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$\bar{\partial}$ Cohomology of (H, C)-Groups

Dedicated to Professor Shigeo Nakano on his 60th birthday

By

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Introduction

In this paper we consider an n-dimensional connected complex Lie group G without nonconstant holomorphic functions (Such a Lie group is called an (H, C)-group). In the previous paper [8] we found a sufficient condition for $H^p(G, \mathcal{O}_G)$ to be finite-dimensional $(p\geqq 1)$, using the resolution: $0 \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,n} \longrightarrow 0$ of $\mathcal{O}_G$, where $\mathcal{A}^{p,q}$ denotes the sheaf of germs of real analytic (p, q)-forms on G. It was not possible to find a necessary and sufficient condition for $H^p(G, \mathcal{O}_G)$ to be finite-dimensional by the method of the paper [8]. Roughly speaking the cause of the above unsuccess is that the resolution by the sheaves of germs of real analytic forms is not good enough to calculate the $\bar{\partial}$ cohomology groups of G.

The purpose of this paper is to establish the cohomology groups $H^p(G, \mathcal{O}_G)$ of an (H, C)-group G $(p\geqq 1)$, using some number theoretical property of G. It is known that every (H, C)-group G has a structure of C^{*p}-principal bundle $\pi : G \longrightarrow T^q_C$ over a q-dimensional complex torus T^q_C $(p+q=n)$ ([14]). We take the subsheaf $\mathcal{H}$ of $\mathcal{A}^{0,0}$ so that $\mathcal{H} := \{f\in \mathcal{A}^{0,0} ; f$ is holomorphic along each fiber of $\pi\}$. First we shall prove a cohomology vanishing theorem for the sheaf $\mathcal{H}$ on G in Section 2. Using the sheaf $\mathcal{H}$, we shall get the resolution:

$$0 \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{H}^{0,0} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{H}^{0,q} \longrightarrow 0$$

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of $\mathcal{O}_G$ in Section 3 to calculate the $\bar{\partial}$ cohomology groups $H^p(G, \mathcal{O}_G)$ $(p \geqq 1)$. In Section 4 we shall find a necessary and sufficient condition for $H^p(G, \mathcal{O}_G)$ to be finite-dimensional and calculate the dimension of $H^p(G, \mathcal{O}_G)$ (Theorem 4.3). We can regard an (H, C)-group G as a quotient group C^n/Γ by a discrete subgroup Γ. The above necessary and sufficient condition is expressed by a Diophantine inequality with respect to the subgroup Γ of C^n. Unless the condition is fulfilled for Γ, then by Theorem 4.3, there exists j $(1 \leqq j \leqq q)$ such that $H^j(G, \mathcal{O}_G)$ is infinite-dimensional. Further we shall prove that $H^p(G, \mathcal{O}_G)$ is not Hausdorff for all p $(1 \leqq p \leqq q)$ (Theorem 4.4). By the theorems in Section 4 the cohomology groups $H^p(G, \mathcal{O}_G)$ of an (H, C)-group G are completely determined by some number theoretical property of G and we have a classification of all (H, C)-groups as follows. Let C^n/Γ be an n-dimensional (H, C)-group. If Γ is generated by R-linearly independent vectors $v_1, \dots, v_{n+q}$, then C^n/Γ is called an (H, C)-group of rank $n+q$ ([11]). Let $\mathcal{T}^{n,q}$ be the set of all n-dimensional (H, C)-groups of rank $n+q$. Then

$$\begin{aligned}\mathcal{T}^{n,q} = \{C^n/\Gamma \in \mathcal{T}^{n,q} \,;\, \dim H^p(C^n/\Gamma, \mathcal{O}) < \infty, \ p \geqq 1\} \\ \cup \{C^n/\Gamma \in \mathcal{T}^{n,q} \,;\, H^p(C^n/\Gamma, \mathcal{O}) \text{ is not Hausdorff for any} \\ p \text{ satisfying } 1 \leqq p \leqq q\} \quad \text{(disjoint)}.\end{aligned}$$

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§ 1. Preliminaries

In this paper we consider an n-dimensional connected complex Lie group G without nonconstant holomorphic functions. Such a Lie group G is said to be a toroid group or an (H, C)-group ([5], [9], [11]). We recall that G is abelian and then G is isomorphic onto C^n/Γ for some discrete subgroup Γ of C^n as a Lie group ([11]). We may assume that Γ is generated by R-linearly independent vectors $\{e_1, \dots, e_n, v_1 = (v_{11}, \dots, v_{1n}), \dots, v_q = (v_{q1}, \dots, v_{qn})\}$ of C^n $(1 \leqq q \leqq n)$, where e_j is the j-th unit vector of C^n. Since every holomorphic function on $G = C^n/\Gamma$ is constant, $\{v_1, \dots, v_q\}$ must satisfy the condition:

$$\max\{|\textstyle\sum_{j=1}^n v_{ij} m_j - m_{n+i}| \,;\, 1 \leqq i \leqq q\} > 0 \tag{1.1}$$

for all $m=(m_1, \ldots, m_n, m_{n+1}, \ldots, m_{n+q}) \in Z^{n+q}-\{0\}$ ([9], [11]). Since $\operatorname{Im} v_1:=(\operatorname{Im} v_{11}, \ldots, \operatorname{Im} v_{1n}), \ldots, \operatorname{Im} v_q:=(\operatorname{Im} v_{q1}, \ldots, \operatorname{Im} v_{qn})$ are R-linearly independent, we may assume $\det[\operatorname{Im} v_{ij}; 1 \leqq i, j \leqq q] \neq 0$ without loss of generality. Throughout this paper we assume that $G=C^n/\Gamma$ and Γ denotes the discrete subgroup satisfying the above assumption and (1.1). Further we use the notations:

$$K_{m,i}:=\sum_{j=1}^n v_{ij}m_j-m_{n+i} \text{ and } K_m:=\max\{|K_{m,i}|; 1 \leqq i \leqq q\}$$

for $m \in Z^{n+q}$. Then from (1.1) we have

$$K_m>0 \quad \text{for all} \quad m \in Z^{n+q}-\{0\}. \tag{1.2}$$

We denote the projection $C^n \ni (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_q) \in C^q$ by $\pi_q: C^n \longrightarrow C^q$. Let $e_i^*:=\pi_q(e_i)$, $v_i^*:=\pi_q(v_i)$ for $1 \leqq i \leqq q$ and $\Gamma^*:=\pi_q(\Gamma)$. Since e_i^*, v_i^* are R-linearly independent, we have a q-dimensional complex torus $T_C^q=C^q/\Gamma^*$.

We recall the following proposition due to [14].

Proposition 1.1. *The projection $\pi_q: C^n \longrightarrow C^q$ induces the C^{*n-q} principal bundle $\pi_q: C^n/\Gamma \ni z+\Gamma \longmapsto \pi_q(z)+\Gamma^* \in T_C^q$ over T_C^q.*

We put

$$\alpha_{ij}:=\begin{cases} \operatorname{Re} v_{ij} & (1 \leqq i \leqq q,\ 1 \leqq j \leqq n) \\ 0 & (q+1 \leqq i \leqq n,\ 1 \leqq j \leqq n) \end{cases}$$

$$\beta_{ij}:=\begin{cases} \operatorname{Im} v_{ij} & (1 \leqq i \leqq q,\ 1 \leqq j \leqq n) \\ \delta_{ij} & (q+1 \leqq i \leqq n,\ 1 \leqq j \leqq n), \end{cases}$$

$[\gamma_{ij}; 1 \leqq i, j \leqq n]:=[\beta_{ij}; 1 \leqq i, j \leqq n]^{-1}$ and $v_i:=\sqrt{-1}\, e_i$ for $q+1 \leqq i \leqq n$. Since $\{e_1, \ldots, e_n, v_1, \ldots, v_n\}$ are R-linearly independent, we have an isomorphism

$$\phi: C^n \ni (z_1, \ldots, z_n) \longmapsto (t_1, \ldots, t_{2n}) \in R^{2n}$$

as a real Lie group, where $(z_1, \ldots, z_n)=\sum_{i=1}^n(t_ie_i+t_{n+i}v_i)$. Then we obtain the relations

$$t_j=x_j-\sum_{i,k=1}^n y_k\gamma_{ki}\alpha_{ij} \text{ and } t_{n+j}=\sum_{i=1}^n y_i\gamma_{ij} \tag{1.3}$$

for $1 \leqq j \leqq n$, where $z_i=x_i+\sqrt{-1}\, y_i$ $(1 \leqq i \leqq n)$. ϕ induces the isomorphism $\phi^{\sim}: C^n/\Gamma \cong T^{n+q} \times R^{n-q}$ as a real Lie group, where T^{n+q} is a $n+q$-dimensional real torus. Henceforth we identify C^n/Γ with the real Lie group $T^{n+q} \times R^{n-q}$ and use the real coordinate system $(t_1, \ldots, t_{2n})$ according to the need. We make the following change of coordinates:

$$\zeta_i := \sum_{j=1}^n z_j \gamma_{ji} \quad (1 \leqq i \leqq n) \text{ in } C^n.$$

Then we can regard $(\zeta_1, \ldots, \zeta_n)$ as a local coordinate system of C^n/Γ and we have global vector fields

$$\frac{\partial}{\partial \bar{\zeta}_i} = \sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial \bar{z}_j}$$

and (0, 1)-forms

$$d\bar{\zeta}_i = \sum_{j=1}^n \gamma_{ji} d\bar{z}_j \quad (1 \leqq i \leqq n)$$

on C^n/Γ. It follows from (1.3) that

$$(1.4) \qquad \frac{\partial}{\partial \bar{\zeta}_i} = \frac{1}{2}\left(\sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right).$$

Then for $q+1 \leqq i \leqq n$ we have

$$(1.5) \qquad \frac{\partial}{\partial \bar{\zeta}_i} = \frac{1}{2}\left(\frac{\partial}{\partial t_i} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right).$$

Let $\mathscr{A}$ be the sheaf of germs of (complex valued) real analytic functions on C^n/Γ and

$$\mathscr{H} := \left\{ f \in \mathscr{A} \mid \frac{\partial f}{\partial \bar{\zeta}_i} = 0 \quad q+1 \leqq i \leqq n \right\}.$$

