Theta(V_t) = V_t^*$ (by the second line of (3.15)) so that $(TV_t)^2 = \mathbf{1}$.

§ 4. Equilibrium States and Associated Representations

There exists an (α_t, β)-KMS state $\hat{\varphi}_\beta$ of $\hat{\mathfrak{A}}$ as a weak accumulation point of the Gibbs state for H_N as $N \to \infty$.

Let $\hat{\Theta}_-$ be the automorphism of $\hat{\mathfrak{A}}$ (the dual action of Θ_-) satisfying

$$\hat{\Theta}_-(T) = -T, \ \hat{\Theta}_-(a) = a \ \ (a \in \mathfrak{A}^{\text{CAR}}). \tag{4.1}$$

Such $\hat{\Theta}_-$ exists as an automorphism of $\hat{\mathfrak{A}}$. Since $H_N \in \mathfrak{A}^{\text{CAR}}$, it is $\hat{\Theta}_-$-invariant and hence

$$\hat{\Theta}_- \alpha_t = \alpha_t \hat{\Theta}_- \tag{4.2}$$

and $\hat{\varphi}_\beta$ is $\hat{\Theta}_-$-invariant.

Since Θ and $\hat{\Theta}_-$ commute, we have the decomposition

$$\hat{\mathfrak{A}} = \sum_{\sigma,\sigma'} \hat{\mathfrak{A}}_{\sigma,\sigma'}, \tag{4.3}$$

$$\hat{\mathfrak{A}}_{\sigma,\sigma'} = \{x \in \hat{\mathfrak{A}};\ \Theta(x) = \sigma x,\ \hat{\Theta}_-(x) = \sigma' x\}, \tag{4.4}$$

where σ and σ' are $+$ or $-$. We have

$$\hat{\mathfrak{A}}_{\sigma+} = \mathfrak{A}^{\text{CAR}}_\sigma,\ \hat{\mathfrak{A}}_{\sigma-} = T\mathfrak{A}^{\text{CAR}}_\sigma. \quad (\sigma = +,\ -) \tag{4.5}$$

By (4.2), $(\hat{\varphi}_\beta + \hat{\varphi}_\beta \circ \hat{\Theta}_-)/2$ is a $\hat{\Theta}_-$-invariant (α_t, β)-KMS state of $\hat{\mathfrak{A}}$ and hence we assume that $\hat{\varphi}_\beta$ is already $\hat{\Theta}_-$-invariant. By (3.6) and $[\Theta, \hat{\Theta}_-] = \mathbf{0}$, we may also assume that $\hat{\varphi}_\beta$ is Θ-invariant. Its restrictions to $\mathfrak{A}^s$ and $\mathfrak{A}^{\text{CAR}}$ are (α_t, β)-KMS states and, as such, are unique by the assumption (2). ([7], [8]) Hence such $\hat{\varphi}_\beta$ is the unique $\hat{\Theta}_-$-invariant extension of the unique (α_t, β)-KMS state of $\mathfrak{A}^{\text{CAR}}$ and at the same time the unique $\Theta\hat{\Theta}_-$-invariant extension of the unique (α_t, β)-KMS state of $\mathfrak{A}^s$. In particular, the unique (α_t, β)-KMS state of $\mathfrak{A}^s$ can be obtained as the restriction (to $\mathfrak{A}^{\text{CAR}}$) of the unique $\hat{\Theta}_-$-invariant extension (to $\hat{\mathfrak{A}}$) of the unique (α_t, β)-KMS state of $\mathfrak{A}^{\text{CAR}}$.

By the Θ- and Θ_--invariance, $\hat{\varphi}_\beta$ is 0 on $\hat{\mathfrak{A}}_{\sigma\sigma'}$ except for $\hat{\mathfrak{A}}_{++} = \mathfrak{A}^{\text{CAR}}_+$ and hence explicitly determined by $\varphi^{\text{CAR}}_\beta$ on $\mathfrak{A}^{\text{CAR}}_+$. The cyclic representation $\hat{\pi}_\beta$ of $\hat{\mathfrak{A}}$ associated with $\hat{\varphi}_\beta$ (on a Hilbert space $\hat{\mathscr{H}}_\beta$ with a cyclic vector $\hat{\Phi}_\beta$ yielding $\hat{\varphi}_\beta$) can also be constructed from the cyclic representations of $\mathfrak{A}^{\text{CAR}}$ as follows:

Let $(\mathscr{H}^{\text{CAR}}_\beta, \pi^{\text{CAR}}_\beta, \Phi_\beta)$ and $(\mathscr{H}^{\text{CAR}}_{\beta,\Theta_-}, \pi^{\text{CAR}}_{\beta,\Theta_-}, \Phi_{\beta,\Theta_-})$ be triplets of the

Hilbert space, the cyclic representation of $\mathfrak{A}^{\mathrm{CAR}}$ and the cyclic vector associated with states $\varphi_\beta^{\mathrm{CAR}}$ and $\varphi_{\beta,\Theta_-}^{\mathrm{CAR}}=\varphi_\beta^{\mathrm{CAR}}\circ\Theta_-$, respectively. The triplet for $\hat{\varphi}_\beta$ can then be constructed by the following formulas:

$$\hat{\mathscr{H}}_\beta=\mathscr{H}_\beta^{\mathrm{CAR}}\oplus\mathscr{H}_{\beta,\Theta_-}^{\mathrm{CAR}}. \tag{4.6}$$

$$\hat{\pi}_\beta(a)=\pi_\beta^{\mathrm{CAR}}(a)\oplus\pi_{\beta,\Theta_-}^{\mathrm{CAR}}(a). \quad (a\in\mathfrak{A}^{\mathrm{CAR}}) \tag{4.7}$$

$$\begin{aligned}\hat{\pi}_\beta(T)&(\pi_\beta^{\mathrm{CAR}}(a)\Phi_\beta\oplus\pi_{\beta,\Theta_-}^{\mathrm{CAR}}(b)\Phi_{\beta,\Theta_-})\\ &=\pi_\beta^{\mathrm{CAR}}(\Theta_-(b))\Phi_\beta\oplus\pi_{\beta,\Theta_-}^{\mathrm{CAR}}(\Theta_-(a))\Phi_{\beta,\Theta_-}.\end{aligned} \tag{4.8}$$

$$\hat{\Phi}_\beta=\Phi_\beta\oplus\mathbf{0}. \tag{4.9}$$

We note that two representations $(\pi_\beta^{\mathrm{CAR}}, \mathscr{H}_\beta^{\mathrm{CAR}})$ and $(\pi_{\beta,\Theta_-}^{\mathrm{CAR}}, \mathscr{H}_{\beta,\Theta_-}^{\mathrm{CAR}})$ are unitarily equivalent due to the following circumstances: Let

$$\alpha_t^0(a)=\lim_{N\to\infty} e^{iH_N^0t}ae^{-iH_N^0t} \quad (a\in\mathfrak{A}^{\mathrm{CAR}}), \tag{4.10}$$

