Publ. RIMS, Kyoto Univ.
20 (1984), 461-489

On the Douady Space of a Compact Complex Space in the Category $\mathscr{C}$, II

By

Akira FUJIKI*

Introduction

This is a continuation of our previous paper [4]. Let $f: X \to S$ be a proper morphism of complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. Then we denote by $D_{X/S}(\mathscr{E})$ the Douady space of flat quotients of $\mathscr{E}$ over S; $D_{X/S}(\mathscr{E})$ represents the functor $\boldsymbol{D}_{X/S}(\mathscr{E})$: $(\mathrm{An}/S)^\circ \to \mathrm{Sets}$ defined by $\boldsymbol{D}_{X/S}(\mathscr{E})(T) :=$ the set of coherent quotients $\mathscr{F}$ of $\mathscr{E}_T$ (the pull-back of $\mathscr{E}$ to $X \times_S T$) such that $\mathscr{F}$ is flat over T.

We say that f is a $\mathscr{C}$-morphism (f belongs to $\mathscr{C}/S$ in the terminology of [4]) (resp. is Moishezon) if X_{red} is a meromorphic image over S of a complex space $\tilde{X}$ which is proper and locally Kähler (resp. locally projective) over S (cf. [4, (2.1) (resp. (1.2))]). Then the purpose of the present paper is to prove the following theorem which generalizes Theorem in [4]:

Theorem. *Let $f: X \to S$ be a proper morphism of complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. Let $b: D_{X/S}(\mathscr{E})_{\mathrm{red}} \to S$ be the natural morphism. Suppose that f is a $\mathscr{C}$-morphism* (*resp. is Moishezon*). *Then for any relatively compact subdomain Q of S and for any irreducible component A of $b^{-1}(Q)$ the induced morphism $b|_A : A \to Q$ is proper and is a $\mathscr{C}$-morphism* (*resp. is Moishezon*).

Corollary. *Let X be a compact complex space in $\mathscr{C}$. Then any irreducible component D_α of the reduced Douady space $D_{X,\mathrm{red}}$ is again compact and belongs to $\mathscr{C}$. If, further, X is Moishezon, D_α is again Moishezon.*

Communicated by S. Nakano, February 10, 1983.
* Department of Mathematics, Yoshida College, Kyoto University, Kyoto 606, Japan.

As is mentioned above, Theorem was shown in [4] in the case where $\mathscr{E}=\mathcal{O}_X$ (so that $D_{X/S}(\mathscr{E})$ reduces to the ordinary relative Douady space $D_{X/S}$ of X over S) and the universal subspace $Z_{X/S}\subseteqq X\times_S D_{X/S}$ restriced over $A\subseteqq D_{X/S}$ is reduced. On the other hand, the properness of $b|_A$ in general was shown in [3, Theorem 5.2] under the assumption that f is a Kähler morphism, and more generally it was asserted in [3] with a sketch of proof in the case where f is a $\mathscr{C}$-morphism. (See however Remark 2 at the end of this paper.) Indeed, Theorem above follows readily from the result of [4] above if we re-examine the proof of Theorem 5.2 in [3] (the part where the general case is reduced to the case where $\mathscr{E}=\mathcal{O}_X$ and $Z_{X/S}$ is reduced over A), except that the proof there contains a gap in the formulation and proof of Lemma 5.8. The main point of our proof is thus nothing but to fill that gap in a way suitable for the proof of the above theorem.

In Section 1 we first construct the universal space of extensions of a given coherent analytic sheaf by another coherent analytic sheaf satisfying some additional conditions, and then construct the natural compactification of it. Then in Section 2 using the results obtained in Section 1 we prove a revised form of [3, Lemma 5.8] (cf. Lemma 12 below) and then prove Theorem above, essentially following the proof of [3, Theorem 5.2]. Also some other corrections to [3] will be given (cf. a remark after Lemma 14 and Remark 2).

We follow mainly the notations and terminologies of [4]. Also we recall the following notation from [3]. Let $f: X\to S$ be a morphism of complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. Let $\nu: T\to S$ be a morphism of complex spaces. Then the pull-back of $\mathscr{E}$ to $X_T:=X\times_S T$ will usually be denoted by $\mathscr{E}_T$ (with ν understood). Further (An/S) denotes the category of complex spaces over S. A reduced and irreducible complex space is called a *complex variety*. A Zariski open subset is in general assumed to be nonempty unless otherwise is explicitly stated. The following result of Frisch [2] will be used without reference throughout the paper; for any f: $X\to S$ and $\mathscr{E}$ as above with S reduced and with f proper there exists a dense Zariski open subset U of S over which $\mathscr{E}$ is f-flat.

§ 1. Space of Completions of a Diagram

1.1. In this subsection we gather some preliminary results.

a) Let T be a complex space and $\mathscr{F}$ a coherent analytic sheaf on T. Let $U \subseteqq T$ be a Zariski open subset. Then a holomorphic section $s \in H^0(U, \mathscr{F})$ of $\mathscr{F}$ over U is said to be *extended meromorphically to* T if there exists a coherent sheaf of ideals $\mathscr{I}$ of $\mathcal{O}_T$ such that $\mathrm{supp}\, \mathcal{O}_T/\mathscr{I} \subseteqq A := T - U$ and that for any open subset V of T and any element $b \in H^0(V, \mathscr{I})$, $b_0 s_0$ extends to a holomorphic section of $\mathscr{F}$ over the whole V, where supp denotes the support and the subscript 0 denotes the restriction to $U \cap V$. By the Hibert Nullstellensatz we get

Lemma 1. *Suppose that there exists an effective Cartier divisor D on T whose support coincides with A. Then s is extended meromorphically to T if and only if for any relatively compact subdomain $W \subseteqq T$ there exist an integer $n > 0$ and an element $s^* \in H^0(W, \mathscr{F}(nD))$ which is an extension of $s|_{U \cap W}$ with respect to the natural isomorphism $\mathscr{F}(nD) \cong \mathscr{F}$ on U where $\mathscr{F}(nD) = \mathscr{F} \otimes \mathcal{O}([D]^n)$ with $[D]$ the holomorphic line bundle defined by D.*

Let T be a complex space. Let $U \subseteqq T$ be a Zariski open subset. Let $\mathscr{F}$, $\mathscr{G}$ be coherent analytic sheaves on T. Then we say that $\mathscr{F}$ and $\mathscr{G}$ are *meromorphically equivalent and isomorphic on* U if there exist coherent analytic sheaves $\mathscr{F}_0, \dots, \mathscr{F}_m$ on T such that 1) $\mathscr{F}_0 = \mathscr{F}$, $\mathscr{F}_m = \mathscr{G}$ and 2) for each $1 \leqq i \leqq m$, there exists a homomorphism $\mathscr{F}_{i-1} \to \mathscr{F}_i$ or $\mathscr{F}_i \to \mathscr{F}_{i-1}$ which is isomorphic over U.

Lemma 2. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ are meromorphically equivalent and isomorphic on U. Let $s \in H^0(U, \mathscr{F})$. Let $s' \in H^0(U, \mathscr{G})$ be the section corresponding to s via the natural isomorphism $\mathscr{F} \cong \mathscr{G}$ on U. Then s is extended meromorphically to T if and only if so is s'.*

Proof. We can reduce the proof immediately to the case where there exists a homomorphism $u: \mathscr{F} \to \mathscr{G}$ which is isomorphic over U. Then the assertion for s clearly implies that for s'. So suppose that s' is extended meromorphically to T. Let $\mathscr{I}'$ be the coherent sheaf of ideals of T as in the above definition of meromorphic extension for $(\mathscr{G}, s')$. Let $\mathscr{C}$ be the cokernel of u. Then the support of $\mathscr{C}$ is contained in A. Hence if $\mathscr{J}$ is the ideal sheaf of annihilators of $\mathscr{C}$, then supp $\mathscr{O}_T/\mathscr{J}\subseteqq A$. We set $\mathscr{I}=\mathscr{I}'\mathscr{J}$. We then show that for any open subset $V\subseteqq T$ and any $b\in H^0(V, \mathscr{I})$ b_0s_0 extends to a holomorphic section of $\mathscr{F}$ over V. Since the problem is local we may assume that V is Stein and b is of the form $b=b_1b_2$, where $b_1\in H^0(V, \mathscr{I}')$ and $b_2\in H^0(V, \mathscr{J})$. First, $b_{10}s_0'$ extends to a holomorphic section, say w, of $\mathscr{G}$ over V. Then $b_2w\in H^0(V, \text{Im } u)$ where Im denotes the image. Hence there exists an element $t\in H^0(V, \mathscr{F})$ which is mapped to b_2w. Then since u is isomorphic on U, $t_0=b_0s_0$.

q. e. d.

b) (cf. [6]). Let S be a complex space and $\mathscr{E}$ a coherent analytic sheaf on S. Then the *linear fiber space* $L(\mathscr{E})$ *over* S *associated to* $\mathscr{E}$ is a complex space over S which represents the functor F: $(\text{An}/S)^\circ\to(\text{Vector Spaces}/\boldsymbol{C})$ defined by

$$F(T):=Hom_{\mathscr{O}_T}(\nu^*\mathscr{E}, \mathscr{O}_T)=H^0(T, (\nu^*\mathscr{E})^*)$$

where $\nu: T\to S$ is any morphism and $*$ denotes taking the dual. From the definition we get

$$(1)\qquad \mathscr{S}(L(\mathscr{E}))\cong\mathscr{E}^*$$

where in general for a linear fiber space L over S, $\mathscr{S}(L)$ denotes the sheaf of germs of holomorphic sections of L. When $\mathscr{E}$ is locally free, we see that $L(\mathscr{E}^*)$ is a vector bundle over S and it represents the functor V: $(\text{An}/S)^\circ\to\text{Sets}$ defined by

$$V(T):=H^0(T, \nu^*\mathscr{E})$$

with ν as above. In this case we call $L(\mathscr{E}^*)$ the *vector bundle associated to the locally free sheaf* $\mathscr{E}$ (against the terminology of [6]).

Further recall that a projective fiber space $\boldsymbol{P}(\mathscr{E})$ over S associated to $\mathscr{E}$ is the complex space over S which represents the functor P: $(\text{An}/S)^\circ\to\text{Sets}$ defined by

$$P(T):=\text{the set of invertible quotients } \mathscr{L} \text{ of } \nu^*\mathscr{E}.$$

Lemma 3. *Let S be a complex space and $\mathscr{E}$ a coherent analytic sheaf on S. Let $\pi: L=L(\mathscr{E})\to S$ be the natural projection and $\alpha\in H^0(L,$*

$(\pi^* \mathscr{E})^*)$ *the universal section. Let* $\bar{\pi}: \boldsymbol{P}=\boldsymbol{P}(\mathscr{E} \oplus \mathscr{O}_S) \rightarrow S$ *be the projective fiber space associated to* $\mathscr{E} \oplus \mathscr{O}_S$. *Then there exists a natural Zariski open inclusion* $L \subseteq \boldsymbol{P}$ *over* S *such that* α *extends meromorphically to* $\boldsymbol{P}$ *(as a section of* $(\bar{\pi}^* \mathscr{E})^*$).

Proof. Let $\beta : \bar{\pi}^*(\mathscr{E} \oplus \mathscr{O}_S) \rightarrow \mathscr{L}$ be the universal quotient on $\boldsymbol{P}$. Let $j : L \rightarrow \boldsymbol{P}$ be the S-morphism induced by the quotient homomorphism $(\alpha + id) : \pi^* \mathscr{E} \oplus \mathscr{O}_{\boldsymbol{P}} (\cong \pi^*(\mathscr{E} \oplus \mathscr{O}_{\boldsymbol{P}})) \rightarrow \mathscr{O}_{\boldsymbol{P}}$ on L together with the universality of $\boldsymbol{P}$. On the other hand, let $s \in H^0(\boldsymbol{P}, \mathscr{L})$ be the image by β of the constant section $(0, 1) \in H^0(\boldsymbol{P}, \bar{\pi}^* \mathscr{E} \oplus \mathscr{O}_{\boldsymbol{P}}) = H^0(\boldsymbol{P}, \bar{\pi}^* \mathscr{E} \oplus \mathscr{O}_S)$. Let D be the divisor of zero of s and $W = \boldsymbol{P} - D$. Then on W, s defines the canonical isomorphism $\psi_s : \mathscr{O}_W \cong \mathscr{L}$ $(\cong \mathscr{O}_{\boldsymbol{P}}(D))$. Then $\psi_s^{-1}(\beta|_{\bar{\pi}^* \mathscr{E}})|_W : \bar{\pi}^* \mathscr{E} \rightarrow \mathscr{O}_W$ defines an S-morphism $\mu : W \rightarrow L$ by the universality of L. It is then easy to see that $j(L) \subseteq W$ and that j and μ are inverse to each other. Now let $\bar{\alpha} : \bar{\pi}^* \mathscr{E} \rightarrow \mathscr{O}_{\boldsymbol{P}}(D)$ be the composite of $\beta|_{\bar{\pi}^* \mathscr{E}}$ and the isomorphism $\mathscr{L} \cong \mathscr{O}_{\boldsymbol{P}}(D)$. Then $\bar{\alpha} \in Hom_{\mathscr{O}_{\boldsymbol{P}}}(\bar{\pi}^* \mathscr{E}, \mathscr{O}_{\boldsymbol{P}}(D)) \cong Hom_{\mathscr{O}_{\boldsymbol{P}}}(\bar{\pi}^* \mathscr{E}, \mathscr{O}_{\boldsymbol{P}})(D)$ is considered as a holomorphic extension of α to $\boldsymbol{P}$. Then the lemma follows from Lemma 1. q. e. d.

c) Let $f : X \rightarrow S$ be a morphism of complex spaces. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be coherent analytic sheaves on X. Let x be any point of X. Consider an exact sequence

$$(*) \qquad \mathscr{O}_X^{\oplus q} \xrightarrow{u} \mathscr{O}_X^{\oplus p} \xrightarrow{\rho} \mathscr{F}_1 \longrightarrow 0$$

defined in a neighborhood of x. Then applying $\mathscr{H}om_{\mathscr{O}_X}(\ , \mathscr{F}_2)$ to $(*)$ we get a homomorphism

$$v : \mathscr{F}_2^{\oplus p} \longrightarrow \mathscr{F}_2^{\oplus q}.$$

Then we say that *a pair* $(\mathscr{F}_1, \mathscr{F}_2)$ *satisfies the condition* (C) *at* x if both Im v and Coker v are f-flat at x for some exact sequence $(*)$ as above. We say that $(\mathscr{F}_1, \mathscr{F}_2)$ *satisfies* (C) *over an open subset* U *of* S if $(\mathscr{F}_1, \mathscr{F}_2)$ satisfies (C) at any point of X_U.