Let $f \in H^0(C^n/\Gamma, \mathscr{A})$. Then we have the following Fourier expansion of f:

$$(1.6) \qquad f(t_1, \ldots, t_{2n}) = \sum_{m \in Z^{n+q}} c^m(t'') \exp 2\pi\sqrt{-1} \langle m, t' \rangle,$$

where $t' := (t_1, \ldots, t_{n+q}) \in T^{n+q}$, $t'' := (t_{n+q+1}, \ldots, t_{2n}) \in R^{n-q}$, $m = (m_1, \ldots, m_{n+q}) \in Z^{n+q}$, $\langle m, t' \rangle = \sum_{i=1}^{n+q} m_i t_i$ and $c^m(t'')$ is real analytic in $t'' \in R^{n-q}$ for any $m \in Z^{n+q}$. We put

$$f^m(t) := c^m(t'') \exp 2\pi\sqrt{-1} \langle m, t' \rangle.$$

It follows from (1.4), (1.5) and (1.6) that

$$(1.7) \qquad \frac{\partial f^m}{\partial \bar{\zeta}_i} = \begin{cases} \pi\left(\sum_{j=1}^n v_{ij} m_j - m_{n+i}\right) f^m = \pi K_{m,i} f^m, & 1 \leqq i \leqq q \\ \dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial c^m(t'')}{\partial t_{n+i}} + 2\pi m_i c^m(t'') \right) \exp 2\pi\sqrt{-1} \langle m, t' \rangle, & q+1 \leqq i \leqq n. \end{cases}$$

Furthermore suppose $f \in H^0(C^n/\Gamma, \mathscr{H})$. Since $\dfrac{\partial f}{\partial \bar{\zeta}_i} = 0$ $(q+1 \leqq i \leqq n)$, we have, by (1.6) and (1.7),

$$(1.8) \qquad f(t) = \sum_{m \in Z^{n+q}} c^m \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \exp 2\pi\sqrt{-1} \langle m, t' \rangle,$$

where c^m is complex constant for any $m \in Z^{n+q}$.

§ 2. Cohomology Groups with Coefficients in the Sheaf $\mathscr{H}$

Let M be a paracompact real analytic manifold and $\mathscr{A}_M$ be the sheaf of germs of real analytic functions on M. By the result of [4] we can regard M as a closed real analytic submanifold of a complex manifold N and M has a Stein neighbourhood basis $\{U_i; i \in I\}$ in N. Since ind. $\lim\{H^p(U_j, \mathcal{O}_N); U_j \supset M\} = 0$, we have $H^p(M, \mathscr{A}_M) = 0$ for $p \geqq 1$ ([10]).

In this section we treat cohomology groups as the following type. Let $\mathscr{F}$ the sheaf $\left\{f(z, t) \in \mathscr{A}_{C\times R}; \dfrac{\partial f}{\partial \bar{z}} = 0\right\}$ on $C\times R$. We wish to consider whether $H^p(C\times R, \mathscr{F})$ vanishes for $p \geqq 1$. Using a power series expansion of a function $f \in \mathscr{A}_{C\times R}$, we can prove that a homomorphism $\dfrac{\partial}{\partial \bar{z}}: \mathscr{A}_{C\times R} \longrightarrow \mathscr{A}_{C\times R}$ is surjective. Then we have an exact sequence $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}_{C\times R} \xrightarrow{\frac{\partial}{\partial \bar{z}}} \mathscr{A}_{C\times R} \longrightarrow 0$. From this exact sequence and the lemma of [6, Lemma 2, p. 25], we can regard $H^p(C\times R, \mathscr{F})$ as $H^p(C, \mathcal{O}^E)$, where $\mathcal{O}^E$ is the sheaf of germs of holomorphic functions with values in the locally convex space $E := H^0(R, \mathscr{A}_R)$. Since E admits no structures of Frechet spaces, then we cannot apply the result of [1] and [2] to $H^p(C\times R, \mathscr{F})$. And since $C\times R$ has no Stein neighbourhood bases in $C\times C$, then we cannot prove the vanishing of $H^p(C\times R, \mathscr{F})$ by the same method of the proof of the theorem: $H^p(M, \mathscr{A}_M) = 0$. To get our purpose in this section, we must investigate a property of Stein open neighbourhood of $C^k\times R^l$ in $C^k\times C^l$.

We will use the following notations in the rest of this paper. For an m-tuple $\xi = (\xi_1, \dots, \xi_m)$, $\|\xi\| := \max\{|\xi_i|; 1 \leqq i \leqq m\}$. And the notation {equalities and inequalities involving functions $h_1, \dots, h_m$} denotes the set of all points in the intersection of the domains of definition of $h_1, \dots, h_m$ satisfying the given equalities and inequalities.

Lemma 2.1. *Let $\pi: S \longrightarrow C^k\times C^l$ be a (unramified Riemann) domain of holomorphy over $C^k\times C^l$ $(k, l \geqq 1)$, $\Delta_r := \{(w_1, \dots, w_l) \in C^l; |w_j - a_j| < r_j, 1 \leqq j \leqq l\}$, where $r = (r_1, \dots, r_l)$, $r_j > 0$ and $(a_1, \dots, a_l) \in C^l$ and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l)$ for $\varepsilon_j \geqq 0$ $(1 \leqq j \leqq l)$. Further assume there exist an open subset V_1 of S and $\delta > 0$ such that $\pi|_{V_1}$ is biholomorphic into $C^k\times C^l$ and $\pi(V_1) \supset (C^k\times \Delta_r) \cup \{\|z\| < \delta\} \times \Delta_{r+\varepsilon}$, where $\Delta_{r+\varepsilon} :=$*

$\{|w_j-a_j|<r_j+\varepsilon_j, 1\leqq j\leqq l\}$. *Then there exists an open subset* V_2 *of* S *with* $V_1\subset V_2$ *such that* $\pi|_{V_2}$ *is biholomorphic into* $C^k\times C^l$ *and* $\pi(V_2)\supset C^k\times\Delta_{r+\varepsilon}$.

Proof. We may assume $a_1=\cdots=a_l=0$. Let $f\in H^0(S, \mathcal{O}_S)$. Then f can be expanded in the power series: $f|_{(\pi|V_1)^{-1}(C^k\times\Delta_r)}(x)=\sum_{\nu,\mu}a_{\nu\mu}(z\circ\pi(x))^\nu(w\circ\pi(x))^\mu$, where $(z\circ\pi(x))^\nu=(z_1\circ\pi(x))^{\nu_1}\cdots(z_k\circ\pi(x))^{\nu_k}$ and $(w\circ\pi(x))^\mu=(w_1\circ\pi(x))^{\mu_1}\cdots(w_l\circ\pi(x))^{\mu_l}$. Then the power series $F(z, w):=\sum_{\nu,\mu}a_{\nu\mu}z^\nu w^\mu$ converges in $(C^k\times\Delta_r)\cup\{\|z\|<\delta\}\times\Delta_{r+\varepsilon}$. We put $D_d:=(\{\|z\|<d\}\times\{|w_j|<r_j, 1\leqq j\leqq l\})\cup(\{\|z\|<\delta\}\times\{|w_j|<r_j+\varepsilon_j, 1\leqq j\leqq l\})$ for $d>\delta$. The envelope of holomorphy of D_d is the smallest logarithmically convex complete Reinhardt domain $\hat{D}_d:=\{\|z\|<d, |w_j|<r_j+\varepsilon_j, \log|w_j|-\log r_j<\frac{\log d-\log\|z\|}{\log d-\log\delta}(\log(r_j+\varepsilon_j)-\log r_j)\}$ which contains D_d (for instance see [13]). Since F converges in D_d for all $d>\delta$, then F can be continued holomorphically in $\hat{D}_d$ for any $d>\delta$. We take any point $(z, w)\in C^k\times\Delta_{r+\varepsilon}$. Then we can find a sufficiently large positive number d_0 such that $\log|w_j|-\log r_j<\frac{\log d_0-\log\|z\|}{\log d_0-\log\delta}(\log(r_j+\varepsilon_j)-\log r_j)$. Then $(z, w)\in\hat{D}_{d_0}$. This implies that F converges in $C^k\times\Delta_{r+\varepsilon}$. Since $\pi: S\longrightarrow C^k\times C^l$ is a domain of holomorphy, we find an open subset V_2 of S satisfying the statements of the lemma (for instance see [6, Theorem 18, p. 55]).

The following lemma asserts that $C\times R$ admits no Stein open neighbourhood bases in $C\times C$. For instance we take an open neighbourhood $V:=\{(z, w)\in C^2; |\mathrm{Im}\, w|<(1+|z|)^{-1}\}$ of $C\times R$ in $C\times C$. Then we cannot find a Stein open subset V^* so that $C\times R\subset V^*\subset V$.

Lemma 2. 2. *Let* $I_j:=\{w_j\in C; \mathrm{Im}\, w_j=0, a_j<\mathrm{Re}\, w_j<b_j\}$, *where* $-\infty\leqq a_j<b_j\leqq\infty$ $(1\leqq j\leqq l)$, $I:=I_1\times\cdots\times I_l\subset R^l\subset C^l$ *and* f *a holomorphic function in a neighbourhood of* $C^k\times I$ *in* $C^k\times C^l$. *Then there exists a Stein open neighbourhood* V *of* I *in* C^l *such that* f *can be continued holomorphically to* $C^k\times V$.