$$H_N^0=H([-N, 0])+H([1, N]). \tag{4.11}$$

Let φ_β^0 be the unique (α_t^0, β)-KMS state of $\mathfrak{A}^{\mathrm{CAR}}$. By Θ_--invariance of H_N^0, $\alpha_t^0\Theta_-=\Theta_-\alpha_t^0$ and hence $\varphi_\beta^0\circ\Theta_-=\varphi_\beta^0$. Let $(\mathscr{H}^0, \pi^0, \Phi^0)$ be the triplet associated with φ_β^0. By the Θ_--invariance of φ_β^0, there exists a unitary operator $U(\Theta_-)$ on $\mathscr{H}^0$ satisfying

$$U(\Theta_-)\pi^0(a)\Phi^0=\pi^0(\Theta_-(a))\Phi^0. \tag{4.12}$$

Due to (3.17), α_t is an inner perturbation of α_t^0 by

$$W=W((-\infty, 0], [1, \infty)). \tag{4.13}$$

Let

$$U(\alpha_t^0)\pi^0(a)\Phi^0=\pi^0(\alpha_t^0(a))\Phi^0, \tag{4.14}$$

$$U(\alpha_t^0)=e^{iH^0t}. \tag{4.15}$$

Then, by theory of inner perturbations, Φ^0 is in the domain of $V=\exp-\beta(H^0+W)/2$ and $\|V\Phi^0\|^{-1}V\Phi^0\equiv\Phi_\beta$ is a cyclic vector giving rise to $\varphi_\beta^{\mathrm{CAR}}(a)=(\Phi_\beta, \pi^0(a)\Phi_\beta)$, whilst $U(\Theta_-)\Phi_\beta=\Phi_{\beta,\Theta_-}$ is a cyclic vector giving rise to $\varphi_{\beta,\Theta_-}^{\mathrm{CAR}}(a)=(\Phi_{\beta,\Theta_-}, \pi^0(a)\Phi_{\beta,\Theta_-})$. Therefore, representations $(\pi^0, \mathscr{H}^0)$, $(\pi_\beta^{\mathrm{CAR}}, \mathscr{H}_\beta^{\mathrm{CAR}})$ and $(\pi_{\beta,\Theta_-}^{\mathrm{CAR}}, \mathscr{H}_{\beta,\Theta_-}^{\mathrm{CAR}})$ are all unitarily equivalent.

§5. Asymptotic Behavior of $\mathfrak{A}^{\mathrm{CAR}}$

Theorem 2. *For* $a, b\in\mathfrak{A}^{\mathrm{CAR}}$,

$$\lim_{t\to\infty}\|[a, \alpha_t(b)]_\Theta\|=0 \tag{5.1}$$

where the graded commutator $[\ ,\ ]_\Theta$ *is defined as follows*:

(5.2) $$[a, b]_\Theta = ab - ba \text{ if } \Theta(a) = a \text{ or } \Theta(b) = b.$$

(5.3) $$[a, b]_\Theta = ab + ba \text{ if } \Theta(a) = -a \text{ and } \Theta(b) = -b.$$

A general element b *is decomposed into a sum* $b = b_+ + b_-$ *of even and odd elements* $b_\pm = (b \pm \Theta(b))/2$ *and the above formula is applied, i, e.*

(5.4) $$[a, b]_\Theta = (ab_+ - b_+a) + (ab_- - b_-\Theta(a)).$$

The proof is based on the following spectral property of K:

Lemma 3. K_γ *has a Lebesgue spectrum on the union of closed intervals* $[-2, -2\gamma]$ *and* $[2\gamma, 2]$ *with a uniform multiplicity* 4.

Proof of Lemma 3. By the Fourier expansion

(5.5) $$\tilde{f}(\theta) \equiv \sum_{l \in \mathbf{Z}} f_l e^{il\theta}, \quad f_l = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta) e^{-il\theta} d\theta,$$

U and U^* become multiplication operators

(5.6) $$(Uf)^\sim(\theta) = e^{-i\theta}\tilde{f}(\theta), \quad (U^*f)^\sim(\theta) = e^{i\theta}\tilde{f}(\theta)$$

and hence K_γ reduces to the matrix

(5.7) $$(K_\gamma h)^\sim(\theta) = \tilde{K}_\gamma(\theta)\tilde{h}(\theta), \quad \tilde{K}_\gamma(\theta) = 2\begin{bmatrix} \cos\theta & -i\gamma\sin\theta \\ i\gamma\sin\theta & -\cos\theta \end{bmatrix}.$$

From its eigenvalues $\pm 2(\cos^2\theta + \gamma^2\sin^2\theta)^{1/2}$, we obtain Lemma 3.

Proof of Theorem 2. By the absolute continuity of the spectrum of K_γ, we have

(5.8) $$\lim_{t\to\infty}[B(h_1)^*, \alpha_t(B(h_2))]_\Theta = \lim_{t\to\infty}(h_1, e^{2JiK_\gamma t}h_2) = 0$$

due to the Riemann-Lebesgue Lemma.

By Lemma 2 of [4], we have the following consequence:

Corollary 4. *For* $a \in \mathfrak{A}^{\mathrm{CAR}}$,

(5.9) $$\text{w-}\lim_{t\to\infty} \hat{\pi}_\beta(\alpha_t(a)) = \hat{\varphi}_\beta(a)\mathbf{1}.$$

In fact, $\varphi_\beta^{\mathrm{CAR}}$ being a unique KMS state, $\pi_\beta(\mathfrak{A}^{\mathrm{CAR}})$ is a factor and φ_β is Θ-invariant. Hence Lemma 2 of [4] implies (5.9) on $\mathscr{H}_\beta^{\mathrm{CAR}}$. The same holds for φ_{β,Θ_-} by the unitary equivalence of π_β and π_{β,Θ_-} (Θ_- commutes with Θ) and hence (5.9) holds also on $\mathscr{H}_{\beta,\Theta_-}^{\mathrm{CAR}}$ and hence on the whole space $\mathscr{H}_{\beta}^{\mathrm{CAR}}$.