Lemma 4. *Suppose that* $\mathscr{F}_2$ *is* f*-flat at* x. *Then the condition that* Im v *and* Coker v *are* f*-flat at* x *is independent of the exact sequence* $(*)$ *above.*

Proof. Let r, s be nonnegative integers with $r \leqq s$. Let $\pi : \mathcal{O}_X^{\oplus s} \to \mathcal{O}_X^{\oplus r}$ be the projection to the first r factors. Let $u' = u \oplus \pi$ and $\rho' = \rho p_1$, where u and ρ are as in $(*)$ and $p_1 : \mathcal{O}_X^{\oplus(r+p)} \to \mathcal{O}_X^{\oplus p}$ is the projection. Then the resulting exact sequence

$$\mathcal{O}_X^{\oplus(s+q)} \xrightarrow{u'} \mathcal{O}_X^{\oplus(r+p)} \xrightarrow{\rho'} \mathscr{F}_1 \longrightarrow 0$$

is called the (r, s)-modification of $(*)$. Let $v' : \mathscr{F}_2^{\oplus(r+p)} \to \mathscr{F}_2^{\oplus(s+q)}$ be the homomorphism defined by u'. Then it is immediate to see that $\operatorname{Ker} v' \cong \operatorname{Ker} v$ and $\operatorname{Coker} v' \cong \operatorname{Coker} v \oplus \mathscr{F}_2^{\oplus(s-r)}$. Hence under our assumption the condition is independent of passing to any (r, s)-modification of $(*)$. Then the lemma follows from the fact that any two exact sequences as $(*)$ are isomorphic (over the identity of $\mathscr{F}_1$) at x if we pass to a suitable (r, s)-modification of each sequence (cf. the proof of Theorem in [7, p. 102]). q. e. d.

Using this lemma we shall show the following:

Lemma 5. *Assume that f is proper. Then there exists a dense Zariski open subset U of S such that $(\mathscr{F}_1, \mathscr{F}_2)$ satisfies the condition* (C) *over U.*

Proof. Take a locally finite open covering $\mathfrak{W} = \{W_\alpha\}$ of X such that on each W_α we get an exact sequence $\mathcal{O}_X^{\oplus q_\alpha} \xrightarrow{u_\alpha} \mathcal{O}_X^{\oplus p_\alpha} \longrightarrow \mathscr{F}_1 \longrightarrow 0$. Let $v_\alpha : \mathscr{F}_2^{\oplus p_\alpha} \longrightarrow \mathscr{F}_2^{\oplus q_\alpha}$ be the homomorphism associated to u_α. Let $T_\alpha = \{x \in W_\alpha ; \operatorname{Im} v_\alpha$ and $\operatorname{Coker} v_\alpha$ are not f-flat at $x)$. Then by Frisch T_α is an analytic subset of W_α such that $f(T_\alpha)$ is 'negligible' in S(cf. [2, Prop. (IV, 14)]). Let U_1 be a Zariski open subset of S over which $\mathscr{F}_2$ is f-flat. Then by Lemma 4 $T_\alpha = T_\beta$ on $W_\alpha \cap W_\beta \cap X_{U_1}$. Hence T_α define a global analytic subset $T(U_1)$ on X_{U_1}. Moreover the closure T of $T(U_1)$ in X is clearly analytic. Let $U_2 = S - f(T)$. Since $T \subseteqq \cup_\alpha T_\alpha$ and W is locally finite, $f(T)$ is negligible in S. Let $U = U_1 \cap U_2$. Then it is clear that $(\mathscr{F}_1, \mathscr{F}_2)$ satisfies the condition (C) over U, which is Zariski open in S. q. e. d.

d) Let $b : X \to Y$ be a morphism of complex spaces. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent analytic sheaves on Y. Then there exists a canonical homomorphism (cf. [5, 0, (6.7.6)])

$$(2) \qquad b^\sharp : b^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(b^* \mathscr{F}, b^* \mathscr{G}).$$

In particular when $\mathscr{G}=\mathcal{O}_Y$ this reduces to

$$b^{\#} : b^*\mathscr{F}^* \longrightarrow (b^*\mathscr{F})^*. \tag{3}$$

Lemma 6. *Let $f : X \to S$ be a morphism of complex spaces and $\mathscr{F}_1$, $\mathscr{F}_2$ coherent analytic sheaves on X. Let $\nu : S' \to S$ be a morphism of complex spaces and $\tilde{\nu} : X_{S'} \to X$ the natural projection. Then the canonical homomorphism*

$$\tilde{\nu}^{\#} : \tilde{\nu}^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathscr{H}om_{\mathcal{O}_{X_{S'}}}(\tilde{\nu}^*\mathscr{F}_1, \tilde{\nu}^*\mathscr{F}_2)$$

above is isomorphic if $(\mathscr{F}_1, \mathscr{F}_2)$ satisfies the condition (C) *over S.*

Proof. Let $X'=X_{S'}$. Let $x' \in X'$ be an arbitrary point. Let $x=\tilde{\nu}(x')$. Take any exact sequence $(*)$ defined in a neighborhood of x (cf. c)). From this we get the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \tilde{\nu}^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_2) & \longrightarrow & \tilde{\nu}^*(\mathscr{F}_2^{\oplus p}) & \longrightarrow & \tilde{\nu}^*(\mathscr{F}_2^{\oplus q}) \\
& \downarrow{\scriptstyle \tilde{\nu}^{\#}} & & \wr\!\downarrow & & \wr\!\downarrow \\
0 \longrightarrow & \mathscr{H}om_{\mathcal{O}_{X'}}(\tilde{\nu}^*\mathscr{F}_1, \tilde{\nu}^*\mathscr{F}_2) & \longrightarrow & (\tilde{\nu}^*\mathscr{F}_2)^{\oplus p} & \longrightarrow & (\tilde{\nu}^*\mathscr{F}_2)^{\oplus q}
\end{array}$$

where the top (resp. bottom) sequence is obtained as the pull-back by $\tilde{\nu}$ of the exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathscr{F}_2^{\oplus p} \xrightarrow{\ v\ } \mathscr{F}_2^{\oplus q}$$

coming from $(*)$ (resp. by applying $\mathscr{H}om_{\mathcal{O}_X}(\ ,\ \tilde{\nu}^*\mathscr{F}_2)$ to the exact sequence $\mathcal{O}_{X'}^{\oplus q} \longrightarrow \mathcal{O}_{X'}^{\oplus p} \longrightarrow \tilde{\nu}^*\mathscr{F}_1 \longrightarrow 0$). Hence the top (resp. bottom) sequence is exact by virtue of the condition (C) (resp. without any assumption on $\mathscr{F}_i$ and f). The lemma follows from this.

1.2. a) Let X be a complex space. Suppose that we are given a diagram

$$\begin{array}{ccccccc}
\mathscr{E}_1 & \xrightarrow{\ b\ } & \mathscr{E}_2 & \xrightarrow{\ c\ } & \mathscr{E}_3 & \longrightarrow & 0 \\
\downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u_3} & & \\
\mathscr{R}_1 & & & & \mathscr{R}_3 & &
\end{array} \tag{4}$$

of coherent analytic sheaves on X, where the horizontal sequence is exact and u_1, u_3 are surjective. Then we set

$$\begin{cases} \mathscr{F}_i = \operatorname{Ker} u_i, \quad i=1, 3 \\ \mathscr{F}_4 = c^{-1}(\mathscr{F}_3)/b(\mathscr{F}_1). \end{cases} \tag{5}$$

Let $\mathscr{K}$ be the kernel of c. Then we get an exact sequence

$$(6) \qquad 0 \longrightarrow \mathscr{K} \longrightarrow c^{-1}(\mathscr{F}_3) \xrightarrow{c} \mathscr{F}_3 \longrightarrow 0.$$

Conversely, $c^{-1}(\mathscr{F}_3)$ is characterized as a submodule of $\mathscr{E}_2$ by (6). Further c induces the natural homomorphism $\bar{c}: \mathscr{F}_4 \longrightarrow \mathscr{F}_3$, and $\bar{c}$ in turn induces a homomorphism

$$(7) \qquad \alpha : \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_4) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3).$$

Definition. A *completion of* (4) is a commutative diagram

$$(8) \qquad \begin{array}{ccccccccc} & & \mathscr{E}_1 & \xrightarrow{b} & \mathscr{E}_2 & \xrightarrow{c} & \mathscr{E}_3 & \longrightarrow & 0 \\ & & \downarrow u_1 & & \downarrow u_2 & & \downarrow u_3 & & \\ 0 & \longrightarrow & \mathscr{R}_1 & \longrightarrow & \mathscr{R}_2 & \longrightarrow & \mathscr{R}_3 & \longrightarrow & 0 \end{array}$$

of exact sequences of coherent analytic sheaves on X, where u_2 is surjective as well as u_1, u_3. We define the isomorphisms of two completions in an obvious way; then the set of isomorphism classes is naturally identified with the set of quotients of $\mathscr{E}_2$ which fit into the commutative diagram (8).

Lemma 7. *Let $id \in Hom_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3)$ be the identity of $\mathscr{F}_3$. Let $M = \alpha^{-1}(id)$, which is an affine linear subspace of the vector space $Hom_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_4)$. Then the set of isomorphism classes of completions of* (4) *is in natural bijective correspondence with the set M.*

Proof. Suppose that we are given a completion (8) of (4). Let $\mathscr{F}_2$ be the kernel of u_2. Then c induces an isomorphism c_0: $\mathscr{F}_2/b(\mathscr{F}_1) \simeq \mathscr{F}_3$. Let $j : \mathscr{F}_2/b(\mathscr{F}_1) \to \mathscr{F}_4 = c^{-1}(\mathscr{F}_3)/b(\mathscr{F}_1)$ be the natural inclusion. Then $jc_0^{-1} : \mathscr{F}_3 \to \mathscr{F}_4$ clearly determines a point of M which depends only on the isomorphism class of (8).