Proof. First we assume $l=1$. Then $I=\{w\in C; \mathrm{Im}\, w=0, a<\mathrm{Re}\, w<b\}$, where $-\infty\leqq a<b\leqq\infty$. We have an open and connected neighbourhood D of $C^k\times I$ in $C^k\times C$ so that f is holomorphic in D. Let $\pi:\hat{D}\longrightarrow C^k\times C$ be the envelope of holomorphy of D which is given

by a Riemann domain over $C^k \times C$. Then there exists a holomorphic injection $j: D \longrightarrow \hat{D}$ such that $\pi \circ j =$identity and the mapping $H^0(\hat{D}, \mathcal{O}_{\hat{D}}) \ni g \longmapsto g \circ j \in H^0(D, \mathcal{O}_D)$ is an isomorphism. Put $U := j(D)$, for $\varepsilon = \pm 1$ $\hat{D}_1^\varepsilon :=$the connected component of $\pi^{-1}(C^k \times \{a < \text{Re } w < b, \varepsilon \text{ Im } w < 0\})$ satisfying $\hat{D}_1^\varepsilon \cap U \neq \phi$ and $D^\varepsilon := \{(z, w) \in D ; a < \text{Re } w < b\} \cup C^k \times \{w \in C ; a < \text{Re } w < b, \varepsilon \text{ Im } w > 0\}$. Then $\pi : \hat{D}_1^\varepsilon \longrightarrow C^k \times C$ is a domain of holomorphy. We identify $(z, w) \in D^\varepsilon$ and $x \in \hat{D}_1^\varepsilon$, if $(z, w) \in D$ and $j(z, w) = x$. We write this identification by $(z, w) \sim x$. Then for $\varepsilon = \pm 1$ we get Riemann domains

$$\pi_\varepsilon : G^\varepsilon : \hat{D}_1^\varepsilon \cup D^\varepsilon / \sim \longrightarrow C^k \times C,$$

where $\pi_\varepsilon(x) := \pi(x)$ if $x \in \hat{D}_1^\varepsilon$ and $\pi_\varepsilon(z, w) := (z, w)$ if $(z, w) \in D^\varepsilon$. Now we consider the case $\varepsilon = 1$. We put

$$G_t^1 := G^1 \cap \pi_1^{-1}(C^k \times \{a + t < \text{Re } w < b - t\}) \text{ for } 0 < t < (b-a)/2.$$

Since G_t^1 is p_4-convex in the sense of [3], $G^1 = \bigcup_{0 < t < (b-a)/2} G_t^1$ is a domain of holomorphy. We take $x_0 \in U$ with $\pi_1(x_0) = (z^0, w^0)$ for some $w^0 \in I$. We put

$$\tau_0 := 1/2 \min\{w^0 - a, \ b - w^0\} > 0$$
$$P_{\tau_0} := C^k \times \{|w - w^0 - \sqrt{-1}\,\tau_0| < \tau_0\}.$$

Then we have $P_{\tau_0} \subset D^1 \subset G^1$. And there exist $\delta_1 > 0$ and an open subset U_1 of G^1 with $x_0 \in U_1$ such that $\pi_1|_{U_1}$ is biholomorphic into $C^k \times C$ and $\pi_1(U_1) \supset \{\|z\| < \delta_1\} \times \{|w - w^0 - \sqrt{-1}\,\tau_0| < \tau_0 + \delta_1\}$. By Lemma 2. 1 we have an open subset U_2 of G^1 with $x_0 \in U_2$ so that $\pi_1|_{U_2}$ is biholomorphic into $C^k \times C$ and

$$\pi_1(U_2) \supset C^k \times \{|w - w^0 - \sqrt{-1}\,\tau_0| < \tau_0 + \delta_1\}.$$

Then $\pi(\hat{D}_1^1 \cap U_2) \supset C^k \times \{|w - w^0| < \delta_1, \text{ Im } w \leqq 0\}$. Applying the above method to the case $\varepsilon = -1$, we get $\delta_2 > 0$ and an open subset U_3 of G^{-1} with $x_0 \in U_3$ so that $\pi_{-1}|U_3$ is biholomorphic and $\pi(\hat{D}_1^{-1} \cap U_3) \supset C^k \times \{|w - w^0| < \delta_2, \text{ Im } w \geqq 0\}$. Then there exists an open neighbourhood U_4 of x_0 in $\hat{D}$ such that $\pi|_{U_4}$ is biholomorphic into $C^k \times C$ and $\pi(U_4) \supset C^k \times \{|w - w^0| < \min\{\delta_1, \delta_2\}\}$. This means that f can be continued holomorphically to $C^k \times V^*$ for some open neighbourhood V^* of I in C. Then we complete the proof in the case $l = 1$. We can prove the assertion of the lemma for $l \geqq 3$ similarly to the case $l = 2$. Then we shall only prove the lemma in the case $l = 2$. Let $I := I_1 \times I_2 =$

$\{(w_1, w_2)\ ;\ \mathrm{Im}\ w_i=0,\ a_i<\mathrm{Re}\ w_i<b_i,\ i=1,\ 2\}\subset R^2\subset C^2$, f a holomorphic function in a neighbourhood E of $C^k\times I$ in $C^k\times C^2$ and $\hat{E}$ the envelope of holomorphy of E. In general $\hat{E}$ is given by a Riemann domain over $C^k\times C^2$. Using the same technique of the proof in the case $l=1$, we may treat $\hat{E}$ as a univalent domain of holomorphy in $C^k\times C^2$ which contains E without loss of generality. We take $(t_1^0,\ t_2^0)\in I_1\times I_2$ and $\delta>0$ satisfying $\delta<\min\{t_i^0-a_i,\ b_i-t_i^0\ ;\ i=1,\ 2\}$ and $\{||z||<\delta\}\times\{|w_i-t_i^0|<\delta,\ i=1,\ 2\}\subset E$. Let $I_i^0:=R\cap\{w_i\in C;\ |w_i-t_i^0|<\delta\}$ $(i=1,\ 2)$, $I_1':==I_1^0\cap\{|w_1-t_1^0|<\delta/3\}$ and $\hat{E}(t_2):=\{(z,\ w_1)\in C^k\times C;\ (z,\ w_1,\ t_2)\in\hat{E},\ \mathrm{Re}\ w_1\in I_1^0\}$ for $t_2\in I_2^0$. And we put

$$\hat{E}(t_2)^{\varepsilon}:=\hat{E}(t_2)\cup C^k\times\{w_1;\varepsilon\ \mathrm{Im}\ w_1\geqq 0,\ \mathrm{Re}\ w_1\in I_1^0\}$$

for $\varepsilon=\pm 1$. Then $\hat{E}(t_2)^{\varepsilon}$ is a domain of holomorphy for $\varepsilon=\pm 1$ and $t_2\in I_2^0$. We have $C^k\times\{|w_1-t_1-\sqrt{-1}(\delta/3)|<\delta/3\}\subset\hat{E}(t_2)^{+1}$ and $\{||z||<\delta\}\times\{|w_1-t_1-\sqrt{-1}(\delta/3)|<(2\delta/3)\}\subset\hat{E}(t_2)^{+1}$ for $t_2\in I_2^0$ and $t_1\in I_1'$. It follows from Lemma 2. 1 that

$$C^k\times\{|w_1-t_1-\sqrt{-1}(\delta/3)|<2\delta/3\}\subset\hat{E}(t_2)^{+1}$$

for $t_2\in I_2^0$ and $t_1\in I_1'$. Similarly we have

$$C^k\times\{|w_1-t_1+\sqrt{-1}(\delta/3)|<2\delta/3\}\subset\hat{E}(t_2)^{-1}$$

for $t_2\in I_2^0$ and $t_1\in I_1'$. We put $V_1^0:=\{w_1;\ |\mathrm{Im}\ w_1|<\delta/3,\ \mathrm{Re}\ w_1\in I_1'\}$. Then we have $C^k\times V_1^0\times I_2^0\subset\hat{E}$. We set

$$\hat{E}_1:=\{(z,\ w_1,\ w_2)\in\hat{E};w_1\in V_1^0,\ \mathrm{Re}\ w_2\in I_2^0\},$$

$$\hat{E}_1^{\varepsilon}:=\hat{E}_1\cup C^k\times V_1^0\times\{w_2;\varepsilon\ \mathrm{Im}\ w_2\geqq 0,\ \mathrm{Re}\ w_2\in I_2^0\}$$

for $\varepsilon=\pm 1$. Since $C^k\times V_1^0\times\{w_2;\ |w_2-t_2^0-\sqrt{-1}(\varepsilon\delta/2)|<\delta/2\}\subset\hat{E}_1^{\varepsilon}$ and $\{||z||<\delta\}\times V_1^0\times\{w_2;\ |w_2-t_2^0-\sqrt{-1}(\varepsilon\delta/2)|<\delta\}\subset\hat{E}_1^{\varepsilon}$ $(\varepsilon=\pm 1)$, it follows from Lemma 2. 1 that

$$C^k\times V_1^0\times\{|w_2-t_2^0|<\delta/2\}\subset\hat{E}.$$

Since $(t_1^0,\ t_2^0)$ is an arbitrary point of I, we find an open subset V of I in C^2 so that f can be continued holomorphically to $C^k\times V$. I has a Stein neighbourhood basis in C^2. Then we may regard V as a Stein open subset of C^2 ([4]).

Lemma 2. 3. *Let I be as in Lemma 2. 2 and f a holomorphic function in a neighbourhood of $C^{*k}\times I$ in $C^{*k}\times C^l$. Then there exists a Stein open neighbourhood V of I in C^l such that f can be continued* *holomorphically to* $C^{*k}\times V$.

Proof. It is sufficient to prove the lemma in the case $k=2$. Let $a_\nu(t):=1/(2\pi\sqrt{-1})^2\int_{|z_1|=1}\int_{|z_2|=1}\dfrac{f(z_1, z_2, t)}{z_1^{\nu_1+1}\,z_2^{\nu_2+1}}\,dz_1dz_2$ for $\nu=(\nu_1, \nu_2)\in Z^2$ and $t\in I$. And we use the notations: $\sum_{(1)}:=\sum_{\nu_1,\nu_2\geqq 0}$, $\sum_{(2)}:=\sum_{\nu_1\geqq 0,\nu_2<0}$, $\sum_{(3)}:=\sum_{\nu_1<0,\nu_2\geqq 0}$, $\sum_{(4)}:=\sum_{\nu_1,\nu_2<0}$ and $f_i(z_1, z_2, t):=\sum_{(i)}a_\nu(t)\,z_1^{\nu_1}z_2^{\nu_2}$. Then we can apply Lemma 2. 2 to each f_i. For instance we take $f_4^*(\hat z_1, \hat z_2, t):=f_4(\hat z_1^{-1}, \hat z_2^{-1}, t)$ which is holomorphic in $(\hat z_1, \hat z_2)\in C^2$. By Lemma 2. 2 f_4^* can be continued to $C^2\times V$ for some Stein open neighbourhood V of I in C^l. Since $f=f_1+f_2+f_3+f_4$, we get the proof of the lemma.