§ 6. Asymptotic Behavior of $T\mathfrak{A}^{\mathrm{CAR}}$

We first obtain the asymptotic behavior of V_t in the following form:

Lemma 5. *The following limit exists (in norm topology) for any $a\in\mathfrak{A}^{\mathrm{CAR}}$ and defines automorphisms $\tilde{\Theta}_{\pm}$ of $\mathfrak{A}^{\mathrm{CAR}}$:*

$$\tilde{\Theta}_{\pm}(a)=\lim_{t\to\pm\infty} V_t a V_t^*. \tag{6.1}$$

The automorphisms so defined satisfy the following relations:

$$(\Theta_-\tilde{\Theta}_{\pm})^2=\mathrm{id.},\quad \tilde{\Theta}_{\pm}\Theta=\Theta\tilde{\Theta}_{\pm}. \tag{6.2}$$

Proof. By (3.15) and (2.6), we have

$$V_t a V_t^*=\Theta_-\alpha_t\Theta_-\alpha_{-t}(a). \tag{6.3}$$

Hence it is enough to prove the norm convergence of $\alpha_t\Theta_-\alpha_{-t}$ on the generating elements $B(h)$ for the existence of (6.1), for the automorphism properties of $\tilde{\Theta}_{\pm}$ and for (6.2). We have

$$\begin{aligned}\alpha_t\Theta_-\alpha_{-t}(B(h)) &= B(e^{2JiK_\gamma t}\theta_- e^{-2JiK_\gamma t}h)\\ &= B(e^{2JiK_\gamma t}e^{-2Ji(\theta_-K_\gamma\theta_-)t}\theta_- h),\end{aligned} \tag{6.4}$$

where

$$\theta_-\begin{bmatrix} f\\ g\end{bmatrix}=\begin{bmatrix}\theta_- f\\ \theta_- g\end{bmatrix},\quad (\theta_- f)_j=\begin{cases} f_j & \text{if } j\geqq 1,\\ -f_j & \text{if } j\leqq 0.\end{cases} \tag{6.5}$$

We have

$$(\theta_- U\theta_- f)_j=\begin{cases} (Uf)_j & \text{if } j\neq 0,\\ -(Uf)_j & \text{if } j=0,\end{cases} \tag{6.6}$$

$$(\theta_- U^*\theta_- f)_j=\begin{cases} (U^*f)_j & \text{if } j\neq 1,\\ -(U^*f)_j & \text{if } j=1.\end{cases} \tag{6.7}$$

Hence $\theta_-K_\gamma\theta_- - K_\gamma$ is at most rank 4. Since K_γ and its unitary transform $\theta_-K_\gamma\theta_-$ have absolutely continuous spectrum by Lemma 3,

$$\omega_{\pm}=\lim_{t\to\pm\infty} e^{2JiK_\gamma t}e^{-2Ji(\theta_-K_\gamma\theta_-)t} \tag{6.8}$$

$$\omega_{\pm}^*=\lim_{t\to\pm\infty} e^{2Ji(\theta_-K_\gamma\theta_-)t}e^{-2JiK_\gamma t} \tag{6.9}$$

both exist (in the strong topology) by Theorem X. 4. 4 (and Theorem

X. 3. 5) of [9]. Thus we have the norm convergence

$$\tilde{\Theta}_{\pm}(B(h))=\lim_{t\to\pm\infty}\Theta_{-}\alpha_{t}\Theta_{-}\alpha_{-t}(B(h))=B(\theta_{-}\omega_{\pm}\theta_{-}h). \tag{6.10}$$

We easily see the relation $\theta_{-}\omega_{\pm}\theta_{-}=\omega_{\pm}^{*}$ from (6. 8) and (6. 9) so that $(\omega_{\pm}\theta_{-})^{2}=\mathbf{1}$.

A key point in the subsequent discussion is the following lemma.

Lemma 6. *There are no non-zero operator* $x\in\pi_{\beta}^{\rm CAR}(\mathfrak{A}_{+}^{\rm CAR})''$ *satisfying*

$$x\pi_{\beta}^{\rm CAR}(a)=\pi_{\beta}^{\rm CAR}(\tilde{\Theta}_{+}(a))x \tag{6.11}$$

for all $a\in\mathfrak{A}^{\rm CAR}$. *The same holds if* $\tilde{\Theta}_{+}$ *is replaced by* $\tilde{\Theta}_{-}$. *Furthermore there are no non-zero* $x\in\pi_{\beta}^{\rm CAR}(\mathfrak{A}_{-}^{\rm CAR})^{-w}$ ($-w$ *denotes the weak closure*) *if* $\tilde{\Theta}_{+}$ *is replaced by* $\tilde{\Theta}_{\pm}\Theta$. (*The same statements hold also for* $\pi_{\beta,\Theta_{-}}^{\rm CAR}\sim\pi_{\beta}^{\rm CAR}$.)

The proof of this Lemma is given in the next section. In the rest of this section, we apply this Lemma to obtain the asymptotic behavior of $\hat{\pi}_{\beta}\{\alpha_{t}(Ta)\}$ for $a\in\mathfrak{A}^{\rm CAR}$.

Lemma 7. *For any* $a\in\mathfrak{A}^{\rm CAR}$,

$$\text{w-}\lim_{t\to\infty}\hat{\pi}_{\beta}(\alpha_{t}(Ta))=\mathbf{0}=\varphi_{\beta}(Ta)\mathbf{1}. \tag{6.12}$$

Proof. We consider two cases $\Theta(a)=\pm a$ separately. We have

$$\hat{\pi}_{\beta}(\alpha_{t}(Ta))=\hat{\pi}_{\beta}(T)\hat{\pi}_{\beta}(V_{t}\alpha_{t}(a)). \tag{6.13}$$

Let $z_{\pm}$ be the weak accumulation point of $\hat{\pi}_{\beta}(V_{t}\alpha_{t}(a))$ as $t\to\pm\infty$. Then

$$z_{\pm}\hat{\pi}_{\beta}(b)=\hat{\pi}_{\beta}(\tilde{\Theta}_{\pm}b)z_{\pm} \tag{6.14}$$

for all $b\in\mathfrak{A}^{\rm CAR}$ if $\Theta(a)=a$ whilst

$$z_{\pm}\hat{\pi}_{\beta}(b)=\hat{\pi}_{\beta}(\tilde{\Theta}_{\pm}\Theta b)z_{\pm} \tag{6.15}$$

for all $b\in\mathfrak{A}^{\rm CAR}$ if $\Theta(a)=-a$. We apply Lemma 6 for $x=z_{\pm}\in\hat{\pi}_{\beta}(\mathfrak{A}_{+}^{\rm CAR})''$ if $\Theta a=a$ and for $x=z_{\pm}\in\hat{\pi}_{\beta}(\mathfrak{A}_{-}^{\rm CAR})^{-w}$ if $\Theta a=-a$ on $\mathscr{H}_{\beta}^{\rm CAR}$ and on $\mathscr{H}_{\beta,\Theta_{-}}^{\rm CAR}$ separately (the restriction of $\hat{\pi}_{\beta}(\mathfrak{A}^{\rm CAR})''$ to $\mathscr{H}_{\beta}^{\rm CAR}$ is $\pi_{\beta}(\mathfrak{A}^{\rm CAR})''$ and the restriction of $\hat{\pi}_{\beta}(\mathfrak{A}^{\rm CAR})''$ to $\mathscr{H}_{\beta,\Theta_{-}}^{\rm CAR}$ is unitarily equivalent to it, so that Lemma 6 is applicable to each restriction) and obtain the conclusion $z_{\pm}=\mathbf{0}$. Hence

$$\text{w-}\lim_{t\to\pm\infty}\hat{\pi}_{\beta}(V_{t}\alpha_{t}(a))=\mathbf{0} \tag{6.16}$$

Thus (6.12) holds. (The second equality is due to the definition of $\hat{\varphi}_\beta$.)