Conversely, given a point $m \in M \subseteq Hom_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_4)$, define $\mathscr{F}_2$ to be the natural inverse image of $m(\mathscr{F}_3) \subseteq \mathscr{F}_4$ in $c^{-1}(\mathscr{F}_3) \subseteq \mathscr{E}_2$. Let $\mathscr{R}_2 = \mathscr{E}_2/\mathscr{F}_2$ and let $u_2 : \mathscr{E}_2 \to \mathscr{R}_2$ be the quotient homomorphism. Since $c(\mathscr{F}_2) = \bar{c}m(\mathscr{F}_3) = \mathscr{F}_3$, c induces a surjection $t : \mathscr{R}_2 \to \mathscr{R}_3$. Further $b(\mathscr{F}_1) = \mathscr{F}_2 \cap b(\mathscr{E}_1)$ so that b induces an inclusion $s : \mathscr{R}_1 \to \mathscr{R}_2$. Then (s, t) is exact and u_i, $1 \leqq i \leqq 3$, t, s fit into a commutative diagram (8), thus giving (an isomorphism class of) a completion of (4). Finally it is readily checked that the above correspondences $(8) \to jc_0$

and $m\to(8)$ are inverses to each other. q. e. d.

b) Let $f:X\to S$ be a proper morphism of complex spaces with S reduced. Suppose that we are given a diagram (4) on X. Let $\nu: T\to S$ be any morphism of complex spaces. Then pulled back by ν (4) gives rise to the diagram

$$(4)_T \qquad \begin{array}{ccccccc} \mathscr{E}_{1,T} & \xrightarrow{b_T} & \mathscr{E}_{2,T} & \xrightarrow{c_T} & \mathscr{E}_{3,T} & \longrightarrow & 0 \\ \downarrow{\scriptstyle u_{1,T}} & & & & \downarrow{\scriptstyle u_{3,T}} & & \\ \mathscr{R}_{1,T} & & & & \mathscr{R}_{3,T} & & \end{array}$$

with the same property as (4). Then we set

$$(5)_T \qquad \begin{cases} \mathscr{F}_{T,i}=\operatorname{Ker} u_{i,T}, \quad i=1,3 \\ \mathscr{F}_{T,4}=c_T^{-1}(\mathscr{F}_{T,3})/b_T(\mathscr{F}_{T,1}). \end{cases}$$

Then we have the natural homomorphisms

$$\begin{cases} v_{i,T}:\mathscr{F}_{i,T}\longrightarrow\mathscr{F}_{T,i}, \quad i=1,3 \\ w_T\ :c^{-1}(\mathscr{F}_3)_T\longrightarrow c_T^{-1}(\mathscr{F}_{T,3}) \end{cases}$$

where $v_{i,T}$ is surjective. Further $v_{1,T}$ and w_T induce a homomorphism

$$v_{4,T}:\mathscr{F}_{4,T}\longrightarrow\mathscr{F}_{T,4}.$$

Lemma 8. *Suppose that $\mathscr{E}_3$ and $\mathscr{R}_3$ are f-flat. Then $v_{3,T}$ and $v_{4,T}$ are isomorphic.*

Proof. For $v_{3,T}$ this is immediate from the flatness of $\mathscr{R}_3$. We shall show that $v_{4,T}$ is isomorphic. Since $v_{1,T}$ is surjective, $b_T(\mathscr{F}_{T,1})=b_Tv_{1,T}(\mathscr{F}_{1,T})$ in $\mathscr{E}_{2,T}$. Hence it suffices to show that w_T is isomorphic. Let $\mathscr{K}^T$ be the kernel of $c_T:\mathscr{E}_{2,T}\to\mathscr{E}_{3,T}$. Then since $\mathscr{E}_3$ is f-*flat*, the natural homomorphism $\mathscr{K}_T\to\mathscr{K}^T$ is isomorphic. The assertion then follows from the following commutative diagram (cf. (6))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K}_T & \longrightarrow & c^{-1}(\mathscr{F}_3)_T & \longrightarrow & \mathscr{F}_{3,T} & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow{\scriptstyle w_T} & & \downarrow\wr\,{\scriptstyle v_{3,T}} & & \\ 0 & \longrightarrow & \mathscr{K}^T & \longrightarrow & c_T^{-1}(\mathscr{F}_{T,3}) & \longrightarrow & \mathscr{F}_{T,3} & \longrightarrow & 0 \end{array}$$

where the top sequence is exact since $\mathscr{F}_3$ is f-flat as well as $\mathscr{E}_3$ and $\mathscr{R}_3$. q. e. d.

Suppose now that we are given a completion (8) of (4). We assume that $\mathscr{R}_3$ is f-flat. Then the pull-back $(8)_T$ of (8);

$$(8)_T \qquad \begin{array}{ccccccccc} & & \mathscr{E}_{1,T} & \longrightarrow & \mathscr{E}_{2,T} & \xrightarrow{c_T} & \mathscr{E}_{3,T} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_{1,T}} & & \downarrow{\scriptstyle u_{2,T}} & & \downarrow{\scriptstyle u_{3,T}} & & \\ 0 & \longrightarrow & \mathscr{R}_{1,T} & \longrightarrow & \mathscr{R}_{2,T} & \longrightarrow & \mathscr{R}_{3,T} & \longrightarrow & 0 \end{array}$$

is again a completion of $(4)_T$ because $\mathscr{R}_{1,T}\longrightarrow\mathscr{R}_{2,T}$ is now injective. Thus we can speak of the functor $G:(\mathrm{An}/S)^\circ\longrightarrow\mathrm{Sets}$ defined by $G(T):=$the set of isomorphism classes of completions of $(4)_T$.

We set

$$\mathscr{G}=\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_4) \quad \text{and} \quad \mathscr{H}=\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3)$$

where $\mathscr{F}_3$ and $\mathscr{F}_4$ are as in (5). Then the homomorphism $\alpha:\mathscr{G}\to\mathscr{H}$ (cf. (7)) induces the natural homomorphism

$$(9) \qquad \beta: f_*\mathscr{G}\longrightarrow f_*\mathscr{H}.$$

In order to have the representability of G we make the following assumption;

(10) $\begin{cases} \mathscr{E}_3, \mathscr{R}_3, \mathscr{G}, \mathscr{H} \text{ are all } f\text{-flat, } (\mathscr{F}_3, \mathscr{F}_4) \text{ and } (\mathscr{F}_3, \mathscr{F}_3) \\ \text{satisfy the condition (C) over } S \text{ and } \dim H^0(X_s, \mathscr{G}_s) \\ \text{and } \dim H^0(X_s, \mathscr{H}_s) \text{ are locally constant on } S. \end{cases}$

Since S is reduced, this implies that

(11) $\qquad f_*\mathscr{G}$ and $f_*\mathscr{H}$ are locally free on S

and further that for any $\nu: T\to S$ as above

(12) $\begin{cases} \text{the natural homomorphism } a_T:\nu^*f_*\mathscr{G}\longrightarrow f_{T*}\mathscr{G}_T \text{ and} \\ a_T':\nu^*f_*\mathscr{H}\longrightarrow f_{T*}\mathscr{H}_T \text{ are isomorphic (cf. [1]).} \end{cases}$

By (11) we can consider the holomorphic vector bundle M (resp. N) associated to $f_*\mathscr{G}$ (resp. $f_*\mathscr{H}$) (cf. (1.1, b)). Let $\gamma: M\to N$ be the bundle homomorphism corresponding to β. Let $\iota\in H^0(S, \mathscr{S}(N))$ correspond to $id\in Hom_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3)$ with respect to the natural isomorphism

$$(13) \quad H^0(S, \mathscr{S}(N))\cong H^0(S, f_*\mathscr{H})\cong H^0(X, \mathscr{H})\cong Hom_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3).$$

Define

$$Y=\gamma^{-1}(\iota(S)).$$

Then Y is an analytic subspace of M and hence is naturally a complex space over S.

Proposition 1. *Under the assumption* (10) *G is represented by the complex space Y above.*

Proof. Let $\nu : T \to S$ be any morphism of complex spaces. First note that since $\mathscr{E}_3$ and $\mathscr{R}_3$ are f-flat, $\mathscr{F}_3$ also is f-flat. Further by Lemma 6 the canonical homomorphism $\tilde{\nu}^{\#} : \mathscr{G}_T \to \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{3,T}, \mathscr{F}_{4,T})$ is isomorphic. Also, by Lemma 8 $v_{3,T}$ and $v_{4,T}$ induce an isomorphism $d_T : \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{3,T}, \mathscr{F}_{4,T}) \to \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,4})$. Together with (12) these induce a natural isomorphism $g_T : \nu^* f_* \mathscr{G} \to f_{T*} \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,4})$. Similarly we get an isomorphism $h_T : \nu^* f_* \mathscr{H} \to f_{T*} \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,3})$.

Using these isomorphisms for $T=Y$ we shall now construct the universal completion of $(4)_Y$ on X_Y. Let $\pi : M \to S$ be the natural projection and set $\delta = \pi|_Y : Y \to S$. Let $\sigma \in H^0(Y, \delta^* f_* \mathscr{G})$ be the restriction to Y of the universal section $\tilde{\sigma} \in H^0(M, \pi^* f_* \mathscr{G})$. Let $\sigma' = g_Y(\sigma) \in H^0(S, f_{Y*} \mathscr{H}om_{\mathcal{O}_{X_Y}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,4})) \cong Hom_{\mathcal{O}_{X_Y}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,4})$. Similarly, let $\iota' = h_Y(\iota) \in Hom_{\mathcal{O}_{X_Y}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,3})$. Then ι' is the identity of $\mathscr{F}_{Y,3}$ and σ' is sent to ι' by the natural homomorphism $Hom_{\mathcal{O}_{X_Y}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,4}) \to Hom_{\mathcal{O}_{X_Y}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,3})$. Therefore by Lemma 7 σ' gives rise to a completion (14) of $(4)_Y$;

$$\begin{array}{ccccccc} & \mathscr{E}_{1,Y} & \longrightarrow & \mathscr{E}_{2,Y} & \longrightarrow & \mathscr{E}_{3,Y} & \longrightarrow 0 \\ & \downarrow{\scriptstyle u_{1,Y}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_{3\,Y}} & \\ 0 \longrightarrow & \mathscr{R}_{1,Y} & \longrightarrow & \mathscr{R} & \longrightarrow & \mathscr{R}_{3,Y} & \longrightarrow 0. \end{array} \tag{14}$$

We claim that (14) is the desired universal completion (up to isomorphisms). Let $\nu : T \to S$ be any morphism of complex spaces with a completion $(8)_T$ of $(4)_T$. Then by Lemma 7 $(4)_T$ defines an element $\lambda' \in Hom_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,4})$ whose image in $Hom_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,3})$ is the identity. Let $\lambda = g_T^{-1}(\lambda') \in H^0(T, \nu^* f_* \mathscr{G})$. Then the image of λ in $H^0(X, \nu^* f_* \mathscr{H})$ by $\nu^*(\beta)$ equals $\nu^*(\iota)$. Hence by the universality of M together with the definition of Y there exists a unique S-morphism $\tau : T \to Y$ such that λ is the pull-back of σ by τ ; $\lambda = \tau^*(\sigma)$. This then implies that $\tau^*(\sigma') = \lambda'$ with respect to the natural isomorphism $\tilde{\tau}^* \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{Y,3}, \mathscr{F}_{Y,4}) \cong \mathscr{H}om_{\mathcal{O}_{X_T}}(\tilde{\tau}^* \mathscr{F}_{Y,3}, \tilde{\tau}^* \mathscr{F}_{Y,4}) \cong \mathscr{H}om_{\mathcal{O}_{X_T}}(\mathscr{F}_{T,3}, \mathscr{F}_{T,4})$, where $\tilde{\tau} : X_T \cong X_Y \times_Y T \to X_Y$ is the natural projection (cf. Lemmas 6 and 8). This in turn is equivalent to saying that $(8)_T$ is isomorphic to the pull-back of (14) by τ, in view of the correspondence of Lemma 7. q. e. d.

1.3. Let $f: X \to S$ be a proper morphism of complex spaces with S reduced. Suppose that we are given a diagram (4) on X. As in 1.2 b) we set $\mathscr{G} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_4)$ and $\mathscr{H} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_3)$ where $\mathscr{F}_3$, $\mathscr{F}_4$ are defined by (5). We fix a Zariski open subset $U \subseteqq S$ with the following property (cf. Lemma 5);

$$(15)\quad \begin{cases} \mathscr{E}_3,\ \mathscr{R}_3,\ \mathscr{G},\ \mathscr{H} \text{ are all } f\text{-flat over } U, (\mathscr{F}_3, \mathscr{F}_4) \text{ and } (\mathscr{F}_3, \mathscr{F}_3) \\ \text{satisfy the condition (C) over } U, \text{ and } \dim H^0(X_s, \mathscr{G}_s) \text{ and} \\ \dim H^0(X_s, \mathscr{H}_s) \text{ are locally constant on } U. \end{cases}$$

We then apply the consideration of 1.2 b) to $f_U: X_U \to U$ and $(4)_U$, the restriction of (4) over U. Let M_0 and N_0 be the vector bundles on U associated to the locally free sheaves $f_*\mathscr{G}|_U$ and $f_*\mathscr{H}|_U$ respectively. Let $\gamma: M_0 \to N_0$ be the bundle homomorphism induced by $\beta: f_*\mathscr{G} \to f_*\mathscr{H}$ restricted to U (cf. (9)). Let $\iota: U \to N_0$ be the holomorphic section defined by the identity of $\mathscr{F}_3$ via (13) for $S=U$. Let $Y = \gamma^{-1}(\iota(U)) \subseteqq M_0$ and $\delta: Y \to U$ the natural morphism. Then by Proposition 1 Y represents the functor $G_U: (\mathrm{An}/U)^\circ \to \mathrm{Sets}$ defined by $G_U(U')$ = the set of isomorphism classes of completions of $(4)_T$ with $T=U'$. Thus we get the universal completion (14) on Y (with S replaced by U there).