Let $\pi_q: C^n/\Gamma\ni(z_1, \dots, z_n)+\Gamma\longmapsto(z_1, \dots, z_q)+\Gamma^*\in T_C^q=C^q/\Gamma^*$ be the C^{*n-q}-principal bundle over T_C^q as in Proposition 1. 1. Since $\alpha_{ij}=0$ for $1\leqq j\leqq n$, $q+1\leqq i\leqq n$ and $\gamma_{ij}=0$ for $q+1\leqq i\leqq n$, $1\leqq j\leqq q$, from (1. 3) it follows that $t_j=x_j-\sum_{s=1}^q\sum_{i=1}^q y_i\gamma_{is}\alpha_{sj}$ and $t_{n+j}=\sum_{i=1}^q y_i\gamma_{ij}$ for $1\leqq j\leqq q$. This relation induces an isomorphism $\sigma: T_C^q\ni(z_1, \dots, z_q)+\Gamma^*\longmapsto(\exp 2\pi\sqrt{-1}\,t_1, \dots, \exp 2\pi\sqrt{-1}\,t_q, \exp 2\pi\sqrt{-1}\,t_{n+1}, \dots, \exp 2\pi\sqrt{-1}\,t_{n+q})\in T^{2q}$, where T^{2q} is a real $2q$-dimensional torus. And we have an isomorphism $\phi: C^n/\Gamma\ni(z_1, \dots, z_n)+\Gamma\longmapsto(\exp 2\pi\sqrt{-1}\,(t_{q+1}+\sqrt{-1}\,t_{n+q+1}), \dots, \exp 2\pi\sqrt{-1}(t_n+\sqrt{-1}\,t_{2n});\ \sigma\circ\pi_q(z_1, \dots, z_n))\in C^{*n-q}\times T^{2q}$ with a commutative diagram:

$$\begin{array}{ccc} C^n/\Gamma & \xrightarrow{\quad\phi\quad} & C^{*n-q}\times T^{2q} \\ \downarrow{\scriptstyle \pi_q} & & \downarrow{\scriptstyle \pi'} \\ T_C^q & \xrightarrow{\quad\sigma\quad} & T^{2q}, \end{array}$$

where $\pi'(\xi, \eta)=\eta$ for $\xi=(\xi_1, \dots, \xi_{n-q})\in C^{*n-q}$ and $\eta\in T^{2q}$. We take the sheaf $\mathscr{F}$ of germs of real analytic functions which is holomorphic in each fibre of π' on $C^{*n-q}\times T^{2q}$, that is

$$\mathscr{F}:=\{f\in\mathscr{A}';\ \frac{\partial f}{\partial\bar\xi_i}=0,\ 1\leqq i\leqq n-q\},$$

where $\mathscr{A}'$ is the sheaf of germs of real analytic functions on $C^{*n-q}\times T^{2q}$. And we consider the sheaf $\mathscr{H}:=\{f\in\mathscr{A};\ \dfrac{\partial f}{\partial\bar\zeta_i}=0\ \ q+1\leqq i\leqq n\}$ on C^n/Γ as in Section 1. Then by (1. 5)

$$(2.1)\qquad \phi^*: H^0(W, \mathscr{F})\ni f\longmapsto f\circ\phi\in H^0(\phi^{-1}(W), \mathscr{H})$$

is an isomorphism for any open subset W of $C^{*n-q}\times T^{2q}$. We put $J:=$

$\{j=(\varepsilon_1,\ldots,\varepsilon_{2q});\varepsilon_i=\pm 1\}$ and for $j=(\varepsilon_1,\ldots,\varepsilon_{2q})\in J$ $U_j:=\{(\exp 2\pi\sqrt{-1}\,t_1,\ldots,\exp 2\pi\sqrt{-1}\,t_q,\exp 2\pi\sqrt{-1}\,t_{n+1},\ldots,\exp 2\pi\sqrt{-1}\,t_{n+q})\in T^{2q};-1/2\leqq t_s,t_{n+s}\leqq 1/2,\ t_{n+s}\neq\varepsilon_{q+s}/4,\ t_s\neq\varepsilon_s/4$ for $1\leqq s\leqq q\}$. Then we have open coverings $\mathfrak{U}:=\{C^{*n-q}\times U_j;j\in J\}$ and $\mathfrak{V}:=\{\phi^{-1}(C^{*n-q}\times U_j);j\in J\}$ of $C^{*n-q}\times T^{2q}$ and C^n/Γ, respectively.

Proposition 2.4. *Let $H^p(\mathfrak{V},\mathscr{H})$ be the p-th Cech cohomology group of the covering $\mathfrak{V}$ of C^n/Γ. Then*

$$H^p(\mathfrak{V},\mathscr{H})=0\quad \textit{for } p\geqq 1.$$

Proof. We have an isomorphism $\phi^*:H^p(\mathfrak{U},\mathscr{F})\longrightarrow H^p(\mathfrak{V},\mathscr{H})$ by (2.1). Then we may prove $H^p(\mathfrak{U},\mathscr{F})=0$ for $p\geqq 1$. We regard T^{2q} as the closed real analytic submanifold $\{(w_1,\ldots,w_{2q});\ |w_i|=1,1\leqq i\leqq 2q\}$ of C^{2q}. Let C be any connected component of $U_{j_0}\cap\ldots\cap U_{j_p}$. Then there exist a rectangular set $I\subset R^{2q}\subset C^{2q}$ as in Lemma 2.2 and an open neighbourhood V of I in C^{2q} such that $\psi: V\ni(u_1,\ldots,u_{2q})\longmapsto(\exp 2\pi\sqrt{-1}\,u_1,\ldots,\exp 2\pi\sqrt{-1}\,u_{2q})\in C^{2q}$ is biholomorphic and $\psi(I)=C$. We take $\{c_{j_0\ldots j_p}\}\in Z^p(\mathfrak{U},\mathscr{F})$. In virtue of such mapping ψ and by Lemma 2.3 there exist a Stein and connected open neighbourhood $U^*_{j_0\ldots j_p}$ of $U_{j_0}\cap\ldots\cap U_{j_p}$ in C^{2q} and a unique holomorphic function $c^*_{j_0\ldots j_p}$ in $C^{*n-q}\times U^*_{j_0\ldots j_p}$ such that $c^*_{j_0\ldots j_p}|C^{*n-q}\times U_{j_0}\cap\ldots\cap U_{j_p}=c_{j_0\ldots j_p}$. Since each U_j admits a Stein neighbourhood basis in C^{2q}, we can choose a Stein neighbourhood $U^\sim_j$ of U_j in C^{2q} so that $U^\sim_{j_0}\cap\ldots\cap U^\sim_{j_p}\subset U^*_{j_0\ldots j_p}$. We take ε $(0<\varepsilon<1)$ satisfying $\Delta_\varepsilon=\{1-\varepsilon<|w_i|<1+\varepsilon,\ 1\leqq i\leqq 2q\}\subset\cup_{j\in J}U^\sim_j$. Then we have $(\{c^*_{j_0\ldots j_p}\};\{C^{*n-q}\times U^\sim_{j_0}\cap\ldots\cap U^\sim_{j_p}\cap\Delta_\varepsilon\})\in Z^p(\{C^{*n-q}\times(U^\sim_j\cap\Delta_\varepsilon)\},\mathscr{O})$. Since $C^{*n-q}\times\Delta_\varepsilon$ is a Stein open set, there exists $\{d^*_{j_0\ldots j_{p-1}}\}C^{p-1}(\{C^{*n-q}\times(U^\sim_j\cap\Delta_\varepsilon)\},\mathscr{O})$ such that $\delta\{d^*_{j_0\ldots j_{p-1}}\}=\{c^*_{j_0\ldots j_p}|C^{*n-q}\times(U^\sim_{j_0}\cap\ldots\cap U^\sim_{j_p}\cap\Delta_\varepsilon)\}$. This completes the proof.

Let $f\in H^0(C^n/\Gamma,\mathscr{H})=Z^0(\mathfrak{V},\mathscr{H})$. From (1.8) we have $f(t)=\sum_{m\in Z^{n+q}}c^m\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m,t'\rangle$,

(2.2) $$f\circ\phi^{-1}(\xi,\eta)=\sum_{m\in Z^{n+q}}c^m\xi_1^{m_{q+1}}\ldots\xi_{n-q}^{m_n}\eta_1^{m_1}\ldots\eta_q^{m_q}\eta_{q+1}^{m_{n+1}}\ldots\eta_{2q}^{m_{n+q}}$$

for $(\xi,\eta)\in C^{*n-q}\times T^{2q}$. Observing the proof of Proposition 2.4 and by Lemma 2.3, we have the following

Proposition 2.5. *There exists $\varepsilon>0$ such that the Laurent series* *expansion*:

$$\sum_{m\in Z^{n+q}} c^m \xi_1^{m_{q+1}} \dots \xi_{n-q}^{m_n} w_1^{m_1} \dots w_q^{m_q} w_{q+1}^{m_{n+1}} \dots w_{2q}^{m_{n+q}}$$

which induced by (2. 2) *converges for all* $(\xi, w) \in C^{*n-q} \times \Delta_\varepsilon$, *where* $\Delta_\varepsilon = \{1-\varepsilon < |w_i| < 1+\varepsilon\}$.

§ 3. A Special Resolution of the Structure Sheaf $\mathcal{O}$ of C^n/Γ

Let U be an open subset of C^l, $\mathscr{A}^\sim$ be the sheaf of germs of real analytic functions on $C^{*k} \times U$. We consider the sheaf $\mathscr{G} := \{f \in \mathscr{A}^\sim ; \dfrac{\partial f(z, w)}{\partial \bar z_j} = 0,\ 1 \leqq j \leqq k\}$, where $(z, w) \in C^{*k} \times U$ and $\mathscr{G}^{0,p} := (1/p! \sum_{1\leqq i_1, \dots, i_p \leqq l} f_{i_1 \dots i_p} d\bar w_{i_1} \wedge \dots \wedge d\bar w_{i_p} ; f_{i_1 \dots i_p} \in \mathscr{G}\}$ for $0 \leqq p \leqq l$. Here any form treated in the rest of this paper is written skew-symmetrically in all indices.

Let $f \in H^0(C^{*k} \times U, \mathscr{G})$ and $w^0 = (w_1^0, \dots, w_l^0) \in U$. By Lemma 2. 3 there exists an open neighbourhood U^0 of w^0 in U such that f has the following expansion:

$$(3.1) \qquad f(z, w) = \sum_{\nu \in Z^k} \sum_{\alpha_i, \beta_i \geqq 0} b_{\nu,\alpha,\beta} z^\nu (w-w^0)^\alpha \overline{(w-w^0)}^\beta$$

which converges for all $(z, w) \in C^{*k} \times U^0$, where $(w-w^0)^\alpha = (w_1 - w_1^0)^{\alpha_1} \dots (w_l - w_l^0)^{\alpha_l}$ and $\overline{(w-w^0)}^\beta = \overline{(w_1-w_1^0)}^{\beta_1} \dots \overline{(w_l - w_l^0)}^{\beta_l}$.