Combining Corollary 4 and Lemma 7, we obtain

$$\text{(6.17)} \qquad \underset{t\to\infty}{\text{w-lim}}\ \hat{\pi}_\beta(\alpha_t(x)) = \hat{\varphi}_\beta(x)\mathbf{1}$$

for all $x\in\hat{\mathfrak{A}}$. Restricting x to $\mathfrak{A}^s$, we have the proof of Theorem 1.

§ 7. Proof of Lemma 6

Assume that a non-zero $x\in\mathfrak{M}_+\equiv\pi_\beta^{\mathrm{CAR}}(\mathfrak{A}_+^{\mathrm{CAR}})^{-w}$ satisfies (6.11). By substituting a^* into a and taking the adjoint of (6.11), we obtain

$$\text{(7.1)} \qquad x^*\pi_\beta^{\mathrm{CAR}}(\tilde{\Theta}_+(a)) = \pi_\beta^{\mathrm{CAR}}(a)x^*.$$

Combining with (6.11), we obtain

$$\text{(7.2)} \qquad x^*x\pi_\beta^{\mathrm{CAR}}(a) = \pi_\beta^{\mathrm{CAR}}(a)x^*x.$$

Therefore $x^*x\in\pi_\beta^{\mathrm{CAR}}(\mathfrak{A})''\cap\pi_\beta^{\mathrm{CAR}}(\mathfrak{A})'$. Since $\pi_\beta^{\mathrm{CAR}}(\mathfrak{A})''$ is a factor, $x^*x=\lambda\mathbf{1}$ with $\lambda>0$. ($\lambda\neq 0$ due to $x\neq\mathbf{0}$) By considering $\lambda^{-1/2}x$ instead of x, we may assume that $x^*x=\mathbf{1}$.

By a similar argument, we obtain $xx^*=c\mathbf{1}$ with $c>0$. Since $c^2\mathbf{1}=(xx^*)^2=x(x^*x)x^*=xx^*$ (by $x^*x=\mathbf{1}$), we have $c=1$, namely x is unitary.

The KMS state $\varphi_\beta^{\mathrm{CAR}}$ of the quasifree motion (3.8) is a quasifree state φ_S with $S=(1+e^{-2JK_\gamma\beta})^{-1}$ where

$$\text{(7.3)} \qquad \varphi_S(B(h_1)^*B(h_2)) = (h_1, Sh_2).$$

(Theorem 3 of [10].)

Let $\mathscr{h}$ denote the space of all $h=\begin{pmatrix} f \\ g\end{pmatrix}$ (the test function space for $B(\cdot)$ of the CAR algebra $\mathfrak{A}^{\mathrm{CAR}}$): $\mathscr{h}=l_2\oplus l_2$. Then the cyclic representation π_β^{CAR} of $\mathfrak{A}^{\mathrm{CAR}}$ on $\mathscr{H}_\beta^{\mathrm{CAR}}$ associated with the quasifree state $\varphi_S(=\varphi_\beta^{\mathrm{CAR}})$ can be viewed as the restriction of an irreducible representation $\pi_{P_S}^1$ of a CAR algebra $\mathfrak{A}_1^{\mathrm{CAR}}$ with the test function space $\mathscr{h}\oplus\mathscr{h}$ of twice size for $B(\cdot)$ on the same representation space $\mathscr{H}_\beta^{\mathrm{CAR}}$, where $B(h\oplus 0)$ of $\mathfrak{A}_1^{\mathrm{CAR}}$ identified with $B(h)$ of $\mathfrak{A}^{\mathrm{CAR}}$ and $\pi_{P_S}^1(B(0\oplus h))$ of $\mathfrak{A}_1^{\mathrm{CAR}}$ identified with $U(\Theta)$ times an element of the commutant of $\pi_\beta^{\mathrm{CAR}}(\mathfrak{A}^{\mathrm{CAR}})$ of the form $J\pi_\beta^{\mathrm{CAR}}(B(h_1))J$ with J denoting the modular conjugation and h_1 depending on h. The cyclic vector Φ_β giving rise to the state $\varphi_S(=\varphi_\beta^{\mathrm{CAR}})$ yield a pure state $\varphi_{P_S}^1$ of $\mathfrak{A}_1^{\mathrm{CAR}}$ characterized by the following (basis) projection operator P_S on $\mathscr{h}\oplus\mathscr{h}$:

$$P_S=\begin{bmatrix} S & \{S(1-S)\}^{1/2} \\ \{S(1-S)\}^{1/2} & 1-S \end{bmatrix}. \tag{7.4}$$

(Lemma 4.5 and proof of Theorem 3 of [10].)

By (6.11), the unitary transformation $\mathrm{Ad}\, x$ on $\mathfrak{A}_1^{\mathrm{CAR}}$ will give rise to a Bogolubov automorphism through the following Bogolubov transformation on $\mathscr{k}\oplus\mathscr{k}$ because $x\in\pi_\beta^{\mathrm{CAR}}(\mathfrak{A}_+^{\mathrm{CAR}})''$ commutes with both $\pi_\beta^{\mathrm{CAR}}(\mathfrak{A}^{\mathrm{CAR}})'$ and $U(\Theta)$:

$$U_+=\begin{bmatrix} \omega_+^* & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{7.5}$$

A necessary and sufficient condition for the Bogolubov automorphism of $\mathfrak{A}_1^{\mathrm{CAR}}$ by U_+ to be implementable in a Fock representation given by a pure state $\varphi_{P_S}^1$ is that $(1-P_S)U_+P_S$ is in the Hilbert Schmidt class (Theorem 7 of [10]) or equivalently (Proof of Theorem 7 of [10])

$$\|P_S-U_+P_SU_+^*\|_{\mathrm{H.S.}}<\infty, \tag{7.6}$$

where H. S. denotes the Hilbert Schmidt norm.

In the present situation, the Bogolubov automorphism is actually implemented by a unitary operator x on $\mathscr{H}_\beta^{\mathrm{CAR}}$. We derive a contradiction by disproving (7.6), thereby showing non-existence of x.

By (7.3) and (7.4), we have

$$P_S=\begin{bmatrix} (1+e^{-2JK_\gamma\beta})^{-1} & (2\cosh JK_\gamma\beta)^{-1} \\ (2\cosh JK_\gamma\beta)^{-1} & (1+e^{2JK_\gamma\beta})^{-1} \end{bmatrix} \tag{7.7}$$

and

$$P_S-U_+P_SU_+^*=\begin{bmatrix} B & s_+^*(\cosh JK_\gamma\beta)^{-1} \\ (\cosh JK_\gamma\beta)^{-1}s_+ & \mathbf{0} \end{bmatrix}, \tag{7.8}$$

where

$$B=(1+e^{-2JK_\gamma\beta})^{-1}-(1+e^{-2J\omega_+^*K_\gamma\omega_+\beta})^{-1}, \tag{7.9}$$

$$s_+=(1-\omega_+). \tag{7.10}$$

We now have

$$\|P_S-U_+P_SU_+^*\|_{\mathrm{H.S.}}^2=\mathrm{tr}\, B^2+2\,\mathrm{tr}(s_+^*(\cosh JK_\gamma\beta)^{-2}s_+). \tag{7.11}$$

Since $\|K_\gamma\|\leqq 2$, the second term is larger than

$$2(\cosh 2J\beta)^{-1}\,\mathrm{tr}\, s_+^*s_+. \tag{7.12}$$

We shall show that this is infinite in the next Lemma, completing the proof for the case of $\tilde{\Theta}_+$. The proof for $\tilde{\Theta}_-$ is obtained exactly in the same manner, using

$$s_- = (\mathbf{1} - \omega_-) \tag{7.13}$$

instead of s_+ and U_- instead of U_+, where U_- is defined by (7.5) with ω_+ replaced by ω_-.