Proposition 2. *In the above notations there exist a projective morphism $\bar{\delta}: \bar{Y} \to S$ and an inclusion $Y \subseteqq \bar{Y}_U$ over U such that Y is a dense Zariski open subset of $\bar{Y}$. Moreover exists a coherent quotient $\bar{u}: \mathscr{E}_{2,\bar{Y}} \to \bar{\mathscr{R}}$ on $X_{\bar{Y}} = X \times_S \bar{Y}$ such that the restriction of $\bar{u}$ to $X_Y = X \times_S Y$ is isomorphic to the quotient $u: \mathscr{E}_{2,Y} \to \mathscr{R}$ in the universal completion (14) on X_Y.*

Proof. a) *The construction of $\bar{Y}$.* Let $\mathscr{M} = f_*\mathscr{G}$ and $\mathscr{N} = f_*\mathscr{H}$. Let $L = L(\mathscr{M}^*)$ (resp. $L' = L(\mathscr{N}^*)$) be the linear fiber space over S associated to the dual $\mathscr{M}^*$ (resp. $\mathscr{N}^*$) of $\mathscr{M}$ (resp. $\mathscr{N}$) (cf. 1.1 b)). Since $\mathscr{M} \cong \mathscr{M}^{**}$ on U, we may identify M_0 with L_U. Let $\boldsymbol{P} = \boldsymbol{P}(\mathscr{M}^* \oplus \mathcal{O}_S)$ (resp. $\boldsymbol{P}' = \boldsymbol{P}(\mathscr{N}^* \oplus \mathcal{O}_S)$) be the projective fiber space over S associated to $\mathscr{M}^* \oplus \mathcal{O}_S$ (resp. $\mathscr{N}^* \oplus \mathcal{O}_S$). Then we have the natural inclusions $L \subseteqq \boldsymbol{P}$ and $L \subseteqq \boldsymbol{P}'$ as Zariski open subsets (cf. Lemma

3). Then the transpose $\alpha^* : \mathscr{N}^* \to \mathscr{M}^*$ of α (cf. (7)) induces a homomorphism $b : L \to L'$ of linear fiber spaces over S which is easily seen to extend to a meromorphic map $\bar{b}: \boldsymbol{P} \to \boldsymbol{P}'$ over S (cf. the proof of Lemma 3 and [8]). Let $\bar{\iota} \in H^0(S, \mathscr{S}(L))$ be the section induced by the identity of $\mathscr{F}_3$ via the natural homomorphism $Hom_{\mathscr{O}_X}(\mathscr{F}_3, \mathscr{F}_3) \cong H^0(S, \mathscr{N}) \to H^0(S, \mathscr{N}^{**}) \cong H^0(S, \mathscr{S}(L'))$ (cf. (1)). Note that $\bar{\iota}$ restricts to ι on U with respect to the natural isomorphism $\mathscr{N} \cong \mathscr{N}^{**}$ on U. Let $\bar{Y}' \cong \bar{b}^{-1}(\bar{\iota}(S))$. Then we define $\bar{Y}$ to be the analytic closure of (i.e., the minimal analytic subspace containing) $Y = \bar{Y}'_U \cap L_U$ in $\bar{Y}'$. Then Y is a dense Zariski open subset of $\bar{Y}$ and the natural morphism $\bar{\delta} : \bar{Y} \to S$ is projective as well as $\boldsymbol{P} \to S$.

b) *Construction of a coherent quotient* $\bar{u} : \mathscr{E}_{2,\bar{Y}} \to \bar{\mathscr{R}}$. Consider the diagram

$$
(4)_{\bar{Y}} \qquad
\begin{array}{ccccccc}
\mathscr{E}_{1,\bar{Y}} & \longrightarrow & \mathscr{E}_{2,\bar{Y}} & \longrightarrow & \mathscr{E}_{3,\bar{Y}} & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle u_{1,\bar{Y}}} & & & & \big\downarrow{\scriptstyle u_{3,\bar{Y}}} & & \\
\mathscr{R}_{1,\bar{Y}} & & & & \mathscr{R}_{3,\bar{Y}} & &
\end{array}
$$

The restriction of $(4)_{\bar{Y}}$ to X_Y admits the natural completion, i.e., the universal completion (14). Hence by Lemma 7, (14) determines a unique section λ_0 of $\mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4})$ on X_Y (cf. $(5)_T$ for $\mathscr{F}_{\bar{Y},i}$).

We first show that this λ_0 is extended meromorphically to the whole $X_{\bar{Y}}$. Clearly, it suffices to show that the holomorphic section λ_1 of $f_{\bar{Y}*}\mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4})$ on Y determined by λ_0 is extended meromorphically to the whole $\bar{Y}$. In view of Lemma 2, for this purpose we have only to show the following assertions.

1) $(\bar{\delta}^*\mathscr{M}^*)^*$ *and* $f_{\bar{Y}*}\mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4})$ *are meromorphically equivalent and isomorphic on* Y, and

2) *the section* $\lambda_2 \in H^0(Y, (\bar{\delta}^*\mathscr{M}^*)^*)$ *corresponding to* λ_1 *by* 1) *is extended meromorphically to* $\bar{Y}$.

Proof of 1). Let $d_1 : \bar{\delta}^*\mathscr{M} \to \bar{\delta}^*(\mathscr{M}^{**})$ be induced by the natural homomorphism $\mathscr{M} \to \mathscr{M}^{**}$. Let $d_2 : \bar{\delta}^*(\mathscr{M}^{**}) \to (\bar{\delta}^*\mathscr{M}^*)^*$ be the canonical homomorphism (3) applied to $\mathscr{F} = \mathscr{M}^*$. Let $d_3 = a_{\bar{Y}} : \bar{\delta}^*\mathscr{M} = \bar{\delta}^* f_* \mathscr{G} \to f_{\bar{Y}*}\mathscr{G}_{\bar{Y}}$ (cf. (12)). Let $d_4 : f_{\bar{Y}*}\mathscr{G}_{\bar{Y}} \to f_{\bar{Y}*}(\mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{3,\bar{Y}}, \mathscr{F}_{4,\bar{Y}}))$ be induced by the canonical homomorphism (2) $\mathscr{G}_{\bar{Y}} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}_3, \mathscr{F}_4)_{\bar{Y}} \to \mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{3,\bar{Y}}, \mathscr{F}_{4,\bar{Y}})$. These d_i are all isomorphic on Y (cf. Lemma 6). Hence $(\bar{\delta}^*\mathscr{M}^*)^*$ and $f_{\bar{Y}*}(\mathscr{H}om_{\mathscr{O}_{X_{\bar{Y}}}}(\mathscr{F}_{3,\bar{Y}}, \mathscr{F}_{4,\bar{Y}}))$ are meromorphically equivalent and isomorphic on Y. Next note that $v_{i,\bar{Y}}:\mathscr{F}_{i,\bar{Y}}\to\mathscr{F}_{\bar{Y},i}$, $i=3, 4$, are isomorphic on X_Y (cf. Lemma 8). Then $v_{3,\bar{Y}}$ and $v_{4,\bar{Y}}$ induce homomorphisms $f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{4,\bar{Y}}))\to f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4}))$ and $f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{4,\bar{Y}}))\to f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4}))$ respectively which are isomorphic on Y. Hence $f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4}))$ and $f_{\bar{Y}*}(\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{3,\bar{Y}}, \mathscr{F}_{4,\bar{Y}}))$, and hence the former and $(\bar{\delta}^*\mathscr{M}^*)^*$ also, are meromorphically equivalent and isomorphic on Y. Thus 1) is proved.

Proof of 2). Let $\pi: L\to S$ and $\bar{\pi}: \boldsymbol{P}\to S$ be the natural projections. Consider the following commutative diagram of coherent analytic sheaves on $\boldsymbol{P}$

$$\begin{array}{ccc} \bar{\pi}^*\mathscr{M} & \xrightarrow{\ \varepsilon\ } & (\bar{\pi}^*\mathscr{M}^*)^* \\ \downarrow r_1 & & \downarrow r_2 \\ \bar{\delta}^*\mathscr{M} & \xrightarrow{\ e\ } & (\bar{\delta}^*\mathscr{M}^*)^* \end{array}$$

which is isomorphic on M_0, where $e=d_2d_1$ and ε is defined analogously and where the vertical arrows are induced by the inclusion $\bar{Y}\subseteqq\boldsymbol{P}$. This gives rise to a commutative diagram of the spaces of global sections

$$\begin{array}{ccc} H^0(M_0, \pi^*\mathscr{M}) & \xrightarrow{\ \varepsilon\ } & H^0(M_0, (\pi^*\mathscr{M}^*)^*) \\ \downarrow r_1 & & \downarrow r_2 \\ H^0(Y, \delta^*\mathscr{M}) & \xrightarrow{\ e\ } & H^0(Y, (\delta^*\mathscr{M}^*)^*). \end{array}$$

Let $\tilde{\sigma}\in H^0(M_0, \pi^*\mathscr{M})$ be the universal section. Then $r_2\varepsilon(\tilde{\sigma})=er_1(\tilde{\sigma})=\lambda_2$ (cf. the proof of Proposition 1). Thus it suffices to show that $\varepsilon(\tilde{\sigma})$ is extended meromorphically to the whole $\boldsymbol{P}$. This, however, follows from Lemma 3 applied to $\mathscr{E}=\mathscr{M}^*$ since $\varepsilon(\tilde{\sigma})$ is the restriction to M_0 of the universal section of $(\pi^*\mathscr{M}^*)^*$ on L. This proves 2).

Next, using the fact that λ_0 is extended meromorphically to $X_{\bar{Y}}$ we shall construct a desired quotient $\bar{u}:\mathscr{E}_{2,\bar{Y}}\to\bar{\mathscr{R}}$. Let $h:\tilde{X}\to X_{\bar{Y}}$ be the blowing up of $X_{\bar{Y}}$ with center $A:=X_{\bar{Y}}-X_Y$, where A is endowed with the reduced structure. Then there exists an effective Cartier divisor D on $\tilde{X}$ whose support coincides with $\tilde{X}-h^{-1}(X_Y)$. Since h is isomorphic on $h^{-1}(X_Y)$ we identify $h^{-1}(X_Y)$ with X_Y and consider X_Y also as a Zariski open subset of $\tilde{X}$. Now consider the pull-back of $(4)_{\bar{Y}}$ to $\tilde{X}$ by h;

$$\begin{array}{ccccccc} \tilde{\mathscr{E}}_1 & \xrightarrow{\tilde{b}} & \tilde{\mathscr{E}}_2 & \xrightarrow{\tilde{c}} & \tilde{\mathscr{E}}_3 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tilde{u}_1} & & & & \downarrow{\scriptstyle \tilde{u}_3} & & \\ \tilde{\mathscr{R}}_1 & & & & \tilde{\mathscr{R}}_3 . & & \end{array}$$

Let $\tilde{\mathscr{F}}_i=\mathrm{Ker}\ \tilde{u}_i$, $i=1, 3$, and $\tilde{\mathscr{F}}_4=\tilde{c}^{-1}(\tilde{\mathscr{F}}_3)/\tilde{b}(\tilde{\mathscr{F}}_1)$. Then by the same argument as in the proof of 1) we see readily that $h^*\mathscr{H}om_{\mathcal{O}_{X_{\bar{Y}}}}(\mathscr{F}_{\bar{Y},3}, \mathscr{F}_{\bar{Y},4})$ and $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{F}}_3, \tilde{\mathscr{F}}_4)$ are meromorphically equivalent and isomorphic over X_Y. Thus the section $\tilde{\lambda}$ of $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{F}}_3, \tilde{\mathscr{F}}_4)$ on X_Y corresponding to $h^*\lambda_0$ is extended meromorphically to $\tilde{X}$ by Lemma 2. Hence by Lemma 1 for any relatively compact subdomain $W\subseteqq\tilde{X}$ we can find an integer $n>0$ and a holomorphic section $\tilde{\lambda}^*=\tilde{\lambda}^*_W$ of $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{F}}_3, \tilde{\mathscr{F}}_4)(nD)$ on $\tilde{W}:=h^{-1}(W)$ such that $\tilde{\lambda}^*$ restricts to $\tilde{\lambda}|_{\tilde{W}\cap X_Y}$ in the obvious sense. Then through the natural isomorphism $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{F}}_3, \tilde{\mathscr{F}}_4)(nD)\cong\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathscr{F}}_3(-nD), \tilde{\mathscr{F}}_4)$, $\tilde{\lambda}^*$ determines a homomorphism $\hat{\lambda}^*:\tilde{\mathscr{F}}_3(-nD)\to\tilde{\mathscr{F}}_4$ on $\tilde{W}$. Let $\tilde{\mathscr{F}}(W)=\zeta^{-1}(\hat{\lambda}^*(\tilde{\mathscr{F}}_3(-nD)))$ where $\zeta:\tilde{c}^{-1}(\tilde{\mathscr{F}}_3)\to\tilde{\mathscr{F}}_4$ is the natural homomorphism. Then if we set $\mathscr{F}=\mathrm{Ker}\ u$, $\tilde{\mathscr{F}}(W)=\mathscr{F}$ on $X_Y\cap\tilde{W}$ as a submodule of $\tilde{\mathscr{E}}_2$. Further $h_*\tilde{\mathscr{F}}(W)$ is a coherent submodule of $h_*\tilde{\mathscr{E}}_2=h_*h^*\mathscr{E}_{2,\bar{Y}}$. Let $a:\mathscr{E}_{2,\bar{Y}}\to h_*h^*\mathscr{E}_{2,\bar{Y}}$ be the natural homomorphism which is isomorphic on X_Y. Then $\bar{\mathscr{F}}(W):=a^{-1}(h_*\tilde{\mathscr{F}}(W))$ is a coherent submodule of $\mathscr{E}_{2,\bar{Y}}$ on W with $\bar{\mathscr{F}}(W)|_{W\cap X_Y}=\mathscr{F}|_{W\cap X_Y}$. Thus we have shown that $\mathscr{F}$ extends locally to a coherent submodule of $\mathscr{E}_{2,\bar{Y}}$ at any point of $\bar{Y}$.