Lemma 3. 1. *Let* $f = 1/p! \sum_{1\leqq i_1, \dots, i_p \leqq l} f_{i_1 \dots i_p} d\bar w_{i_1} \wedge \dots \wedge d\bar w_{i_p} \in H^0(C^{*k} \times U, \mathscr{G}^{0,p})$ *with* $\bar\partial f = 0$ $(p \geqq 1)$. *For any* $w^0 \in U$ *choose an open neighbourhood* U^0 *of* w^0 *so that any* $f_{i_1 \dots i_p}$ *can be expanded in* $C^{*k} \times U^0$ *as in* (3. 1). *Then there exists* $g^0 \in H^0(C^{*k} \times U^0, \mathscr{G}^{0,p-1})$ *such that* $\bar\partial g^0 = f$.

Proof. Let m be the least integer such that the explicit representation of f in coordinate form involves only the conjugate differentials $d\bar w_1, \dots, d\bar w_m$. The proof will be by induction on m. First we consider $m = p$. Then $f = f_{12 \dots p} d\bar w_1 \wedge \dots \wedge d\bar w_p$ and we have an expansion $f_{12 \dots p} = \sum a_{\nu\alpha\beta} z^\nu (w-w^0)^\alpha \overline{(w-w^0)}^\beta$ in $C^{*k} \times U^0$ as in (3. 1). Since $\bar\partial f = 0$, $f_{12 \dots p}$ must be holomorphic in $w_{p+1}, \dots, w_l$. Putting $g_{12 \dots p-1} := \sum a_{\nu\alpha\beta}/(\beta_p + 1) z^\nu (w-w^0)^\alpha \overline{(w-w^0)}^\beta \overline{(w_p - w_p^0)}$ and $g := g_{12 \dots p-1} d\bar w_1 \wedge \dots \wedge d\bar w_{p-1}$, $g_{12 \dots p-1}$ is also holomorphic in $w_{p+1}, \dots w_l$ and $\bar\partial g = f$. Using the standard argument for the Dolbeault lemma (for instance see [6, the proof of Theorem 3, p. 27]), we can complete the proof.

Observing the proof of Proposition 2. 4, we have the following lemma.

Lemma 3. 2. *Let* $\{U_i\}$ *be a locally finite open covering of* $U\subset C^l$ *and each* U_i *is a rectangular open subset* $\{a_j^{(i)}<\mathrm{Re}\ w_j<b_j^{(i)},\ c_j^{(i)}<\mathrm{Im}\ w_j<d_j^{(i)};\ 1\leqq j\leqq l\}$ *for some* $a_j^{(i)}$, $b_j^{(i)}$, $c_j^{(i)}$ *and* $d_j^{(i)}\in R$. *Take the open covering* $\mathfrak{U}:=\{C^{*k}\times U_i\}$ *of* $C^{*k}\times U$. *Then* $H^p(\mathfrak{U},\ \mathscr{G}^{0,s})=0$ *for* $p\geqq 1$, $0\leqq s\leqq l$.

By Lemmata 3. 1 and 3. 2 we have the following lemma.

Lemma 3. 3. *Let* U *be a Stein open subset of* C^l *and* $f\in H^0(C^{*k}\times U,\ \mathscr{G}^{0,p})$ *with* $\bar\partial f=0$. *Then there exists* $g\in H^0(C^{*k}\times U,\ \mathscr{G}^{0,p-1})$ *such that* $\bar\partial g=f$.

Proof. By Lemma 3. 1 we have a Stein covering $\{U_i\}$ and $g_i\in H^0(C^{*k}\times U_i,\ \mathscr{G}^{0,p-1})$ with $\bar\partial g_i=f$. We put $h_{i_0i_1}^{(1)}:=g_{i_1}-g_{i_0}$. Then $\bar\partial h_{i_0i_1}^{(1)}=0$. Further by Lemma 3. 1 we get $g_{i_0i_1}^{(1)}\in H^0(C^{*k}\times U_{i_0i_1},\ \mathscr{G}^{0,p-2})$ with $\bar\partial g_{i_0i_1}^{(1)}=h_{i_0i_1}^{(1)}$, where we use the notation $U_{i_0\cdots i_s}:=U_{i_0}\cap\ldots\cap U_{i_s}$. We set $\{h_{i_0i_1i_2}^{(2)}\}:=\delta\{g_{i_0i_1}^{(1)}\}$. Then $\bar\partial h_{i_0i_1i_2}^{(2)}=0$. Inductively we find sequences $\{g_{i_0\cdots i_s}^{(s)}\}\in C^s(\mathfrak{U},\ \mathscr{G}^{0,p-s-1})$ for $1\leqq s\leqq p$ and $\{h_{i_0\cdots i_s}^{(s)}\}\in C^s(\mathfrak{U},\ \mathscr{G}^{0,p-s})$ for $1\leqq s\leqq p$, $\mathfrak{U}:=\{C^{*k}\times U_i\}$ so that $\bar\partial h_{i_0\cdots i_s}^{(s)}=0$, $\{h_{i_0\cdots i_s}^{(s)}\}=\delta\{g_{i_0\cdots i_{s-1}}^{(s-1)}\}$ and $\bar\partial g_{i_0\cdots i_s}^{(s)}=h_{i_0\cdots i_s}^{(s)}$. Since $\{h_{i_0\cdots i_p}^{(p)}\}\in Z^p(\mathfrak{U},\ \mathscr{O})$ and $\mathfrak{U}$ is a Stein covering of the Stein open set $C^{*k}\times U$, then there exists $\{f_{i_0\cdots i_{p-1}}^{(p-1)}\}\in C^{p-1}(\mathfrak{U},\ \mathscr{O})$ such that $\{h_{i_0\cdots i_p}^{(p)}\}=\delta\{f_{i_0\cdots i_{p-1}}^{(p-1)}\}$. Then $\{g_{i_0\cdots i_{p-1}}^{(p-1)}-f_{i_0\cdots i_{p-1}}^{(p-1)}\}\in Z^{p-1}(\mathfrak{U},\ \mathscr{G}^{0,0})$. By Lemma 3. 2 we get $\{f_{i_0\cdots i_{p-2}}^{(p-2)}\}\in C^{p-2}(\mathfrak{U},\ \mathscr{G}^{0,0})$ so that $\{g_{i_0\cdots i_{p-1}}^{(p-1)}-f_{i_0\cdots i_{p-1}}^{(p-1)}\}=\delta\{f_{i_0\cdots i_{p-2}}^{(p-2)}\}$. We have $\bar\partial\{g_{i_0\cdots i_{p-1}}^{(p-1)}-f_{i_0\cdots i_{p-1}}^{(p-1)}\}=\delta\{\bar\partial f_{i_0\cdots i_{p-2}}^{(p-2)}\}=\{\bar\partial g_{i_0\cdots i_{p-1}}^{(p-1)}\}=\{h_{i_0\cdots i_{p-1}}^{(p-1)}\}=\delta\{g_{i_0\cdots i_{p-2}}^{(p-2)}\}$. Then $\{g_{i_0\cdots i_{p-2}}^{(p-2)}-\bar\partial f_{i_0\cdots i_{p-2}}^{(p-2)}\}\in Z^{p-2}(\mathfrak{U},\ \mathscr{G}^{0,1})$. Repeating the above argument, finally we find $\{f_i^{(0)}\}\in B^0(\mathfrak{U},\ \mathscr{G}^{0,p-2})$ so that $h_{i_0i_1}^{(1)}=g_{i_1}-g_{i_0}=\bar\partial f_{i_1}^{(0)}-\bar\partial f_{i_0}^{(0)}$. We put $g:=g_i-\bar\partial f_i^{(0)}$. Then $g\in H^0(C^{*k}\times U,\ \mathscr{G}^{0,p-1})$ and $\bar\partial g=f$.

Now we need the sheaf $\mathscr{H}=\{f\in\mathscr{A};\dfrac{\partial f}{\partial\bar\zeta_j}=0\quad q+1\leqq j\leqq n\}$ on C^n/Γ defining in Section 1. Further we consider the sheaf

$$\mathscr{H}^{0,p}:=\{1/p!\sum_{1\leqq i_1\cdots,i_p\leqq q}f_{i_1\cdots i_p}d\bar\zeta_{i_1}\wedge\ldots\wedge d\bar\zeta_{i_p};f_{i_1\cdots i_p}\in\mathscr{H}\}$$

of germs of $\mathscr{H}$-forms of type $(0, p)$ which involves only the differentials $d\bar{\zeta}_1, \dots, d\bar{\zeta}_q$.

Proposition 3. 4. *The sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{H}^{0,0} \xrightarrow{\bar{\partial}} \mathscr{H}^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{H}^{0,q} \longrightarrow 0$$

is exact. And $H^p(C^n/\Gamma, \mathcal{O})$ is isomorphic to the quotient space $\{f; f \in H^0(C^n/\Gamma, \mathscr{H}^{0,p}), \bar{\partial}f=0\} / \{\bar{\partial}g; g \in H^0(C^n/\Gamma, \mathscr{H}^{0,p-1})\}$ for $p \geqq 1$.