In the case of $x \in \pi_\beta^{\mathrm{CAR}}(\mathfrak{A}_-^{\mathrm{CAR}})''$, x anticommutes with $U(\Theta)$ and hence

$$\begin{aligned} x\pi_{P_S}^1(B(\mathbf{0}\oplus h))x^* &= -\pi_{P_S}^1(B(\mathbf{0}\oplus h)) \\ &= U(\Theta)\pi_{P_S}^1(B(\mathbf{0}\oplus h))U(\Theta)^*. \end{aligned} \tag{7.14}$$

Here the second equality is due to the circumstance that $\pi_{P_S}^1(B(\mathbf{0}\oplus h))$ is the product of $U(\Theta)$ with $J\pi_\beta^{\mathrm{CAR}}(B(h_1))J$ and $U(\Theta)$ commutes with the modular conjugation J. Since

$$\begin{aligned} x\pi_{P_S}^1(B(h\oplus\mathbf{0}))x^* &= \pi_\beta^{\mathrm{CAR}}(\tilde{\Theta}_\pm\Theta(B(h))) \\ &= U(\Theta)\pi_{P_S}^1(B(\omega_\pm^* h\oplus\mathbf{0}))U(\Theta)^*, \end{aligned} \tag{7.15}$$

we have the situation that $\mathrm{Ad}(U(\Theta)x)$ induces the Bogolubov automorphism of $\mathfrak{A}_1^{\mathrm{CAR}}$ given by $U_\pm$. Therefore the same contradiction arises also in this case and the proof is complete, once we prove the following:

Lemma 8. $\mathrm{tr}\ s_+^* s_+ = \mathrm{tr}\ s_-^* s_- = \infty$.

Proof. Let $\tilde{f}(\theta)$ be defined as before and $\tilde{h}(\theta) = \begin{pmatrix} \tilde{f}(\theta) \\ \tilde{g}(\theta) \end{pmatrix}$. Let $r_\pm^\gamma(\theta)$ and $k_\gamma(\theta)$ be defined as follows:

$$r_+^0(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad r_-^0(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad k_0(\theta) = \cos\theta. \tag{7.16}$$

$$r_\pm^\gamma(\theta) = (2k_\gamma(\theta))^{-1} \begin{bmatrix} k_\gamma(\theta) \pm \cos\theta & \mp i\gamma\sin\theta \\ \pm i\gamma\sin\theta & k_\gamma(\theta) \mp \cos\theta \end{bmatrix}, \tag{7.17}$$

$$k_\gamma(\theta) = (\cos^2\theta + \gamma^2\sin^2\theta)^{1/2}. \quad (\gamma \neq 0) \tag{7.18}$$

The two operators $r_\pm^\gamma(\theta)$ are spectral projections of $\tilde{K}_\gamma(\theta)$ satisfying

$$r_+^\gamma(\theta) + r_-^\gamma(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.19}$$

$$\tilde{K}_\gamma(\theta) r_\pm^\gamma(\theta) = \pm 2k_\gamma(\theta) r_\pm^\gamma(\theta). \tag{7.20}$$

Let h_j $(j=1, 2)$ have only finite number of non-zero components. Then $\tilde{h}_j(\theta)$ consists of polynomials of $e^{i\theta}$ and $e^{-i\theta}$ and hence is an entire function with a period 2π. The operators $r_{\pm}^{\gamma}(\theta)$ as well as $k_{\gamma}(\theta)$ are holomorphic near the real axis and have a period 2π. We shall compute the limit of

$$(7.21)\qquad (h_1,\ e^{i2JK_\gamma t}\frac{1-\theta_-}{2}e^{-i2JK_\gamma t}h_2)$$
$$=\lim_{\varepsilon\to+0}\sum_{\sigma,\sigma'}\int_0^{2\pi}\overline{(r_\sigma^\gamma\tilde{h}_1)}(\theta_1)I_{\gamma\varepsilon}^{\sigma\sigma'}(\theta_1,\ t)\frac{d\theta_1}{2\pi},$$

$$(7.22)\qquad I_{\gamma\varepsilon}^{\sigma\sigma'}(\theta_1,\ t)=\int_0^{2\pi}e^{4Ji(\sigma k_\gamma(\theta_1)-\sigma' k_\gamma(\theta_2))t}F_\varepsilon(\theta_2-\theta_1)\,r_{\sigma'}^\gamma(\theta_2)\,\tilde{h}_2(\theta_2)\frac{d\theta_2}{2\pi},$$

as $t\to\pm\infty$ (which will be (7.27)), where σ and σ' are $+$ or $-$ and

$$(7.23)\qquad F_\varepsilon(\theta_2-\theta_1)=(1-e^{i(\theta_2-\theta_1)-\varepsilon})^{-1}=\sum_{l=0}^{-\infty}e^{i\theta_1 l}e^{-i\theta_2 l}e^{\varepsilon l}.$$

(We have used the fact that $(1-\theta_-)/2$ is the multiplication of the characteristic function $\chi_-(l)$ for $(-\infty,\ 0]$ and is a limit of the multiplication operator θ_-^ε of $e^{\varepsilon l}\chi_-(l)$ as $\varepsilon\to+0$.)

First, note that L_2 norm of $I_{\gamma\varepsilon}^{\sigma\sigma'}$ is bounded by $||h_2||$ due to $||\theta_-^\varepsilon||=1$ and $||r_{\sigma'}^\gamma||=1$. Hence a small interval of θ_1 gives only a small correction which tends to 0 as the relevant interval vanishes.

Second, by the periodicity, we may shift the range of θ_2 integration so that it is centered around θ_1. F_ε is then smooth and bounded even in the limit of $\varepsilon\to 0$ except for a neighbourhood (of any desired small length) of $\theta_2=\theta_1$. Hence the contribution from outside a small neighbourhood of $\theta_2=\theta_1$ tends to 0 as $t\to\pm\infty$ by the Riemann-Lebesgue Lemma. This will then imply that the contribution to (7.21) also tends to 0 by the dominated convergence theorem.