Now we define the submodule $\mathscr{F}\langle A\rangle$ of $\mathscr{E}_{2,\bar{Y}}$, which is defined on the whole $X_{\bar{Y}}$ and which extends $\mathscr{F}$, by the following condition; a holomorphic section s of $\mathscr{E}_{2,\bar{Y}}$ defined on an open subset B of $X_{\bar{Y}}$ is a section of $\mathscr{F}\langle A\rangle$ if and only $s|_{X_Y\cap B}\in H^0(X_Y\cap B, \mathscr{F})$. Since on any W as above $\mathscr{F}\langle A\rangle=(\bar{\mathscr{F}}(W)|_{X_Y})\langle A\rangle$ (with the right hand side defined in the same way) and $\bar{\mathscr{F}}(W)$ is coherent on W it follows that $\mathscr{F}\langle A\rangle$ is a coherent submodule of $\mathscr{E}_{2,\bar{Y}}$ on the whole $X_{\bar{Y}}$. (On W, $\mathscr{F}\langle A\rangle$ is characterized by the exact sequence $0\to\bar{\mathscr{F}}(W)\to\mathscr{F}\langle A\rangle\to\mathscr{H}^0_A(\mathscr{E}_{2,\bar{Y}}/\bar{\mathscr{F}}(W))\to 0$.) Thus if we set $\bar{\mathscr{R}}=\mathscr{E}_{2,\bar{Y}}/\mathscr{F}\langle A\rangle$, then the natural homomorphism $\bar{u}:\mathscr{E}_{2,\bar{Y}}\to\bar{\mathscr{R}}$ satisfies the condition of the proposition. q. e. d.

1.4. Let $f: X\to B$ be a proper morphism of complex spaces. Let m

$\geqq 1$ be an integer. Suppose that for each $0\leqq k\leqq m$ we are given coherent analytic sheaves $\mathscr{E}_k$, $\mathscr{E}^k$, and a coherent quotient $u_k:\mathscr{E}_k\to\mathscr{R}_k$ on X such that 1) $\mathscr{E}^0=\mathscr{E}_0$ and 2) $\mathscr{E}_k$, $\mathscr{E}^k$ fit into an exact sequence

$$\mathscr{E}_k \longrightarrow \mathscr{E}^k \longrightarrow \mathscr{E}^{k-1} \longrightarrow 0, \qquad k\geqq 1.$$

Let U be a complex variety and $\tau: U\to B$ be a morphism of complex spaces. Suppose that for each $1\leqq k\leqq m$ there exists a commutative diagram of exact sequences

$$(16)_k \qquad \begin{array}{ccccccccc} & & \mathscr{E}_{k,U} & \longrightarrow & \mathscr{E}^k_U & \longrightarrow & \mathscr{E}^{k-1}_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_{k,U}} & & \downarrow{\scriptstyle u^k} & & \downarrow{\scriptstyle u^{k-1}} & & \\ 0 & \longrightarrow & \mathscr{R}_{k,U} & \longrightarrow & \mathscr{R}^k & \longrightarrow & \mathscr{R}^{k-1} & \longrightarrow & 0 \end{array}$$

of coherent analytic sheaves on X_U with all u^k surjective as well as $u_{k,U}$, where $(u^0:\mathscr{E}^0_U\to\mathscr{R}^0)=(u_{0,U}:\mathscr{E}^0_U\to\mathscr{R}_{0,U})$ by definition. Then we have the following :

Lemma 9. *We can find* 1) *a complex variety A which is locally projective over B* (*cf.* [4, (1.2)]), 2) *a dense open subset V of U and a B-morphism $\eta: V\to A$ and* 3) *a coherent quotient $u:\mathscr{E}^m_A\to\mathscr{R}$ of $\mathscr{E}^m_A$, such that a*) *the pull-back of u to $X_V=X_A\times_A V$ via η is isomorphic to the restriction of $u^m:\mathscr{E}^m_U\to\mathscr{R}^m$ to X_V and b*) *A is the analytic closure of $\eta(V)$ in A.*

Proof. Replacing B by the analytic closure B_0 of $\tau(U)$ we may assume that $B=B_0$. In particular B is reduced. Now we prove the lemma by induction on $m\geqq 1$. So suppose that the lemma is true for the data above with $0\leqq k\leqq m-1$. Then we can find a complex variety A' which is locally projective over B, a dense open subset $V'\subseteqq U$, a B-morphism $\eta': V'\to A'$ and a coherent quotient $u': \mathscr{E}^{m-1}_{A'}\to\mathscr{R}'$ of $\mathscr{E}^{m-1}_{A'}$ such that a)$'$ the pull-back of u' via η' is isomorphic to u^{m-1} restricted to $X_{V'}$ and b)$'$ the analytic closure of $\eta'(V')$ in A' coincides with A'. Here, when $m=1$ (the beginning of the induction) we set $V'=U$, $A'=B$, $\eta'=\tau$ and $u'=u_0$. Now we consider the diagram

$$(17) \qquad \begin{array}{ccccccc} \mathscr{E}_{m,A'} & \longrightarrow & \mathscr{E}^m_{A'} & \longrightarrow & \mathscr{E}^{m-1}_{A'} & \longrightarrow & 0 \\ \downarrow{\scriptstyle u_{m,A'}} & & & & \downarrow{\scriptstyle u'} & & \\ \mathscr{R}_{m,A'} & & & & \mathscr{R}' & & \end{array}$$

Let U' be a Zariski open subset of A' which satisfies the condition (15) for $f_{A'}: X_{A'}\to A'$ and for (17) (instead of $f: X\to S$ and (4)). Let Y' be the complex space over U' which represents the functor $G_{U'}: (\mathrm{An}/U')^{\circ}\to\mathrm{Sets}$; $G_{U'}(U'')=$ the set of isomorphism classes of completions of $(17)_{U''}$ (cf. Proposition 1). Then $X_{Y'}=X_{U'}\times_{U'}Y'$ carries the universal completion

$$(18)\qquad \begin{array}{ccccccccc} & & \mathscr{E}_{m,Y'} & \longrightarrow & \mathscr{E}^{m}_{Y'} & \longrightarrow & \mathscr{E}^{m-1}_{Y'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_{m,Y'}} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u'_{Y'}} & & \\ 0 & \longrightarrow & \mathscr{R}_{m,Y'} & \longrightarrow & '\mathscr{R} & \longrightarrow & \mathscr{R}'_{Y'} & \longrightarrow & 0 \end{array}$$

of $(17)_{Y'}$. Let $\delta: Y'\to U'$ be the natural morphism. Let $\bar{\delta}: \bar{Y}\to A'$ be the projective morphism which extends δ obtained in Proposition 2. By that proposition v extends to a coherent quotient $\bar{v}: \mathscr{E}^{m}_{\bar{Y}'}\to{}'\bar{\mathscr{R}}$ on $X_{\bar{Y}'}$. Let $V=\eta'^{-1}(U')$. Then by b)$'$ V is dense in V' and hence also in U. On the other hand, by a)$'$ $(16)_m$ (restricted over V') is regarded as a completion of the pull-back $(17)_{V'}$ of (17) to $X_{V'}$ via η'. Hence by the universality of Y', $\eta'|_V$ lifts to a unique morphism $\eta: V\to Y'$ such that the pull-back of (18) to X_V via η is isomorphic to $(16)_{m,V}$. Thus if we define A by the analytic closure of $\eta(V)$ in $\bar{Y}'$ and $u: \mathscr{E}^{m}_{A}\to\mathscr{R}$ by the restriction of $\bar{v}$ on X_A, then the conditions a) and b) are obviously fulfilled (cf. [4, (1.2.1)] for the local projectivity). q. e. d.

Remark 1. Actually we can show that η is a restriction of a meromorphic map $U\to A$ over B, and hence we can take the above V as a Zariski open subset of U.

§ 2. Proof of Theorem

2.1. We begin with two lemmas which are used in [3] without explicit proof.

Lemma 10. *Let $g: Z\to T$ be a proper flat morphism of complex spaces. Suppose that Z is (i.e., Z_{red} is) pure dimensional and T is irreducible. Let $\mathscr{F}$ be a g-flat coherent analytic sheaf on X. Let T' be a complex variety and $\eta: T'\to T$ a morphism. Then the followings hold true.* 1) *If $Z_{T'}$ is reduced, then Z itself is reduced. If, further, $Z_{T'}$ is* *irreducible, then so is Z.* 2) *Suppose that $Z_{T'}$ is reduced and $\mathscr{F}_{T'}$ is torsion free. Then $\mathscr{F}$ also is torsion free.*

Proof. Write $Z'=Z_{T'}$ and $\mathscr{F}'=\mathscr{F}_{T'}$. Let $g': Z'\to T'$ be the natural morphism. Suppose first that Z' is reduced. Since T' is a variety and g' is flat, Z' is pure dimensional as well as Z. Then by 3)$\to$ 1) of [3, Lemma 1.4] $Z'_{t'}$ is reduced for some $t'\in T'$ and hence $Z_{\eta(t')}$ also is reduced. Then by 1)$\to$3) of the same lemma Z is reduced. If further $\mathscr{F}'$ is torsion free, then by 3)$\to$1) of [3, Lemma 5.6] $\mathscr{F}'_{t''}$ is a torsion free $\mathcal{O}_{Z_{\eta(t'')}}$-module for some $t''\in T'$. Then by 1)$\to$ 3) of the same lemma $\mathscr{F}$ is torsion free.

Next suppose that Z' is reduced and irreducible. Let $t'\in T'$ be as above and set $t=\eta(t')$ so that Z_t is reduced. Let $Z_1,\dots,Z_r$ be the irreducible components of Z. We have to show that $r=1$. We use the same argument as in [4, Prop. 3]. By our assumption, for any i $Z_i\supseteqq Z_t$, and hence $Z_{i,t}=Z_t$, since $g(Z_i)=T$. Let $Z_{ij}=Z_i\cap Z_j$ and $T_{ij}=\{t\in T\,;\ \dim Z_{ij,t}\geqq \dim Z-\dim T\}$. Let $T_0=\cup_{i\neq j}T_{ij}$. Then we can take a holomorphic map $h: H\to T$ of the unit disc $H:=\{d\in \boldsymbol{C}\,;\ |d|<1\}$ into T such that $h(0)=t$ and $h^{-1}(T_0)=\{0\}$. Set $\tilde{Z}=Z_H$ and $\tilde{Z}_i=Z_{i,H}$. Then by [3, Lemma 1.4] $\tilde{Z}$, and hence $\tilde{Z}_i$ also, are reduced since $\tilde{Z}_0$ is reduced. Hence $\tilde{Z}_i\to H$ is flat as well as $\tilde{Z}\to H$. Let χ be any Hermitian form on $\tilde{Z}$ (cf. [3, Def. 1.2]) and χ_i the restriction of χ to Z_i. Then by [3, Cor. 3.3] the positive functions

$$\lambda(d):=\int_{\tilde{Z}_d}\chi_d, \qquad \lambda_i(d):=\int_{\tilde{Z}_{i,d}}\chi_{i,d}$$

are continuous on H. (Note that by taking h suitably, we may assume that $\tilde{Z}_d$ and $\tilde{Z}_{i,d}$ are all reduced for $d\in H$.) Hence

$$\lim_{d\to 0}\sum_{i=1}^{r}\lambda_i(d)=\sum_{i=1}^{r}\lambda_i(0)=\sum_{i=1}^{r}\int_{\tilde{Z}_{i,0}}\chi_{i,0}$$
$$=r\int_{\tilde{Z}_0}\chi_0=r\lambda(0)=r\lim_{d\to 0}\lambda(d).$$

On the other hand, by our choice of h, $\lambda(d)=\sum_{i=1}^{r}\lambda_i(d)$ for any $d\neq 0$. This is possible only when $r=1$. q. e. d.