Proof. We can regard $(\zeta_1, \dots, \zeta_n)$ as a local coordinate system of C^n/Γ. Let f be a germ belonging to a stalk H_{ζ^0} at $\zeta^0 \in C^n/\Gamma$. f has an expansion

$$f = \sum_{\alpha,\beta} a_{\alpha\beta}(\zeta_{q+1}, \dots, \zeta_n)(\zeta_1 - \zeta_1^0)^{\alpha_1} \cdots (\zeta_q - \zeta_q^0)^{\alpha_q} \overline{(\zeta_1 - \zeta_1^0)}^{\beta_1} \cdots \overline{(\zeta_q - \zeta_q^0)}^{\beta_q}$$

which is similar to (3. 1) and converges in a small neighbourhood U^0 of ζ^0 in C^n/Γ, where $a_{\alpha\beta}$ is holomorphic in $\zeta_{q+1}, \dots, \zeta_n$. Applying the method of the proof of Lemma 3. 1 to this expansion of $f \in \mathscr{H}_{\zeta^0}$, we can prove the exactness of the sequence of the proposition. We put $\mathrm{Ker}\, \bar{\partial}_k := \mathrm{Ker}\{\bar{\partial}: \mathscr{H}^{0,k} \longrightarrow \mathscr{H}^{0,k+1}\}$ and $\mathrm{Im}\, \bar{\partial}_k := \mathrm{Im}\{\bar{\partial}: \mathscr{H}^{0,k} \longrightarrow \mathscr{H}^{0,k+1}\}$. Then we have $\mathrm{Im}\, \bar{\partial}_k = \mathrm{Ker}\, \bar{\partial}_{k+1}$ and the short exact sequences

$$(3.2) \qquad 0 \longrightarrow \mathrm{Ker}\, \bar{\partial}_k \longrightarrow \mathscr{H}^{0,k} \longrightarrow \mathrm{Im}\, \bar{\partial}_k \longrightarrow 0 \quad \text{for } 0 \leqq k \leqq q.$$

Let $\mathfrak{V} = \{\phi^{-1}(C^{*n-q} \times U_j)\}$ be the same locally finite covering of C^n/Γ as in Proposition 2. 4. Since $\phi^{-1}(C^{*n-q} \times U_j)$ is biholomorphic to $C^{*n-q} \times U_j$, it follows from Lemma 3. 3 that

$$\bar{\partial}: C^p(\mathfrak{V}, \mathscr{H}^{0,k}) \longrightarrow C^p(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_k)$$

is an epimorphism. Then we have an exact sequence

$$0 \longrightarrow C^p(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_k) \longrightarrow C^p(\mathfrak{V}, \mathscr{H}^{0,k}) \longrightarrow C^p(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_k) \longrightarrow 0.$$

From (3. 2) there exists a long exact sequence $0 \longrightarrow H^0(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_k) \longrightarrow H^0(\mathfrak{V}, \mathscr{H}^{0,k}) \longrightarrow H^0(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_k) \longrightarrow H^1(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_k) \longrightarrow H^1(\mathfrak{V}, \mathscr{H}^{0,k}) \longrightarrow \cdots$. Using this exact sequence and the result of Proposition 2. 4, we have $H^s(\mathfrak{V}, \mathscr{H}^{0,k}) = 0$ for $s \geqq 1$, $H^p(C^n/\Gamma, \mathcal{O}) = H^p(\mathfrak{V}, \mathcal{O}) = H^p(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_0) = H^{p-1}(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_0)$ and $H^{p-k}(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_k) = H^{p-k-1}(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_k)$ for $0 \leqq k \leqq p-1$. Then we obtain

$$H^p(C^n/\Gamma, \mathcal{O}) \cong H^1(\mathfrak{V}, \mathrm{Ker}\, \bar{\partial}_{p-1}) \cong$$
$$H^0(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_{p-1}) / \mathrm{Im}\{\bar{\partial}: H^0(\mathfrak{V}, \mathscr{H}^{0,p-1}) \longrightarrow H^0(\mathfrak{V}, \mathrm{Im}\, \bar{\partial}_{p-1})\}.$$

This coincides with the quotient space asserten in this proposition.

Remark. By Proposition 3. 4 we obtain $H^p(C^n/\Gamma, \mathcal{O})=0$ for $p\geqq q+1$. This comes from the result of [7] directly, since we showed in [7] that C^n/Γ is strongly $(q+1)$-complete in the sense of Andreotti and Grauert.

§ 4. $\bar{\partial}$ Cohomology Groups of (H, C)-Groups

Let $f\in H^0(C^n/\Gamma, \mathscr{H})$. By (1. 8) we have the Fourier expansion: $f=\sum_{m\in Z^{n+q}} c^m \exp(-2\pi\sum_{i=q+1}^n m_i t_{n+1}) \exp 2\pi\sqrt{-1}\langle m, t'\rangle$. For $m=(m_1, \ldots, m_{n+q})\in Z^{n+q}$ we put $m':=(m_1, \ldots, m_q, m_{n+1}, \ldots, m_{n+q})$, $m'':=(m_{q+1}, \ldots, m_n)$, $\|m'\|:=\max\{|m_i|, |m_{n+i}|; 1\leqq i\leqq q\}$ and $\|m''\|:=\max\{|m_j|; q+1\leqq j\leqq n\}$. Then we have the following

Lemma 4. 1. *The following conditions on a sequence* $\{c^m\in C; m\in Z^{n+q}\}$ *are equivalent.*

(a) *The Fourier expansion* $\sum_{m\in Z^{n+q}} c^m \exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i}) \exp 2\pi\sqrt{-1}$ $\langle m, t'\rangle$ *converges to a function in* $H^0(C^n/\Gamma, \mathscr{H})$.

(b) *There exists* $\varepsilon>0$ *such that for all* $a>0$
$C(a):=\sup\{|c^m| \exp(\varepsilon\|m'\|+a\|m''\|); m\in Z^{n+q}\}<\infty$.

Proof. We first prove (a) $\Longrightarrow$ (b). Put $f(t):=\sum_{m\in Z^{n+q}} c^m \exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i}) \exp 2\pi\sqrt{-1}\langle m, t'\rangle\in H^0(C^n/\Gamma, \mathscr{H})$, $(w_1, \ldots, w_{2q}):=(\exp 2\pi\sqrt{-1}\, t_1, \ldots, \exp 2\pi\sqrt{-1}\, t_q, \exp 2\pi\sqrt{-1}\, t_{n+1}, \ldots, \exp 2\pi\sqrt{-1}\, t_{n+q})\in\{|w_i|=1, 1\leqq i\leqq 2q\}\subset C^{2q}$ and $\xi_i:=\exp 2\pi\sqrt{-1}(t_i+\sqrt{-1}\, t_{n+q+i})\in C^*$, $1\leqq i\leqq n-q$. Then by Proposition 2. 5 we have $\delta>0$ so that $f^*(\xi, w):=\sum_{m\in Z^{n+q}} c^m\xi^{m''}w^{m'}$ is holomorphic in $(\xi, w)\in C^{*n-q}\times\{1-\delta<|w_j|<1+\delta\}$, where $\xi^{m''}:=\xi_1^{m_{q+1}}\cdots\xi_{n-q}^{m_n}$, $w^{m'}:=w_1^{m_1}\cdots w_q^{m_q}w_{q+1}^{m_{n+1}}\cdots w_{2q}^{m_{n+q}}$. Put $\varepsilon:=1/2\min\{-\log(1-\delta), \log(1+\delta)\}$. Then for any $a>0$ $\sup\{|c^m|\, |\xi_1|^{m_{q+1}}\cdots|\xi_{n-q}|^{m_n}|w_1|^{m_1}\cdots|w_q|^{m_q}|w_{q+1}|^{m_{n+1}}\cdots|w_{2q}|^{m_{n+q}}; m\in Z^{n+q}, \exp(-a)\leqq|\xi_i|\leqq\exp a, \exp(-\varepsilon)\leqq|w_i|\leqq\exp\varepsilon\}<\infty$. Conversely assume (b) holds. Then $\sum_{m\in Z^{n+q}} c^m\xi^{m''}w^{m'}$ converges uniformly on every compact subset of $C^{*n-q}\times\{\exp(-\varepsilon)<|w_j|<\exp\varepsilon\}$. This implies (a).

For $m\in Z^{n+q}$ we use the notation: $\|m^*\|:=\max\{|m_i|, 1\leqq i\leqq n\}$.

Lemma 4. 2. *The following conditions* (0) *and* (1) *are equivalent.*

(0) *For any* $\varepsilon>0$ *there exists a positive number* $a=a(\varepsilon)$ *such that* $\sup_{m\neq 0} \exp(-\varepsilon\|m'\|-a\|m''\|)/K_m<\infty$.

(1) *There exists* $a>0$ *such that* $\sup_{m\neq 0} \exp(-a\|m^*\|)/K_m<\infty$.

Proof. Since $K_m=\max_{1\leqq i\leqq q}\sqrt{(\sum_{j=1}^n \operatorname{Re} v_{ij}m_j-m_{n+i})^2+(\sum_{j=1}^n \operatorname{Im} v_{ij}m_j)^2}$ and the $q\times q$-matrix $[\operatorname{Im} v_{ij}; 1\leqq i, j\leqq q]$ is non-singular, we can find $C_1, C_2>0$ so that $S:=\{m; m\neq 0, |K_m|\leqq 1\}\subset\{m; m\neq 0, \|m'\|\leqq C_1+C_2\|m''\|\}$. We shall show that the statement (1) is equivalent to the following

(*) There exists $a>0$ such that $\sup_{m\neq 0}\exp(-a\|m''\|)/K_m<\infty$.

Since $\|m''\|\leqq\|m^*\|$, the implication $(*)\Longrightarrow(1)$ is trivial. Assume (1) holds. We have $\|m^*\|\leqq\|m'\|+\|m''\|\leqq C_1+(C_2+1)\|m''\|$ for $m\in S$. We put $b:=(C_2+1)a>0$. Then $\sup_{m\in S}\exp(-b\|m''\|)/K_m\leqq\exp(bC_1/(C_2+1))\sup_{m\in S}\exp(-a\|m^*\|)/K_m<\infty$. This implies (*) holds. We prove $(0)\Longrightarrow(*)$. Assume (0) holds. We get $a>0$ such that $\sup_{m\neq 0}\exp(-\|m'\|-a\|m''\|)/K_m<\infty$. We have $\exp(-(C_1+C_2\|m''\|-a\|m''\|)/K_m\leqq\exp(-\|m'\|-a\|m''\|)/K_m$ for $m\in S$. This implies the statement (*) holds. The implication $(*)\Longrightarrow(0)$ is trivial.

Remark. The condition (0) depends on our assumption that $\det[\operatorname{Im} v_{ij}; 1\leqq i, j\leqq q]\neq 0$. But the condition (1) is independent on that.