By the holomorphy, we may shift the θ_2-integration by $\pm i\eta(\theta_2)$ ($\eta(\theta_2)\geqq 0$) in the neighbourhood of $\theta_2=\theta_1$. The shift by $+i\eta(\theta_2)$ does not cause any change to the integral, whilst the shift by $-i\eta(\theta_2)$ yields an additional term (for $\varepsilon<\eta(\theta_2)$), which is in the limit of $\varepsilon\to 0$ given by

$$(7.24)\qquad e^{4Ji\sigma k_\gamma(\theta_1)(1-\delta_{\sigma\sigma'})t}r_{\sigma'}^\gamma(\theta_1)\,\tilde{h}_2(\theta_1)\equiv A_t^{\sigma\sigma'}(\theta_1).$$

Let

$$(7.25)\qquad \bar{\sigma}=\bar{\sigma}_{Jt}(\theta)\equiv\mathrm{sign}\,(Jt(d/d\theta)k_\gamma(\theta)).$$

Then the θ_2-integral after the shift by $-i\sigma'\bar{\sigma}\eta(\eta>0)$ tends to 0 as $\varepsilon\to+0$ and $t\to\infty$ (with a definite sign of t) due to the large t exponential damping. The set of θ_1 for which $(d/d\theta)k_\gamma(\theta)=0$ at

$\theta=\theta_1$ is of measure 0 and can be neglected. Therefore we have

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\{I_{\gamma\varepsilon}^{\sigma\sigma'}(\theta_1, t)-\delta_{\sigma',\bar{\sigma}_{Jt}(\theta_1)}A_t^{\sigma\sigma'}(\theta_1)\}=0. \tag{7.26}$$

Since terms in (7.26) have uniformly bounded L_2 norms, we can use these estimates of $I_{\gamma\varepsilon}^{\sigma\sigma'}$ in evaluating (7.21).

If $\sigma\neq\sigma'$, then the exponential oscillation of $A_t^{\sigma\sigma'}$ makes (7.21) vanish in the limit of $t\to\infty$. Hence we obtain

$$(h_1, q_\pm h_2)=\sum_\sigma (r_\sigma^\gamma\tilde{h}_1, \tilde{q}_{\pm\sigma}^J r_\sigma^\gamma\tilde{h}_2), \tag{7.27}$$

where $q_\pm=(1-\omega_\pm\theta_-)/2$ and

$$(\tilde{q}_\pm^J h)^\sim(\theta)=\begin{cases}\tilde{h}(\theta) & \text{if } \pm J(d/d\theta)k_\gamma(\theta)>0,\\ 0 & \text{otherwise.}\end{cases} \tag{7.28}$$

The sign function $\sigma_\gamma(\theta)=\text{sign }k'_\gamma(\theta)$ is given by

$$\begin{aligned}\sigma_\gamma(\theta)&=-\text{sign}(\cos\theta\ \sin\theta)\\ &=\begin{cases}+ & \text{if } -\pi/2<\theta<0 \pmod{\pi},\\ - & \text{if } 0<\theta<\pi/2 \pmod{\pi}\end{cases}\end{aligned} \tag{7.29}$$

if $\gamma\neq 0$ and

$$\begin{aligned}\sigma_\gamma(\theta)&=-\text{sign}(\sin\theta)\\ &=\begin{cases}+ & \text{if } -\pi<\theta<0 \pmod{2\pi},\\ - & \text{if } 0<\theta<\pi \pmod{2\pi}\end{cases}\end{aligned} \tag{7.30}$$

if $\gamma=0$. For each θ, $q_\pm$ selects either r_+^γ or r_-^γ and hence

$$(h_1, q_\pm h_2)=\int(\tilde{h}_1(\theta), r_{\pm\sigma(J)\sigma_\gamma(\theta)}^\gamma(\theta)\tilde{h}_2(\theta))\,d\theta/(2\pi), \tag{7.31}$$

where $\sigma(J)=\text{sign }J$.

We can now compute

$$\text{tr } s_+^* s_+=\text{tr}(2-\omega_+-\omega_+^*). \tag{7.32}$$

Let $t_\pm=(\mathbf{1}\pm\theta_-)/2$. Since $\omega_+^*=\theta_-\omega_+\theta_-$

$$\begin{aligned}\text{tr } s_+^*s_+&=2\text{tr}\{t_+(\mathbf{1}-\omega_+\theta_-)t_++t_-(\mathbf{1}+\omega_+\theta_-)t_-\}\\ &=4\text{tr}\{t_+q_+t_++t_-(\mathbf{1}-q_+)t_-\}.\end{aligned} \tag{7.33}$$

The trace can be split into the trace of the 2×2 matrices and the trace on l_2. Since the matrix traces of r_+^γ and $\mathbf{1}-r_+^\gamma=r_-^\gamma$ are both 1, the trace in (7.33) is equal to the trace of $t_++t_-=\mathbf{1}$ on l_2, which is infinite. This completes the proof of Lemma 8.

Corollary 9. $q_{+}+q_{-}=1$, $\omega_{-}=-\omega_{+}$, $\tilde{\Theta}_{+}=\tilde{\Theta}_{-}\Theta$.

Remark 10. By $\Gamma K_{\gamma}\Gamma=-K_{\gamma}$, we have $\Gamma r_{\pm}^{\gamma}\Gamma=r_{\mp}^{\gamma}$ for the multiplication operator $r_{\pm}^{\gamma}$ of $r_{\pm}^{\gamma}(\theta)$. Since Γ changes θ to $-\theta$ (due to $\Gamma(f\oplus g)=\bar{g}\oplus\bar{f}$ and $\bar{f}(\theta)=\bar{f}(-\theta)$), we have $\Gamma q_{\pm}\Gamma=q_{\pm}$. Actually this is required in order that $\omega_{\pm}\theta_{-}=1-2q_{\pm}$ induces Bogolubov automorphisms.

§ 8. Twisted Asymptotic Abelian Property

The weak asymptotic property (6. 17) implies

$$\text{w-lim}[y, \hat{\pi}_{\beta}(\alpha_t(x))]=0$$

for any $x\in\hat{\mathfrak{A}}$ and any operator y on the representation space $\hat{\mathscr{H}}_{\beta}$. On the other hand, such an asymptotic property has been derived in the case of one-sided XY-model from a twisted asymptotic abelian property (in norm) on the level of C^*-algebra $\mathfrak{A}^s$. We now discuss this problem for the two-sided XY-model.