Lemma 11. *Let $f: X\to S$ be a morphism of complex spaces and Z a subspace of X. Let $\mathscr{E}$ be a coherent analytic sheaf on X. Let $\bar{\phi}: T\to S$*

be a morphism of complex spaces. Let $\mathscr{I}$ *be the ideal sheaf of* Z *in* X *and* $\mathscr{I}_T$ *that of* Z_T *in* X_T. *Then for each* $k \geqq 0$ *there exist a natural isomorphism* $\lambda_k : \psi^*(\mathscr{E}/\mathscr{I}^k\mathscr{E}) \to \psi^*\mathscr{E}/\mathscr{I}_T^k\psi^*\mathscr{E}$ *and a natural surjective homomorphism* $\mu_k : \psi^*(\mathscr{I}^k\mathscr{E}/\mathscr{I}^{k+1}\mathscr{E}) \to \mathscr{I}_T^k\psi^*\mathscr{E}/\mathscr{I}_T^{k+1}\psi^*\mathscr{E}$, *where* $\psi : X_T \to X$ *is the natural projection.*

Proof. First, λ_k is defined by the requirement that the following diagram of exact sequences be commutative

$$\begin{array}{ccccccc} \psi^*(\mathscr{I}^k\otimes\mathscr{E}) & \longrightarrow & \psi^*\mathscr{E} & \longrightarrow & \psi^*(\mathscr{E}/\mathscr{I}^k\mathscr{E}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle b_k} & & \| & & \downarrow{\scriptstyle \lambda_k} & & \\ \psi^*\mathscr{I}^k\otimes\psi^*\mathscr{E} & \longrightarrow & \psi^*\mathscr{E} & \longrightarrow & \psi^*\mathscr{E}/\mathscr{I}_T^k\psi^*\mathscr{E} & \longrightarrow & 0 \end{array}$$

where b_k is the inverse of the canonical isomorphism (cf. [6, 0, (4.3.3)]) and the top (resp. the bottom) sequence is obtained by pulling back by ψ the exact sequence $\mathscr{I}^k\otimes\mathscr{E} \to \mathscr{E} \to \mathscr{E}/\mathscr{I}^k\mathscr{E} \to 0$ (resp. by tensoring the sequence $\psi^*\mathscr{I}^k \to \mathscr{O}_{X_T} \to \mathscr{O}_{X_T}/\mathscr{I}_T^k \to 0$ with $\psi^*\mathscr{E}$). The isomorphy of λ_k is then clear.

Next, we define μ_k by the requirement that the following diagram of exact sequences be commutative

$$\begin{array}{ccccccc} \psi^*(\mathscr{I}^{k+1}\otimes\mathscr{E}) & \longrightarrow & \psi^*(\mathscr{I}^k\otimes\mathscr{E}) & \longrightarrow & \psi^*(\mathscr{I}^k\mathscr{E}/\mathscr{I}^{k+1}\mathscr{E}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle c_{k+1}} & & \downarrow{\scriptstyle c_k} & & \downarrow{\scriptstyle \mu_k} & & \\ \mathscr{I}_T^{k+1}\psi^*\mathscr{E} & \longrightarrow & \mathscr{I}_T^k\psi^*\mathscr{E} & \longrightarrow & \mathscr{I}_T^k\psi^*\mathscr{E}/\mathscr{I}_T^{k+1}\psi^*\mathscr{E} & \longrightarrow & 0 \end{array}$$

where c_k is the composite of b_k above and the natural surjection $\psi^*\mathscr{I}^k\otimes\psi^*\mathscr{E} \to \mathscr{I}_T^k\psi^*\mathscr{E}$ and where the top sequence is obtained from the exact sequence $\mathscr{I}^{k+1} \to \mathscr{I}^k \to \mathscr{I}^k/\mathscr{I}^{k+1} \to 0$ by applying to it $\psi^*(\ \otimes\mathscr{E})$, taking the natural isomorphism $\mathscr{I}^k\mathscr{E}/\mathscr{I}^{k+1}\mathscr{E} \cong \mathscr{I}^k/\mathscr{I}^{k+1}\otimes\mathscr{E}$ into account. The surjectivity of μ_k is clear. q. e. d.

2.2. Now we come to the reformulation of [3, Lemma 5.8] mentioned in the introduction. We say that a meromorphic map $g : Z \to Y$ is *generically surjective* if the image of Z contains a dense open subset of Y. Moreover we employ the following notations through the end of this paper.

Notation. Let $f : X \to S$ be a morphism of complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. We denote by $u_{X/S}(\mathscr{E}) : \mathscr{E}_D \to \mathscr{R}_{X/S}(\mathscr{E})$ a fixed universal quotient (homomorphism) on $X_D := X \times_S D$ (defined up to automorphisms of $\mathscr{R}_{X/S}(\mathscr{E})$), where $D = D_{X/S}(\mathscr{E})$. Let $Z_{X/S}$

$\subseteqq X_D := X\times_S D$ be the universal subspace where $D=D_{X/S}$. Then as in [3] for any irreducible component D_α of $D_{X/S}(\mathscr{E})_{\mathrm{red}}$ we shall denote $X_{D_\alpha} := X\times_S D_\alpha$, $\mathscr{E}_{D_\alpha}$, $(\mathscr{R}_{X/S}(\mathscr{E}))_{D_\alpha}$ simply by X_α, $\mathscr{E}_\alpha$, $\mathscr{R}_\alpha$ respectively, and further, when $\mathscr{E}=\mathscr{O}_X$, denote $Z_{X/S}\times_S D_\alpha$ by Z_α. By $\hat{D}_{X/S}$ we denote the union of those irreducible components D_α of $D_{X/S,\mathrm{red}}$ for which Z_α is reduced. We set $\dim X/S=\dim X-\dim S$ and $\dim \mathscr{E}/S =\dim \operatorname{supp}(\mathscr{E})-\dim S$, where supp denotes the support.

Lemma 12. *Let $f:X\to S$ be a proper morphism of complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. Then for every irreducible component D_α of $D_{X/S}(\mathscr{E})_{\mathrm{red}}$ with $\dim \mathscr{R}_\alpha/D_\alpha=q\geqq 0$, there exist* 1) *an irreducible component T of $\hat{D}_{X/S}$ such that $\dim Z_T/T=q$ where $Z=Z_{X/S}$,* 2) *coherent analytic sheaves $\mathscr{E}_k$, $0\leqq k\leqq n$, on Z_T,* 3) *an irreducible component B_k of $D_{Z_T/T}(\mathscr{E}_k)_{\mathrm{red}}$ for each k such that $Z_T\times_T B_k$ is reduced,* 4) *a complex variety A which is locally projective over $B_0\times_T\cdots\times_T B_n$ and finally* 5) *a generically surjective meromorphic S-map $h: A\to D_\alpha$.*

Proof. Let $\mathscr{J}$ be the ideal sheaf of annihilators of $\mathscr{R}_\alpha$ on X_α. Let $\operatorname{supp}\mathscr{R}_\alpha$ be the support of $\mathscr{R}_\alpha$. Then define the subspace $S(\mathscr{R}_\alpha)$ of X_α by $S(\mathscr{R}_\alpha)=(\operatorname{supp}\mathscr{R}_\alpha, \mathscr{O}/\mathscr{J})$. Let $\mathscr{I}$ be the ideal sheaf of $\operatorname{supp}\mathscr{R}_\alpha$ on X_α. Define $\mathscr{R}_k=\mathscr{I}^k\mathscr{R}_\alpha/\mathscr{I}^{k+1}\mathscr{R}_\alpha$, $k\geqq 0$, where $\mathscr{I}^0=\mathscr{O}\equiv \mathscr{O}_{X_\alpha}$. Then we have the following commutative diagram of exact sequences on X_α

$$(19)_k\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}^k\mathscr{E}_\alpha/\mathscr{I}^{k+1}\mathscr{E}_\alpha & \longrightarrow & \mathscr{E}_\alpha/\mathscr{I}^{k+1}\mathscr{E}_\alpha & \longrightarrow & \mathscr{E}_\alpha/\mathscr{I}^k\mathscr{E}_\alpha & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_k} & & \downarrow{\scriptstyle u^{k+1}} & & \downarrow{\scriptstyle u^k} & & \\ 0 & \longrightarrow & \mathscr{R}_k & \longrightarrow & \mathscr{R}_\alpha/\mathscr{I}^{k+1}\mathscr{R}_\alpha & \longrightarrow & \mathscr{R}_\alpha/\mathscr{I}^k\mathscr{R}_\alpha & \longrightarrow & 0 \end{array}$$

with the vertical arrows surjective. Take $n>0$ sufficiently large so that $\mathscr{J}\supseteqq\mathscr{I}^{n+1}$ over U_0 for some Zariski open subset U_0 of D_α. Let U be a Zariski open subset of D_α such that $U\subseteqq U_0$ and that $\mathscr{O}/\mathscr{I}$ and $\mathscr{R}_k$, $0\leqq k\leqq n$, are all flat over U. Then by $(19)_k$ we see that $\mathscr{R}_\alpha/\mathscr{I}^k\mathscr{R}_\alpha$ also are flat over U for $1\leqq k\leqq n+1$. Now since $\mathscr{O}/\mathscr{I}$ is flat over U, by the universality of $D_{X/S}$ we have a unique S-morphism $\phi: U\to D_{X/S}$ such that

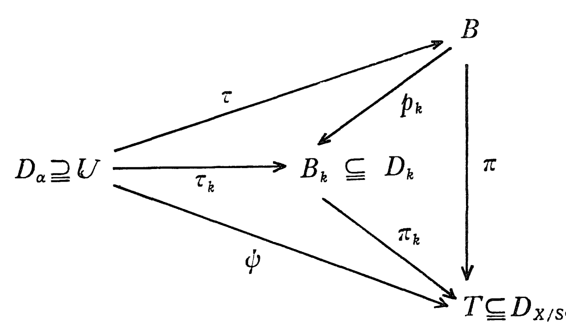
$$(20)\qquad (\operatorname{supp}\mathscr{R}_\alpha)_U\cong Z_U := Z\times_{D_{X/S}}U.$$

Let T be any irreducible component of $D_{X/S,\mathrm{red}}$ containing $\phi(U)$.

Then by (20) and Lemma 10 Z_T is reduced, so $T \subseteqq \hat{D}_{X/S}$.

Let $\mathscr{I}_T$ be the ideal sheaf of Z_T in X_T and $\mathscr{E}_k = \mathscr{I}_T^k \mathscr{E}_T / \mathscr{I}_T^{k+1} \mathscr{E}_T$, regarded as a coherent analytic sheaf on Z_T. We then consider the relative Douady space $D_k := D_{Z_T/T}(\mathscr{E}_k)$ associated with the pair $(Z_T/T, \mathscr{E}_k)$. Then by (20), Lemma 11 and $(19)_k$, $\mathscr{R}_k$, restricted over U, are flat quotients of $(\mathscr{E}_k)_U \equiv \tilde{\psi}^* \mathscr{E}_k$, where $\tilde{\psi} : Z_U \cong Z_T \times_T U \to Z_T$ is the natural morphism. Then by the universality of D_k, ψ lifts to a unique morphism $\tau_k : U \to D_k$ such that $\hat{u}_k : (\mathscr{E}_k)_U \to \mathscr{R}_k$ is isomorphic to the pull-back via τ_k of the universal quotient $u_{Z_T/T}(\mathscr{E}_k) : \mathscr{E}_{k,D_k} \to \mathscr{R}_{Z_T/T}(\mathscr{E}_k)$. Let B_k be any irreducible component of $D_{k,\mathrm{red}}$ containing $\tau_k(U)$. Then again by Lemma 10 $Z_T \times_T B_k$ is reduced. Let $v_k : \mathscr{E}_{k,B_k} \to {}_k\mathscr{R}$ be the restriction of $u_{Z_T/T}(\mathscr{E}_k)$ over B_k. Define $\tau = \tau_0 \times_T \cdots \times_T \tau_n : U \to B := B_0 \times_T \cdots \times_T B_n$. Let $\pi_k : B_k \to T$, $\pi : B \to T$, $p_k : B \to B_k$ be the natural projections, and $\tilde{\pi}_k : Z_{B_k} \to Z_T$, $\tilde{\pi} : Z_B \to Z_T$, $\tilde{p}_k : Z_B \to Z_{B_k}$ the induced morphisms.

We shall now apply Lemma 9 to our situation ;



In the notation of 1.4 first we set $m = n$ and

$$(21) \quad \begin{cases} (f : X \to B) = (f_B : X_B \to B) \\ (\mathscr{E}_k, \mathscr{E}^k) = (\tilde{\pi}^* \mathscr{E}_k, \tilde{\pi}^*(\mathscr{E}_T / \mathscr{I}_T^{k+1} \mathscr{E}_T)) \\ (u_k : \mathscr{E}_k \to \mathscr{R}_k) = (\tilde{p}_k^*(v_k) : \tilde{\pi}^* \mathscr{E}_k = \tilde{p}_k^*(\tilde{\pi}_k^* \mathscr{E}_k) \to \tilde{p}_k^*({}_k\mathscr{R}))^{1)}. \end{cases}$$

Then $\mathscr{E}^0 = \mathscr{E}_0$ and we have a natural exact sequence $\mathscr{E}_k \to \mathscr{E}^k \to \mathscr{E}^{k-1} \to 0$. We further set

$$\begin{cases} (\tau : U \to B) = (\tau : U \to B) \\ (u^k : \mathscr{E}_U^k \to \mathscr{R}^k) = (u_U^{k+1} : (\mathscr{E}_\alpha / \mathscr{I}^{k+1} \mathscr{E}_\alpha)_U \to (\mathscr{R}_\alpha / \mathscr{I}^{k+1} \mathscr{R}_\alpha)_U) \end{cases}$$

where we identified $\tilde{\tau}^*(\tilde{\pi}^*(\mathscr{E}_T / \mathscr{I}_T^{k+1} \mathscr{E}_T)) \cong \tilde{\psi}^*(\mathscr{E}_T / \mathscr{I}_T^{k+1} \mathscr{E}_T)$ with

1) We consider $\mathscr{E}_k$, $\mathscr{E}^k$, $\mathscr{R}_k$ as coherent analytic sheaves on X_B with respect to the natural inclusion $Z_B := Z_T \times_T B \subseteqq X_B$, where these sheaves are zero outside Z_B.