Let $\rho=1/p!\sum_{1\leqq i_1,\dots,i_p\leqq q}\rho_{i_1\dots i_p}d\bar{\zeta}_{i_1}\wedge\dots\wedge d\bar{\zeta}_{i_p}\in H^0(C^n/\Gamma, \mathscr{H}^{0,p})$. We expand each $\rho_{i_1\dots i_p}$ as in (1. 8) : $\rho_{i_1\dots i_p}=\sum_{m\in Z^{n+q}}b^m_{i_1\dots i_p}\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle$. We put $\rho^m_{i_1\dots i_p}:=b^m_{i_1\dots i_p}\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle$, $\rho^m:=1/p!\sum_{1\leqq i_1,\dots,i_p\leqq q}b^m_{i_1\dots i_p}\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle\ d\bar{\zeta}_{i_1}\wedge\dots\wedge d\bar{\zeta}_{i_p}$. Then $\rho=\sum_{m\in Z^{n+q}}\rho^m$. Suppose ρ is $\bar{\partial}$-exact. Namely there exists a $(0, p-1)$-form $\lambda=\sum_{m\in Z^{n+q}}\lambda^m\in H^0(C^n/\Gamma, \mathscr{H}^{0,p-1})$ such that $\rho=\bar{\partial}\lambda$. Then we have $\rho^m=\bar{\partial}\lambda^m$ for any $m\in Z^{n+q}$. We write $\lambda^m=1/(p-1)!\sum_{1\leqq i_1,\dots,i_{p-1}\leqq q}\lambda^m_{i_1\dots i_{p-1}}d\bar{\zeta}_{i_1}\wedge\dots\wedge d\bar{\zeta}_{i_{p-1}}$ and $\lambda^m_{i_1\dots i_{p-1}}=d^m_{i_1\dots i_{p-1}}\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle$. The equation $\rho^m=\bar{\partial}\lambda^m$ implies

(4.1) $$\rho^m_{i_1\cdots i_p}=\sum_{k=1}^p(-1)^{k+1}\frac{\partial\lambda^m_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial\bar{\zeta}_{i_k}}.$$

Combining (4.1) with (1.7), we have for any $m\in Z^{n+q}$

(4.2) $$b^m_{i_1\cdots i_p}=\sum_{k=1}^p(-1)^{k+1}\pi K_{m,i_k}d^m_{i_1\cdots\hat{i}_k\cdots i_p}.$$

Now suppose $\phi=\sum_{m\in Z^{n+q}}\phi^m\in H^0(C^n/\Gamma,\ \mathscr{H}^{0,p})$ is $\bar{\partial}$-closed. From $\bar{\partial}\phi=0$ and (4.2) it follows that

(4.3) $$\pi\sum_{k=1}^{p+1}(-1)^{k+1}K_{m,i_k}c^m_{i_1\cdots\hat{i}_k\cdots i_{p+1}}=0,$$

where we denote $\phi^m=1/p!\sum_{1\leqq i_1,\ldots,i_p\leqq q}c^m_{i_1\cdots i_p}\exp(-2\pi\sum_{i=q+1}^n m_it_{n+i})$ $\exp 2\pi\sqrt{-1}\ \langle m,\ t'\rangle\ d\bar{\zeta}_{i_1}\wedge\ldots\wedge d\bar{\zeta}_{i_p}$.

We put

$$i(m):=\min\{i\ ;\ |K_{m,i}|=K_m,\quad 1\leqq i\leqq q\}$$

and the indices $(i(m),\ i_1,\ldots,\ i_p)$ in the place of $(i_1,\ldots,\ i_{p+1})$ of the formula (4.3), then we have

(4.4) $$\pi K_{m,i(m)}c^m_{i_1\cdots i_p}=\pi\sum_{k=1}^p(-1)\ K^{k+1}K_{m,i_k}c^m_{i(m)i_1\cdots\hat{i}_k\cdots i_p}.$$

Since $K_m>0$ for $m\neq0$ by (1.2), we can put

(4.5) $$\psi^m:=1/\pi(p-1)!\sum_{1\leqq i_1\cdots i_{p-1}\leqq q}c^m_{i(m)i_1\cdots i_{p-1}}/K_{m,i(m)}$$
$$\exp(-2\pi\sum_{i=q+1}^n m_it_{n+i})\exp 2\pi\sqrt{-1}\ \langle m,\ t'\rangle\ d\bar{\zeta}_{i_1}\wedge\ldots\wedge d\bar{\zeta}_{i_{p-1}}$$

for $m\neq0$. Observing (4.2), it follows from (4.4) and (4.5) that

(4.6) $$\phi^m=\bar{\partial}\psi^m\ \text{for any}\ m\neq0.$$

Remark. For any $\bar{\partial}$-closed $(0,\ p)$-form $\phi=\sum\phi^m$ we have always a formal solution $\sum_{m\neq0}\psi^m$ for the $\bar{\partial}$-equation $\bar{\partial}\sum_{m\neq0}\psi^m=\sum_{m\neq0}\phi^m$ by (4.6).

Here we need to topologize $H^0(C^n/\Gamma,\ \mathscr{H})$. Let $\mathscr{A}(R)$ be the vector space of real analytic functions on R. We regard R as a closed real analytic submanifold of C under the natural inclusion. We take a compact subset K of R and an open and connected neighbourhood U_j of K in C, $1\leqq j\leqq\infty$ satisfying $U_{j+1}\Subset U_j$ and $\cap_jU_j=K$. Let $\mathscr{A}(K)$ be the vector space of real analytic functions in a neighbourhood of K in R. We denote by $\mathscr{H}(U_j)$ the space of bounded holomorphic functions on U_j, $j\geqq1$. Put $||f||:=\sup\{|f(z)|\ ;\ z\in U_j\}$, $f\in\mathscr{H}(U_j)$. This norm makes $\mathscr{H}(U_j)$ into a Banach space.

By the inductive limit: $\mathscr{A}(K)=\text{ind}\lim \mathscr{H}(U_j)$ we regard $\mathscr{A}(K)$ as a (D, F, S)-space. The restriction mapping $\mathscr{A}(K_1)\longrightarrow\mathscr{A}(K_2)$, $K_2\subset K_1$ induces the projective limit: $\mathscr{A}(R)=\text{proj}\lim\mathscr{A}(K)$. It is known that the above locally convex topology on $\mathscr{A}(R)$ is complete and semi-Montel. Similarly to the topology of $\mathscr{A}(R)$ we can make the vector space $H^0(C^n/\Gamma, \mathscr{A})$ into a locally convex space. Then $H^0(C^n/\Gamma, \mathscr{H})$ is regarded as a closed subspace of $H^0(C^n/\Gamma, \mathscr{A})$ and itself a locally convex space. And we have the locally convex topology of $H^0(C^n/\Gamma, \mathscr{H}^{0,p})$ induced by $H^0(C^n/\Gamma, \mathscr{H})$. Further by Proposition 3. 4 we have the locally convex topology of $H^p(C^n/\Gamma, \mathscr{O})$, using the quotient topology.

The following theorem gives a characterization of an (H, C)-group C^n/Γ whose cohomology groups $H^p(C^n/\Gamma, \mathscr{O})$ $(p\geqq 1)$ are finite-dimensional.

Theorem 4. 3. *Let C^n/Γ be an (H, C)-group, where Γ is generated by $\{e_1,\dots, e_n, v_1,\dots, v_q\}$, $K_{m,i}:=\sum_{j=1}^n v_{ij}m_j-m_{n+i}$ $(1\leqq i\leqq q)$ and $K_m:=\max\{|K_{m,i}|;\ 1\leqq i\leqq q\}$ for $m\in Z^{n+q}$. Then the following statements* (1), (2), (3) *and* (4) *are equivalent.*

(1) *There exists $a>0$ such that*
$$\sup_{m\neq 0}\ \exp(-a\|m^*\|)/K_m<\infty,$$
where $\|m^*\|=\max\{|m_i|;\ 1\leqq i\leqq n\}$.

(2) $$\dim H^p(C^n/\Gamma, \mathscr{O})=\begin{cases}\dfrac{q!}{(q-p)!\,p!} & \text{if } 1\leqq p\leqq q\\ 0 & \text{if } p>q.\end{cases}$$

(3) $\dim H^p(C^n/\Gamma, \mathscr{O})<\infty$ *for any* $p\geqq 1$.

(4) $\bar{\partial}(H^0(C^n/\Gamma, \mathscr{H}^{0,p-1}))$ *is a closed subspace of* $H^0(C^n/\Gamma, \mathscr{H}^{0,p})$ *for any* $p\geqq 1$.

Proof. Assume (1) holds. Then by Lemma 4. 2 we may suppose that the statement (0) of Lemma 4. 2 holds. We take a $\bar{\partial}$-closed form $\phi=1/p!\sum_{m\in Z^{n+q}}\sum_{1\leqq i_1,\dots,i_p\leqq q}\phi^m_{i_1\cdots i_p}d\bar{\zeta}_{i_1}\wedge\dots\wedge d\bar{\zeta}_{i_p}\in H^0(C^n/\Gamma, \mathscr{H}^{0,p})$, where $\phi^m_{i_1\cdots i_p}=c^m_{i_1\cdots i_p}\exp(-2\pi\sum_{i=q+1}^n m_i t_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle$. By Lemma 4. 1 there exists $\varepsilon_0>0$ such that for any $a>0$ $C(a):=\sup\{|c^m_{i_1\cdots i_p}|\exp(\varepsilon_0\|m'\|+a\|m''\|);\ m\in Z^{n+q}\}<\infty$ $(1\leqq i_1,\dots, i_p\leqq q)$. By the statement (0) of Lemma 4. 2 we find $a_0>0$ such that

$$\sup_{m\neq 0} \exp(-\varepsilon_0/2\|m'\|-a_0\|m''\|)/K_m=C_0<+\infty.$$

Then for any $a>0$, $m\neq 0$ and $1\leqq i_1,\ldots,i_p\leqq q$ $|c^m_{i_1\cdots i_p}|\exp(\varepsilon_0/2\|m'\|+a\|m''\|)/K_m\leqq C_0|c^m_{i_1\cdots i_p}|\exp(\varepsilon_0\|m'\|+(a+a_0)\|m''\|)\leqq C_0C(a+a_0)<\infty$. This means that $\sum_{m\neq 0}\psi^m$ given by (4.5) converges to a $(0, p-1)$-form ψ in $H^0(C^n/\Gamma, \mathscr{H}^{0,p-1})$. And by (4.6) we have $\phi-\bar\partial\psi=\phi^0+\sum_{m\neq 0}(\phi^m-\bar\partial\psi^m)=\sum_{1\leqq i_1,\cdots,i_p\leqq q} c^0_{i_1\cdots i_p}d\bar\zeta_{i_1}\wedge\ldots\wedge d\bar\zeta_{i_p}$. This shows (2) holds. It is obvious that $(2)\Longrightarrow(3)\Longrightarrow(4)$. Finally we prove $(4)\Longrightarrow(1)$. By Lemma 4.2 we may prove that (4) implies the statement (0) of Lemma 4.2 instead of $(4)\Longrightarrow(1)$. Suppose that $\{K_m; m\in Z^{n+q}\}$ doesn't satisfy the statement (0) of Lemma 4.2. Then there exists $\varepsilon_1>0$ such that we can choose $\{m_\nu;\nu\geqq 1\}\subset Z^{n+q}-\{0\}$ satisfying $\exp(-\varepsilon_1\|m'_\nu\|-\nu\|m''_\nu\|)/K_{m_\nu}\geqq\nu$ for any $\nu\geqq 1$. We put