Theorem 11. *For* Q_1, $Q_2\in\mathfrak{A}^s$, *the following holds*:

(8. 1) $$\lim_{t\to\infty}\|[Q_1, \alpha_t(Q_2)]\|=0 \quad if\ \Theta(Q_1)=Q_1,\ \Theta(Q_2)=Q_2.$$

(8. 2) $$\lim_{t\to\pm\infty}\|Q_1\alpha_t(Q_2)-\Theta_{-}\tilde{\Theta}_{+}(\alpha_t(Q_2))Q_1\|=0 \quad if\ \Theta(Q_1)=-Q_1,\ \Theta(Q_2)=Q_2.$$

(8. 3) $$\lim_{t\to\pm\infty}\|Q_1\alpha_t(Q_2)-\alpha_t(Q_2)\Theta_{-}\tilde{\Theta}_{+}(Q_1)\|=0 \quad if\ \Theta(Q_1)=Q_1,\ \Theta(Q_2)=-Q_2.$$

Proof. This is an immediate consequence of Theorem 2, Corollary 9, (3. 14) and Lemma 5. For (8. 2), note that $\Theta_{-}\alpha_t=\alpha_t\alpha_{-t}\Theta_{-}\alpha_t$, $\alpha_{-t}\Theta_{-}\alpha_t\to\Theta_{-}\tilde{\Theta}_{\pm}$ as $t\to\mp\infty$ and $\tilde{\Theta}_{-}a=\tilde{\Theta}_{+}a$ for $a\in\mathfrak{A}_{+}^{s}=\mathfrak{A}_{+}^{\mathrm{CAR}}$ due to Corollary 9.

Note that $\Theta_{-}\tilde{\Theta}_{\pm}=\lim_{t\to\pm\infty}\alpha_t\Theta_{-}\alpha_{-t}$ implies the commutativity of $\Theta_{-}\tilde{\Theta}_{\pm}$ with α_t.

Remark 12. (6. 17) may be viewed as a consequence of (8. 1), (8. 3) and Lemma 6.

Remark 13. Since α_t commutes with Θ as well as $\Theta_-\Theta_\pm$, both of which commute with each other, it might be thought that Theorem 13 has an extension to Θ-odd Q's and possibly the result could be formulated in terms of a $\boldsymbol{Z}_4(=\boldsymbol{Z}_2\times\boldsymbol{Z}_2)$ −graded commutator (two $\boldsymbol{Z}_2$ refering to Θ and $\Theta_-\tilde{\Theta}_\pm$). However it is impossible to extend $\Theta_-\tilde{\Theta}_\pm$ to a $*$-automorphism of $\mathfrak{A}$ due to the following reason:

Let ψ be an extension of $\Theta_-\tilde{\Theta}_+$ (or $\Theta_-\tilde{\Theta}_-$) to $\mathfrak{A}$. First we prove that $\psi^2\equiv\gamma$ is either an identity or $\hat{\Theta}_-$ on the basis of $\gamma(a)=a$ for all $a\in\mathfrak{A}^{\mathrm{CAR}}$. Let

(8.4) $$\gamma(T)=s+Tt,\ s,t\in\mathfrak{A}^{\mathrm{CAR}}.$$

From $T^*=T$ and $T^2=\mathbf{1}$, we obtain

(8.5) $$s^*=s,\ t^*=TtT=\Theta_-(t),$$

(8.6) $$s^2+t^*t=\mathbf{1},\ \Theta_-(s)t+ts=\mathbf{0}.$$

From $TaT=\Theta_-(a)$ and $\gamma(a)=a$ for $a\in\mathfrak{A}^{\mathrm{CAR}}$, we have $\gamma(T)a\gamma(T)=\gamma(TaT)=\Theta_-(a)$ and hence

(8.7) $$sas+t^*\Theta_-(a)t=\Theta_-(a),$$

(8.8) $$tas+\Theta_-(sa)t=\mathbf{0}.$$

By substituting sa into a of (8.7) and using (8.8) and (8.6), we obtain

(8.9) $$\Theta_-(sa)=(s^2-t^*t)as=(2s^2-\mathbf{1})as.$$

Setting $a=s^{n-1}$, we obtain

(8.10) $$\Theta_-(s^n)=(2s^2-\mathbf{1})s^n.$$

Substituting (8.10) for $n=1$ and 2 into $\Theta_-(s^2)=\Theta_-(s)^2$, we obtain $s^2(2s^2-\mathbf{1})(\mathbf{1}-s^2)=\mathbf{0}$. Substituting $(\mathbf{1}-s^2)=t^*t$, we obtain $A^*A=\mathbf{0}$ for $A=ts^2(2s^2-\mathbf{1})$ and hence $A=\mathbf{0}$. It then implies $BB^*=\mathbf{0}$ for $B=ts(2s^2-\mathbf{1})=t\Theta_-(s)$ and hence $B=\mathbf{0}$. This implies $t^*s=\Theta_-(B)=\mathbf{0}$ and hence $st=(t^*s)^*=\mathbf{0}$. Substituting (8.10) with $n=1$ into the second equation of (8.6), we obtain $(2s^2-\mathbf{1})st+ts=\mathbf{0}$ and hence $ts=\mathbf{0}$. Hence $(\mathbf{1}-s^2)s^2=t^*ts^2=\mathbf{0}$. Thus s^2 is an orthogonal projection. By substituting sa into a of (8.8), using this result and applying Θ_-, we obtain

(8.11) $$\mathbf{0}=s^2a\Theta_-(t)=s^2at^*.$$

Since the UHF algebra $\mathfrak{A}^{\mathrm{CAR}}$ is simple, (8.11) implies $s=\mathbf{0}$ or $t=\mathbf{0}$. If $t=\mathbf{0}$, (8.7) implies that Θ_- is an inner automorphism of

$\mathfrak{A}^{\mathrm{CAR}}$. Since Θ_- is a Bogolubov transformation given by θ_-, and since $\mathbf{1}\pm\theta_-$ is not in the trace class (they are twice infinite projections), Θ_- is not inner (Theorem 5 and Definition 8.1 of [10]). Thus the alternative $t=\mathbf{0}$ is impossible.

The alternative $s=\mathbf{0}$ implies $t^*t=\mathbf{1}$. Since $\mathbf{1}=\Theta_-(t^*t)=tt^*$, t is a unitary. (8.7) and (8.5) then imply that $t=\pm\mathbf{1}$ (since $\mathfrak{A}^{\mathrm{CAR}}$ has a trivial center) and hence $\gamma=\mathrm{id}$ or $\gamma=\hat{\Theta}_-$.

Next, we set $\psi(T)=s+Tt$. We still have (8.5) and (8.6). Since $\psi^2=\gamma=\mathrm{id}$, or $\hat{\Theta}_-$ and $\psi(a)=\Theta_-\tilde{\Theta}_+(a)$ for $a\in\mathfrak{A}^{\mathrm{CAR}}$, we obtain

$$(8.12)\qquad \Theta_-\tilde{\Theta}_+(s)+s\Theta_-\tilde{\Theta}_+(t)=\mathbf{0},\quad t\Theta_-\tilde{\Theta}_+(t)=\pm\mathbf{1}.$$

From $\psi(T)a\psi(T)=\Theta_-\tilde{\Theta}_+^2(a)$ for $a\in\mathfrak{A}^{\mathrm{CAR}}$, we obtain

$$(8.13)\qquad sas+t^*\Theta_-(a)t=\Theta_-\tilde{\Theta}_+^2(a),$$

$$(8.14)\qquad tas+\Theta_-(sa)t=\mathbf{0}.$$

By (8.12), $\Theta_-\tilde{\Theta}_+(t)t=\Theta_-\tilde{\Theta}_+(t\Theta_-\tilde{\Theta}_+(t))=\pm\mathbf{1}$ and hence t has an inverse $\pm\Theta_-\tilde{\Theta}_+(t)$. Substituting t^{-1} times (8.14) into as of (8.13) and dividing by t from the right, we obtain

$$(8.15)\qquad (-st^{-1}\Theta_-(s)+t^*)\Theta_-(a)=\Theta_-\tilde{\Theta}_+^2(a)t^{-1}.$$

By setting $a=\mathbf{1}$, and substituting the resulting expression into (8.15), we obtain

$$(8.16)\qquad t^{-1}\Theta_-(a)=\Theta_-\tilde{\Theta}_+^2(a)t^{-1}.$$

Substituting $\Theta_-(a)$ into a, we see that $\Theta_-\tilde{\Theta}_+^2\Theta_-$ must be inner. We now prove that this is impossible.