$(\mathscr{E}_\alpha/\mathscr{I}^{k+1}\mathscr{E}_\alpha)_U \cong \tilde{\phi}^*\mathscr{E}_T/\mathscr{I}^{k+1}\tilde{\phi}^*\mathscr{E}_T$ with respect to the natural isomorphism $\tilde{\phi}^*(\mathscr{E}_T/\mathscr{I}^{k+1}\mathscr{E}_T) \cong \tilde{\phi}^*\mathscr{E}_T/\mathscr{I}^{k+1}\tilde{\phi}^*\mathscr{E}_T$ given in Lemma 9 [2]. With this definition $u^0 = u_{0,T}$ by the definition of τ_0 and u_0, and we get the commutative diagram $(16)_k$ on X_U in view of $(19)_k$.

Thus we can apply Lemma 9 to these data; we can find a complex variety A which is locally projective over B, a dense open subset $V \subseteqq U$, a B-morphism $\eta: V \to A$ which lifts $\tau|_V$, and a coherent quotient $u(A): \mathscr{E}^n_A \cong (\mathscr{E}^n_B)_A \to \mathscr{R}(A)$ on $X_A = X \times_S A$, where $\mathscr{E}^n$ is defined by (21) such that a) $u^{n+1}: \mathscr{E}_\alpha/\mathscr{I}^{n+1}\mathscr{E}_\alpha \to \mathscr{R}_\alpha/\mathscr{I}^{n+1}\mathscr{R}_\alpha$ restricted to X_V is isomorphic to the pull-back of $u(A)$ via η and b) A is the analytic closure of $\eta(V)$ in A. Let $W \subseteqq A$ be a Zariski open subset over which $\mathscr{R}(A)$ is flat. Then by the universality of $D_{X/S}(\mathscr{E})$ there exists a unique morphism $h_0: W \to D_{X/S}(\mathscr{E})$ over S such that the composite quotient $\mathscr{E}_A \to \mathscr{E}^n_A \to \mathscr{R}(A)$ restricted to X_W is isomorphic to the pull-back of the universal quotient $u_{X/S}(\mathscr{E}): \mathscr{E}_D \to \mathscr{R}_{X/S}(\mathscr{E})$ by h_0 where $D = D_{X/S}(\mathscr{E})$. Then by the condition b) above $W_0 := \eta^{-1}(W)$ is a nonempty Zariski open subset of V and by the condition a) $h_0(\eta|_{W_0})$ is the identity of W_0. (Note that $\mathscr{E}_\alpha/\mathscr{I}^{n+1}\mathscr{E}_\alpha = \mathscr{E}_\alpha$ on X_V.) Hence $W_0 \subseteqq h_0(W)$, and then since W is irreducible, $h_0(W) \subseteqq D_\alpha$. Finally by [4, Lemma 4] h_0 extends to a meromorphic map $h: A \to D_\alpha$ which is generically surjective. This completes the construction of all the objects required in the lemma. q. e. d.

2.3. For the proof of Theorem, besides Lemma 12 above we need Lemma 5.9 and Lemma 5.7 of [3], which we shall quote here as Lemma 13 and Lemma 14 respectively.

Lemma 13. *Let $f: X \to S$ be a proper flat morphism of reduced complex spaces and $\mathscr{E}$ a coherent analytic sheaf on X. Let $q = \dim X/S$. Then for any irreducible component D_α of $D_{X/S}(\mathscr{E})_{\mathrm{red}}$ such that X_α is reduced, there exist* 1) *irreducible components T_i, $1 \leqq i \leqq m$, of $\hat{D}_{X/S}$ such that $Z_i := Z_{X/S} \times_S T_i$ are irreducible,* 2) *for each i subvariety Y_i of $D_{X_i/T_i}(\mathscr{E}_i)_{\mathrm{red}}$, where $X_i = X_{T_i}$ and $\mathscr{E}_i = \mathscr{E}_{T_i}$, such that either* a) $\dim \mathscr{R}_i/Y_i < q$ *or* b) *$\tilde{Z}_i := Z_i \times_{T_i} Y_i$ is reduced, $\mathscr{R}_i$ is a torsion free $\mathscr{O}_{\tilde{Z}_i}$-module and $Y_i \subseteqq D_{Z_i/T_i}(\bar{\mathscr{E}}_i)$ with respect to the natural inclusion $D_{Z_i/T_i}(\bar{\mathscr{E}}_i) \subseteqq D_{X_i/T_i}(\mathscr{E}_i)$*

2) $\bar{\tau}$ is the natural morphism $Z_T \times_T U \to Z_T \times_T B$ induced by τ.

where $\mathscr{R}_i = (\mathscr{R}_{X_i/T_i}(\mathscr{E}_i))_{Y_i}$ *and* $\bar{\mathscr{E}}_i = \mathscr{E}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{Z_i}$, 3) *an analytic subvariety* N *of* $Y_1 \times_S \dots \times_S Y_m$, *and finally* 4) *a generically surjective meromorphic map* $h : N \to D_\alpha$ *over* S.

The newly added statement '$Y_i \subseteqq D_{Z_i/T_i}(\bar{\mathscr{E}}_i)$' above is in fact shown in the final part of the proof of Lemma 5.9 of [3].

Lemma 14. *Let* $f : X \to S$ *be a proper flat morphism of complex varieties and* $\mathscr{E}$ *a coherent analytic sheaf on* X. *Let* $r > 0$ *be an integer and* $Y = X \times_S G_r(\mathscr{E})$, *where* $G_r(\mathscr{E})$ *is the Grassmann variety over* X *of locally free quotients of rank* r *cf* $\mathscr{E}$ (*cf.* [6]), *regarded naturally as a complex space over* S. *Then for any irreducible component* D_α *of* $D_{X/S}(\mathscr{E})_{\mathrm{red}}$ *such that* X_α *is reduced and that* $\mathscr{R}_\alpha$ *is torsion free of rank* r *on* X_α *there exist an analytic subset* E_α *of* $\hat{D}_{Y/S}$ *and a bimeromorphic map* $\tau : E_\alpha \to D_\alpha$ *over* S.

Here we remark that the proof of this lemma in [3] uses [3, Lemma 5.5], but unfortunately its proof is incomplete. (Perhaps not true as is stated there.) So we shall formulate and prove another version of it (Proposition 3 below), which is enough for Lemma 14 above as follows immediately from the proof of Lemma 5.7 in [3].

Let T be a complex variety and $h : Y \to T$ be a morphism of complex spaces with Y reduced. Let Y_i, $1 \leqq i \leqq m$, be the irreducible components of Y. Then we say that Y *is pure dimensional over* T if Y is pure dimensional and each Y_i is mapped surjectively onto T. In this case $\dim Y_i/T = \dim Y/T$ is independent of i.

To state the proposition we introduce some notations. Let $f : X \to S$, $f' : X' \to S$ be proper morphisms of complex spaces. Let $Z \subseteqq X \times_S X'$ be a subspace. Then we set

$$M_0 = \{s \in S \; ; \; X_s \text{ is reduced and } Z_s \subseteqq X_s \times X'_s \text{ is a graph of a meromorphic map } X_s \to X'_s\}.$$

Assume further that there exists an S-morphism $g : X' \to X$. Then we further set

$$M = \{s \in S \; ; \; X_s \text{ is reduced and } Z_s \text{ is a graph of a meromorphic section of } g_s : X'_s \to X_s\}.$$

Proposition 3. *Let $f: X\to S$, $f': X'\to S$, $Z\subseteqq X\times_S X'$ and $g: X'\to X$ be as above. Suppose that both X and Z are reduced and S is a variety. Suppose further that both X and Z are pure dimensional over S. Then*

1) *Z is a graph of a meromorphic map $X\to X'$ over S if and only if there exists a Zariski open subset $U\subseteqq S$ such that for any $s\in U$, X_s is reduced and Z_s is a graph of a meromorphic map $X_s\to X'_s$.*

2) *Let $h: Z\to S$ be the natural morphism. Suppose that both f and h are flat. Then M_0 is Zariski open in S (possibly empty) and M is locally closed with respect to the Zariski topology of S.*

First we make the following obvious remark. Let $g: \tilde{Y}\to Y$ be a morphism of complex spaces. Then

$$\text{(22)}\qquad \begin{cases} g \text{ is isomorphic if and only if 1) } \dim_{\tilde{y}} g^{-1}g(\tilde{y})=0 \\ \text{for any } \tilde{y}\in\tilde{Y} \text{ and 2) the natural homomorphism} \\ \iota_g: \mathcal{O}_Y\to g_*\mathcal{O}_{\tilde{Y}} \text{ is isomorphic.}\end{cases}$$

The essential part of the proof of the proposition is contained the following

Lemma 15. *Let S be a complex variety. Let $f: X\to S$, $h: Z\to S$ be morphisms of reduced complex spaces and $\pi: Z\to X$ an S-morphism. Suppose that X and Z are pure dimensional over S. Let $U_f:=\{s\in S\,;\, f$ is flat along X_s and X_s is reduced$\}$ and $U_h:=\{s\in S\,;\, h$ is flat along Z_s and Z_s is reduced$\}$. Then the set $N:=\{s\in U_f\cap U_h; \pi_s: Z_s\to X_s$ is bimeromorphic$\}$ is Zariski open in S. Moreover N is nonempty if and only if π is bimeromorphic.*

Proof. First we note that U_f and U_h are Zariski open by [3, Lemma 1.5]. We set $U_0=U_f\cap U_h$. Let $p=\dim Z/S$ and $q=\dim X/S$. Then

(23) Z_s (resp. X_s) has pure dimension p (resp. q) for any $s\in U_0$.

Let $\tilde{A}=\{z\in Z\,;\,\dim_z\pi^{-1}\pi(z)>0\}$ and $A=\pi(\tilde{A})$. Then clearly $\tilde{A}_s=\{z\in Z_s\,;\,\dim_z\pi_s^{-1}\pi_s(z)>0\}$. Consider now the exact sequence

$$\text{(24)}\qquad 0\longrightarrow\mathcal{K}\longrightarrow\mathcal{O}_X\xrightarrow{\iota_\pi}\pi_*\mathcal{O}_Z\longrightarrow\mathcal{M}\longrightarrow 0$$

of coherent analytic sheaves on X where $\mathcal{K}$ (resp. $\mathcal{M}$) is the kernel (resp. image) of ι_π. Let $B=\operatorname{supp}\mathcal{M}$ and $C=\operatorname{supp}\mathcal{K}$. Let $W_1=X-A$, $W_2=X-(A\cup B)$ and $W_3=X-(A\cup B\cup C)$. Let $\tilde{W}_i=\pi^{-1}(W_i)$

and $\pi_i=\pi|_{\tilde{W}_i}:\tilde{W}_i\to W_i$. Then $\pi_1:\tilde{W}_1\to W_1$ is a finite morphism, and $\pi_3:\tilde{W}_3\to W_3$ is an isomorphism in view of (22). In fact, W_3 is the maximal Zariski open subset of X with this property. Further

$$B_s=\text{supp }\pi_{s*}\mathcal{O}_{Z_s}/\iota_{\pi_s}\mathcal{O}_{X_s}\text{ on } W_1\cap X_s. \tag{25}$$

This can be seen as follows. By the restriction to each fiber X_s (24) gives the exact sequence

$$\mathcal{O}_{X_s}\xrightarrow{(\iota\pi)_s}(\pi_*\mathcal{O}_Z)_s\longrightarrow\mathscr{M}_s\longrightarrow 0.$$

On the other hand, since π_1 is finite, on $X_s\cap W_1$ there exists a natural isomorphism between $(\iota_\pi)_s:\mathcal{O}_{X_s}\to(\pi_*\mathcal{O}_Z)_s$ and $\iota_{\pi_s}:\mathcal{O}_{X_s}\to\pi_{s*}\mathcal{O}_{Z_s}$. Thus supp $\pi_{s*}\mathcal{O}_{Z_s}/\iota_{\pi_s}\mathcal{O}_{X_s}=$ supp $\mathscr{M}_s$ and (25) follows. Next we shall see that

$$C_s=\text{supp (Ker }\iota_{\pi_s})\text{ on } W_2\cap f^{-1}(U_h). \tag{26}$$

Indeed, on W_2 (24) reduces to $0\to\mathscr{K}\to\mathcal{O}_X\to\pi_*\mathcal{O}_Z\to 0$ and on $W_1\cap f^{-1}(U_h)$, $\pi_*\mathcal{O}_Z$ is f-flat. Thus for any $s\in U_h$ the restriction of (24) to X_s gives the exact sequence

$$0\longrightarrow\mathscr{K}_s\longrightarrow\mathcal{O}_{X_s}\xrightarrow{\iota_{\pi_s}}\pi_{s*}\mathcal{O}_{Z_s}\longrightarrow 0$$

on $W_2\cap X_s$. Thus supp (Ker ι_{π_s}) = supp $\mathscr{K}_s$ and (26) follows.