$$\delta^m:=\begin{cases}\exp(-\varepsilon_1\|m'_\nu\|-\nu\|m''_\nu\|)/K_{m_\nu} & \text{if } m=m_\nu \text{ for some } \nu\geqq 1.\\ 0 & \text{otherwise}\end{cases}$$

and $\psi^m:=\delta^m\exp(-2\pi\sum_{i=q+1}^n m_it_{n+i})\exp 2\pi\sqrt{-1}\langle m, t'\rangle$ for any $m\in Z^{n+q}$. Since $\bar\partial\psi^m=\sum_{j=1}^q\pi K_{m_\nu,j}\exp(-\varepsilon_1\|m'_\nu\|-\nu\|m''_\nu\|)\exp(-2\pi\sum_{i=q+1}^n m_it_{n+i})/K_{m_\nu}\exp 2\pi\sqrt{-1}\langle m, t'\rangle d\bar\zeta_j$, if $m=m_\nu$ for some $\nu\geqq 1$ and $|K_{m_\nu,j}/K_{m_\nu}|\leqq 1$, then $\sum_m\bar\partial\psi^m$ converges to a form $\phi\in H^0(C^n/\Gamma, \mathscr{H}^{0,1})$. By the choice of the sequence $\{m_\nu\}$, $\sum_m\psi^m$ cannot converge to any function in $H^0(C^n/\Gamma, \mathscr{H}^{0,0})$. Suppose $\phi=\bar\partial\lambda$ for some $\lambda=\sum_m\lambda^m\in H^0(C^n/\Gamma, \mathscr{H}^{0,0})$, then $\lambda^m=\psi^m$ for $m\neq 0$. It is a contradiction. Then $\phi=\lim_{N\to\infty}\bar\partial(\sum_{\|m\|<N}\psi^m)$ belongs not to $\bar\partial(H^0(C^n/\Gamma, \mathscr{H}^{0,0}))$, but to the closure of $\bar\partial(H^0(C^n/\Gamma, \mathscr{H}^{0,0}))$ in $H^0(C^n/\Gamma, \mathscr{H}^{0,0})$. This contradicts the statement (4).

By the above proof of the implication $(4)\Longrightarrow(1)$, if $\{K_m; m\in Z^{n+q}\}$ doesn't satisfy the statement (1) of Theorem 4.3, then $H^1(C^n/\Gamma, \mathcal{O})$ is a non-Hausdorff locally convex space and then infinite-dimensional. Further in the above situation we shall prove that $H^p(C^n/\Gamma, \mathcal{O})$ are also non-Hausdorff spaces for all p satisfying $2\leqq p\leqq q$.

Theorem 4.4. *Every (H, C)-group C^n/Γ satisfies either of the following statements (a) and (b).*

(*a*) *$H^p(C^n/\Gamma, \mathcal{O})$ is finite-dimensional for any p.*

(*b*) *$H^p(C^n/\Gamma, \mathcal{O})$ is a non-Hausdorff locally convex space for any p satisfying $1\leqq p\leqq q$.*

Further the statement (*b*) *is equivalent to the following*
(*c*) $\sup_{m\neq 0} \exp(-a\|m^*\|)/K_m=\infty$ *for any* $a>0$,
where $\|m^*\|=\max\{|m_i|;\ 1\leqq i\leqq n\}$.

Proof. By Lemma 4. 2 and Theorem 4. 3, we must prove (b) holds on the assumtion that $\{K_m\}$ doesn't satisfy the statement (0) of Lemma 4. 2. We choose $\varepsilon_1>0$, the sequence $\{m_\nu\}$ and δ^m as in the proof of $(4)\Longrightarrow(1)$ in Theorem 4. 3. We can find i_0 so that $1\leqq i_0\leqq q$ and $\sup\{\nu;\ |K_{m_\nu,i_0}|=K_{m_\nu}\}=\infty$. We may assume $i_0=q$ without loss of generality. We take a $(0, p-1)$-form

$$\psi^m:=\delta^m \exp(-2\pi\textstyle\sum_{i=q+1}^n m_it_{n+i}) \exp 2\pi\sqrt{-1}\ \langle m,\ t'\rangle\ d\bar\zeta_1\wedge\dots\wedge d\bar\zeta_{p-1}.$$

Then $\sum_m \psi^m$ cannot converge to any $(0, p-1)$-form in $H^0(C^n/\Gamma, \mathscr{H}^{0,p-1})$. On the other hand $\sum_m \bar\partial\psi^m$ converges to a $(0, p)$-form $\phi=\sum_m \sum_{j=p}^q \pi K_{m,j}\delta^m \exp(-2\pi\sum_{i=q+1}^n m_it_{n+i}) \exp 2\pi\sqrt{-1}\ \langle m,\ t'\rangle\ d\bar\zeta_j\wedge d\bar\zeta_1\wedge\dots\wedge d\bar\zeta_{p-1}\neq 0$. Suppose $\phi=\bar\partial\lambda$ for some $\lambda=\sum_m \lambda^m\in H^0(C^n/\Gamma, \mathscr{H}^{0,p-1})$. Then $\bar\partial\psi^m=\bar\partial\lambda^m$. We write

$$\lambda^m=1/(p-1)!\ \textstyle\sum b^m_{i_1\cdots i_{p-1}} \exp(-2\pi\sum_{i=q+1}^n m_it_{n+i})\ \exp 2\pi\sqrt{-1}\ \langle m,\ t'\rangle\ d\bar\zeta_{i_1}\wedge\dots\wedge d\bar\zeta_{i_{p-1}}.$$

Comparing the term of $\bar\partial\psi^m$ with that of $\bar\partial\lambda^m$ involving only the exterior differential $d\bar\zeta_q\wedge d\bar\zeta_1\wedge\dots\wedge d\bar\zeta_{p-1}$ of type $(0, p)$, we have $\pi K_{m_\nu,q}b^{m_\nu}_{12\cdots p-1}+\sum_{i=1}^{p-1}(-1)^i\pi K_{m_\nu,i}b^{m_\nu}_{q1\cdots\hat{i}\cdots p-1}=\pi K_{m_\nu,q}\delta^{m_\nu}$. Then $\delta^{m_\nu}=b^{m_\nu}_{12\cdots p-1}+\sum_{i=1}^{p-1}(-1)^i b^{m_\nu}_{q1\cdots\hat{i}\cdots p-1}K_{m_\nu,i}/K_{m_\nu,q}$ for $\nu\geqq 1$. Since $\sup\{\nu;\ |K_{m_\nu,q}|=K_{m_\nu}\}=\infty$, we can choose a subsequence $\{m_{\tilde\nu}\}$ of $\{m_\nu\}$ so that $|K_{m_{\tilde\nu},i}/K_{m_{\tilde\nu},q}|\leqq 1$ for any $1\leqq i\leqq q$ and that

$$\lim_{\nu\to\infty}(b^{m_{\tilde\nu}}_{12\cdots p-1}+\textstyle\sum_{i=1}^{p-1}(-1)^i b^{m_{\tilde\nu}}_{q1\cdots\hat{i}\cdots p-1})K_{m_{\tilde\nu},i}/K_{m_{\tilde\nu},q}=0.$$

This contradicts that $\lim_{\nu\to\infty}\delta^{m_{\tilde\nu}}=\infty$. Hence ϕ belongs not to $\bar\partial(H^0(C^n/\Gamma, \mathscr{H}^{0,p-1}))$ but to the closure of $\bar\partial(H^0(C^n/\Gamma, \mathscr{H}^{0,p-1}))$.

Remark. When the author was making the preprint for this paper, he got the following information which was given by S. Takeuchi. Independently C. Vogt [15] showed in his Dissertation that the statements (a) and (b) are equivalent.

(a) There exist $C>0$ and $a>0$ such that $K_m\geqq C\exp(-a\|m^*\|)$.
(b) $\dim H^1(C^n/\Gamma,\ \mathscr{O})<\infty$.

By Theorem 4. 3 and 4. 4 we have the following

Corollary 4. 5. *The statements* (1), (2), (3) *and* (4) *in Theorem* 4. 3 *are equivalent to each of the following statements* (5) *and* (6).

(5) *For some* p $(1\leqq p\leqq q)$ $\dim H^{p}(C^{n}/\Gamma, \mathcal{O})<\infty$.

(6) *For some* p $(1\leqq p\leqq q)$ $\bar{\partial}(H^{0}(C^{n}/\Gamma, \mathcal{H}^{0,p-1}))$ *is a closed subspace of* $H^{0}(C^{n}/\Gamma, \mathcal{H}^{0,p})$.

Remark. We constructed an example of an (H, C)-group C^{n}/Γ so that $H^{1}(C^{n}/\Gamma, \mathcal{O})$ is not Hausdorff ([7]). By Corollary 4. 5 we can show $H^{p}(C^{n}/\Gamma, \mathcal{O})$ are not Hausdorff for this (H, C)-group C^{n}/Γ $(2\leqq p\leqq q)$.

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Supplementary notes:

After this paper was submitted, the referee informed the author the following results.

By another method L. C. Piccinini showed implicitly that $C\times R$ admits no Stein neighbourhood bases in $C\times C$ in the article:

[16] Non surjectivity of $\partial^2/\partial x^2+\partial^2/\partial y^2$ as an operator on the space of analytic functions on R^3, *Lecture Notes of the Summer College on Global Analysis*, Trieste, August 1972,

and C. Vogt has published the paper:

[17] Two remarks concerning toroidal groups, *Manuscripta Math.*, **41** (1983), 217-232.

In [17] he showed independently that the statements (1) and (2) of Theorem 4.3 of this paper are equivalent.