The necessary and sufficient condition for $\Theta_-\tilde{\Theta}_+^2\Theta_-$ to be inner is that $\omega_+^2-\mathbf{1}$ is in the trace class and $\det\omega_+^2=\mathbf{1}$ or $\omega_+^2+\mathbf{1}$ is in the trace class and $\det(-\omega_+^2)=-1$ by Theorem 5 of [10]. We shall exclude the first case by showing that $(\omega_+^2-\mathbf{1})$ or equivalently $(\omega_+^2-\mathbf{1})\theta_-$ is not in the trace class and the second case by showing $\det(-\omega_+^2)=1$ if $\omega_+^2+\mathbf{1}$ is in the trace class.

Since $(\omega_-\theta_-)q_\pm=\mp q_\pm$ (also see Corollary 9), we have

$$(8.17)\qquad \begin{aligned}(\omega_+^2-\mathbf{1})\theta_-&=(\omega_+\theta_-)\theta_-(\omega_+\theta_-)-\theta_-\\&=-2(q_+\theta_-q_-+q_-\theta_-q_+).\end{aligned}$$

We shall prove that $(\omega_+^2-\mathbf{1})\theta_-$ is not in the trace class by proving that it is even not in the Hilbert-Schmidt class. By (8.17),

$$(8.18)\quad \|(\omega_+^2-\mathbf{1})\theta_-\|_{\mathrm{H.S.}}^2=4(\|q_+\theta_-q_-\|_{\mathrm{H.S.}}^2+\|q_-\theta_-q_+\|_{\mathrm{H.S.}}^2)$$
$$=8\|q_-\theta_-q_+\|_{\mathrm{H.S.}}^2$$
$$=(8/\pi^2)\sum_{\sigma,\sigma'}\lim_{\varepsilon\to 0}\int_{\Delta_{-\sigma}}d\theta_1\int_{\Delta_{\sigma'}}d\theta_2|F_\varepsilon(\theta_2-\theta_1)|^2G_{\sigma'\sigma}(\theta_2,\theta_1),$$

$$(8.19)\quad G_{\sigma'\sigma}(\theta_2,\ \theta_1)\equiv \mathrm{tr}(r_{\sigma'}^{\gamma}(\theta_2)r_\sigma^\gamma(\theta_1)),$$

where F_ε is given by (7.23), σ and σ' are $+$ or $-$, Δ_σ is the set of all θ for which $\sigma_\gamma(\theta)=\sigma$ (cf. (7.29) and (7.30)) and $r_\sigma^\gamma(\theta)$ is defined by (7.16) and (7.17).

For $\sigma=\sigma'$, (8.19) tends to 1 as $\theta_2-\theta_1$ tends to 0. In this case, θ_1 and θ_2 belongs to disjoint regions Δ_σ and $\Delta_{-\sigma}$. Hence we may set $\varepsilon=0$. Since $|2F_0(\theta_2-\theta_1)|^2=\{\sin(\theta_2-\theta_1)/2\}^{-2}$ is not integrable (relative to $d\theta_1 d\theta_2$) near $\theta_1=\theta_2$ ($\theta_1\in\Delta_\sigma$, $\theta_2\in\Delta_{-\sigma}$), and each term in the sum of (8.18) is positive, we have

$$(8.20)\quad \|(\omega_+^2-\mathbf{1})\theta_-\|_{\mathrm{H.S.}}^2=\infty.$$

Finally we prove $\det(-\omega_+^2)=1$ if $\omega_+^2+\mathbf{1}$ is in the trace class. By $\Gamma\omega_+\Gamma=\omega_+$, the multiplicity of the non-real eigenvalue α of ω_+ is the same as that of $\bar{\alpha}$. Let J be the componentwise complex conjugation of $l_2\oplus l_2$. Then (3.12) shows $JK_\gamma=K_\gamma J$. Since $J\theta_-=\theta_-J$ we have $J\omega_+J=\omega_-=-\omega_+$ by (6.8) and Corollary 9. Therefore the multiplicity of the eigenvalues ± 1 of ω_+ is the same. Since ω_+ is unitary, we obtain $\det(-\omega_+^2)=1$ if $\omega_+^2+\mathbf{1}$ is in the trace class (so that ω_+ has a pure point spectrum and $\det(-\omega_+^2)$ is definable). This proves the impossibility of extending $\Theta_-\tilde{\Theta}_+$ to an automorphism of $\mathfrak{A}$.

Since Θ is an automorphism of $\mathfrak{A}$, the same conclusion holds for $\Theta_-\tilde{\Theta}_-=\Theta_-\tilde{\Theta}_+\Theta$.

References

[1] Bratteli, O. and Robinson, D. W., *Operator algebras and quantum statistical mechanics* II, Springer, 1981.
[2] Emch, G. G. and Radin, C., Relaxation of local thermal deviations from equilibrium, *J. Math. Phys.*, **12** (1971), 2043-2046.
[3] Robinson, D. W., Return to Equilibrium, *Comm. Math. Phys.*, **31** (1973), 171-189.
[4] Araki, H. and Barouch, E., On the dynamics and ergodic properties of the *XY*-model, *J. Statist. Phys.*, **31** (1983), 327-345.
[5] Lieb, E., Schultz, T. and Mattis, D., Two soluble models of an antiferromagnetic chain, *Annals of Phys.*, **16** (1961), 407-466.
[6] Kishimoto, A., Dissipations and derivations, *Comm. Math. Phys.*, **47** (1976), 25-32.
[7] Araki, H., On uniqueness of KMS states of one-dimensional quantum lattice systems,

Comm. Math. Phys., **44** (1975), 1-7.
[8] Kishimoto, A., On uniqueness of KMS states of one-dimensional quantum lattice systems, *Comm. Math. Phys.*, **47** (1976), 167-170.
[9] Kato, T., *Perturbation theory for linear operators*, Springer, 1966.
[10] Araki, H., On quasifree states of CAR and Bogoliubov automorphisms, *Publ. RIMS Kyoto Univ.*, **6** (1970), 385-442.