Now set $T_A=\{s\in S\,;\,\dim\tilde{A}_s\geqq p\}$, $T_B=\{s\in S\,;\,\dim B_s\geqq q\}$, and $T_C=\{s\in S\,;\,\dim C_s\geqq q\}$. Let $U_1=S-(T_A\cup T_B\cup T_C)$. Then the first assertion follows from the following:

Claim. $N=U_0\cap U_1$.

Proof. It is clear that $N\subseteqq U_0$. Let $s\in N$ be arbitrary. Then $\pi_s:Z_s\to X_s$ is bimeromorphic. In particular $s\notin T_A$ by (23) and hence $W_1\cap X_s$ is dense in X_s. It follows then that $s\notin T_B$ by (25) and so $W_2\cap X_s$ is also dense in X_s. Then by (26) $s\notin T_C$ either. Thus $s\in U_1$ and we have proved that $N\subseteqq U_0\cap U_1$. Conversely, let $s\in U_0\cap U_1$ be an arbitrary point. Then by (23) $\tilde{A}_s$ is nowhere dense in Z_s and then π_s is generically finite. Thus $\pi_s(\tilde{A}_s)=A_s$ is nowhere dense in X_s. Hence $X_s\cap W_3$ is dense in X_s and $Z_s\cap\tilde{W}_3$ in Z_s. Thus π_s is bimeromorphic and $s\in N$.

By the above claim it follows that $N\neq\emptyset$ if and only if $U_1\neq\emptyset$. Further by the pure dimensionality assumption the latter condition is equivalent to the condition that W_3 is dense in X and $\tilde{W}_3$ is dense in Z. From this the desired equivalence follows. q. e. d.

Proof of Proposition 3. Let $\pi: Z\to X$ be the natural morphism. Let N be defined as in Lemma 15 applied to π. 1) If Z is a graph of a meromorphic map, π is bimeromorpnic. Then it suffices to set $U=N$. Conversely suppose that a Zariski open subset U satisfying the condition of 1) exists. Restricting U, we may assume that f, h are flat over U and Z_s is reduced for any $s\in S$ (cf. [3]). Since Z_s is a graph of a meromorphic map if and only if $\pi_s: Z_s\to X_s$ is bimeromorphic, U is contained in N. Hence $N\neq\emptyset$ and then π is bimeromorphic by Lemma 15.

2) Since both f and h are flat, $N=\{s\in S\ ;\ X_s,\ Z_s$ are reduced, and $\pi_s: Z_s\to X_s$ is bimeromorphic$\}$. Then clearly $N\subseteqq M_0$. Conversely, if $s\in M_0$, then Z_s is reduced since X_s is reduced. Thus $s\in N$. Hence $N=M_0$ and M_0 is Zariski open. Once this is established, the proof for M is the same as that of Lemma 5.5 2) in [3].

2.4. Using Lemmas 12, 13, and 14 above and Theorem in [4] (cf. Introduction) we shall prove Theorem along the line of *Proof of Theorem* 5.2 in [3].

Proof of Theorem. First we assume that f is a $\mathscr{C}$-morphism. For the functorial properties of $\mathscr{C}$-morphisms used below we refer to [4. (2.4)]. The statement of Theorem is clearly equivalent to the following :

Let D_α be any irreducible component of $D_{X/S}(\mathscr{E})_{\mathrm{red}}$. Then for any irreducible component $D^i_{\alpha,Q}$ of $D_{\alpha,Q}:=D_\alpha\times_S Q$, the natural morphism $b^i_{\alpha,Q}: D^i_{\alpha,Q}\to Q$ (is proper and) is a $\mathscr{C}$-morphism.

We prove Theorem in this form by induction on $q=\dim\mathscr{R}_\alpha/S$. If $q=-1$ (i. e., supp $\mathscr{R}_\alpha=\emptyset$), then $\mathscr{R}_\alpha=\{0\}$ so that $D_\alpha\cong S$ and hence the theorem is clearly true. So assume that $q\geqq 0$ in what follows.

Step 1. We may assume that f is flat, $\dim X/S=q$, and that both X and X_α are reduced.

Proof. Suppose that the theorem is true in this case. We look at Lemma 12 applied to f, $\mathscr{E}$ and D_α. We use the notations of that lemma. First of all, since T is an irreducible component of $\hat{D}_{X/S}$ and $\rho_T : Z_T \to T$ is a $\mathscr{C}$-morphism (being induced by the projection $f_T : X_T \to T$), for any irreducible component T_Q^i of T_Q the natural morphism $T_Q^i \to Q$ is a $\mathscr{C}$-morphism by Theorem in [4]. Moreover, we note that T_Q^i is actually relatively compact in T as we see immediately by applying the above argument to any relatively compact $Q' \subseteqq S$ with $Q \Subset Q'$. (The same kind of remark applies also to the other spaces defined below, though we do not mention it explicitly.) On the other hand, ρ_T is flat, $\dim Z_T/T = q$ and for any $0 \leqq k \leqq n$ $Z_T \times_T B_k$ is reduced. Thus Theorem is true for $(\rho_T : Z_T \to T, \mathscr{E}_k, B_k)$ for each k by our assumption. Hence by the above remark for any irreducible component $B_k^i := B_{k,Q}^i$ of $B_{k,Q}$ the natural morphism $B_k^i \to T_Q$ is a $\mathscr{C}$-morphism. Hence for any $I = (i_0, \dots, i_m)$ $B_0^{i_0} \times_{T_Q} \cdots \times_{T_Q} B_n^{i_n} \to T_Q$ is a $\mathscr{C}$-morphism, and so the composite map

$$A_Q^I \longrightarrow B_0^{i_0} \times_{T_Q} \cdots \times_{T_Q} B_n^{i_n} \longrightarrow T_Q \longrightarrow Q$$

also is a $\mathscr{C}$-morphism, where A_Q^I is the inverse image of $B_0^{i_0} \times_{T_Q} \cdots \times_{T_Q} B_n^{i_n}$ in A. From this it follows that $b_{\alpha.Q}^i$ is a $\mathscr{C}$-morphism by Lemma 12.

Step. 2. In addition to the conditions of Step 1 we may further assume that both X and S are irreducible and that $\mathscr{R}_\alpha$ is torsion free on X_α.

Proof. Suppose that Theorem is proved under this assumption. We observe Lemma 13 applied to the given f, $\mathscr{E}$ and D_α and use the notations of that lemma. First since $T_i \subseteqq \hat{D}_{X/S}$, for any irreducible component $T_i^\gamma := T_{i,Q}^\gamma$ of $T_{i,Q}$, the natural morphism $T_i^\gamma \to Q$ is a $\mathscr{C}$-morphism by Theorem in [4]. Hence for any $(\gamma_1, \dots, \gamma_m)$ the induced morphism $T_1^{\gamma_1} \times_Q \cdots \times_Q T_m^{\gamma_m} \to Q$ also is a $\mathscr{C}$-morphism. Next we shall show that for any irreducible component $Y_i^\mu := Y_{i,Q}^\mu$ of $Y_{i,Q}$ the natural morphism $Y_i^\mu \to T_{i,Q}$ is a $\mathscr{C}$-morphism, and hence that for any $M = (\mu_1, \dots, \mu_m)$ the induced morphism $N_Q^M := N_Q \cap (Y_1^{\mu_1} \times_Q \cdots \times_Q Y_m^{\mu_m}) \to T_Q := T_{1.Q} \times_Q \cdots \times_Q T_{m.Q}$ also is a $\mathscr{C}$-morphism. This would then imply that the composite map $N_Q^M \to T_Q \to Q$ is again a

$\mathscr{C}$-morphism. Hence by Lemma 13 $b^i_{\alpha,Q}$ would also be a $\mathscr{C}$-morphism.

Now we show that

(27) $Y^{\#}_i \to T_{i,Q}$ is a $\mathscr{C}$-morphism.

Suppose first that $\dim \mathscr{R}_i/Y_i = q$. Let Y_{β_i} be any irreducible component of $D_{Z_i/T_i}(\bar{\mathscr{E}}_i)_{\mathrm{red}}$ containing Y_i. Then we claim that the triple $(\rho_i : Z_i \to T_i, \bar{\mathscr{E}}_i, Y_{\beta_i})$ satisfies the condition of Step 2. First, by Lemma 13 Z_i and T_i are varieties, ρ_i is flat and $\dim Z_i/T_i = \dim \tilde{Z}_i/Y_i = \dim \mathscr{R}_i/Y_i = q$. Further by Lemma 10 $\tilde{Z}_{\beta_i} := Z_i \times_{T_i} Y_{\beta_i}$ is reduced since $\tilde{Z}_i = \tilde{Z}_{\beta_i}|_{Y_i}$ is reduced. Similarly, $\mathscr{R}_{\beta_i} := \mathscr{R}_{Z_i/T_i}(\bar{\mathscr{E}}_i)_{\beta_i}$ is a torsion free $\mathscr{O}_{\tilde{Z}_{\beta_i}}$-module by Lemma 10 since $\tilde{Z}_{\beta_i}$ is pure dimensional as well as Z_i (cf. [4, Lemma 3]) and $\mathscr{R}_i = \mathscr{R}_{Z_i/T_i}(\bar{\mathscr{E}}_i)_{Y_i}$ is torsion free. Thus our claim is proved. Hence from our assumption (27) follows. Next assume that $\dim \mathscr{R}_i/Y_i < q$. Then $\dim \mathscr{R}_{\beta_i}/Y_{\beta_i} < q$ also by the flatness of $\mathscr{R}_{\beta_i}$ over Y_{β_i}. Hence by induction hypothesis (27) again follows.

Step 3. By Step 2 we may assume that f and D_α satisfy the condition of Step 2. This time we observe Lemma 14 applied to our f, $\mathscr{E}$ and D_α. We use the notations of that lemma. Since $E_\alpha \subseteqq \hat{D}_{Y/S}$ and $Y \to S$ is a $\mathscr{C}$-morphism ($G_r(\mathscr{E}) \to X$ being projective), by Theorem in [4] for any irreducible component $E^i_{\alpha,Q}$ of $E_{\alpha,Q}$ the natural morphism $E^i_{\alpha,Q} \to Q$ is a $\mathscr{C}$-morphism. Then the theorem follows from Lemma 14 because $D^i_{\alpha,Q}$ is a bimeromorphic image of some $E^i_{\alpha,Q}$.

In the case where f is Moishezon, the above proof works without any change if we replace '$\mathscr{C}$-morphism' by 'Moishezon morphism' there. (See (1.5) and Proposition 1 of [4] for the functorial properties instead of [4, (2.4)]). q. e. d.

Remark 2. In [3, Theorem 5.2] we asserted that any irreducible component of $D_{X/S}(\mathscr{E})_{\mathrm{red}}$ is proper over the whole S without restricting to a relatively compact subdomain Q as above under the assumption that f is a Kähler morphism. However, this seems to be incorrect; the gap in the proof lies in the final statement of Lemma 5.7 of [3] based on the false claim that the composite map $G_r(\mathscr{E}) \to X \to S$ is again Kähler in the notation of Lemma 12, (which is in general true only after restricted over Q as above (cf. [4, §2])).

At the end of [3] we also asserted that Theorem is true also for

$f: X \to S$ in loc-$\mathscr{C}/S$ in the sense of [4, (2.3)] (in $\mathscr{C}/S$ in the sense of [3]). But this is obviously false, e.g., for the fiber bundle $f: X \to C$ constructed in [3, Remark 4.3], though the proof indicated there applies to a $\mathscr{C}$-morphism in the sence of [4], i.e., of this paper, as we have seen above.

References

[1] Banica, C. and Stanasila, O., *Algebraic methods in the global theory of complex spaces*, John Wiley & Sons and Editura Academiei, 1976.

[2] Frisch, J., Points de platitude d'un morphisme d'espaces analytiques complexes, *Inventiones math.*, **4** (1967), 118-138.

[3] Fujiki, A., Closedness of the Douady spaces of compact Kähler spaces, *Publ. RIMS, Kyoto Univ.*, **14** (1978), 1-52.

[4] ———, On the Douady space of a compact complex space in the category C, *Nagoya Math. J.*, **85** (1982), 189-211.

[5] Grothendieck, A. and Dieudonné, J., Eléments de géométrie algebrique I, *Publ. Math. IHES*, **4** (1960).

[6] Grothendieck, A., Technique de construction en géométrie analytique, *Seminaire H. Cartan*, 13*e année*, **12**, 1960/61.

[7] Gunning, R.C. and Rossi, H., *Analytic functions of several complex variables*, Prentice Hall, Englewood Cliffs, N.J., 1965.

[8] Rossi, H., Picard varieties of an isolated singular point, *Rice Univ. Studies*, **54** (1968), 63-73.